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***On the Invariants of Finite Nilpotent Groups\****

By Hisasi MORIKAWA

1. If  $x_1, \dots, x_n$  are independent variables over a field  $k$  operated on by a group  $G$  of linear transformations of finite order, and if  $x_1^\sigma, \dots, x_n^\sigma$  are the linear functions into which the variables are changed by a linear transformation of the group, then a rational function  $F(x_1, \dots, x_n)$  of variables is called an invariant of the group, if  $F(x_1, \dots, x_n) = F(x_1^\sigma, x_2^\sigma, \dots, x_n^\sigma)$  for each linear transformation  $\sigma$  of the group. Under what condition is the field of invariants birational? This is a very difficult question for general group  $G$  and field  $k$ .

In the present note, calculating the complete systems of generators of the invariants inductively, we shall prove the birationality of the fields of invariants for the particular type of representations of nilpotent groups. Our result is the following:

**Theorem.** *Let  $G$  be a finite nilpotent group of exponent  $N$  and let  $k$  be a field containing  $N$ -th roots of unity, where we assume that the characteristic of  $k$  is coprime to  $N$ . Let  $G = G_0 > G_1 > \dots > G_{r+1} = \{e\}$  be the descending chain of normal subgroups of  $G$  such that  $G_i/G_{i+1}$  is a cyclic subgroup of  $G/G_{i+1}$  ( $i=0, 1, \dots, r$ ). Let  $\lambda_i$  be the natural homomorphism of  $G$  onto  $G/G_i$  and let  $\{M_i(\sigma)\}$  be the regular representation of  $G/G_i$  ( $i=1, 2, \dots, r+1$ ). Put*

$$M(\sigma) = \underbrace{M_1(\lambda_1(\sigma)) + \dots + M_1(\lambda_1(\sigma))}_{t_1} \oplus \underbrace{M_2(\lambda_2(\sigma)) + \dots + M_2(\lambda_2(\sigma))}_{t_2} \oplus \dots \oplus \underbrace{M_{r+1}(\sigma) + \dots + M_{r+1}(\sigma)}_{t_{r+1}}, \text{ where } t_i \geq 1 \text{ (} i=1, 2, \dots, r+1 \text{)}.$$

*Then the field of invariants of  $\{M(\sigma); \sigma \in G\}$  is birational.*

2. The proof of the theorem.

We shall prove the theorem by the induction on  $r$ . For the case  $r+1=1$  the theorem is evidently true. We assume that it is true for the case the length of the above descending chain of normal subgroups is less than  $r+1$ .

\*) The note was prepared while the author was a Yukawa Fellow at Osaka University.

1) We mean by birationality the purely transcendence over  $k$ .

We denote by  $\alpha_1, \dots, \alpha_{n_r}$  all the elements of  $G_r$  and by  $\bar{\tau}_1, \dots, \bar{\tau}_{m_r}$  all the elements of  $G/G_r$ . We denote by  $[\bar{\tau}_l]$  a representative of  $\bar{\tau}_l$  in  $G$ , where we assume that  $[\bar{e}] = e$ . Let  $\{C_{\bar{\tau}_i, \bar{\tau}_j}\}$  be the factor set of  $G/G_r$  in  $G_r$  which defines  $G$  and let  $\mathcal{X}$  be a generator of the character group of  $G_r$ . Let  $x_1, \dots, x_m$  be the variables on which  $\{M(\sigma) : \sigma \in G\}$  operates, where  $m$  is the degree of  $\{M(\sigma)\}$ . We divide the variables  $\{x_1, \dots, x_m\}$  into two parts  $\{x_1, \dots, x_{m'}\}$  and  $\{x_{m'+1}, \dots, x_m\}$  such that

$$(x_1^\sigma, \dots, x_{m'}^\sigma) = (x_1, \dots, x_{m'}) (M_1(\lambda_1(\sigma)) + \dots + M_r(\lambda_r(\sigma)))$$

and

$$(x_{m'+1}^\sigma, \dots, x_m^\sigma) = (x_{m'+2}, \dots, x_m) (M_{r+1}(\sigma) + \dots + M_r(\sigma)).$$

We denote by  $(\sigma_1, \dots, \sigma_n)$  the system of elements of  $G$  such that  $(\sigma_1\sigma, \dots, \sigma_n\sigma) = (\sigma_1, \dots, \sigma_n) M_{r+1}(\sigma)$ . Put  $x_{1,\sigma_1} = x_{m'+2}, x_{1,\sigma_2} = x_{m'+2}, \dots, x_{1,\sigma_n} = x_{m'+n}, x_{2,\sigma_1} = x_{m'+n+1}, \dots, x_{2,\sigma_n} = x_{m'+2n}, \dots, x_{h,\sigma_1} = x_{m-n+1}, \dots, x_{h,\sigma_n} = x_m$ . Then we have  $x_{i,\sigma_j}^\sigma = x_{i,\sigma_j\sigma}$  ( $i=1, 2, \dots, h; \sigma \in G$ ). We denote by  $k(x_1, \dots, x_{m'})^{G/G_r}$  the subfield of  $k(x_1, \dots, x_{m'})$  consisting of elements which are fixed by all automorphisms induced by  $M_1(\lambda_1(\sigma)) \oplus \dots \oplus M_r(\lambda_r(\sigma))$  ( $\sigma \in G$ ). Then the factor system  $\mathcal{X}(C_{\bar{\tau}_i, \bar{\tau}_j}, k(x_1, \dots, x_{m'}); \bar{\tau}_i, \bar{\tau}_j \in G/G_r)$  defines a normal simple algebra over  $k(x_1, \dots, x_{m'})^{G/G_r}$ . Since this algebra splits in  $k(x_1, \dots, x_{m'})$ , there exists a system  $\{a(\bar{\tau}_l); \bar{\tau}_l \in G/G_r\}$  of elements of  $k(x_1, \dots, x_{m'})$  such that

$$(1) \quad \mathcal{X}(C_{\bar{\tau}_j, \bar{\tau}_l}) = \frac{a(\bar{\tau}_j)^{\bar{\tau}_l} a(\bar{\tau}_l)}{a(\bar{\tau}_j \bar{\tau}_l)}$$

and  $a(\bar{e}) = 1$ .

We put

$$(2) \quad t_{\gamma, i, \sigma} = \sum_{\alpha \in G_r} \mathcal{X}^i(\alpha^{-1}) x_{\gamma, \sigma \alpha},$$

$$(3) \quad u_{\gamma, i, \bar{\tau}_j} = \frac{t_{\gamma, i, [\bar{\tau}_j]}}{t_{1, i, \sigma}}$$

and

$$(4) \quad w_{\gamma, i, \bar{\tau}_j} = \frac{1}{a(\bar{\tau}_j)^i} \frac{t_{\gamma, i, [\bar{\tau}_j]}}{\sum_i \frac{1}{a(\bar{\tau}_l)^i} t_{1, i, [\bar{\tau}_l]}}$$

Then we have

$$(5) \quad (t_{\gamma, i, \sigma})^{\sigma'} = t_{\gamma, i, \sigma \sigma'},$$

$$(6) \quad (t_{\gamma, i, \sigma})^{\alpha_l} = \mathcal{X}^i(\alpha_l) t_{\gamma, i, \sigma},$$

$$(7) \quad (u_{\gamma,i,\bar{\tau}_j})^{\alpha_l} = u_{\gamma,i,\bar{\tau}_j},$$

$$(8) \quad (w_{\gamma,i,\bar{\tau}_j})^{\tau_l} = w_{\gamma,i,\bar{\tau}_j\bar{\tau}_l},$$

$$(9) \quad w_{1,i,\bar{e}} = \frac{1}{\sum \frac{1}{a(\bar{\tau}_l)^i} u_{1,i,\bar{\tau}_l}}$$

$$(10) \quad w_{\gamma,i,\bar{\tau}_j} = \frac{\frac{1}{a(\bar{\tau}_j)^i} u_{\gamma,i,\bar{\tau}_j}}{\sum \frac{1}{a(\bar{\tau}_l)^i} u_{1,i,\bar{\tau}_l}}$$

and

$$(11) \quad u_{\gamma,i,\bar{\tau}_j} = a(\bar{\tau}_j)^i \frac{w_{\gamma,i,\bar{\tau}_j}}{w_{1,i,\bar{e}}}$$

Next, denoting by  $k(x_1, \dots, x_{m'}, x_{1,\sigma_1}, \dots, x_{h,\sigma_n})^{G_r}$  the subfield to  $k(x_1, \dots, x_{m'}, x_{1,\sigma_1}, \dots, x_{h,\sigma_n})$  consisting of elements  $\xi$  such that  $\xi^\alpha = \xi (\alpha \in G_r)$ , we observe that  $k(x_1, \dots, x_{m'}, x_{1,\sigma_1}, x_{h,\sigma_n})^{G_r}$  is generated by monomials

$$t_{\gamma_1,i_1,\sigma_{j_1}} \cdots t_{\gamma_e,i_e,\sigma_{j_e}}$$

satisfying  $i_1 + \dots + i_e \equiv 0 \pmod{n_r}$  over  $k(x_1, \dots, x_{m'})$  and  $k(x_1, \dots, x_{m'}, x_{1,\alpha_1}, \dots, x_{h,\alpha_{n_r}})^{G_r}$  is generated by

$$(12) \quad f_i = t_{1,i,e} t_{1,1,e}^{nr-i} \quad (i = 1, 2, \dots, h)$$

over  $k(x_1, \dots, x_{m'})$ . Hence we have

$$(13) \quad k(x_1, \dots, x_{m'}, x_{1,\sigma_1}, \dots, x_{h,\sigma_n})^{G_r} = k(x_1, \dots, x_{m'}, f_1, \dots, f_{n_r}, u_{1,\bar{\tau}_1}, \dots, u_{h,n_r,\bar{\tau}_{m'}}).$$

Putting

$$(14) \quad v_i = \sum_{j=1}^{m_r} f_i^{\bar{\tau}_j}$$

we observe that  $v_i = (\sum_{j=1}^{m_r} u_{1,i,\bar{\tau}_j} u_{1,1,\bar{\tau}_j}^{nr-i}) f_i$  and

$$(15) \quad k(x_1, \dots, x_{m'}, x_{1,\sigma_1}, \dots, x_{h,\sigma_n}) = k(x_1, \dots, x_{m'}, v_1, \dots, v_{n_r}, u_{1,1,\bar{\tau}_1}, \dots, u_{h,n_r,\bar{\tau}_{m'}}).$$

If we put

$$(16) \quad y_{\gamma,i,\bar{\tau}_j} = w_{\gamma,i,\bar{\tau}_j} \quad \begin{pmatrix} i = 1, 2, \dots, n_r \\ j = 1, 2, \dots, m_r \\ \gamma = 1, 2, \dots, h \end{pmatrix}$$

and

$$(17) \quad y_{1,i,\bar{\tau}_j} = w_{1,i,\bar{\tau}_j} v_i \quad \begin{pmatrix} i = 1, 2, \dots, n_r \\ j = 1, 2, \dots, m_r \end{pmatrix}$$

then we have

$$(18) \quad (y_{\gamma,i,\bar{\tau}_j})^{\bar{\tau}_l} = y_{\gamma,i,\bar{\tau}_j \bar{\tau}_l}$$

and

$$(19) \quad y_{1,i,\bar{e}} = \frac{v_i}{\sum_l \frac{1}{a(\bar{\tau}_l)^i} u_{1,i,\bar{\tau}_l}}$$

Hence we get

$$(20) \quad \sum_l \frac{1}{a(\bar{\tau}_l)^i} u_{1,i,\bar{\tau}_l} = \frac{v_i}{y_{1,i,\bar{e}}}$$

By virtue of (11) we observe that

$$(21) \quad \frac{1}{a(\bar{\tau}_j)^i} u_{\gamma,i,\bar{\tau}_j} = \frac{y_{\gamma,i,\bar{\tau}_j} v_i}{y_{1,i,\bar{e}}} \quad (\gamma = 2, 3, \dots, h)$$

$$(22) \quad \frac{1}{a(\bar{\tau}_j)^i} u_{1,i,\bar{\tau}_j} = \frac{y_{1,i,\bar{\tau}_j}}{y_{1,i,\bar{e}}}$$

and

$$(23) \quad v_i = \sum_l y_{1,i,\bar{\tau}_l}$$

Therefore we can observe that  $y_{1,l,\bar{\tau}_1}, \dots, y_{h,n_r,\bar{\tau}_{m_r}}$  are independent variables over  $k(x_1, \dots, x_{m'})$  and

$$(24) \quad \begin{aligned} & k(x_1, \dots, x_{m'}, x_{1,\sigma_1}, \dots, x_{h,\sigma_n})^{G_r} \\ &= k(x_1, \dots, x_{m'}, v_1, \dots, v_{n_r}, u_{1,l,\bar{\tau}_1}, \dots, u_{h,n_r,\bar{\tau}_{m_r}}) \\ &= k(x_1, \dots, x_{m'}, y_{1,l,\bar{\tau}_1}, \dots, y_{h,n_r,\bar{\tau}_{m_r}}). \end{aligned}$$

Hence, by virtue of (18),  $G/G_r$  operates on  $k(x_1, \dots, x_m)^{G_r}$  similarly as  $G$  on  $k(x_1, \dots, x_m)$ . By virtue of the assumption of the induction, we observe that the subfield  $k((x_1, \dots, x_m)^{G_r})^{G/G_r}$  in  $k(x_1, \dots, x_m)^{G_r}$  consisting of element  $\xi$  such that  $\xi^\sigma = \xi$  ( $\sigma \in G/G_r$ ) is birational. On the other hand  $k(x_1, \dots, x_m)^{G_e}$  is the field of  $G$ . This proves the theorem.

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