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P. HALL'S STRANGE FORMULA FOR ABELIAN p-GROUPS

Dedicated to professor Tsuyoshi Ohyama's 60-th birthday

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1. Introduction

The purpose of this paper is to study some summations over non-ssomorphic abelian p-groups. In particular, for a finite group (or a group which has finitely many solutions of the equation $x^{p^n}=1$ for each $n\geq 1$) G, we study two Dirichlet series as follows:

$$S_{G^p}^{A}(z) := \sum_{A}{}' s(A, G)|A|^{-z},$$
$$H_{G^p}^{A}(z) := \sum_{A}{}' \frac{h(A, G)}{|\mathrm{Aut}\,A|}|A|^{-z},$$

where the summation is taken over a complete set of representatives of isomorphism classes of finite abelian p-groups and

$$s(A, G) := \#\{A_1\leq G \mid A_1\cong A\},$$
$$h(A, G) := |\mathrm{Hom}(A, G)|.$$

(The above series $S_{G^p}^{A}(z)$ and $H_{A^p}^{G}(z)$ are called the zeta functions of Sylow and Frobenius type in the paper [Yo91] because they appeared in the study of Sylow's third theorem and Frobenius' theorem on the number of solutions of the equation $x^n=1$ on a finite group.)

The main theorem states a relation between them:

Theorem 3.1.

$$H_{G^p}^{A}(z)/S_{G^p}^{A}(z) = \prod_{m=1}^{\infty}(1-p^{-m-z})^{-1}.$$

In particular, the left hand side is independent of G.

The proof of this theorem is based on the LDU-decomposition of the Hom-set matrix of the category of finite abelian p-groups and on the generating functions related to the Hom-set matrix. See [Yo 87], [Yo 91].

As a corollary, we get P. Hall's strange formula ([Ha 38]):

Corollary 4.4.

$$\sum_A{}' \frac{1}{|A|} = \sum_A{}' \frac{1}{|\mathrm{Aut}\, A|},$$

where the summation is taken over all non-isomorphic abelian p-groups.

In his paper [Ha 38], P. Hall proved this formula as follows: Since the number of isomorphism classes of abelian groups of order p^n equals the partition number $p(n)$,

$$\sum_A{}' \frac{1}{|A|} = \sum_{n=0}^{\infty} \frac{p(n)}{p^n} = \frac{1}{f_\infty(1/p)}.$$

Here for $0 \leq n \leq \infty$, we define

$$f_n(x) := (1-x)(1-x^2)\cdots(1-x^n).$$

Thus an identity of Euler ([An 76,] Corollary 2.2) gives

$$\sum_A{}' \frac{1}{|A|} = \sum_{n=0}^{\infty} \frac{(1/p)^n}{f_n(1/p)}. \tag{1}$$

Let A be an abelian group of order p^n and of type

$$(1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n}),$$

that is, A is the direct product of λ_1 cyclic groups of order p, λ_2 cylic groups of order p^2, and so on. Let

$$\mu_i := \lambda_i + \lambda_{i+1} + \cdots,$$

so that

$$\mu_1 \geq \mu_2 \geq \cdots \geq 0, \quad \mu_1 + \mu_2 + \cdots = n.$$

Then the order of the automorphism group of A is given by

$$|\mathrm{Aut}\, A| = \prod_{i \geq 1} f_{\mu_i - \mu_{i+1}}(1/p)\, p^{{\mu_i}^2}.$$

Thus in order to prove Corollary 4.4, it will suffice to show that

$$\frac{x^n}{f_n(x)} = \sum_{\Sigma \mu_i = n} \frac{x^{{\mu_1}^2 + {\mu_2}^1 + \cdots}}{f_{\mu_1 - \mu_2}(x) f_{\mu_1 - \mu_{3(x)\cdots}}(x) \cdots}, \tag{2}$$

where the summation is taken over all $(\mu) = (\mu_1, \mu_2, \cdots)$ such that $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ and $\mu_1 + \mu_2 + \cdots = n$.

P. Hall proved (2) by a combinatorial method in his paper ([Ha 38]). His formula (3) implies another strange formula:

Corollary 4.3. *For any $n \geq 0$,*

$$\sum_{|A|=p^n}' \frac{1}{|\mathrm{Aut}\, A|} = \sum_{\mathrm{rk}(A)=n}' \frac{1}{|A|}, \tag{3}$$

where the summations are taken over all non-isomorphic abelian p-groups of order p^n, resp. of rank n.

In fact, the RHS of (3) equals the n-th term of the RHS of (1).

As a more general corollary of the main theorem, we have the following identity, which is not found in P. Hall's papers [Ha 38], [Ha 40]:

Corollary 4.2. *Let C be an abelian group of order p^n. Then*

$$\sum_{|A|=p^m}' \frac{|\mathrm{Epi}(C, A)|}{|\mathrm{Aut}\, A|} = \sum_{\mathrm{rk}(A)=m-n}' \frac{1}{|A|}, \tag{4}$$

where Epi(C,A) *denotes the set of epimorphic homomorphisms from C to A.*

We can develop a similar theory for the summations on any category of groups which is closed under subgroups and homomorphic images. Furthermore, in the case of elementary abelian p-groups, it is related with a topological property of p-subgroup complex of a finite group G. See [Yo 91].

I would like to thank the referee of this paper for reading carefully it and pointing out many errors.

2. The Hom-set matrix

In this section, we study the relation between the number of homomorphisms from abelian p-groups into a finite group G and the number of abelian p-subgroups in G. The categroy of finite groups has the unique epi-mono-factorization property, and so we can apply the method of [Yo 87] to this category

Let A,B,G be finite groups, and define

$$\begin{aligned} h(A, G) &:= |\mathrm{Hom}(A, G)|, \\ s(A, G) &:= \sharp\{H \leq G \mid H \simeq A\}, \\ q(A, B) &:= \sharp\{A' \trianglelefteq A \mid A/A' \simeq B\}, \\ d(A, B) &:= \begin{cases} |\mathrm{Aut}\, A| & \text{if } A \simeq B \\ 0 & \text{otherwise.} \end{cases}, \end{aligned}$$

where Aut A denotes the automorphism group of A. Then $q(A, B) \cdot |\mathrm{Aut}\, B|$ (resp. $|\mathrm{Aut}\, A| \cdot s(A, B)$) equals the number of epimorphic (resp. monomorphic) homomorphisms from A to B. We view these families of integers

$(h(A,G))$, $(s(A,G))$, $(q(A,G))$, $(d(A,B))$ as matrices H, A, Q, D, respectively, indexed by isomorphism classes of finite groups.

Lemma 2.1. *$H=QDS$ as matrices, that is,*

$$|\mathrm{Hom}(A,G)| = \sum_C{}' \#\{A'\trianglelefteq A \mid A/A'\cong C\}\cdot|\mathrm{Aut}\, C|\cdot s(C,G)\,, \tag{1}$$

where Σ' denotes the summation over a set of complete representatives of isomorphism classes of finite groups.

Proof. Since each homomorphism from A to G induces a unique epimorphism from A onto its image in G, we have that

$$|\mathrm{Hom}(A,G)| = \sum_{H\leq G} |\mathrm{Epi}(A,H)|\,,$$

where $\mathrm{Epi}(A,H)$ denotes the set of epimorphisms from A to H. By the homomorphism theorem,

$$|\mathrm{Epi}(A,H)| = \#\{B\trianglelefteq A \mid A/B\cong H\}\cdot|\mathrm{Aut}\, H| = q(A,H)\,d(H,H)\,,$$

and hence

$$|\mathrm{Hom}(A,G)| = \sum_C{}' \#\{A'\trianglelefteq A \mid A/A'\cong C\}\cdot|\mathrm{Aut}\, C|\cdot s(C,G)\,,$$

as required. □

REMARK. Arranging the isomorphism classes of finite groups in order of the orders, Q(resp. S) makes a lower (resp. upper) uni-triangular matrix. Thus this lemma give an LDU-decomposition of the hom-set matrix H. Similar decomposition holds for any locally finite category with unique epi-mono factorization property. See [Yo 87].

Let p be a prime. We are mainly interested in abelian-groups. First of all, we introduce the *Möbius function* of abelain p-groups:

$$\mu(B) := \begin{cases} (-1)^r p^{\binom{r}{2}} & \text{if } B\cong C_p^r \\ 0 & \text{else.}\end{cases} \tag{2}$$

Then the following lemma is well-known:

Lemma 2.2. *For an abelian p-group A,*

$$\sum_{B\leq A} \mu(B) = \begin{cases} 1 & \textit{if } A=1 \\ 0 & \textit{else.}\end{cases}$$

For the proof, refer to P. Hall [Ha 36] (2.7), [St 86] Example 3.10.2, [Mc 79], p. 97. Note that it suffices to prove it only for an elementary abelian group A. We can now prove the inversion formula.

Proposition 2.3. *Let A be a p-group and G a finite group. Then*

$$s(A, G)=\frac{1}{|\mathrm{Aut}\, A|}\sum_{B\leq A}\mu(B)h(A/B, G)\,. \tag{3}$$

Proof. By lemma 2.1 we have that

$$\begin{aligned}\mathrm{RHS}&=\sum_{B\leq A}\frac{\mu(B)}{|\mathrm{Aut}\, A|}\sum_{C}{}' q(A/B, C)\,|\mathrm{Aut}\, C\,|\,s(C, G)\\&=\sum_{C}{}'\frac{|\mathrm{Aut}\, C\,|}{|\mathrm{Aut}\, A\,|}\Big(\sum_{B\leq A}\mu(B)\, q(A/B, C)\Big)\, s(C, G)\,,\end{aligned}$$

where Σ' denotes the summation over a complete set of representatives of all finite abelian p-gropus. Thus it will suffice to show that

$$\sum_{B\leq A}\mu(B)\, q(A/B, C)=\begin{cases}1 & \text{if } A\cong C\\ 0 & \text{else.}\end{cases} \tag{4}$$

But this is proved as follows:

$$\begin{aligned}\mathrm{LHS}&=\sum_{B\leq A}\mu(B)\cdot\#\{D/B\leq A/B\,|\,A/D\cong C\}\\&=\sum_{\substack{F\leq A\\ :A/D\cong C}}\Big(\sum_{B\leq D}\mu(B)\Big)\\&=\begin{cases}1 & \text{if } A\cong C\\ 0 & \text{else}\end{cases}\end{aligned}$$

by Lemma 2.2, as required. Hence the proposition was proved. □

3. Zeta functions of Sylow and Frobenius type

Throughout this section, G denotes a finite group, and $A, B, C\cdots$ denote abelian p-groups. Furthermore, Σ' means a summation over a complete set of isomorphism classes of abelian p-groups. As before, let $s(A, G)$ be the number of subgroups of G isomorphic to A, and let $h(A, G)$ be the number of homomorphisms from A to G.

For any abelian p-groups A, B, C, we define the *Hall polynomial* $g^A_{B,C}$ by

$$g^A_{B,C} := \#\{B_i\leq A\,|\,B_1\cong B,\, A/B_1\cong C\}\,.$$

See [Mc 79] for the general theory of Hall polynomials.

We now define the zeta functions of Sylow and Frobenius types as follows:

$$S^{Ap}_G(z) := \sum_{A}{}' s(A, G)\,|A|^{-z}\,, \tag{1}$$

$$H^{Ap}_G(z) := \sum_{A}{}'\frac{h(A, G)}{|\mathrm{Aut}\, A|}|A|^{-z}\,, \tag{2}$$

where, of course, A runs over a complete set of representatives of isomorphism classes of abelian p-groups. Clearly,

$$S_G^{Ap}(z) \in \boldsymbol{Z}[p^{-z}], \quad H_G^{Ap}(z) \in \boldsymbol{Q}[[p^{-y}]] .$$

The following theorem is the main theorem of this paper:

Theorem 3.1. *For any finite group G,*

$$\frac{H_G^{Ap}(z)}{S_G^{Ap}(z)} = \prod_{m=1}^{\infty} (1-p^{-m-z})^{-1} \qquad (\mathrm{Re}\, z > -1) . \tag{3}$$

In particular, the left hand side is independent of the finite group G.

To prove this theorem, we need the following lemma due to P. Hall [Ha 40] (10). The proof given here is different from the one by P. Hall. See Corollary 4.5.

Lemma 3.2. (P.Hall). *For any finite abelian p-group B, C,*

$$\sum_A{}' \frac{1}{|\mathrm{Aut}\, A|} g_{B,C}^A = \frac{1}{|\mathrm{Aut}\, B| \cdot |\mathrm{Aut}\, C|} , \tag{4}$$

where A runs over non-isomorphic abelian p-groups.

Proof. Define a set

$$\mathcal{E}_A(C, B) := \{(f, g) \mid 1 \to B \xrightarrow{f} A \xrightarrow{g} C \to 1\}$$

and a map

$$\begin{aligned} \pi &: \mathcal{E}_A(C, B) \to \{B_1 \leq A \mid A/B_1 \cong C\} \\ &: \quad (f, g) \quad \to f(B). \end{aligned}$$

Clearly, π is surjective, and if $\pi(f, g) = \pi(f', g')$, then there exist unique $\tau \in \mathrm{Aut}\, B$ and $\rho \in \mathrm{Aut}\, C$ such that $f' = f\tau$, $g' = \rho g$. Thus the cardinality of each fiber equals $|\mathrm{Aut}\, B| \cdot |\mathrm{Aut}\, C|$, and so

$$g_{B,C}^A = \frac{|\mathcal{E}_A(C, B)|}{|\mathrm{Aut}\, B| \cdot |\mathrm{Aut}\, C|} . \tag{5}$$

Next we let Aut A act on $\mathcal{E}_A(C, B)$ by

$$\sigma \cdot (f, g) := (\sigma f, g\sigma^{-1})$$

for $\sigma \in \mathrm{Aut}\, A$, $(f, g) \in \mathcal{E}_A(C, B)$. There exists a bijective correspondence between the stabilizer of (f, g) and the group homomorphisms:

$$(\mathrm{Aut}\, A)_{(f,g)} \leftrightarrow \mathrm{Hom}\,(C, B) ,$$

by $\sigma\leftrightarrow\eta$, where $a\cdot f(\eta(g(a)))=\sigma(a)$, and so the cardinality of each Aut A-orbit equals

$$|\mathrm{Aut}\,A|/|\mathrm{Hom}(C,B)|\ .$$

Thus

$$|\mathcal{E}_A(C,B)|=\frac{|\mathrm{Aut}\,A|}{|\mathrm{Hom}(C,B)|}\cdot|\mathcal{E}_A(C,B)/\mathrm{Aut}\,A|\ . \tag{6}$$

On the other hand, by the definition of the equivalence of module-extensions, we have that

$$|\mathrm{Ext}^1(C,B)|=\sum_A{}'\ |\mathcal{E}_A(C,B)/\mathrm{Aut}\,A| \tag{7}$$

By (5), (6), (7), we have that

$$\sum_A{}'\frac{1}{|\mathrm{Aut}\,A|}g^A_{B,C}=\frac{|\mathrm{Ext}^1(C,B)|}{|\mathrm{Aut}\,B|\cdot|\mathrm{Aut}\,C|\cdot|\mathrm{Hom}(C,B)|}\ .$$

It remains to prove that

$$|\mathrm{Ext}^1(C,B)|\ =\ |\mathrm{Hom}(\mathrm{C},B)|. \tag{8}$$

Since there exists an exact sequence $1\to F\to F\to C\to 1$, where F is a finitely generated free abelian group, the long exact seqeunce gives an exact sequence

$$\begin{aligned}0\to\mathrm{Hom}(C,B)\to\mathrm{Hom}(F,B)\to\mathrm{Hom}(F,B)\\ \to\mathrm{Ext}^1(C,B)\to\mathrm{Ext}^1(F,B)=0\,,\end{aligned}$$

which implies that $|\mathrm{Ext}^1(C,B)|=|\mathrm{Hom}(C,B)|$. This proves the lemma. □

Lemma 3.3. *Let $\mu(B)$ denote the Mobius function of finite abelian p-groups. Then*

$$\sum_B{}'\frac{\mu(B)}{|\mathrm{Aut}\,B|}|B|^{-z}=\prod_{m=1}^{\infty}(1-p^{-m-z}),\qquad \mathrm{Re}\,z>-1\,, \tag{9}$$

where B runs over non-ssomorphic abelian p-groups.

Proof. Since

$$\mu(B)=\begin{cases}(-1)^n p^{\binom{n}{2}} & \text{if } B\cong C_p^n\\ 0 & \text{if } B \text{ is not elementary abelian,}\end{cases}$$

we have

$$\begin{aligned}\mathrm{LHS}&=\sum_{u=0}^{\infty}\frac{(-1)^n p^{\binom{n}{2}}}{|\mathrm{GL}(n,p)|}p^{-nz}\\ &=\sum_{n=0}^{\infty}\frac{(-p^{-z})^n}{(p-1)(p^2-1)\cdots(p^n-1)}\end{aligned}$$

Thus the required identity follows from the q-binomial theorem ([An 76], Theorem 2.1):

$$\sum_{n=0}^{\infty} \frac{x^n}{(p-1)(p^2-1)\cdots(p^n-1)} = \prod_{m=1}^{\infty} (1+p^{-m}x), \quad |x|<p. \tag{10}$$

□

Proof of Theorem 3.1. The inversion formula Proposition 2.3 gives

$$\begin{aligned} s(A, G) &= \frac{1}{|\mathrm{Aut}\, A|} \sum_{B\le A} \mu(B)h(A/B, G) \\ &= \frac{1}{|\mathrm{Aut}\, A|} \sum_{B,C}{}' \mu(B)h(C, G)g^A_{B,C}. \end{aligned}$$

Thus

$$\begin{aligned} S_G^{A_p}(z) &= \sum_A{}' s(A, G)|A|^{-z} \\ &= \sum_{A,B,C}{}' \frac{\mu(B)h(C, G)}{|\mathrm{Aut}\, A|} g^A_{B,C}|A|^{-z} \\ &= \sum_{B,C}{}' \left(\sum_A{}' \frac{g^A_{B,C}}{|\mathrm{Aut}\, A|}\right) \mu(B)h(C, G)|B|^{-z}|C|^{-z}. \end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned} S_G^{A_p}(z) &= \sum_{B,C}{}' \frac{1}{|\mathrm{Aut}\, B|\cdot|\mathrm{Aut}\, C|} \mu(B)h(C, G)|B|^{-z}|C|^{-z} \\ &= \left(\sum_B{}' \frac{\mu(B)}{|\mathrm{Aut}\, B|}|B|^{-z}\right)\cdot\left(\sum_C{}' \frac{h(C, G)}{|\mathrm{Aut}\, C|}|C|^{-z}\right). \end{aligned}$$

By lemma 3.3, we conclude that

$$S_G^{A_p}(z) = \prod_{m=1}^{\infty} (1-p^{-m-z})\cdot H_G^{A_p}(z).$$

This proves the theorem. □

4. Corollaries

Theorem 3.1 implies some identities as a special cases.

Corollary 4.1. *Let $\mathcal{A}_p(G)$ denote the set of abelian p-subgroups of a finite group G. Then*

$$\frac{1}{|\mathcal{A}_p(G)|} \sum_A{}' \frac{h(A, G)}{|\mathrm{Aut}\, A|} = \sum_A{}' \frac{1}{|A|}, \tag{1}$$

where the summation is over non-isomorphic abelian p-groups.

Proof. Put $z=0$ in Theorem 3.1. $\square$

Corollary 4.2. *Let C be an abelian group of order p^n. Then for any $m\geq 0$,*

$$\sum_{|A|=p^m}' \frac{|\mathrm{Epi}(C,A)|}{|\mathrm{Aut}\,A|} = \sum_{\mathrm{rk}(A)=m-n}' \frac{1}{|A|},$$

where Epi(C, A) *denotes the set of epimorphic homomorphisms from C to A.*

Proof. By Lemma 2.1, we have

$$H_G(z) = \sum_A' \frac{h(A,G)}{|\mathrm{Aut}\,A|}|A|^{-z} = \sum_C' \sum_A' \frac{|\mathrm{Epi}(A,C)|}{|\mathrm{Aut}\,A|}|A|^{-z}s(C,G). \tag{2}$$

The following formula is well-known ([An 76]), (2.1.1)):

$$\prod_{m=1}^{\infty}(1-xp^{-m})^{-1} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p(k,n)p^{-k}x^n$$

$$= \sum_A' \frac{1}{|A|}x^{\mathrm{rk}(A)}, \tag{3}$$

where $p(k,n)$ denotes the number of partitions of k into n parts and, of course, it is equal to the number of non-siomorphic abelian p-groups of order p^k and of rank n. Thus Theorem 3.1 yields that

$$H_G(z) = S_G(z)\cdot\prod_{m=1}^{\infty}(1-p^{-z-m})^{-1}$$

$$= \sum_C' s(C,G)|C|^{-z}\cdot\sum_A' \frac{1}{|A|}p^{-z\,\mathrm{rk}(A)}$$

$$= \sum_C' \left(\sum_A' \frac{1}{|A|}p^{-z\,\mathrm{rk}(A)}\right)|C|^{-z}s(C,G). \tag{4}$$

Thus comparing (2) and (4), an easy induction argument yields that

$$\sum_A' \frac{|\mathrm{Epi}(A,C)|}{|\mathrm{Aut}\,A|}|A|^{-z} = \sum_A' \frac{1}{|A|}p^{-z\cdot\mathrm{rk}(A)}|C|^{-z}. \tag{5}$$

We now assume that $|C|=p^n$. Then the right hand side of (5) is

$$\mathrm{RHS} = \sum_A' \frac{1}{|A|}p^{-z\cdot(\mathrm{rk}(A)+n)}$$

$$= \sum_{m=0}^{\infty}\sum_{\mathrm{rk}(A)=m-n}' \frac{1}{|A|}.$$

Hence

$$\sum_{|A|=p^m}' \frac{|\mathrm{Epi}(A,C)|}{|\mathrm{Aut}\,A|} = \sum_{\mathrm{rk}(A)=m-n}' \frac{1}{|A|},$$

as required. □

As a special case of this corollary, we have the following identites (refer to [Ha 40], (7)).

Corollary 4.3. *For any $n \geq 0$,*

$$\sum_{|A|=p^n}' \frac{1}{|\mathrm{Aut}\, A|} = \sum_{\mathrm{rk}(A)=n}' \frac{1}{|A|}, \tag{6}$$

where the summations are taken over all non-isomorphic abelian p-groups of order p^n, resp. of rank n.

Corollary 4.3 implies P. Hall's strange tormula ([Ha 38]):

Corollary 4.4 (P.Hall): $\sum_A' \frac{1}{|\mathrm{Aut}\, A|} = \sum_A' \frac{1}{|A|}$.

The final corollary is the fundamental theorem of finite abelian groups.

Corollary 4.5. *Any finite abelian p-group is a direct product of soem cyclic groups.*

Proof. The formula which P. Hall proved in his paper [Ha 38] is

$$\sum_{A:\,\mathrm{known}}' \frac{1}{|\mathrm{Aut}\, A|} = \prod_{m=1}^{\infty} (1-p^{-m})^{-1} = \sum_{A:\,\mathrm{known}}' \frac{1}{|A|},$$

where A runs over all non-isomorphic abelian p-groups which are direct products of some cyclic groups. On the other hand, the formula proved in this section is just

$$\sum_{A:\,\mathrm{all}}' \frac{1}{|\mathrm{Aut}\, A|} = \sum_{A:\,\mathrm{known}}' \frac{1}{|A|}, \tag{7}$$

where in the first summation, A runs over *all* non-isomorphic abelain p-groups. Comparing the above two formulae, we conclude that all abelian p-groups are known. □

REMARK. In the proof of (7), we used only the property that a finite abelian group A has a free resolution of the type

$$1 \to F \to F \to A \to 1.$$

See (8) in the proof of Lemma 3.2.

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