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YANG-MILLS CONNECTIONS AND MODULI SPACE

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0. Introduction

There are many researches on deformations of Yang-Mills connections over 4-dimensional manifolds. In this paper, we generalize the results into higher dimensional cases. In 4-dimensional case, the Hodge $*$ -operator acts on $\Lambda^2 M$ and the notion of (anti-)self dual connection is introduced, which brings beautiful results in Atiyah, Singer and Hitchin [1]. Therefore, in higher dimensional cases, we have to assume some properties on the base riemannian manifold. In [1], it is already pointed out that if M is a (2-dimensional) complex manifold, an anti-self dual connection defines a holomorphic structure of the bundle. Itoh [6] considers in full this situation, which we will generalize by the notion of "Einstein holomorphic connection" over a Kähler manifold. However, the moduli space of Yang-Mills connections over a higher dimensional Kähler manifold may have many singularities, and probably we can not expect that the moduli space becomes a manifold.

The fundamental notions in this paper come from [1], and fundamental idea comes from Koiso [9]. It is remarkable that the results for the moduli space of Einstein metrics and that of Yang-Mills connections are quite analogous. In fact we will get the following results.

Theorem 2.7 (c.f. [9, Theorem 3.1]). *The local pre-moduli space is a finite dimensional real analytic set.*

Corollary 6.5 (c.f. [9, Theorem 10.5]). *If the initial structure (Einstein metric or Yang-Mills connection) is compatible with a complex structure, then also around structures are compatible with some complex structures.*

Theorem 9.3 (c.f. [9, Theorem 12.3]). *Under some assumption, the local pre-moduli space has a canonical Kähler structure.*

However, there is an important difference. For Einstein metrics, we have no effective obstruction spaces for deformation ([9, Proposition 5.4]), but for Yang-Mills connections we have one (Theorem 6.9).

1. Yang-Mills connections

Let (M, g) be a compact riemannian manifold, G a compact Lie group, P a principal G -bundle over M . Denote by \mathfrak{g} the Lie algebra of G and by G_P (resp. \mathfrak{g}_P) the associated fiber bundle $P \times_{\text{Ad}_G} G$ (resp. $P \times_{\text{Ad}_G} \mathfrak{g}$). The space \mathcal{C} of all connections of P is an affine space whose standard vector space is $C^\infty(\Lambda^1 \otimes \mathfrak{g}_P)$, where Λ^p denotes the vector bundle of p -forms on M (see [1, p430]). We fix an effective representation $G \rightarrow GL(V)$ and identify a connection of P with a covariant derivation on $P \times_G V$ or $P \times_G \text{End}(V)$. In this sence, for a connection ∇ of P and an element A of $C^\infty(\Lambda^1 \otimes \mathfrak{g}_P)$ the curvature tensor is transformed as

$$(1.0.1) \quad R^{\nabla+A} = R^\nabla + d^\nabla A + [A \wedge A],$$

where d^∇ and $[\cdot \wedge \cdot]$ are defined by

$$(1.0.2) \quad (d^\nabla A)(X, Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X)$$

and

$$(1.0.3) \quad [A \wedge B](X, Y) = \frac{1}{2} ([A(X), B(Y)] - [A(Y), B(X)]).$$

We fix a G -invariant inner product on \mathfrak{g} . Then the vector bundle \mathfrak{g}_P admits a canonical fiber inner product (\cdot, \cdot) and the vector space $C^\infty(\mathfrak{g}_P)$ admits a (global) inner product $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|$ the L_2 -norm defined by $\langle \cdot, \cdot \rangle$. Define an action integral F_{YM} for connections by

$$(1.0.4) \quad F_{YM}(\nabla) = \frac{1}{2} \|R^\nabla\|^2.$$

DEFINITION 1.1. The function F_{YM} on \mathcal{C} is called *the Yang-Mills functional*, its Euler-Lagrange equation is called *the Yang-Mills equation* and its solution is called a *Yang-Mills connection*.

Let us represent the Yang-Mills equation by a tensor equation. Let ∇_t be a 1-parameter family of connections on P and set $\nabla = \nabla_0$ and $A = (d/dt)_0 \nabla_t$. Then

$$\begin{aligned} \frac{d}{dt} \Big|_0 F_{YM}(\nabla_t) &= \left\langle \frac{d}{dt} \Big|_0 R^{\nabla_t}, R^\nabla \right\rangle \\ &= \langle d^\nabla A, R^\nabla \rangle = \langle A, (d^\nabla)^* R^\nabla \rangle \\ &= 2 \langle A, \delta^\nabla R^\nabla \rangle, \end{aligned}$$

where $(\cdot)^*$ denotes the formal adjoint and the operator δ^∇ from $C^\infty(\Lambda^p \otimes \mathfrak{g}_P)$ to $C^\infty(\Lambda^{p-1} \otimes \mathfrak{g}_P)$ is defined by

$$(1.1.1) \quad (\delta^\nabla s)_{i_1 \dots i_{p-1}} = -\nabla^l s_{li_1 \dots i_{p-1}}.$$

Thus Yang-Mills equation becomes

$$(1.1.2) \quad E_{YM}(\nabla) \equiv \delta^\nabla R^\nabla = 0.$$

Next, we consider infinitesimal deformations of a Yang-Mills connection. From now on, we enlarge the space of C^∞ -sections to the space of H^s -sections, and denote by $H^s(E)$ the space of all H^s -sections of a fiber bundle E over M , where s is assumed to be sufficiently large. H^s -norm will be denoted by $\|\cdot\|_s$. The completion of the space \mathcal{C} etc. with respect to H^s -topology will be denoted by \mathcal{C}^s etc.

DEFINITION 1.2. Let ∇ be a Yang-Mills connection. A solution of the equation

$$(1.2.1) \quad (E_{YM})'_\nabla(A) = 0$$

is called a *Yang-Mills infinitesimal deformation*, where $'$ denotes the Fréchet derivative. The space of all Yang-Mills H^s -infinitesimal deformations is denoted by $YMID^s(\nabla)$.

Lemma 1.3. $(E_{YM})'_\nabla(A) = \delta^\nabla d^\nabla A + \text{tr}[R^\nabla, A],$

where $\text{tr}[R^\nabla, A]_i = g^{kl}[R^\nabla_{ki}, A_l]$.

Proof.
$$\begin{aligned} \frac{d}{dt} \Big|_0 (\delta^\nabla_t R^\nabla_t)_i &= - \frac{d}{dt} \Big|_0 \nabla'_i R^\nabla_t{}^i \\ &= -[A^t, R^\nabla_t{}^i] - \nabla^t \left(\frac{d}{dt} \Big|_0 R^\nabla_t{}^i \right), \end{aligned}$$

where $A = \frac{d}{dt} \Big|_0 \nabla_t$.

Q.E.D.

The automorphism group $\mathcal{G} = C^\infty(G_P)$ of the bundle P is called the gauge group of P , and it acts on \mathcal{C} by pull-back as

$$(1.3.1) \quad \gamma^* \nabla = \nabla + \gamma^{-1} \nabla \gamma \quad (\gamma \in \mathcal{G}, \nabla \in \mathcal{C}).$$

If ∇ is a Yang-Mills connection, then $\gamma^* \nabla$ is so. In particular, if γ_t is a 1-parameter family of gauge transformations such that $\gamma_0 = \text{id}_P$, then $(d/dt)_0 \gamma_t^* \nabla = \nabla((d/dt)_0 \gamma_t)$ becomes a Yang-Mills infinitesimal deformation of ∇ .

DEFINITION 1.4. Let ∇ be a Yang-Mills connection. A Yang-Mills infinitesimal deformation is said to be *trivial* if it coincides with ∇v for some $v \in H^{s+1}(\mathfrak{g}_P)$. A Yang-Mills infinitesimal deformation is said to be *essential* if it is orthogonal to all trivial Yang-Mills infinitesimal deformations. The space of all essential Yang-Mills infinitesimal deformations is denoted by $YMEID(\nabla)$.

By definition, a Yang-Mills infinitesimal deformation $A \in H^s(\Lambda^1 \otimes \mathfrak{g}_P)$ is essential if and only if $\langle \nabla v, A \rangle = 0$ for any $v \in H^{s+1}(\mathfrak{g}_P)$, which is equivalent to that $\delta^\nabla A = 0$. Thus the defining equation of the space $YMEID(\nabla)$ becomes

$$(1.4.1) \quad \delta^\nabla d^\nabla A + \text{tr}[R^\nabla, A] = 0$$

and

$$(1.4.2) \quad \delta^\nabla A = 0.$$

This system is elliptic, and so the space $YMEID(\nabla)$ is finite dimensional and each element is C^∞ .

The following lemma will be used later.

Lemma 1.5. *For any connection ∇ , equality*

$$(1.5.1) \quad \delta^\nabla E_{YM}(\nabla) = 0$$

and decomposition

$$(1.5.2) \quad H^s(\Lambda^1 \otimes \mathfrak{g}_P) = \text{Im}(\nabla | H^{s+1}) \oplus \text{Ker } \delta^\nabla$$

hold. If ∇ is a Yang-Mills connection, then the sequence

$$(1.5.3) \quad C^\infty(\mathfrak{g}_P) \xrightarrow{\nabla} C^\infty(\Lambda^1 \otimes \mathfrak{g}_P) \xrightarrow{(E_{YM})'_\nabla} C^\infty(\Lambda^1 \otimes \mathfrak{g}_P) \xrightarrow{\delta^\nabla} C^\infty(\mathfrak{g}_P)$$

is an elliptic complex. In particular, the following decompositions as Hilbert space hold.

$$(1.5.4) \quad H^s(\Lambda^1 \otimes \mathfrak{g}_P) = \text{Im}(E_{YM}' | H^{s+2}) \oplus YMEID \oplus \text{Im}(\nabla | H^{s+1}),$$

$$(1.5.5) \quad \text{Ker}(E_{YM}' | H^s) = YMEID \oplus \text{Im}(\nabla | H^{s+1}),$$

$$(1.5.6) \quad \text{Ker}(\delta^\nabla | H^s) = YMEID \oplus \text{Im}(E_{YM}' | H^{s+2}).$$

Proof. Equality (1.5.1) is easy to check directly, but here we show it using an idea from variation. Since the function F_{YM} on \mathcal{C} is invariant under the action of the group \mathcal{Q} , we see that

$$(F_{YM})'_\nabla(\nabla v) = 0 \quad \text{for any } v \in C^\infty(\mathfrak{g}_P),$$

$$\text{i.e.,} \quad \langle E_{YM}(\nabla), \nabla v \rangle = 0,$$

which implies (1.5.1). Decomposition (1.5.2) follows from Lemma 13.1. Let ∇ be a Yang-Mills connection. Then the space $\text{Im} \nabla$ is the space of trivial infinitesimal deformations, hence $(E_{YM})'_\nabla \circ \nabla = 0$. From equality (1.5.1) we derive the equality

$$(\delta^\nabla)' \cdot E_{YM}(\nabla) + \delta^\nabla (E_{YM})'_\nabla = 0.$$

Thus sequence (1.5.3) is a complex, and its ellipticity is easy to check. Therefore we have decomposition

$$(1.5.7) \quad H^s(\Lambda^1 \otimes \mathfrak{g}_P) = \text{Im}(\nabla | H^{s+1}) \oplus \text{Im}((E_{YM})'_{\nabla} | H^{s+2}) \\ \oplus \text{Ker } \nabla^* \cap \text{Ker}(E_{YM})'_{\nabla}.$$

Here we have $\nabla^* = \delta^{\nabla}$ and so we get (1.5.4) if we show that $(E_{YM})'_{\nabla}$ is self-adjoint. But we see

$$\langle (E_{YM})'_{\nabla}(A), B \rangle = (\text{Hess } F_{YM})(A, B),$$

regarding F_{YM} as a function on C^s , hence $(E_{YM})'_{\nabla}$ is symmetric with respect to $\langle \cdot, \cdot \rangle$. Since the space $\text{Im}(\nabla | H^{s+1})$ is closed in $H^s(\Lambda^1 \otimes \mathfrak{g}_P)$, decomposition (1.5.5) is reduced to the definition of the space $YMEID(\nabla)$. If we remark that the space $\text{Ker}(\delta^{\nabla} | H^s)$ is the orthogonal complement of the space $\text{Im}(\nabla | H^{s+1})$, then (1.5.6) follows from (1.5.4). Q.E.D.

2. Moduli space of Yang-Mills connections

To define ‘‘local pre-moduli space’’, we need some preparation. We use some basic facts on C^{ω} -maps in Hilbert space category. (See Lemmas 13.2, 13.3.)

Remark that the space $\mathcal{G}^s = H^s(G_P)$ is a C^{ω} - (infinite dimensional) Lie group. In fact, if we take a complexification G^C of G and set $G_{\mathbb{F}} = P \times_{\text{Ad}_G} G^C$, then to multiply and to get inverse element are extended to maps: $H^s(G_{\mathbb{F}}) \times H^s(G_{\mathbb{F}}) \rightarrow H^s(G_{\mathbb{F}})$, $H^s(G_{\mathbb{F}}) \rightarrow H^s(G_{\mathbb{F}})$ so that the restriction on each fiber is holomorphic. Therefore, by Lemma 13.3, they are C^{ω} .

Let ∇ be a connection and \mathcal{G}_{∇}^s the group of isotropy, i.e., $\mathcal{G}_{\nabla}^s = \{\gamma \in \mathcal{G}^s; \nabla \gamma = 0\}$ (see (1.3.1)). Since ∇ is an elliptic operator, we see that $\mathcal{G}_{\nabla}^s = \mathcal{G}_{\nabla}^{\infty}$ and so we simply denote it by \mathcal{G}_{∇} . The exponential map $\exp: \mathfrak{g} \rightarrow G$ defines a C^{ω} -map $\exp^s: H^s(\mathfrak{g}_P) \rightarrow \mathcal{G}^s$ (by Lemma 13.3) and we can easily check that the quotient space $\mathcal{G}_{\nabla} \backslash \mathcal{G}^s$ admits a C^{ω} -structure and that there exists a local cross section $\chi^s: \mathcal{G}_{\nabla} \backslash \mathcal{G}^s \rightarrow \mathcal{G}^s$ so that the domain U^s is uniform on s , i.e., equations $U^{s+1} = (\mathcal{G}_{\nabla} \backslash \mathcal{G}^{s+1}) \cap U^s$ and $\chi^{s+1} = \chi^s | U^{s+1}$ hold for any s . Define a C^{ω} -map

$$\mathcal{A}^s: U^{s+1} \times (\nabla + \text{Ker}(\delta^{\nabla} | H^s)) \rightarrow C^s$$

by
$$\mathcal{A}^s(u, \nabla_1) = \chi^{s+1}(u) * \nabla_1.$$

Its derivative at $([\text{id}], \nabla)$ is given by

$$(v, A) \rightarrow \nabla(\chi'_{[\text{id}]}(v)) + A,$$

and is bijective by decomposition (1.5.2). Therefore there exists a local inverse map $(\mathcal{A}^s)^{-1} = q^s \times p^s: C^s \rightarrow U^{s+1} \times (\nabla + \text{Ker}(\delta^{\nabla} | H^s))$. By an analogous way with Ebin’s Slice theorem in [4, Theorem 7.1], we get the following

Proposition 2.1. *Let $\nabla \in C^\infty$. There exist a neighbourhood U^{s+1} of [id] in $\mathcal{Q}_\nabla \setminus \mathcal{Q}^{s+1}$, a neighbourhood V^s of ∇ in $\nabla + \text{Ker}(\delta^\nabla | H^s)$ and a neighbourhood W^s of ∇ in C^s so that*

$$\mathcal{A}^s: U^{s+1} \times V^s \rightarrow W^s$$

is a C^ω -diffeomorphism. Moreover if $\gamma \in \mathcal{Q}_\nabla$ then $\gamma^(V^s) = V^s$, and $\gamma^*(V^s) \cap V^s \neq \phi$ if and only if $\gamma \in \mathcal{Q}_\nabla$.*

Proof. Only the last statement is not shown. Since \mathcal{Q}_∇ is a compact group and preserves $\text{Ker } \gamma^\nabla$, taking $\cap_{\gamma \in \mathcal{Q}_\nabla} \gamma^*(V^s)$ if necessary, we may assume that $\gamma^*(V^s) = V^s$ if $\gamma \in \mathcal{Q}_\nabla$. We now show that if $\gamma^*(V^s) \cap V^s \neq \phi$ then $\gamma \in \mathcal{Q}_\nabla$. If $[\gamma]$ belongs to U^{s+1} , then bijectivity of \mathcal{A}^s implies that $\gamma \in \mathcal{Q}_\nabla$. Hence we assume that for any V^s there is $\gamma \in \mathcal{Q}^{s+1}$ such that $\gamma^*(V^s) \cap V^s \neq \phi$ but $[\gamma] \notin U^{s+1}$. This means that there are a sequence $\{\gamma_i\}$ in \mathcal{Q}^{s+1} and sequences $\{\nabla_{1i}\}$ and $\{\nabla_{2i}\}$ in $(\nabla + \text{Ker}(\delta^\nabla | H^s))$ which converge to ∇ such that $\gamma_i^* \nabla_{1i} = \nabla_{2i}$ and $[\gamma_i] \notin U^{s+1}$. Then by the following lemma, a subsequence of $\{\gamma_i\}$ converges to an element γ_∞ in \mathcal{Q}^{s+1} , and so $\gamma_\infty \in \mathcal{Q}_\nabla$ and $[\gamma_i] \in U^{s+1}$ for some i , which contradicts the assumption. Q.E.D.

Lemma 2.2. *Let $\{\gamma_i\}$, $\{\nabla_{1i}\}$ and $\{\nabla_{2i}\}$ be as above. Then a subsequence of $\{\gamma_i\}$ converges in \mathcal{Q}^{s+1} .*

Proof. The equation $\gamma_i^* \nabla_{1i} = \nabla_{2i}$ is equivalent to the equation $\gamma_i^{-1} \nabla_{1i} \gamma_i = \nabla_{2i} - \nabla_{1i}$. Set $A_i = \nabla_{1i} - \nabla$ and $B_i = \nabla_{2i} - \nabla$. Then we see that $\nabla \gamma_i = \gamma_i(B_i - A_i) - A_i \gamma_i$. In general, we have

$$\|\gamma_i\|_t < C_1 \|\nabla \gamma_i\|_{t-1} + C_2 \|\gamma_i\|_0$$

for some real number C_1 and C_2 , and $\|\gamma_i\|_0 < C_3$ since G is compact. Therefore

$$\|\gamma_i\|_t < C_1 \|\gamma_i(B_i - A_i) - A_i \gamma_i\|_{t-1} + C_2 \cdot C_3.$$

Since the multiplication: $H^s \times H^t \rightarrow H^t$ for $t \leq s$ is continuous (see [12, Section 9]), we see that

$$\|\gamma_i\|_t < C_1 \|\gamma_i\|_{t-1} (\|A_i\|_s + \|B_i\|_s) + C_2 \cdot C_3 \quad (t-1 \leq s).$$

Thus we see by induction that the sequence $\|\gamma_i\|_{s+1}$ is bounded, and so a subsequence of $\{\gamma_i\}$ converges in H^s , which we replace by $\{\gamma_j\}$. Then we have

$$\nabla(\gamma_i - \gamma_j) = (\gamma_i(B_i - A_i) - A_i \gamma_i) - (\gamma_j(B_j - A_j) - A_j \gamma_j),$$

and so

$$\|\gamma_i - \gamma_j\|_{s+1} < C_4 \|\gamma_i - \gamma_j\|_s + C_5 \|\gamma_i - \gamma_j\|_0$$

for some C_4 and C_5 , and $\{\gamma_j\}$ is a Cauchy sequence in H^{s+1} -topology. Q.E.D.

DEFINITION 2.3. The manifold V^s in Proposition 2.1 is called *the slice at ∇* and is denoted by \mathcal{S}_∇^s .

A priori, the slice may degenerate for $s \rightarrow \infty$. But we have following lemmas, which say that we can take slices “uniformly” and they are “natural”.

Lemma 2.4. *Let $t \geq s$ and set $U^{t+1} = U^{s+1} \cap (\mathcal{Q}_\nabla \setminus \mathcal{Q}^{t+1})$, $V^t = V^s \cap C^t$ and $W^t = W^s \cap C^t$. Then Proposition 2.1 holds when s is replaced by t .*

Proof. It is sufficient to prove for $t = s + 1$. The map

$$\mathcal{A}^{s+1}: U^{s+2} \times V^{s+1} \rightarrow W^{s+1}$$

is a C^∞ -injective immersion.

(surjectivity) Let $\nabla_1 \in W^{s+1}$. Then there is $\gamma \in \pi^{-1}(U^{s+1})$ so that $\gamma^* \nabla_1 \in V^s$. Set $A_1 = \nabla_1 - \nabla$ and $A_2 = \gamma^* \nabla_1 - \nabla$. Then $A_1 \in H^{s+1}(\mathfrak{g}_P)$, $A_2 \in H^s(\mathfrak{g}_P)$, and $\nabla \gamma = \gamma A_2 - A_1 \gamma$. Since $\delta^\nabla A_2 = 0$, we have

$$\delta^\nabla \nabla \gamma = \text{tr}(\nabla \gamma \otimes A_2) - \delta^\nabla(A_1 \gamma),$$

where $\nabla \gamma \otimes A_2 \in H^s$ and $\delta^\nabla(A_1 \gamma) \in H^s$. Thus $\gamma \in \mathcal{Q}^{s+2}$.

(surjectivity of derivative) Let $u_0 \in U^{s+2}$ and $\nabla + A_0 \in V^{s+1}$. Then the derivative of the map \mathcal{A}^s is given by

$$\begin{aligned} & (\mathcal{A}^s)'_{(u_0, \nabla + A_0)}(u', A') \\ &= \chi(u_0)^* \{ \nabla(\varphi'(u')) + [A_0, \varphi'(u')] + A' \} \end{aligned}$$

where φ is defined by $\varphi(u) = \chi(u) \cdot \chi(u_0)^{-1}$. Let B be any element of $H^{s+1}(\mathfrak{g}_P)$. Then there are $u' \in T_{u_0} U^{s+1}$ and $A' \in T_{A_0} V^s$ so that

$$\chi(u_0)^* \{ \nabla(\varphi'(u')) + [A_0, \varphi'(u')] + A' \} = B.$$

This implies that

$$\delta^\nabla \nabla(\varphi'(u')) = \delta^\nabla(\chi(u_0)^{-1} * B - [A_0, \varphi'(u')]).$$

where the right hand side belongs to H^{s-1} . Thus $\varphi'(u') \in H^{s+1}$, and so the right hand side belongs to H^s , and $\varphi'(u') \in H^{s+2}$. Therefore $u' \in H^{s+2}$ and $A' \in H^{s+1}$.

Q.E.D.

Lemma 2.5. *Let $\nabla_1 \in \mathcal{S}_\nabla^s$. If there is $\gamma \in \mathcal{Q}^{s+1}$ such that $\gamma^* \nabla_1 \in C^t$, then $\nabla_1 \in \mathcal{S}_\nabla^t$. In particular, if $\gamma^* \nabla_1 \in C^\infty$, then $\nabla_1 \in C^\infty$.*

Proof. Let $\{\gamma_i\}$ be a sequence in \mathcal{Q}^{t+1} which converges to γ in H^{s+1} -topology. Then $\gamma_i^{-1} * \gamma^* \nabla_1 \rightarrow \nabla_1$ in C^s , and so for some i $\gamma_i^{-1} * \gamma^* \nabla_1$ belongs to W^s in Proposition 2.1. But here $\gamma_i^{-1} * \gamma^* \nabla_1 \in C^t$. Therefore by Lemma 2.4 $\gamma_i^{-1} * \gamma^* \nabla_1 \in W^t$, and so $\pi(\gamma \gamma_i^{-1}) \in U^{t+1}$ and $\nabla_1 \in \mathcal{S}_\nabla^t$. Q.E.D.

Corollary 2.6. *Let $\nabla_1 \in \mathcal{S}_\nabla^s$ be a Yang-Mills connection. Then $\nabla_1 \in \mathcal{S}_\nabla^s$.*

Proof. By Theorem 12.1, ∇_1 satisfies the condition in Lemma 2.5. Q.E.D.

Theorem 2.7. *Let ∇ be a Yang-Mills connection. There are a neighbourhood U^s of ∇ in \mathcal{S}_∇^s and a closed C^ω -submanifold Z of U^s whose tangent space at ∇ coincides with $YMEID(\nabla)$ such that the set $YMLPM(\nabla)$ of all Yang-Mills connections in U^s is a real analytic set of Z . Moreover, the spaces Z and $YMLPM(\nabla)$ do not depend on s .*

Proof. Set $\varphi^s = E_{YM}|_{\mathcal{S}_\nabla^s}$. Then by (1.5.2) we see

$$\text{Im } \varphi^{s'} = E_{YM'}(\text{Ker}(\delta^\nabla|_{H^s})) = \text{Im}(E_{YM'}|_{H^s}).$$

On the other hand, from (1.5.4) and (1.5.5) we have

$$H^{s-2}(\Lambda^1 \otimes \mathfrak{g}_P) = \text{Im}(E_{YM'}|_{H^s}) \oplus \text{Ker}(E_{YM'}|_{H^{s-2}}).$$

Let p^s (resp. q^s) be the projectoin to the first (resp. second) component. Then the C^ω -map $p^s \circ \varphi^s$ has surjective derivative at ∇ and by the implicit function theorem there is a neighbourhood U^s of ∇ in \mathcal{S}_∇^s so that the set $Z = \{\nabla_1 \in U^s \mid p^s \circ \varphi^s(\nabla_1) = 0\}$ is a C^ω -submanifold of U^s . The tangent space $T_\nabla Z$ coincides with the space $YMEID(\nabla)$ and the set $YMLPM(\nabla)$ is the zero of the map $q^s \circ \varphi^s$ on Z .

Next we have to show that if we set $Z^t = Z \cap \mathcal{S}_\nabla^t$ and $U^t = U^s \cap \mathcal{S}_\nabla^t$ for $t \geq s$ then Z^t coincides with Z as manifold and $p^t \circ \varphi^t$ has surjective derivative at any point of Z^t . Let $\nabla + A \in Z$. Then by the definition of Z and Lemma 1.5 we have

$$\delta^\nabla A = 0, \quad E_{YM'}(E_{YM}(\nabla + A)) = 0.$$

Since this is an elliptic system, A is C^∞ , and so $Z^t = Z$ as set. Let $\nabla_1 \in Z^t$. Since $p^s \circ \varphi^s$ has surjective derivative at ∇_1 , for any $A \in \text{Im}(E_{YM'}|_{H^t})$ there are $B \in \text{Ker}(\delta^\nabla|_{H^s})$ and $C \in \text{Ker}(E_{YM'}|_{H^s})$ so that $(\varphi^s)'_{\nabla_1}(B) = A + C$. Then

$$E_{YM'} \circ (\varphi^s)'_{\nabla_1}(B) = E_{YM'}(A) \in H^{t-4},$$

and $\delta^\nabla B = 0$. Therefore $B \in H^t$, which implies that $p^t \circ \varphi^t$ has surjective derivative at ∇_1 , and so Z^t is a closed C^ω -submanifold of U^t . Moreover, the identity: $Z^t \rightarrow Z$ is bijective and its derivative also, hence is a diffeomorphism. Q.E.D.

DEFINITION 2.8. The set $YMLPM(\nabla)$ is called *the local pre-moduli space of Yang-Mills connections around ∇* and the set Z is called its *support manifold*.

We may summarize results as

Theorem 2.9. *Let ∇ be a Yang-Mills connection. The local pre-moduli space $YMLPM(\nabla)$ of Yang-Mills connections has the following properties. a)*

$YMLPM(\nabla) \subset S_\nabla^\infty$. b) If ∇_1 is a Yang-Mills connection sufficiently close to ∇ , then there is $\gamma \in \mathcal{G}^{s+1}$ so that $\gamma^*\nabla_1 \in YMLPM(\nabla)$. c) If $\gamma^*YMLPM(\nabla) \cap YMLPM(\nabla) \neq \emptyset$ for $\gamma \in \mathcal{G}^{s+1}$, then $\gamma^*\nabla = \nabla$, i.e., $\nabla\gamma = 0$.

REMARK 2.10. The global moduli space $\mathcal{G} \setminus \{\text{Yang-Mills connections}\}$ is locally homeomorphic with the coset space $\mathcal{G}_\nabla \setminus YMLPM(\nabla)$. Since \mathcal{G}_∇ is a compact Lie group, almost all local properties of the global moduli space is reduced to that of $YMLPM(\nabla)$.

Corollary 2.11. (1) Let ∇ be a Yang-Mills connection. If $YMEID(\nabla) = 0$, then $[\nabla]$ is isolated in the global moduli space of Yang-Mills connections. (2) The Yang-Mills functional F_{YM} is constant on the space $YMLPM$, and locally constant on the global moduli space. (3) If a connection ∇ minimizes the functional F_{YM} on \mathcal{C} , then any Yang-Mills connection sufficiently close to ∇ also minimizes F_{YM} . (4) Any Yang-Mills connection sufficiently close to a flat connection is flat. (5) Let M be 4-dimensional. Any Yang-Mills connection sufficiently close to a self-dual (resp. anti self-dual) connection is self-dual (resp. anti self-dual).

Proof. (1) The assumption implies that the support manifold Z is a point. (2), (3) The set $YMLPM$ forms a real analytic set and its points are critical points of F_{YM} . (4) A connection ∇ is flat if and only if $F_{YM}(\nabla) = 0$. (5) A connection ∇ is (anti) self-dual if and only if $F_{YM}(\nabla)$ coincides with a topological invariant of the principal bundle (see [1, p. 432]). Q.E.D.

3. The obstruction for deformations

We have shown that the local pre-moduli space $YMLPM(\nabla)$ is a real analytic set of the support manifold. Therefore we want to know when $YMLPM$ coincides with the support manifold. In this section we introduce a notion which will be used later.

Let \mathcal{P} be an open set of a Hilbert space, \mathcal{Q} and \mathcal{R} Hilbert spaces. Let $E: \mathcal{P} \rightarrow \mathcal{Q}$ and $I: \mathcal{P} \times \mathcal{Q} \rightarrow \mathcal{R}$ be C^∞ -maps and define $I_p: \mathcal{Q} \rightarrow \mathcal{R}$ for each fixed $p \in \mathcal{P}$.

DEFINITION 3.1. If I_p is linear for each $p \in \mathcal{P}$ and $I_p(E(p)) = 0$ for all $p \in \mathcal{P}$, then I is called an identity for E .

If I is an identity for E and $E(p) = 0$, then we see that $I_p \circ E'_p = 0$, i.e., $\text{Im } E'_p \subset \text{Ker } I_p$.

DEFINITION 3.2. Let I be an identity for E and assume that $E(p) = 0$. The space $\text{Ker } I_p / \text{Im } E'_p$ is called the obstruction space for E -deformations of p with respect to I .

Lemma 3.3. Let I be an identity for E and $p \in E^{-1}(0)$. If the obstruction space $\text{Ker } I_p / \text{Im } E'_p$ vanishes, then the set $E^{-1}(0)$ around p forms a manifold whose

tangent space at p coincides with $\text{Ker } E'_p$, provided that one of the following conditions is satisfied. (1) The map E is C^∞ . (2) The space $\text{Im } I_p$ is closed in \mathcal{R} .

Proof. By a similar way as in Proof of Theorem 2.7, we see that there exists the "support manifold" Z whose tangent space at p coincides with $\text{Ker } E'_p$ such that $E^{-1}(0)=(p_C \circ E|Z)^{-1}(0)$ around p , where p_C is the projection to a complement C of $\text{Im } E'_p$ in \mathcal{Q} . Set $\hat{E}=E|Z: Z \rightarrow C$ and $\hat{I}=I|(Z \times C): Z \times C \rightarrow \mathcal{R}$. It is enough to prove that $\hat{E}=0$. Remark that $\text{Ker } \hat{I}_p = \text{Ker } I_p \cap C = \text{Im } E'_p \cap C = 0$ and so \hat{I}_p is injective. Assume condition (2). Then $\text{Im } \hat{I}_p = I_p(C) = I_p(\text{Im } E'_p \oplus C) = \text{Im } I_p$, and so $\text{Im } \hat{I}_p$ is closed in \mathcal{R} , hence \hat{I}_p is an isomorphism from C into \mathcal{R} . Therefore \hat{I}_{p_1} is injective if $p_1 \in Z$ is sufficiently close to p . But here we know that $\hat{I}_{p_1}(\hat{E}(p_1))=0$. Thus $\hat{E}(p_1)=0$.

Next we assume condition (1) and show the r -th derivative $\hat{E}^{(r)}$ vanishes for all $r \geq 0$ by induction. By taking r -th derivative of the identity $\hat{I}_{p_i}(\hat{E}(p_i))=0$ and setting $v = \frac{d}{dt} \Big|_0 p_i$, we get

$$\hat{I}_p(\hat{E}_p^{(r)}(v, \dots, v)) = -\sum_{i=1}^r \binom{r}{i} \left(\frac{d}{dt}\right)^i \Big|_0 \hat{I}_{p_i} \cdot \left(\frac{d}{dt}\right)^{r-i} \Big|_0 \hat{E}(p_i).$$

By induction we may assume that the right hand side vanishes, and so the left hand side vanishes. But we know that \hat{I}_p is injective, hence $\hat{E}_p^{(r)}=0$. Q.E.D.

REMARK 3.4. This Lemma essentially is "Kuranishi's method" ([8]).

4. The deformation of Yang-Mills connection caused by a deformation of base metric

We want apply Lemma 3.3 to a deformations of Yang-Mills connection. Unfortunately, it is not possible if we use Yang-Mills equation itself. In 6, we will introduce the notion of "Einstein holomorphic connection" and apply Lemma 3.3.

Now, by equation (1.5.1), δ^∇ is an identity for E_{YM} , and the obstruction space

$$\text{Ker } \delta^\nabla / \text{Im } E_{YM}'_\nabla \cong \overline{YMEID}(\nabla)$$

by equation (1.5.6).

Proposition 4.1. *Let ∇ be a Yang-Mills connection. The obstruction space for E_{YM} -deformation of ∇ with respect to δ^∇ is isomorphic with the space $\overline{YMEID}(\nabla)$ of essential infinitesimal deformations.*

Hence we apply Lemma 3.3 to the situation that we deform the metric g on M and Yang-Mills connection follows it. Denote by \mathcal{M}^s the space of all H^s -riemannian metrics on M and define maps E, I by

$$E: \mathcal{M}^{s+1} \times C^{s+2} \rightarrow H^s(\Lambda^1 \otimes \mathfrak{g}_P); \quad (g, \nabla) \rightarrow \delta_g^\nabla R^\nabla,$$

$$I: \mathcal{M}^{s+1} \times C^{s+2} \times H^s(\Lambda^1 \otimes \mathfrak{g}_P) \rightarrow H^s(\mathfrak{g}_P); \quad (g, \nabla, \varphi) \rightarrow \delta^\nabla \varphi.$$

Then I is an identity for E .

Theorem 4.2. *Let ∇ be a Yang-Mills connection over (M, g) , i.e., $E(g, \nabla) = 0$. If $YMEID(\nabla) = 0$, then for any deformation g_t of g there exists a 1-parameter family of connections ∇_t so that each ∇_t is a Yang-Mills connection with respect to g_t , provided that $|t|$ is small.*

Proof. The obstruction space for E -deformation of (g, ∇) with respect to I coincides with the space

$$\text{Ker } \delta_g^\nabla / \text{Im } E'_{(g, \nabla)},$$

which is a quotient space of the space $YMEID(\nabla)$ by equation (1.5.6), and vanishes by assumption. Therefore by Lemma 3.3 the set $E^{-1}(0)$ around (g, ∇) forms a manifold whose tangent space at (g, ∇) coincides with $\text{Ker } E'_{(g, \nabla)}$. But here the projection map from $\text{Ker } E'_{(g, \nabla)}$ to $T_g \mathcal{M}^{s+1}$ is surjective, which completes the proof by the implicit function theorem. In fact, for any $h \in T_g \mathcal{M}^{s+1}$, we get $E'_{(g, \nabla)}(h, 0) \in \text{Ker } \delta_g^\nabla$, therefore by equation (1.5.6) and assumption, there is $A \in T_\nabla C^{s+2}$ such that $\text{Ker } E'_{(g, \nabla)}(h, 0) = E'_{YM' \nabla}(A)$, i.e., $E'_{(g, \nabla)}(h, -A) = 0$. Q.E.D.

5. Holomorphic structures

Let M be a compact complex manifold and P^G a principal G^C -bundle, where G^C is a complexification of G . A G^C -invariant almost complex structure on P^G is called an almost holomorphic structure of P^G . If it is integrable, then it is called a holomorphic structure. An almost holomorphic structure can be regarded as a first order differential operator

$$(5.0.1) \quad \bar{\partial}: C^\infty(\mathfrak{g}_P^C) \rightarrow C^\infty(\Lambda^{0,1} \otimes \mathfrak{g}_P^C),$$

and it is a holomorphic structure if and only if the torsion $T(\bar{\partial})$ of $\bar{\partial}$ vanishes, where $T(\bar{\partial}) \in C^\infty(\Lambda^{0,2} \otimes \mathfrak{g}_P^C)$ is defined by

$$(5.0.2) \quad T(\bar{\partial})(X, Y)v \equiv \bar{\partial}_X \bar{\partial}_Y v - \bar{\partial}_Y \bar{\partial}_X v - \bar{\partial}_{[X, Y]} v.$$

An almost holomorphic structure $\bar{\partial}$ extends to the operators

$$(5.0.3) \quad \bar{\partial}^p: C^\infty(\Lambda^{0,p} \otimes \mathfrak{g}_P^C) \rightarrow C^\infty(\Lambda^{0,p+1} \otimes \mathfrak{g}_P^C),$$

and if $\bar{\partial}$ is a holomorphic structure, then they defines an elliptic complex and the cohomology groups

$$(5.0.4) \quad H^{0,p}(\mathfrak{g}_P^C) \equiv \text{Ker } \bar{\partial}^p / \text{Im } \bar{\partial}^{p+1}$$

are defined.

We can study deformations of holomorphic structures by a similar way as deformations of complex structures on manifold (c.f. [13, pp 172–176]), but we use here notations similar with [6].

The space $\mathcal{A}\mathcal{H}$ of all almost holomorphic structures forms an affine space with standard vector space $C^\infty(\Lambda^{0,1} \otimes \mathfrak{g}_\mathbb{F}^c)$, and the complex gauge group $\mathcal{G}^c = C^\infty(G_\mathbb{F}^c)$ acts on it. Let $\mathcal{G}_\mathfrak{g}^c$ be the group of isotropy, i.e., $\mathcal{G}_\mathfrak{g}^c = \{\gamma \in \mathcal{G}^c \mid \gamma^* \bar{\partial} = \bar{\partial}\}$. Then the H^{s+1} -gauge group $\mathcal{G}^{c,s+1}$ acts on $\mathcal{A}\mathcal{H}^s$ holomorphically and the coset space $\mathcal{G}_\mathfrak{g}^c \backslash \mathcal{G}^{c,s+1}$ forms a complex analytic manifold. The following proposition is proved by a similar manner as Proposition 2.1.

Proposition 5.1. *Let $\bar{\partial} \in \mathcal{A}\mathcal{H}$. There exist a neighbourhood $S_\mathfrak{g}^{c,s}$ of $\bar{\partial}$ in $\bar{\partial} + \text{Ker}(\bar{\partial}^* | H^s)$, a neighbourhood $U^{c,s+1}$ of $[id]$ in $\mathcal{G}_\mathfrak{g}^c \backslash \mathcal{G}^{c,s+1}$ and a neighbourhood $W^{c,s}$ of $\bar{\partial}$ in $\mathcal{A}\mathcal{H}^s$ so that the action*

$$\mathcal{A}^{c,s}: U^{c,s+1} \times S_\mathfrak{g}^{c,s} \rightarrow W^{c,s}$$

becomes a complex analytic diffeomorphism. Here, the formal adjoint $\bar{\partial}^$ of $\bar{\partial}$ is defined by some (and fixed) hermitian inner product of $\mathfrak{g}_\mathbb{F}^c$.*

Let $\bar{\partial}$ be a holomorphic structure of P^c . For $A \in C^\infty(\Lambda^{0,1} \otimes \mathfrak{g}_\mathbb{F}^c)$ we see that $T(\bar{\partial} + A) = T(\bar{\partial}) + \bar{\partial}^1 A + [A \wedge A]$. Therefore the equation of infinitesimal deformation of holomorphic structure of $\bar{\partial}$ is given by

$$(5.1.1) \quad \bar{\partial}^1 A = 0.$$

The space of all essential infinitesimal deformations of $\bar{\partial}$ is given by

$$(5.1.2) \quad EHID(\bar{\partial}) = \text{Ker } \bar{\partial}^1 \cap \text{Ker } \bar{\partial}^*.$$

By a similar way as Theorem 2.7, we have

Theorem 5.2. *Let $\bar{\partial}$ be a holomorphic structure. There are a neighbourhood $U^{c,s}$ of $\bar{\partial}$ in $S_\mathfrak{g}^{c,s}$ and a complex analytic submanifold Z^c of $U^{c,s}$ so that the set of all H^s -holomorphic structures in $U^{c,s}$ forms a complex analytic set of Z^c .*

DEFINITION 5.3. The set of all H^s -holomorphic structures in $U^{c,s}$ is called *the local pre-moduli space of holomorphic structures around $\bar{\partial}$* and denoted by $HLPM(\bar{\partial})$. The manifold Z^c is called its *support manifold*.

Equalities (5.1.2) and (5.0.4) mean that the space $HEID(\bar{\partial})$ is canonically isomorphic to $H^{0,1}(\mathfrak{g}_\mathbb{F}^c)$. Moreover, for any $\bar{\partial} \in \mathcal{A}\mathcal{H}$ we have

$$(5.3.1) \quad \bar{\partial}^2 T(\bar{\partial}) = 0,$$

which means that $\bar{\partial}^2$ is an identity for T . Therefore, by Lemma 3.3, we get

Theorem 5.4. *Let $\bar{\delta}$ be a holomorphic structure. If $H^{0,2}(\mathfrak{g}_P^{\mathbb{C}}) = 0$, then the space $HLPM(\bar{\delta})$ forms a (complex) manifold whose tangent space at $\bar{\delta}$ coincides with the space $HEID(\bar{\delta})$.*

REMARK 5.5. Since the action of $\mathcal{G}^{c,s+1}$ on \mathcal{AA}^s is complex analytic, the complex structure of the above space $HLPM(\bar{\delta})$ is canonical. I.e., if $\bar{\delta}_1 \in HLPM(\bar{\delta})$, then the “projection map”: $HLPM(\bar{\delta}_1) \rightarrow HLPM(\bar{\delta})$ defined by Proposition 5.1 is complex analytic.

REMARK 5.6. The space $HLPM(\bar{\delta})$ has similar properties as $YMLPM(\nabla)$ in Theorem 2.9. But property (c) does not hold for $HLPM(\bar{\delta})$, because G^c is not compact. Therefore the quotient space $\mathcal{G}_P^c \backslash HLPM(\bar{\delta})$ is not necessarily identified with an open set of global moduli space of holomorphic structures.

6. Einstein holomorphic connections

Let (M, g) be a compact Kähler manifold, ω its Kähler form. Then the $(0, 1)$ component of a connection ∇ on P is a almost holomorphic structure $\bar{\delta}$ of P^c . Since $T(\bar{\delta})$ coincides with the $(0, 2)$ component of R^∇ , $\bar{\delta}$ is a holomorphic structure if and only if R^∇ is of type $(1, 1)$.

DEFINITION 6.1. A connection ∇ of P is said to be *holomorphic* if the $(0, 1)$ component of ∇ is a holomorphic structure, or equivalently, if R^∇ is of type $(1, 1)$. (Remark that this definition is not exactly the same with [6].)

Denote by R_H^∇ (resp. R_S^∇) the hermitian (resp. skew-hermitian) part of R^∇ . Elements of Lie algebra \mathfrak{z} of the center $Z(G)$ of G define parallel sections of $C^\infty(\mathfrak{g}_P)$, and are denoted also by \mathfrak{z} .

DEFINITION 6.2. A holomorphic connection ∇ is called an *Einstein holomorphic connection* if $(\omega, R^\nabla) \in \mathfrak{z}$ as section.

For example, if $G = U(r)$, a connection ∇ is an Einstein holomorphic connection if and only if ∇ is an Einstein hermitian connection for some holomorphic structure.

Lemma 6.3 (Itoh, Personal communication). *An Einstein holomorphic connection takes the minimum value of the Yang-Mills functional F_{YM} on C . Conversely a connection which takes the value is an Einstein holomorphic connection.*

Proof. Let ∇ be a connection of P and consider the characteristic classes of P . For each $c \in \mathfrak{z}$, the classes represented by (c, R^∇) and $\text{Tr}(R^\nabla \wedge R^\nabla)$ do not depend on ∇ , and so the values $\int_M (c, R^\nabla) \wedge \omega^{n-1}$ and $\int_M \text{Tr}(R^\nabla \wedge R^\nabla) \wedge \omega^{n-2}$ are constant for ∇ . Therefore there are $c_0 \in \mathfrak{z}$ and a real number C such that equalities

$$(6.3.1) \quad \langle \omega \otimes c, R^\nabla \rangle = \langle c_0, c \rangle$$

and

$$(6.3.2) \quad \|R_H^\nabla\|^2 - \|R_S^\nabla\|^2 - \|(\omega, R^\nabla)\|^2 = C$$

hold for all ∇ . Let $(\omega, R^\nabla) = c_1 + v$ where $c_1 \in \mathfrak{z}$ and v is orthogonal to \mathfrak{z} with respect to the global inner product. Then

$$\langle c_0, c \rangle = \langle c, (\omega, R^\nabla) \rangle = \langle c, c_1 \rangle$$

for any $c \in \mathfrak{z}$. Therefore $c_1 = c_0$ and

$$\begin{aligned} \|R^\nabla\|^2 &= \|R_H^\nabla\|^2 + \|R_S^\nabla\|^2 \\ &= C + 2\|R_S^\nabla\|^2 + \|(\omega, R^\nabla)\|^2 \\ &\geq C + \|(\omega, R^\nabla)\|^2 \\ &= C + \|c_0\|^2 + \|v\|^2 \\ &\geq C + \|c_0\|^2. \end{aligned}$$

The equality $\|R^\nabla\|^2 = C + \|c_0\|^2$ holds if and only if $R_S^\nabla = 0$ and $v = 0$, i.e., $(\omega, R^\nabla) \in \mathfrak{z}$. Q.E.D.

REMARK 6.4. We saw that if ∇ is an Einstein holomorphic connection then $(\omega, R^\nabla) = c_0$.

Corollary 6.5. *All Einstein holomorphic connections are Yang-Mills connections. Conversely all Yang-Mills connections which are sufficiently close to an Einstein holomorphic connection are Einstein holomorphic connections.*

Proof. Easy to see by Corollary 2.11 (3).

Q.E.D.

Next we consider infinitesimal deformations of Einstein holomorphic connections. Define a map $E_{EH}: C^s \rightarrow H^{s-1}(\Lambda^{0,2} \otimes \mathfrak{g}_P^{\mathbb{C}}) \oplus H^{s-1}(\mathfrak{g}_P)$ by

$$(6.5.1) \quad \nabla \rightarrow (p^{0,2}R^\nabla, (\omega, R^\nabla) - c_0),$$

where $p^{0,r}$ is the projection map from Λ^r to $\Lambda^{0,r}$ and $c_0 \in \mathfrak{z}$ is defined in Proof of Lemma 6.3. By Remark 6.4, a connection ∇ is an Einstein holomorphic connection if and only if $E_{EH}(\nabla) = 0$.

DEFINITION 6.6. Let ∇ be an Einstein holomorphic connection. An element A of $H^s(\Lambda^1 \otimes \mathfrak{g}_P)$ is called an *Einstein holomorphic infinitesimal deformation of ∇* if $E_{EH}'_\nabla(A) = 0$. An Einstein holomorphic infinitesimal deformation is said to be *essential* if it is orthogonal to all trivial infinitesimal deformations of ∇ , and the space of all Einstein holomorphic essential infinitesimal deformations is denoted by $EHEID(\nabla)$.

By a similar way as Theorem 2.7, we get

Theorem 6.7. *Let ∇ be an Einstein holomorphic connection. There are a neighbourhood U^s of ∇ in S_∇^s and a closed C^ω -submanifold Z of U^s whose tangent space at ∇ coincides with $EHEID(\nabla)$ such that the set $EHLPM(\nabla)$ of all Einstein holomorphic connections in U^s is a real analytic set of Z .*

Moreover, the combination of an obvious inclusion: $EHLPM(\nabla) \subset YMLPM(\nabla)$ and the converse inclusion $YMLPM(\nabla) \subset EHLPM(\nabla)$ by Corollary 6.5 means that $EHLPM(\nabla) = YMLPM(\nabla)$. Let ∇ be a connection. Define a map $I_\nabla: H^{s-1}(\Lambda^{0,2} \otimes \mathfrak{g}_P^c) \oplus H^{s-1}(\mathfrak{g}_P) \rightarrow H^{s-2}(\Lambda^{0,3} \otimes \mathfrak{g}_P^c) \oplus \mathfrak{z}$ by

$$I_\nabla(P, \eta) = (p^{0,3}(d^\nabla P), \mathfrak{z}\text{-part of } \eta).$$

Lemma 6.8. *The map I is an identity for E_{EH} .*

Proof. For any ∇ , we see

$$p^{0,3}(d^\nabla(p^{0,2}R^\nabla)) = p^{0,3}(d^\nabla R^\nabla) = 0,$$

and (6.3.1) means that \mathfrak{z} -part of $(\omega, R^\nabla) - c_0$ vanishes.

Q.E.D.

Therefore if ∇ is an Einstein holomorphic connection and $\text{Ker } I_\nabla / \text{Im } E_{EH}'_\nabla$ vanishes, then the local pre-moduli space $EHLPM(\nabla)$ of Einstein holomorphic connections forms a manifold with tangent space $EHEID(\nabla)$ at ∇ .

Theorem 6.9. *Let ∇ be an Einstein holomorphic connection. In general, the space $EHLPM(\nabla)$ forms a real analytic set of the support manifold Z whose tangent space at ∇ is isomorphic with the cohomology group $H^{0,1}(M, \mathfrak{g}_P^c)$. If $H^{0,2}(M, \mathfrak{g}_P^c) = 0$ and $H^0(M, \mathfrak{g}_P^c) \cong \mathfrak{z}^c$, then the space $EHLPM(\nabla)$ coincides with the support manifold.*

Proof. We must show that $EHEID(\nabla) \cong H^{0,1}(M, \mathfrak{g}_P^c)$ and $\text{Ker } I_\nabla / \text{Im } E_{EH}'_\nabla \cong H^{0,2}(M, \mathfrak{g}_P^c) \oplus H^0(M, \mathfrak{g}_P) / \mathfrak{z}$, where $H^0(M, \mathfrak{g}_P)$ denotes the vector space of all parallel sections of \mathfrak{g}_P . First we see that the sequence

$$(6.9.1) \quad C^\infty(\mathfrak{g}_P) \xrightarrow{\nabla} C^\infty(\Lambda^1 \otimes \mathfrak{g}_P) \xrightarrow{E_{EH}'_\nabla} C^\infty(\Lambda^{0,2} \otimes \mathfrak{g}_P^c) \oplus C^\infty(\mathfrak{g}_P) \xrightarrow{\text{pr} \circ I_\nabla} C^\infty(\Lambda^{0,3} \otimes \mathfrak{g}_P^c)$$

is an elliptic complex. Therefore

$$(6.9.2) \quad (\text{Ker } I_\nabla / \text{Im } E_{EH}'_\nabla) \oplus \mathfrak{z} \cong \text{Ker } (\text{pr} \circ I_\nabla) / \text{Im } E_{EH}'_\nabla \\ \cong \text{Ker } (\text{pr} \circ I_\nabla) \cap \text{Ker } (E_{EH}'_\nabla)^*.$$

Let $(P, \eta) \in \text{Ker } (\text{pr} \circ I_\nabla) \cap \text{Ker } (E_{EH}'_\nabla)^*$. We easily see that

$$(6.9.3) \quad (\omega, d^\nabla A) = 4 \text{Re}(\sqrt{-1} \nabla^{\bar{\alpha}} A_{\bar{\alpha}}).$$

Thus $(P, \eta) \in \text{Ker } (E_{EH}'_\nabla)^*$ means that

$$(6.9.4) \quad \langle (P, \eta), (p^{0,2}(d^\nabla A), 4\operatorname{Re}(\sqrt{-1} \nabla^{\bar{\alpha}} A_{\bar{\alpha}})) \rangle = 0$$

for all $A \in C^\infty(\Lambda^1 \otimes \mathfrak{g}_P)$, from which we have

$$(6.9.5) \quad -\nabla^{\bar{\beta}} P_{\bar{\beta}\bar{\alpha}} + 2\sqrt{-1} \nabla_{\bar{\alpha}} \eta = 0.$$

Here we know that $\nabla^{\bar{\alpha}} \nabla^{\bar{\beta}} P_{\bar{\beta}\bar{\alpha}} = 0$ since ∇ is Einstein. Therefore we see that

$$(6.9.6) \quad -\nabla^{\bar{\beta}} P_{\bar{\beta}\bar{\alpha}} = 0 \quad \text{and} \quad \nabla \eta = 0.$$

Combining with the assumption that $(P, \eta) \in \operatorname{Ker}(\operatorname{pr} \circ I_\nabla)$, we see that P is harmonic and η is parallel. The converse is obvious, and we get

$$(6.9.7) \quad \operatorname{Ker}(\operatorname{pr} \circ I_\nabla) \cap \operatorname{Ker}(E_{EH'} \nabla)^* \simeq H^{0,2}(M, \mathfrak{g}_P^{\mathbb{C}}) \oplus H^0(M, \mathfrak{g}_P).$$

Let $A \in EHEID(\nabla)$. Then by definition and equality (6.9.3) we get

$$(6.9.8) \quad \nabla_{\bar{\alpha}} A_{\bar{\beta}} - \nabla_{\bar{\beta}} A_{\bar{\alpha}} = 0,$$

$$(6.9.9) \quad \nabla^{\bar{\alpha}} A_{\bar{\alpha}} \in C^\infty(\mathfrak{g}_P)$$

and

$$(6.9.10) \quad \nabla^{\bar{\alpha}} A_{\bar{\alpha}} + \nabla^{\alpha} A_{\alpha} = 0.$$

Thus we see that $p^{0,1}A$ is harmonic and so the first isomorphism holds. Q.E.D.

The above results are resumed as follows.

Theorem 6.10. *Let ∇ be an Einstein holomorphic connection. The space $EHLPM(\nabla)$ coincides with the space $YMLPM(\nabla)$ around ∇ , which is a real analytic set of the support manifold Z whose tangent space at ∇ coincides with the space $EHEID(\nabla)$. If $H^{0,2}(M, \mathfrak{g}_P^{\mathbb{C}}) = 0$ and $H^0(M, \mathfrak{g}_P^{\mathbb{C}}) = \mathfrak{z}^{\mathbb{C}}$, then the space $EHLPM(\nabla)$ coincides with the support manifold Z .*

REMARK 6.11. The above statement suggests the equality $YMEID(\nabla) = EHEID(\nabla)$, which in fact holds.

7. The deformation of Einstein holomorphic connections caused by a deformation of complex structure of the base manifold

In this section we discuss on deformations of Einstein holomorphic connections in a similar situation as in section 4. Let (M, J, g) be the base Kähler manifold and (J_t, g_t) be a one-parameter family of Kähler structure such that $(J_0, g_0) = (J, g)$. Define maps E, I

$$E: (-\varepsilon, \varepsilon) \times C^s \rightarrow H^{s-1}(\Lambda^{0,2} \otimes \mathfrak{g}_P^{\mathbb{C}}) \oplus H^{s-1}(\mathfrak{g}_P),$$

$$I_{(t, \nabla)}: H^{s-1}(\Lambda^{0,2} \otimes \mathfrak{g}_P^{\mathbb{C}}) \oplus H^{s-1}(\mathfrak{g}_P) \rightarrow H^{s-2}(\Lambda^{0,3} \otimes \mathfrak{g}_P^{\mathbb{C}}) \oplus \mathfrak{z}$$

by

$$E(t, \nabla) = (p^{0,2}R^{\nabla t}, (\omega_t, R^{\nabla t})_t - c_t)$$

and

$$I_{(t, \nabla)}(P, \eta) = (p^{0,3}(d^{\nabla t}P), \mathfrak{z}\text{-part of } \eta),$$

where all operators and $c_t \in \mathfrak{z}$ depending on base Kähler structure are defined by (J_t, g_t) . Then we know that I is an identity for E .

Theorem 7.1. *Let ∇ be an Einstein holomorphic connection on (M, J, g) . If $H^{0,2}(M, \mathfrak{g}_P^c) = 0$ and $H^0(M, \mathfrak{g}_P) = \mathfrak{z}$, then for any deformation (J_t, g_t) of Kähler structures of (J, g) there exists a one-parameter family of connections ∇_t of P so that each ∇_t is an Einstein holomorphic connection over (M, J_t, g_t) , provided that $|t|$ is sufficiently small. Moreover, each local pre-moduli space $EHLPM(\nabla_t)$ over (M, J_t, g_t) forms a manifold of the same dimension.*

Proof. The obstruction space for E -deformation of $(0, \nabla)$ with respect to I coincides with the space

$$\text{Ker } I_{\nabla} / \text{Im } E'_{(0, \nabla)},$$

where I_{∇} is introduced before Lemma 6.8. It is a quotient space of the space $\text{Ker } I_{\nabla} / \text{Im } E'_{EH'_{\nabla}}$ and vanishes by assumption. Therefore by Lemma 3.3 the set $E^{-1}(0)$ around $(0, \nabla)$ forms a manifold whose tangent space at $(0, \nabla)$ is given by $\text{Ker } E'_{(0, \nabla)}$. But here the projection map from $\text{Ker } E'_{(0, \nabla)}$ to $T_0(-\varepsilon, \varepsilon)$ is surjective, which completes the proof by the implicit function theorem. In fact, for $u \in T_0(-\varepsilon, \varepsilon)$, we get $E'_{(0, \nabla)}(u, 0) \in \text{Ker } I_{\nabla}$, therefore by assumption there is $A \in T_{\nabla} \mathcal{C}^s$ such that $E'_{(0, \nabla)}(u, 0) = E'_{EH'_{\nabla}}(A)$, i.e., $E'_{(0, \nabla)}(u, -A) = 0$. Q.E.D.

8. Einstein holomorphic connections and holomorphic structures

For a connection ∇ of P we denote by $\Psi(\nabla)$ the $(0, 1)$ -part of ∇ , which is an almost holomorphic structure of P^c . Remark that the map Ψ commutes with the action of the gauge group \mathcal{G} . Therefore Ψ induces a map from the moduli space of Einstein holomorphic connections to the moduli space of holomorphic structures. This map locally corresponds to a map $\varphi: EHLPM(\nabla) \rightarrow HLPM(\bar{\delta})$, where ∇ is an Einstein holomorphic connection and $\bar{\delta} = \Psi(\nabla)$.

Theorem 8.1. *Let ∇ be an Einstein holomorphic connection. If $H^0(M, \mathfrak{g}_P) \cong \mathfrak{z}$, then the map $p \circ \Psi$ gives a bijection between $EHLPM(\nabla)$ and $HLPM(\Psi(\bar{\delta}))$ around ∇ , where the map $p: W^{c,s} \rightarrow S_{\Psi(\nabla)}^{c,s}$ is defined by Proposition 5.1.*

Proof. Set $\mathcal{E} = \{\nabla_1 \in S_{\nabla}^s; (\omega, R^{\nabla_1}) - c_0 = 0\}$. The derivative of the map $f: \nabla_1 \rightarrow (\omega, R^{\nabla_1}) - c_0$ at ∇ is given by

$$A \rightarrow 2\sqrt{-1} (\nabla^{\bar{a}} A_{\bar{a}} - \nabla^a A_a) .$$

Set $A_{\bar{a}} = \sqrt{-1} \nabla_{\bar{a}} \psi$ for $\psi \in H^{s+1}(\mathfrak{g}_P)$. Then

$$A \in T_{\nabla} \mathcal{S}_{\nabla}^s$$

and

$$2\sqrt{-1} (\nabla^{\bar{a}} A_{\bar{a}} - \nabla^a A_a) = 2\nabla^* \nabla \psi .$$

Therefore the image of the derivative of the map f from \mathcal{S}_{∇}^s is closed in $H^{s-1}(\mathfrak{g}_P)$, and coincides with the orthogonal complement of $H^0(M, \mathfrak{g}_P)$. Therefore by assumption and Lemma 6.8, the map f from \mathcal{S}_{∇}^s to the orthogonal complement of \mathfrak{z} has surjective derivative, from which we see that \mathcal{E} is a manifold whose tangent space at ∇ coincides with the space

$$\{A \in H^s(\Lambda^1 \otimes \mathfrak{g}_P); -\nabla^{\bar{a}} A_{\bar{a}} = 0\} .$$

Since the derivative of the map $p \circ \Psi$ from \mathcal{E} is nothing but the correspondence: $A \rightarrow (0, 1)$ -part of A , $p \circ \psi$ gives a local diffeomorphism from \mathcal{E} to $\mathcal{S}_{\bar{\delta}}^s$. If $\nabla_1 \in EHLPM(\nabla)$ then $p \circ \Psi(\nabla_1) \in HLPM(\Psi(\nabla))$, conversely, if $\bar{\delta}_1 \in HLPM(\Psi(\nabla))$ then $(p \circ \Psi|_{\mathcal{E}})^{-1}(\bar{\delta}_1)$ is Einstein holomorphic by definition of \mathcal{E} . Q.E.D.

REMARK 8.2. Theorem 8.1 and Theorem 5.4 give another proof of Theorem 6.10.

Combining with Theorem 6.10, we get the following

Theorem 8.3. *Let ∇ be an Einstein holomorphic connection and set $\bar{\delta} = \Psi(\nabla)$. Then there exists a natural correspondence*

$$YMLPM(\nabla) = EHLPM(\nabla) \rightarrow HLPM(\bar{\delta}) ,$$

where \rightarrow is an injection, and becomes a bijection if $\text{Ker } \nabla = \mathfrak{z}$.

9. A structure on the moduli space

Let ∇ be an Einstein holomorphic connection and set $\bar{\delta} = \Psi(\nabla)$. Assume that $H_{\bar{\delta}}^0(\mathfrak{g}_{\bar{\delta}}) = \mathfrak{z}^C$ and $H_{\bar{\delta}}^2(\mathfrak{g}_{\bar{\delta}}) = 0$. Then the manifolds $EHLPM(\nabla)$ and $HLPM(\bar{\delta})$ are isomorphic by Theorem 8.1, and become complex manifolds by Theorem 5.4. The complex structures are realized by the almost complex structures given by multiplying $\sqrt{-1}$ on $T_{\bar{\delta}} HLPM(\bar{\delta})$ and \tilde{J} on $T_{\nabla} EHLPM(\nabla)$, where \tilde{J} is defined by $(\tilde{J}A)_i = -A_j J^j_i$. In fact, we see that

$$\Psi(\tilde{J}A) = (\tilde{J}A)^{(0,1)} = -A_{\bar{j}} J^{\bar{j}}_{\bar{a}} = \sqrt{-1} A_{\bar{a}} = \sqrt{-1} \Psi(A) .$$

On the other hand, the space \mathcal{C}^s has the riemannian metric $\langle \cdot, \cdot \rangle$, which is

invariant under the action of \mathcal{G}^{s+1} . Therefore the manifold $EHLPM(\nabla)$ has a canonical riemannian metric, which is given as follows. Let $\nabla_1 \in EHLPM(\nabla)$ and $A, B \in T_{\nabla_1} EHLPM(\nabla)$. The elements A and B are Einstein holomorphic infinitesimal deformations of ∇_1 , and are decomposed into the essential parts A_E, B_E and trivial parts A_T, B_T (see (1.5.2)). We define the inner product of A and B by $\langle A_E, B_E \rangle$. From Lemma 13.1, we see that this inner product becomes a C^∞ -riemannian metric.

DEFINITION 9.1. The above riemannian metric on $EHLPM(\nabla)$ is called *the natural riemannian metric*.

REMARK 9.2. Let ∇_1 and ∇_2 be Einstein holomorphic connections and assume that there are $\nabla_0 \in EHLPM(\nabla_1)$ and $\gamma \in \mathcal{G}^{s+1}$ such that $\gamma^*\nabla_0 \in EHLPM(\nabla_2)$. Then for each $\nabla \in EHLPM(\nabla_1)$ sufficiently close to ∇_0 there is $\gamma \in \mathcal{G}^{s+1}$ so that $\gamma^*\nabla \in EHLPM(\nabla_2)$, and this correspondence: $\nabla \rightarrow \gamma^*\nabla$ becomes an isometry. Therefore we may say that the canonical riemannian metric is independent of ∇ .

Theorem 9.3. *Let ∇ be an Einstein holomorphic connection and set $\bar{\delta} = \Psi(\nabla)$. If $H_{\mathfrak{g}}^0(\mathfrak{g}_P^C) = \mathfrak{z}^C$ and $H_{\mathfrak{g}}^2(\mathfrak{g}_P^C) = 0$, then the canonical riemannian metric on $EHLPM(\nabla)$ is a Kähler metric with respect to the complex structure on $HLPM(\bar{\delta})$.*

Proof. We easily see that the canonical riemannian metric is a hermitian metric. We have to show that the Kähler form is closed. We replace ∇ by ∇_0 and denote by ∇ elements of $HLPM(\nabla_0)$ regarded as variable. Consider the fiber bundle $p: P \times EHLPM \rightarrow EHLPM$. In general, a diffeomorphism from a fiber to another fiber which commutes with the action of G and fixes M pull backs a G -invariant structure, and so if a vector field v on $P \times EHLPM$ is p -projectable, G -invariant and $\pi^*v = 0$, where π is the projection to M , then the Lie derivation \mathcal{L}_v on a family of G -invariant structures is defined. For example,

$$\mathcal{L}_v \nabla \equiv \left. \frac{d}{ds} \right|_0 (\exp sv)^* \nabla.$$

If we decompose v into the P -part v_P and the $EHLPM$ -part v_M , we see that

$$\mathcal{L}_v \nabla = v_M[\nabla] + L_{v_P} \nabla.$$

Now, we denote the almost complex structure on $EHLPM$ by J^E , the canonical riemannian metric by g^E and the Kähler form by ω^E . Decompose $v \in T(EHLPM)$ into v_E and v_T so that $\mathcal{L}_{v_E} \nabla$ is essential and $\mathcal{L}_{v_T} \nabla$ is trivial. This decomposition is not unique, but we may assume that it depends C^∞ -ly on v by Lemma 13.1. Then we see that

$$\begin{aligned} \mathcal{L}_{(J^E v)_E} \nabla &= \tilde{J} \mathcal{L}_{v_E} \nabla, \\ g^E(v, w) &= \langle \mathcal{L}_{v_E} \nabla, \mathcal{L}_{w_E} \nabla \rangle, \\ \omega^E(v, w) &= g^E(J^E v, w) = \langle \tilde{J} \mathcal{L}_{v_E} \nabla, \mathcal{L}_{w_E} \nabla \rangle, \end{aligned}$$

where \tilde{J} is defined in the first paragraph of this section. We may assume that $[v, w]=[w, z]=[z, v]=0$ without loss of generality, and see that

$$\begin{aligned} (d\omega^E)(v, w, z) &= v \cdot \omega^E(w, z) + \text{alternating terms} \\ &= v \cdot \langle \tilde{J} \mathcal{L}_{w_B} \nabla, \mathcal{L}_{z_B} \nabla \rangle + \text{alt} . \\ &= \langle \tilde{J} \mathcal{L}_{v_B} \mathcal{L}_{w_B} \nabla, \mathcal{L}_{z_B} \nabla \rangle + \langle \tilde{J} \mathcal{L}_{w_B} \nabla, \mathcal{L}_{v_B} \mathcal{L}_{z_B} \nabla \rangle + \text{alt} . \\ &= -\langle \tilde{J} \mathcal{L}_{v_B} \mathcal{L}_{w_B} \nabla, \tilde{J} \mathcal{L}_{z_B} \nabla \rangle + \langle \mathcal{L}_{v_B} \mathcal{L}_{z_B}, \tilde{J} \mathcal{L}_{w_B} \nabla \rangle + \text{alt} . \\ &= -\langle [\mathcal{L}_{v_B}, \mathcal{L}_{w_B}] \nabla, \tilde{J} \mathcal{L}_{z_B} \nabla \rangle + \text{alt} . \\ &= -\langle \mathcal{L}_{[v_B, w_B]} \nabla, \tilde{J} \mathcal{L}_{z_B} \nabla \rangle + \text{alt} . \end{aligned}$$

But here $p_*[v_E, w_E]=[v, w]=0$ and so $[v_E, w_E]$ is vertical, which implies that $\mathcal{L}_{[v_B, w_B]} \nabla$ is trivial. Q.E.D.

10. Example I

Let M be a flat torus T^2 , P the trivial principal $U(2)$ -bundle and ∇_0 the canonical connection of P . ∇_0 is a flat connection, and so an Einstein holomorphic connection. Therefore, by Lemma 6.3, all Einstein holomorphic connections of P are flat. Fix a point x in M and an element p in P_x . Any closed curve $c(c(0)=c(1)=x)$ in M is horizontally lifted to a curve \tilde{c} in P so that $\tilde{c}(0)=p$, and we get an element $\tilde{c}(1)$ in P_x . Let a be an element of $U(2)$ such that $\tilde{c}(1)=p \cdot a$. Since ∇ is flat, this mapping: $c \rightarrow a$ induces a homomorphism: $\pi_1(M) \rightarrow U(2)$, defined by $[c] \rightarrow a$. Taking generators $\{[c_1], [c_2]\}$ of $\pi_1(M)$, we get corresponding elements $\{a_1, a_2\}$ in $U(2)$ such that $a_1^{-1} \cdot a_2^{-1} \cdot a_1 \cdot a_2 = \text{id}$. Denote by $f(\nabla)$ this pair (a_1, a_2) . We see that by a gauge transformation η of P , $f(\nabla)=(a_1, a_2)$ is transformed as

$$(10.1) \quad f(\eta^* \nabla) = (b^{-1} \cdot a_1 \cdot b, b^{-1} \cdot a_2 \cdot b),$$

where $b \in U(2)$ is defined by $\eta(x) \cdot p = p \cdot b$.

Thus the global moduli space of Einstein holomorphic connections is identified with the quotient space $\{\text{commuting pair in } U(2) \times U(2)\} / \sim$, where \sim is defined by $(b^{-1} \cdot a_1 \cdot b, b^{-1} \cdot a_2 \cdot b) \sim (a_1, a_2)$ for $b \in U(2)$. By diagonalization, this space becomes the space $T^2 \times T^2 / \sim$, where

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix} \right) \sim \left(\begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix}, \begin{pmatrix} \gamma' & 0 \\ 0 & \delta' \end{pmatrix} \right)$$

if and only if they coincide or $\beta' = \alpha, \alpha' = \beta, \delta' = \gamma$ and $\gamma' = \delta$.

On the other hand, the space $EHEID(\nabla_0)$ is the space of harmonic sections of $\Lambda^1 \otimes \mathfrak{u}(2)$, and is isomorphic with $\mathbb{R}^2 \otimes \mathfrak{u}(2)$. Let $A \in EHEID(\nabla_0)$ and consider the connection $\nabla_0 + A$. Since $\nabla_0 A = 0$, we see that

$$E_{EH}(\nabla_0 + A) = (0, 2[\overline{A^{(0,1)}}, A^{(0,1)}]),$$

and

$$(\omega, d^{\nabla_0}B) = 2\sqrt{-1}(\nabla_0^{\bar{a}}B_{\bar{a}} - \nabla_0^a B_a),$$

which implies that $\nabla_0 + A$ is an element of the support manifold of $EHLPM(\nabla_0)$. Thus we see that *the support manifold is locally isomorphic with $\mathbf{R}^2 \otimes \mathfrak{u}(2)$* . Moreover $\nabla_0 + A$ belongs to $EHLPM(\nabla_0)$ if and only if

$$[\overline{A^{(0,1)}}, A^{(0,1)}] = 0.$$

Therefore *the space $EHLPM(\nabla_0)$ is a proper subset of the support manifold*. Moreover, the group $\mathcal{G}_{\nabla_0} \cong U(2)$ acts on the space $EHLPM(\nabla_0)$ analogously as (10.1), and we see that

$$\mathcal{G}_{\nabla_0} \backslash EHLPM(\nabla_0) \cong \mathbf{R}^2 \times \mathbf{R}^2 / \sim.$$

By a similar way we see that the space $HEID(\bar{\partial}_0)$ is canonically isomorphic with the space $\mathbf{C} \otimes \mathfrak{gl}(2, \mathbf{C})$, and $\bar{\partial}_0 + HEID(\bar{\partial}_0)$ is the support manifold of $HLPM(\bar{\partial}_0)$. In this case, the space $HLPM(\bar{\partial}_0)$ is an open set of the support manifold. We can see more details as follows. The group $\mathcal{G}_{\bar{\partial}_0}^{\mathbf{C}}$ acts on the space $HLPM(\bar{\partial}_0)$, and

$$\mathcal{G}_{\bar{\partial}_0}^{\mathbf{C}} \backslash HLPM(\bar{\partial}_0) \cong GL(2, \mathbf{C}) \backslash \mathfrak{gl}(2, \mathbf{C}),$$

whose elements are classified using Jordan's normal form. An element of $\mathfrak{gl}(2, \mathbf{C})$ corresponds to an Einstein holomorphic connection if and only if it is diagonalizable. Thus

$$\mathcal{G}_{\nabla_0} \backslash EHLPM(\nabla_0) \cong \mathcal{G}_{\bar{\partial}_0}^{\mathbf{C}} \backslash HLPM(\bar{\partial}_0).$$

Remark that the space $\mathcal{G}_{\bar{\partial}_0}^{\mathbf{C}} \backslash HLPM(\bar{\partial}_0)$ is *not* a Hausdorff space. In fact, any neighbourhood of the element $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ in $U(2)$ implies some $\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix}$ ($t \neq 0$), which is conjugate with $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

11. Example II

Let (M, g) be an Einstein-Kähler manifold with Ricci tensor $= e \cdot g$, ∇ an Einstein holomorphic connection and $\bar{\partial} = \Psi(\nabla)$. Then we can see that

$$\begin{aligned} (11.0.1) \quad \{(\bar{\partial}^* \bar{\partial} + 2\bar{\partial} \bar{\partial}^*) A\}_{\bar{a}} &= (\nabla^* \nabla A)_{\bar{a}} + e A_{\bar{a}} + 2[R^{\nabla^{\bar{\beta}}}_{\bar{a}}, A_{\bar{\beta}}] \\ &= 2\{-\nabla^{\bar{\beta}} \nabla_{\bar{\beta}} A_{\bar{a}} + e A_{\bar{a}} + [R^{\nabla^{\bar{\beta}}}_{\bar{a}}, A_{\bar{\beta}}]\} \end{aligned}$$

for $\mathfrak{g}_{\bar{\beta}}^{\mathbf{C}}$ -valued (0,1)-form A ,

$$(11.0.2) \quad \left\{ \left(\frac{2}{3} \bar{\partial}^* \bar{\partial} + 2 \bar{\partial} \bar{\partial}^* \right) A \right\}_{\bar{\alpha} \bar{\beta}} \\ = (\nabla^* \nabla A)_{\bar{\alpha} \bar{\beta}} + 2e A_{\bar{\alpha} \bar{\beta}} + 2[R^{\nabla \bar{\gamma}}_{\bar{\alpha}}, A_{\bar{\gamma} \bar{\beta}}] + 2[R^{\nabla \bar{\gamma}}_{\bar{\beta}}, A_{\bar{\alpha} \bar{\gamma}}]$$

for \mathfrak{g}_P^C -valued (0,2)-form A . Therefore, to see whether $H_{\mathfrak{g}}^1$ and $H_{\mathfrak{g}}^2$ vanish, we have to get eigenvalues of these operators.

Let M be a homogeneous space K/H and P the principal G -bundle $K \times_{\rho} G$, where ρ is a homomorphism: $H \rightarrow G$. Then we have

$$G_P = K \times_{\text{Ad } \rho} G, \quad \mathfrak{g}_P = K \times_{\text{Ad } \rho} \mathfrak{g}.$$

As usual, we identify $C^\infty(\mathfrak{g}_P)$ with $C^\infty(K, \mathfrak{g})_H$. Let $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ be an H -invariant decomposition and define a differential operator $D: C^\infty(K, \mathfrak{g})_H \rightarrow C^\infty(K, \mathfrak{m}^* \otimes \mathfrak{g})_H$ by

$$(D\mathfrak{s})(X) = (X\mathfrak{s}).$$

Then this operator D gives a covariant derivative of \mathfrak{g}_P , which is identified with the standard connection ∇ of P . Let C_K (resp. C_H) be the Cassimir operator of the K -module (resp. H -module) $C^\infty(K, \mathfrak{g})_H$. We can check that

$$\nabla^* \nabla = C_K - C_H$$

and

$$R^\nabla(X, Y) = -\rho[X, Y] \quad \text{for } X, Y \in \mathfrak{m}.$$

(See e.g., [10, Proposition 5.3].)

Therefore the eigenvalues of operators (11.0.1) and (11.0.2) are calculated explicitly by the representation theory. The calculation is easy but complicated, and we omit the detail. See e.g. [10, §7].

Let $M = P^n(\mathbf{C}) = SU(n+1)/S(U(n) \times U(1))$ and P the unitary frame bundle of T^+M . Then $\mathfrak{g} = \mathfrak{m}^- \otimes \mathfrak{m}^+$, and the operator (11.0.1) has only positive eigenvalues. Thus $H_{\mathfrak{g}}^1(M, \mathfrak{g}_P^C) = 0$.

Proposition 11.1. *The standard connection of the unitary frame bundle of $T^+P^n(\mathbf{C})$ is isolated in the moduli space.*

Next, let P be the unitary frame bundle of the symmetric tensor product S^2T^+M of T^+M . Then $\mathfrak{g} = (S^2\mathfrak{m}^-) \otimes (S^2\mathfrak{m}^+)$. In this case the operator (11.0.1) has 0 as an eigenvalue, and all eigenvalues of the operator (11.0.2) are positive. Moreover, we can easily check that $H_{\mathfrak{g}}^0(M, \mathfrak{g}_P^C) = \mathfrak{z}^C$. Thus by Theorem 6.10, we get the following

Proposition 11.2. *The local pre-moduli space around the standard connection of the unitary frame bundle of $S^2T^+P^n(\mathbf{C})$ ($n \geq 2$) forms a non-trivial manifold.*

12. Regularity of Yang-Mills connections

In this section we consider not a family of connections but one connection. Let ∇ be a Yang-Mills $C^{2+\alpha}$ -connection of P ($0 < \alpha < 1$). I.e., if we represent ∇ by a local frame $\{\xi_\rho\}$ of \mathfrak{g}_P as

$$\nabla_{\partial_i} \xi_\rho = \Gamma_{i\rho}^q \xi_q$$

then $\Gamma_{i\rho}^q$ are $C^{2+\alpha}$. A local section ξ of \mathfrak{g}_P is said to be harmonic if $\nabla^* \nabla \xi = 0$. The defining equation of harmonic section is a linear elliptic differential equation with $C^{1+\alpha}$ -coefficients. Therefore we can take a local frame by harmonic sections, which are $C^{3+\alpha}$ ([2, p. 228 Theorem 1]). The coefficients $\Gamma_{i\rho}^q$ with respect to the frame are $C^{2+\alpha}$. But we know that $\{\Gamma_{i\rho}^q\}$ satisfies Yang-Mills equation:

$$g^{kl} \partial_k (\partial_l \Gamma_{i\rho}^q - \partial_i \Gamma_{l\rho}^q) + \text{lower terms} = 0,$$

and harmonic equation

$$g^{kl} \partial_k \Gamma_{l\rho}^q + \text{lower terms} = 0,$$

which is quasi-linear elliptic system with C^∞ -coefficients. Thus $\Gamma_{i\rho}^q$ are C^∞ ([11, Theorem 6.8.1]). If (M, g) is a C^∞ -riemannian manifold, then $\Gamma_{i\rho}^q$ are C^∞ ([11, Theorem 6.7.6]).

Theorem 12.1. *Let (M, g) be a C^∞ (resp. C^ω) riemannian manifold and ∇ a Yang-Mills C^3 -connection. Then there exists a C^3 -gauge transformation γ so that $\gamma^* \nabla$ is C^∞ (resp. C^ω).*

Corollary 12.2. *Let (M, g) be a simply connected C^ω -riemannian manifold. Let ∇_1 and ∇_2 be Yang-Mills connections on M . Assume that there is an open set U of M and a gauge transformation γ on U such that $\gamma^* \nabla_1 = \nabla_2$. Then γ extends to a global gauge transformation $\tilde{\gamma}$ so that $\tilde{\gamma}^* \nabla_1 = \nabla_2$ on M .*

Proof. We may assume that $\gamma = id$ on U and ∇_1 is C^ω . For $x \in U$ and $y \in M$, take a joining geodesic $c: [0, 1] \rightarrow M$ and a C^ω -tubular neighbourhood $V \cong (-\varepsilon, 1 + \varepsilon) \times D^{n-1}$ of $c[0, 1]$. Take a C^ω -frame of \mathfrak{g}_P on $\{0\} \times D^{n-1}$ and take the parallel extension $\{\xi_\rho\}$ (resp. $\{\tilde{\xi}_\rho\}$) for the direction $(-\varepsilon, 1 + \varepsilon)$ with respect to ∇_1 (resp. ∇_2). Let $\tilde{\gamma}$ be the gauge transformation on V which transforms $\{\xi_\rho\}$ to $\{\tilde{\xi}_\rho\}$. Since ∇_2 is C^ω with respect to $\{\tilde{\xi}_\rho\}$, $\tilde{\gamma}^{-1*} \nabla_2$ is C^ω with respect to $\{\xi_\rho\}$. But here $\tilde{\gamma} = id$ on U , which implies that $\tilde{\gamma}^{-1*} \nabla_2 = \nabla_1$ on V by analyticity. Moreover the extension of γ to $\tilde{\gamma}$ is unique and well-defined since M is simply connected. Q.E.D.

REMARK 12.3. This is an analogy of the unique extension theorem of Einstein metrics in [3, Section 5].

13. Some basic lemmas

Lemma 13.1 ([8, Lemma 4.3]). *Let v_t be a family of volume elements on M , E_t, F_t families of vector bundles over M with fiber metrics g_t^E, g_t^F and $Q_t: C^\infty(E_t) \rightarrow C^\infty(F_t)$ a family of differential operators of order k with injective symbol. Assume that $v_t, E_t, F_t, g_t^E, g_t^F$ and Q_t depend C^∞ -ly (resp. real analytically) on t . That is, there are bundle isomorphism $e_t: E_0 \rightarrow E_t$ and $f_t: F_0 \rightarrow F_t$ such that the coefficients of $e_t^* g_t^E, f_t^* g_t^F$ and $(f_t^{-1})_* \circ Q_t \circ (e_t)_*$ depend C^∞ -ly (resp. real analytically) on t . Then the dimension of the space $\text{Ker } Q_t$ is upper semicontinuous. If the dimension of the space $\text{Ker } Q_t$ is constant, then the decompositions*

$$(13.1.1) \quad H^s(E_t) = Q_t^*(H^{s+k}(F_t)) \oplus \text{Ker } Q_t,$$

$$(13.1.2) \quad H^s(F_t) = Q_t(H^{s+k}(E_t)) \oplus \text{Ker } Q_t^*$$

depend C^∞ -ly (resp. real analytically) on t , where Q_t^ is the formal adjoint operator of Q_t with respect to g_t^E, g_t^F and v_t . Moreover the isomorphisms*

$$(13.1.3) \quad Q_t^* + 1: Q_t(H^{s+2k}(E_t)) \oplus \text{Ker } Q_t \rightarrow H^s(E_t),$$

$$(13.1.4) \quad Q_t + 1: Q_t^*(H^{s+2k}(F_t)) \oplus \text{Ker } Q_t^* \rightarrow H^s(F_t)$$

also depend C^∞ -ly (resp. real analytically) on t .

Lemma 13.2 ([4, Theorem 3.12]). *In the real analytic category in Banach spaces, the implicit function theorem holds.*

Lemma 13.3 ([8, Lemma 13.7]). *Let E and F be vector bundles over M and E^C, F^C their complexifications. Let f be a C^∞ -cross section of E and $\psi: E \rightarrow F$ a fiber preserving C^∞ -map defined on an open set of E which contains the image of f . Assume that ψ has an extension to a fiber preserving map $\psi^C: E^C \rightarrow F^C$ defined on an open set of E^C such that the restriction ψ_x^C to each fiber E_x^C is holomorphic. Then the map $\Psi: H^s(E) \rightarrow H^s(F)$ defined by*

$$(13.3.1) \quad \Psi(u) = \Psi \circ u,$$

defined on an open neighbourhood of f , is real analytic provided that $s > [n/2] + 1$.

References

- [1] M.F. Atiyah, N.J. Hitchin and I.M. Singer: *Self-duality in four-dimensional riemannian geometry*, Proc. Roy. Soc. London A. **362** (1978), 425–461.
- [2] L. Bers, F. John and M. Schechter: *Partial differential equations*, Interscience Publishers, New York, 1964.
- [3] D.M. DeTurck and J.L. Kazdan: *Some regularity theorems in riemannian geo-*

- metry*, Ann. Sci. École Norm. Sup. **14** (1981), 249–260.
- [4] D.G. Ebin: *The manifold of riemannian metrics*, Global Analysis, Proc. Sympos. Pure Math. **15** (1968), 11–40.
- [5] S. Fučík, J. Nečas, J. Souček and V. Souček: *Spectral analysis of nonlinear operators*, Lecture Notes in Math., vol. 346, Springer, Berlin-Heidelberg-New York, 1973.
- [6] M. Itoh: *The moduli space of Yang-Mills connections over a Kähler surface is a complex manifold*, Osaka J. Math. **22** (1985), 845–862.
- [7] S. Kobayashi: *Curvature and stability of vector bundles*, Proc. Japan Acad. A. **58** (1982), 158–162.
- [8] M. Kuranishi: *New proof for the existence of locally complete families of complex structures*, Proc. of the conference on complex analysis, Minneapolis, 1964, 142–154, Springer-Verlag, New York.
- [9] N. Koiso: *Einstein metrics and complex structures*, Invent. Math. **73** (1983), 71–106.
- [10] N. Koiso: *Rigidity and stability of Einstein metrics - the case of compact symmetric spaces*, Osaka J. Math. **17** (1980), 51–73.
- [11] C.B. Morrey, Jr.: *Multiple integrals in the calculus of variations*, Springer-Verlag, Berlin, 1966.
- [12] R.S. Palais: *Foundation of global non-linear analysis*, Benjamin, New York, 1968.
- [13] D. Sundararaman: *Moduli, deformations and classifications of compact complex manifolds*, Research Notes in Math., Pitman Advanced Publishing Program, Boston-London-Melbourne, 1980.

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