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<th><strong>Title</strong></th>
<th>Yang-Mills connections and moduli space</th>
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<td><strong>Author(s)</strong></td>
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</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 24(1) P.147-P.171</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1987</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/3978">https://doi.org/10.18910/3978</a></td>
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<tr>
<td><strong>DOI</strong></td>
<td>10.18910/3978</td>
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Osaka University
0. Introduction

There are many researches on deformations of Yang-Mills connections over 4-dimensional manifolds. In this paper, we generalize the results into higher dimensional cases. In 4-dimensional case, the Hodge *-operator acts on $\Lambda^2 M$ and the notion of (anti-)self dual connection is introduced, which brings beautiful results in Atiyah, Singer and Hitchin [1]. Therefore, in higher dimensional cases, we have to assume some properties on the base riemannian manifold. In [1], it is already pointed out that if $M$ is a (2-dimensional) complex manifold, an anti-self dual connection defines a holomorphic structure of the bundle. Itoh [6] considers in full this situation, which we will generalize by the notion of "Einstein holomorphic connection" over a Kähler manifold. However, the moduli space of Yang-Mills connections over a higher dimensional Kähler manifold may have many singularities, and probably we can not expect that the moduli space becomes a manifold.

The fundamental notions in this paper come from [1], and fundamental idea comes from Koiso [9]. It is remarkable that the results for the moduli space of Einstein metrics and that of Yang-Mills connections are quite analogous. In fact we will get the following results.

**Theorem 2.7** (c.f. [9, Theorem 3.1]). The local pre-moduli space is a finite dimensional real analytic set.

**Corollary 6.5** (c.f. [9, Theorem 10.5]). If the initial structure (Einstein metric or Yang-Mills connection) is compatible with a complex structure, then also around structures are compatible with some complex structures.

**Theorem 9.3** (c.f. [9, Theorem 12.3]). Under some assumption, the local pre-moduli space has a canonical Kähler structure.

However, there is an important difference. For Einstein metrics, we have no effective obstruction spaces for deformation ([9, Proposition 5.4]), but for Yang-Mills connections we have one (Theorem 6.9).
1. Yang-Mills connections

Let \((M, g)\) be a compact riemannian manifold, \(G\) a compact Lie group, \(P\) a principal \(G\)-bundle over \(M\). Denote by \(\mathfrak{g}\) the Lie algebra of \(G\) and by \(G_r\) (resp. \(\mathfrak{g}_r\)) the associated fiber bundle \(P \times_{\mathfrak{g}_r} G\) (resp. \(P \times_{\mathfrak{g}_r} \mathfrak{g}\)). The space \(C\) of all connections of \(P\) is an affine space whose standard vector space is \(C^\infty(\wedge^2 \otimes \mathfrak{g}_r)\), where \(\wedge^p\) denotes the vector bundle of \(p\)-forms on \(M\) (see [1, p430]). We fix an effective representation \(G \to GL(V)\) and identify a connection of \(P\) with a covariant derivation on \(P \times \mathfrak{g}V\) or \(P \times \mathfrak{g}End(V)\). In this sense, for a connection \(\nabla\) of \(P\) and an element \(A\) of \(C^\infty(\Lambda^1 \otimes \mathfrak{g}_r)\) the curvature tensor is transformed as

\[
R^\nabla + A = R^\nabla + d^\nabla A + [A \wedge A],
\]

where \(d^\nabla\) and \([\cdot \wedge \cdot]\) are defined by

\[
(d^\nabla A)(X, Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X)
\]

and

\[
[A \wedge B](X, Y) = \frac{1}{2} ([A(X), B(Y)] - [A(Y), B(X)]).
\]

We fix a \(G\)-invariant inner product on \(\mathfrak{g}\). Then the vector bundle \(\mathfrak{g}_r\) admits a canonical fiber inner product \((\cdot, \cdot)\) and the vector space \(C^\infty(\mathfrak{g}_r)\) admits a (global) inner product \(\langle \cdot, \cdot \rangle\). We denote by \(|| \cdot ||\) the \(L^2\)-norm defined by \(\langle \cdot, \cdot \rangle\). Define an action integral \(F_{YM}\) for connections by

\[
F_{YM}(\nabla) = \frac{1}{2} ||R^\nabla||^2.
\]

**Definition 1.1.** The function \(F_{YM}\) on \(C\) is called the Yang-Mills functional, its Euler-Lagrange equation is called the Yang-Mills equation and its solution is called a Yang-Mills connection.

Let us represent the Yang-Mills equation by a tensor equation. Let \(\nabla_t\) be a 1-parameter family of connections on \(P\) and set \(\nabla = \nabla_0\) and \(A = (d/dt)_0 \nabla_t\). Then

\[
\left. \frac{d}{dt} \right|_0 F_{YM}(\nabla_t) = \left. \frac{d}{dt} \right|_0 R^{\nabla_t}, R^\nabla
\]

\[
= \langle d^\nabla A, R^\nabla \rangle = \langle A, (d^\nabla)^* R^\nabla \rangle
\]

\[
= 2\langle A, d^\nabla R^\nabla \rangle,
\]

where \((\cdot)^*\) denotes the formal adjoint and the operator \(d^\nabla\) from \(C^\infty(\Lambda^2 \otimes \mathfrak{g}_r)\) to \(C^\infty(\Lambda^3 \otimes \mathfrak{g}_r)\) is defined by

\[
(\delta^\nabla s)_{i_{-1}^{\cdots}i_{-p}} = -\nabla^i_{i_{-1}^{\cdots}i_{-p}}.
\]
Thus Yang-Mills equation becomes

\[ E_{YM}(\nabla) \equiv \delta^* R^\alpha = 0 \]

Next, we consider infinitesimal deformations of a Yang-Mills connection. From now on, we enlarge the space of \( C^\infty \)-sections to the space of \( H^s \)-sections, and denote by \( H^s(E) \) the space of all \( H^s \)-sections of a fiber bundle \( E \) over \( M \), where \( s \) is assumed to be sufficiently large. \( H^s \)-norm will be denoted by \( || \cdot ||_s \). The completion of the space \( \mathcal{C} \) etc. with respect to \( H^s \)-topology will be denoted by \( \mathcal{C}' \) etc.

**Definition 1.2.** Let \( \nabla \) be a Yang-Mills connection. A solution of the equation

\[ (E_{YM})^\prime(A) = 0 \]

is called a Yang-Mills infinitesimal deformation, where \( \cdot \) denotes the Fréchet derivative. The space of all Yang-Mills \( H^s \)-infinitesimal deformations is denoted by \( \text{YMID}^s(\nabla) \).

**Lemma 1.3.** \( (E_{YM})^\prime(A) = \delta^* d^\alpha A + \text{tr}[R^\alpha, A] \)

where \( \text{tr}[R^\alpha, A] = g^{ii}[R^\alpha_{ii}, A] \).

Proof. \[ \frac{d}{dt} \bigg|_0 (\delta^* d^\alpha R^\alpha) = -\frac{d}{dt} \bigg|_0 \nabla^i R^\alpha_{ii} \]
\[ = -[A^t, R^\alpha_{ii}] - \nabla^i \left( \frac{d}{dt} \bigg|_0 R^\alpha_{ii} \right), \]
where \( A = \frac{d}{dt} \nabla^i \). Q.E.D.

The automorphism group \( \mathcal{G} = C^\infty(G_P) \) of the bundle \( P \) is called the gauge group of \( P \), and it acts on \( \mathcal{C} \) by pull-back as

\[ \gamma^* \nabla = \nabla + \gamma^{-1} \nabla \gamma \quad (\gamma \in \mathcal{G}, \nabla \in \mathcal{C}). \]

If \( \nabla \) is a Yang-Mills connection, then \( \gamma^* \nabla \) is so. In particular, if \( \gamma_t \) is a 1-parameter family of gauge transformations such that \( \gamma_0 = \text{id}_P \), then \( (d/dt)_0 \gamma^* \nabla = \nabla((d/dt)_0 \gamma_t) \) becomes a Yang-Mills infinitesimal deformation of \( \nabla \).

**Definition 1.4.** Let \( \nabla \) be a Yang-Mills connection. A Yang-Mills infinitesimal deformation is said to be trivial if it coincides with \( \nabla v \) for some \( v \in H^{s+1}(g_P) \). A Yang-Mills infinitesimal deformation is said to be essential if it is orthogonal to all trivial Yang-Mills infinitesimal deformations. The space of all essential Yang-Mills infinitesimal deformations is denoted by \( \text{YMEID}(\nabla) \).
By definition, a Yang-Mills infinitesimal deformation \( A \in H'(\Lambda^1 \otimes g) \) is essential if and only if \( \langle \nabla v, A \rangle = 0 \) for any \( v \in H^{+1}(g) \), which is equivalent to that \( \delta^v A = 0 \). Thus the defining equation of the space \( \text{YMEID}(\nabla) \) becomes

\[
\delta^v d^v A + \text{tr}[R^v, A] = 0
\]

and

\[
\delta^v A = 0 .
\]

This system is elliptic, and so the space \( \text{YMEID}(\nabla) \) is finite dimensional and each element is \( C^\infty \).

The following lemma will be used later.

**Lemma 1.5.** For any connection \( \nabla \), equality

\[
\delta^v E_{YM}(\nabla) = 0
\]

and decomposition

\[
H'(\Lambda^1 \otimes g) = \text{Im}(\nabla | H^{+1}) \oplus \text{Ker} \delta^v
\]

hold. If \( \nabla \) is a Yang-Mills connection, then the sequence

\[
C^\infty(g) \xrightarrow{\nabla} C^\infty(\Lambda^1 \otimes g) \xrightarrow{(E_{YM})_v} C^\infty(\Lambda^1 \otimes g) \xrightarrow{\delta^v} C^\infty(g)
\]

is an elliptic complex. In particular, the following decompositions as Hilbert space hold.

\[
H'(\Lambda^1 \otimes g) = \text{Im}(E_{YM}' | H^{+2}) \oplus \text{YMEID} \oplus \text{Im}(\nabla | H^{+1}) ,
\]

\[
\text{Ker}(E_{YM}' | H') = \text{YMEID} \oplus \text{Im}(\nabla | H^{+1}) ,
\]

\[
\text{Ker}(\delta^v | H') = \text{YMEID} \oplus \text{Im}(E_{YM}' | H^{+2}) .
\]

**Proof.** Equality (1.5.1) is easy to check directly, but here we show it using an idea from variation. Since the function \( F_{YM} \) on \( C \) is invariant under the action of the group \( G \), we see that

\[
(F_{YM})|_v(\nabla v) = 0 \quad \text{for any} \quad v \in C^\infty(g) ,
\]

i.e.,

\[
\langle E_{YM}(\nabla), \nabla v \rangle = 0 ,
\]

which implies (1.5.1). Decomposition (1.5.2) follows from Lemma 13.1. Let \( \nabla \) be a Yang-Mills connection. Then the space \( \text{Im} \nabla \) is the space of trivial infinitesimal deformations, hence \( (E_{YM})_v \circ \nabla = 0 \). From equality (1.5.1) we derive the equality

\[
(\delta^v)' \cdot E_{YM}(\nabla) + \delta^v(E_{YM})_v = 0 .
\]
Thus sequence (1.5.3) is a complex, and its ellipticity is easy to check. Therefore we have decomposition

\[(1.5.7) \quad H^s(\Lambda^1 \otimes \mathfrak{g}_P) = \text{Im}(\nabla | H^{s+1}) \oplus \text{Im}((E_{YM})_\gamma^* | H^{s+2}) \oplus \text{Ker} \nabla^* \cap \text{Ker}(E_{YM})_\gamma^* .\]

Here we have $\nabla^* = \delta^\gamma$ and so we get (1.5.4) if we show that $(E_{YM})_\gamma^*$ is self-adjoint. But we see

\[\langle (E_{YM})_\gamma(A), B \rangle = (\text{Hess} F_{YM})(A, B) ,\]

regarding $F_{YM}$ as a function on $C^\omega$, hence $(E_{YM})_\gamma^*$ is symmetric with respect to $\langle , \rangle$. Since the space $\text{Im}(\nabla | H^{s+1})$ is closed in $H^s(\Lambda^1 \otimes \mathfrak{g}_P)$, decomposition (1.5.5) is reduced to the definition of the space $YMEID(\nabla)$. If we remark that the space $\text{Ker}(\delta^\gamma | H^s)$ is the orthogonal complement of the space $\text{Im}(\nabla | H^{s+1})$, then (1.5.6) follows from (1.5.4).

Q.E.D.

2. Moduli space of Yang-Mills connections

To define "local pre-moduli space", we need some preparation. We use some basic facts on $C^\omega$-maps in Hilbert space category. (See Lemmas 13.2, 13.3.)

Remark that the space $\mathcal{Q}^s = H^s(G_P)$ is a $C^\omega$-(infinite dimensional) Lie group. In fact, if we take a complexification $G^c$ of $G$ and set $G^c = P \times \text{Ad}_G G$, then to multiply and to get inverse element are extended to maps: $H^s(G^c) \times H^s(G^c) \to H^s(G^c)$, $H^s(G^c) \to H^s(G^c)$ so that the restriction on each fiber is holomorphic. Therefore, by Lemma 13.3, they are $C^\omega$.

Let $\nabla$ be a connection and $\mathcal{Q}_\nabla$ the group of isotropy, i.e., $\mathcal{Q}_\nabla = \{ \gamma \in \mathcal{Q}^s; \nabla \gamma = 0 \}$ (see (1.3.1)). Since $\nabla$ is an elliptic operator, we see that $\mathcal{Q}_\nabla = \mathcal{Q}$ and so we simply denote it by $\mathcal{Q}$. The exponential map $\exp: g \to G$ defines a $C^\omega$-map $\exp: H^s(\mathfrak{g}_P) \to \mathcal{Q}$ (by Lemma 13.3) and we can easily check that the quotient space $\mathcal{Q} \backslash \mathcal{Q}^s$ admits a $C^\omega$-structure and that there exists a local cross section $\chi^s: \mathcal{Q} \backslash \mathcal{Q}^s \to \mathcal{Q}$ so that the domain $\mathcal{U}^s$ is uniform on $s$, i.e., equations $\mathcal{U}^{s+1} = (\mathcal{Q} \backslash \mathcal{Q}^{s+1}) \cap \mathcal{U}^s$ and $\chi^{s+1} = \chi^s | U^{s+1}$ hold for any $s$. Define a $C^\omega$-map

\[\mathcal{H}^s: U^{s+1} \times (\nabla + \text{Ker}(\delta^\gamma | H^s)) \to C^s\]

by

\[\mathcal{H}^s(u, \nabla) = \chi^{s+1}(u) \star \nabla_1 .\]

Its derivative at $([\text{id}], \nabla)$ is given by

\[(v, A) \to \nabla(\chi_{[\text{id}])(v)) + A ,\]

and is bijective by decomposition (1.5.2). Therefore there exists a local inverse map $(\mathcal{H}^s)^{-1} = q^s \times p^s: C^s \to U^{s+1} \times (\nabla + \text{Ker}(\delta^\gamma | H^s))$. By an analogous way with Ebin's Slice theorem in [4, Theorem 7.1], we get the following
Proposition 2.1. Let \( \nabla \in C^\infty \). There exist a neighbourhood \( U^{s+1} \) of \([id]\) in \( G \setminus G^{s+1} \), a neighbourhood \( V^* \) of \( \nabla \) in \( \nabla + \text{Ker}(\delta^*|H') \) and a neighbourhood \( W^* \) of \( \nabla \) in \( C^s \) so that

\[
\mathcal{A}: U^{s+1} \times V^* \to W^*
\]
is a \( C^\infty \)-diffeomorphism. Moreover if \( \gamma \in \mathcal{G}_\tau \) then \( \gamma^*(V^*) = V^* \) and \( \gamma^*(V^*) \cap V^* \neq \phi \) if and only if \( \gamma \in \mathcal{G}_\tau \).

Proof. Only the last statement is not shown. Since \( \mathcal{G}_\tau \) is a compact group and preserves \( \text{Ker} \gamma^* \), taking \( \cap_{\gamma \in \mathcal{G}_\tau} \gamma^*(V^*) \) if necessary, we may assume that \( \gamma^*(V^*) = V^* \) if \( \gamma \in \mathcal{G}_\tau \). We now show that if \( \gamma^*(V^*) \cap V^* \neq \phi \) then \( \gamma \in \mathcal{G}_\tau \). If \([\gamma]\) belongs to \( U^{s+1} \), then bijectivity of \( \mathcal{A}^* \) implies that \( \gamma \in \mathcal{G}_\tau \). Hence we assume that for any \( V^* \) there is \( \gamma \in \mathcal{G}^{s+1} \) such that \( \gamma^*(V^*) \cap V^* \neq \phi \) but \([\gamma] \notin U^{s+1} \). This means that there are a sequence \( \{\gamma_i\} \) in \( \mathcal{G}^{s+1} \) and sequences \( \{V_{il}\} \) and \( \{V_{il}\} \) in \( (\nabla + \text{Ker}(\delta^*|H')) \) which converge to \( \nabla \) such that \( \gamma_i^* V_{il} = V_{2i} \) and \([\gamma_i] \notin U^{s+1} \).

Then by the following lemma, a subsequence of \( \{\gamma_i\} \) converges to an element \( \gamma_* \) in \( \mathcal{G}^{s+1} \), and so \( \gamma_* \in \mathcal{G}_\tau \) and \([\gamma_i] \in U^{s+1} \) for some \( i \), which contradicts the assumption. Q.E.D.

Lemma 2.2. Let \( \{\gamma_i\} \), \( \{V_{1i}\} \) and \( \{V_{2i}\} \) be as above. Then a subsequence of \( \{\gamma_i\} \) converges in \( \mathcal{G}^{s+1} \).

Proof. The equation \( \gamma_i^* V_{1i} = V_{2i} \) is equivalent to the equation \( \gamma_i^* V_{1i} = \nabla_{V_{1i}}^\nabla_{V_{2i}} \). Set \( A_i = \nabla_{V_{1i}}^\nabla_{V_{2i}} \) and \( B_i = \nabla_{V_{1i}}^\nabla_{V_{2i}} \). Then we see that \( \nabla \gamma_i = \gamma_i(B_i - A_i) - A_i \gamma_i \). In general, we have

\[
||\gamma_i|| < C_1 ||\gamma_i||_{l-1} + C_2 ||\gamma_i||_l
\]
for some real number \( C_1 \) and \( C_2 \), and \( ||\gamma_i||_l < C_3 \) since \( G \) is compact. Therefore

\[
||\gamma_i||_l < C_1 ||\gamma_i||_{l-1} + C_2 ||\gamma_i||_l + C_2 \cdot C_3
\]

Since the multiplication: \( H^s \times H^s \to H^s \) for \( t \leq s \) is continuous (see [12, Section 9]), we see that

\[
||\gamma_i||_l < C_1 ||\gamma_i||_{l-1} + C_2 \cdot C_3 \quad (t - 1 \leq s).
\]

Thus we see by induction that the sequence \( ||\gamma_i||_{l+1} \) is bounded, and so a subsequence of \( \{\gamma_i\} \) converges in \( H^s \), which we replace by \( \{\gamma_i\} \). Then we have

\[
\nabla(\gamma_i - \gamma_j) = (\gamma_i(B_i - A_i) - A_i \gamma_i) - (\gamma_j(B_j - A_j) - A_j \gamma_j)
\]

and so

\[
||\gamma_i - \gamma_j||_{l+1} < C_4 ||\gamma_i - \gamma_j||_l + C_5 ||\gamma_i - \gamma_j||_l
\]

for some \( C_4 \) and \( C_5 \), and \( \{\gamma_i\} \) is a Cauchy sequence in \( H^{s+1} \)-topology. Q.E.D.
Definition 2.3. The manifold $V'$ in Proposition 2.1 is called the slice at $V$ and is denoted by $S'_V$.

A priori, the slice may degenerate for $s \to \infty$. But we have following lemmas, which say that we can take slices "uniformly" and they are "natural".

Lemma 2.4. Let $t \geq s$ and set $U^{t+1} = U^{t+1} \cap (\mathcal{O} \setminus \mathcal{O}^{t+1})$, $V' = V' \cap C'$ and $W' = W' \cap C'$. Then Proposition 2.1 holds when $s$ is replaced by $t$.

Proof. It is sufficient to prove for $t = s + 1$. The map

$$\mathcal{A}^{t+1} : U^{t+2} \times V^{t+1} \to W^{t+1}$$

is a $C^\infty$-injective immersion.

(surjectivity) Let $\nabla_1 \in W^{t+1}$. Then there is $\gamma \in \pi^{-1}(U^{t+1})$ so that $\gamma \ast \nabla_1 \in V'$. Set $A_1 = \nabla_1 - \nabla$ and $A_2 = \gamma \ast \nabla_1 - \nabla$. Then $A_1 \in H^{t+1}(\mathfrak{g}_p)$, $A_2 \in H'(\mathfrak{g}_p)$, and $\nabla \gamma = \gamma A_2 - A_1 \gamma$. Since $\delta^s A_2 = 0$, we have

$$\delta^s \nabla \gamma = \text{tr}(\nabla \gamma \otimes A_2) - \delta^s (A_1 \gamma),$$

where $\nabla \gamma \otimes A_2 \in H'$ and $\delta^s (A_1 \gamma) \in H'$. Thus $\gamma \in \mathcal{O}^{t+2}$.

(surjectivity of derivative) Let $u_0 \in U^{t+2}$ and $\nabla + A_0 \in V^{t+1}$. Then the derivative of the map $\mathcal{A}^t$ is given by

$$(\mathcal{A}^t)'(u_0, \nabla + A_0) (u', A') = \chi(u_0) \ast \{\nabla(\varphi'(u')) + [A_0, \varphi'(u')] + A'\}$$

where $\varphi$ is defined by $\varphi(u) = \chi(u) \ast \chi(u_0)^{-1}$. Let $B$ be any element of $H^{t+1}(\mathfrak{g}_p)$. Then there are $u' \in T_{u_0}U^{t+1}$ and $A' \in T_{A_0}V'$ so that

$$\chi(u_0) \ast \{\nabla(\varphi'(u')) + [A_0, \varphi'(u')] + A'\} = B.$$

This implies that

$$\delta^s \nabla(\varphi'(u')) = \delta^s (\chi(u_0)^{-1} * B - [A_0, \varphi'(u')]),$$

where the right hand side belongs to $H^{t-1}$. Thus $\varphi'(u') \in H^{t+1}$, and so the right hand side belongs to $H'$, and $\varphi'(u') \in H'^{t+2}$. Therefore $u' \in H'^{t+2}$ and $A' \in H'^{t+1}$.

Q.E.D.

Lemma 2.5. Let $\nabla_1 \in S'_V$. If there is $\gamma \in \mathcal{O}^{t+1}$ such that $\gamma \ast \nabla_1 \in C'$, then $\nabla_1 \in S'_V$. In particular, if $\gamma \ast \nabla_1 \in C^\infty$, then $\nabla_1 \in C^\infty$.

Proof. Let $\{\gamma_i\}$ be a sequence in $\mathcal{O}^{t+1}$ which converges to $\gamma$ in $H^{t+1}$-topology. Then $\gamma_i \ast \gamma \ast \nabla_1 \to \nabla_1$ in $C'$, and so for some $i$ $\gamma_i \ast \gamma \ast \nabla_1$ belongs to $W'$ in Proposition 2.1. But here $\gamma_i \ast \gamma \ast \nabla_1 \in C'$. Therefore by Lemma 2.4 $\gamma_i \ast \gamma \ast \nabla_1 \in W'$, and so $\pi(\gamma \gamma_i) \in U^{t+1}$ and $\nabla_1 \in S'_V$.

Q.E.D.
Corollary 2.6. Let $\nabla_1 \in S^\Psi_\nabla$ be a Yang-Mills connection. Then $\nabla_1 \in S^\Psi_\nabla$.

Proof. By Theorem 12.1, $\nabla_1$ satisfies the condition in Lemma 2.5. Q.E.D.

Theorem 2.7. Let $\nabla$ be a Yang-Mills connection. There are a neighbourhood $U^s$ of $\nabla$ in $S^\Psi_\nabla$ and a closed $C^\omega$-submanifold $Z$ of $U^s$ whose tangent space at $\nabla$ coincides with $YMEID(\nabla)$ such that the set $YMLPM(\nabla)$ of all Yang-Mills connections in $U^s$ is a real analytic set of $Z$. Moreover, the spaces $Z$ and $YMLPM(\nabla)$ do not depend on $s$.

Proof. Set $\varphi^s = E_{YM}|S^\Psi_\nabla$. Then by (1.5.2) we see

$$\text{Im } \varphi^s = E_{YM}'(\text{Ker}(\delta^s|H^t)) = \text{Im}(E_{YM}'|H^t).$$

On the other hand, from (1.5.4) and (1.5.5) we have

$$H^{t-i}(A^1 \otimes \text{Gr}) = \text{Im}(E_{YM}'|H') \oplus \text{Ker}(E_{YM}'|H^{t-i}).$$

Let $p^i$ (resp. $q^i$) be the projection to the first (resp. second) component. Then the $C^\omega$-map $p^i \circ \varphi^s$ has surjective derivative at $\nabla$ and by the implicit function theorem there is a neighbourhood $U^s$ of $\nabla$ in $S^\Psi$ so that the set $Z = \{\nabla_1 \in U^s | p^i \circ \varphi^s(\nabla_1) = 0\}$ is a $C^\omega$-submanifold of $U^s$. The tangent space $T_\nabla Z$ coincides with the space $YMEID(\nabla)$ and the set $YMLPM(\nabla)$ is the zero of the map $q^i \circ \varphi^s$ on $Z$.

Next we have to show that if we set $Z^t = Z \cap S^\Psi_\nabla$ and $U^t = U^s \cap S^\Psi_\nabla$ for $t \geq s$ then $Z^t$ coincides with $Z$ as manifold and $p^i \circ \varphi^t$ has surjective derivative at any point of $Z^t$. Let $\nabla + A \in Z$. Then by the definition of $Z$ and Lemma 1.5 we have

$$\delta^s A = 0, \quad E_{YM}'|V(\delta^s|H^t)) = 0.$$

Since this is an elliptic system, $A$ is $C^\omega$, and so $Z^t = Z$ as set. Let $\nabla_1 \in Z^t$. Since $p^i \circ \varphi^t$ has surjective derivative at $\nabla_1$, for any $A \in \text{Im}(E_{YM}'|H^t)$ there are $B \in \text{Ker}(\delta^s|H^t)$ and $C \in \text{Ker}(E_{YM}'|H^t)$ so that $(\varphi^t)'_c(B) = A + C$. Then

$$E_{YM}'|V(\varphi^t)'_c(B) = E_{YM}'|V(A) \in H^{t-i},$$

and $\delta^s B = 0$. Therefore $B \in H^t$, which implies that $p^i \circ \varphi^t$ has surjective derivative at $\nabla_1$, and so $Z^t$ is a closed $C^\omega$-submanifold of $U^t$. Moreover, the identity: $Z^t \to Z$ is bijective and its derivative also, hence is a diffeomorphism. Q.E.D.

Definition 2.8. The set $YMLPM(\nabla)$ is called the local pre-moduli space of Yang-Mills connections around $\nabla$ and the set $Z$ is called its support manifold.

We may summarize results as

Theorem 2.9. Let $\nabla$ be a Yang-Mills connection. The local pre-moduli space $YMLPM(\nabla)$ of Yang-Mills connections has the following properties. a)
(b) If \( \nabla_1 \) is a Yang-Mills connection sufficiently close to \( \nabla \), then there is \( \gamma \in \mathcal{G}^{+1} \) so that \( \gamma^* \nabla_1 \in \text{YMLPM}(\nabla) \). (c) If \( \gamma^* \text{YMLPM}(\nabla) \cap \text{YMLPM}(\nabla) \neq \emptyset \) for \( \gamma \in \mathcal{G}^{+1} \), then \( \gamma^* \nabla = \nabla \), i.e., \( \nabla \gamma = 0 \).

**Remark 2.10.** The global moduli space \( \mathcal{G} \setminus \{ \text{Yang-Mills connections} \} \) is locally homeomorphic with the coset space \( \mathcal{G}_\tau \setminus \text{YMLPM}(\nabla) \). Since \( \mathcal{G}_\tau \) is a compact Lie group, almost all local properties of the global moduli space is reduced to that of \( \text{YMLPM}(\nabla) \).

**Corollary 2.11.** (1) Let \( \nabla \) be a Yang-Mills connection. If \( \text{YMID}(\nabla) = 0 \), then \( [\nabla] \) is isolated in the global moduli space of Yang-Mills connections. (2) The Yang-Mills functional \( F_{YM} \) is constant on the space \( \text{YMLPM} \), and locally constant on the global moduli space. (3) If a connection \( \nabla \) minimizes the functional \( F_{YM} \) on \( C \), then any Yang-Mills connection sufficiently close to \( \nabla \) also minimizes \( F_{YM} \). (4) Any Yang-Mills connection sufficiently close to a flat connection is flat. (5) Let \( M \) be 4-dimensional. Any Yang-Mills connection sufficiently close to a self-dual (resp. anti self-dual) connection is self-dual (resp. anti self-dual).

**Proof.** (1) The assumption implies that the support manifold \( Z \) is a point. (2), (3) The set \( \text{YMLPM} \) forms a real analytic set and its points are critical points of \( F_{YM} \). (4) A connection \( \nabla \) is flat if and only if \( F_{YM}(\nabla) = 0 \). (5) A connection \( \nabla \) is (anti) self-dual if and only if \( F_{YM}(\nabla) \) coincides with a topological invariant of the principal bundle (see [1, p. 432]). Q.E.D.

### 3. The obstruction for deformations

We have shown that the local pre-moduli space \( \text{YMLPM}(\nabla) \) is a real analytic set of the support manifold. Therefore we want to know when \( \text{YMLPM} \) coincides with the support manifold. In this section we introduce a notion which will be used later.

Let \( \mathcal{P} \) be an open set of a Hilbert space, \( \mathcal{Q} \) and \( \mathcal{R} \) Hilbert spaces. Let \( E: \mathcal{P} \to \mathcal{Q} \) and \( I: \mathcal{P} \times \mathcal{Q} \to \mathcal{R} \) be \( C^\infty \)-maps and define \( I_\rho: \mathcal{Q} \to \mathcal{R} \) for each fixed \( \rho \in \mathcal{P} \).

**Definition 3.1.** If \( I_\rho \) is linear for each \( \rho \in \mathcal{P} \) and \( I_\rho(E(\rho))=0 \) for all \( \rho \in \mathcal{P} \), then \( I \) is called an identity for \( E \).

If \( I \) is an identity for \( E \) and \( E(\rho)=0 \), then we see that \( I_\rho \circ E_\rho=0 \), i.e., \( \text{Im} \ E_\rho \subset \text{Ker} \ I_\rho \).

**Definition 3.2.** Let \( I \) be an identity for \( E \) and assume that \( E(\rho)=0 \). The space \( \text{Ker} \ I_\rho/\text{Im} \ E_\rho \) is called the obstruction space for \( E \)-deformations of \( \rho \) with respect to \( I \).

**Lemma 3.3.** Let \( I \) be an identity for \( E \) and \( \rho \in E^{-1}(0) \). If the obstruction space \( \text{Ker} \ I_\rho/\text{Im} \ E_\rho \) vanishes, then the set \( E^{-1}(0) \) around \( \rho \) forms a manifold whose
tangent space at \( p \) coincides with \( \text{Ker} \ E'_p \), provided that one of the following conditions is satisfied. (1) The map \( E \) is \( C^\omega \). (2) The space \( \text{Im} \ I_p \) is closed in \( \mathcal{R} \).

Proof. By a similar way as in Proof of Theorem 2.7, we see that there exists the "support manifold" \( Z \) whose tangent space at \( p \) coincides with \( \text{Ker} \ E'_p \) such that \( E^{-1}(0)=(p_c \circ E|Z)^{-1}(0) \) around \( p \), where \( p_c \) is the projection to a complement \( C \) of \( \text{Im} \ E'_p \) in \( \mathcal{Q} \). Set \( \hat{E}=E|Z: Z \to C \) and \( \hat{I}=I|(Z \times C): Z \times C \to \mathcal{R} \). It is enough to prove that \( \hat{E}=0 \). Remark that \( \text{Ker} \ I_p = \text{Ker} \ I_p \cap C = \text{Im} \ E'_p \cap C = 0 \) and so \( \hat{I}_p \) is injective. Assume condition (2). Then \( \text{Im} \ I_p = I_p(C) = I_p(\text{Im} E'_p(\oplus C)) = \text{Im} I_p \), and so \( I_p \) is closed in \( \mathcal{R} \), hence \( \hat{I}_p \) is an isomorphism from \( C \) into \( \mathcal{R} \). Therefore \( \hat{I}_p \) is injective if \( p_1 \in Z \) is sufficiently close to \( p \). But here we know that \( \hat{I}_p(\hat{E}(p_1))=0 \). Thus \( \hat{E}(p_1)=0 \).

Next we assume condition (1) and show the \( r \)-th derivative \( \hat{E}^{(r)} \) vanishes for all \( r \geq 0 \) by induction. By taking \( r \)-th derivative of the identity \( \hat{I}_{p_1}(\hat{E}(p_1))=0 \) and setting \( v=\frac{d}{dt} \big|_{0} p_1 \), we get

\[
\hat{I}_{p_1}(\hat{E}^{(r)}(v, \ldots, v)) = -\sum_{i=1}^{r}(\frac{d}{dt})^i \big|_{0} \hat{I}_{p_1}(\frac{d}{dt})^{r-i} \big|_{0} \hat{E}(p_1).
\]

By induction we may assume that the right hand side vanishes, and so the left hand side vanishes. But we know that \( \hat{I}_p \) is injective, hence \( \hat{E}^{(r)}=0 \). Q.E.D.

**Remark 3.4.** This Lemma essentially is "Kuranishi's method" ([8]).

### 4. The deformation of Yang-Mills connection caused by a deformation of base metric

We want apply Lemma 3.3 to a deformations of Yang-Mills connection. Unfortunately, it is not possible if we use Yang-Mills equation itself. In 6, we will introduce the notion of "Einstein holomorphic connection" and apply Lemma 3.3.

Now, by equation (1.5.1), \( \delta^v \) is an identity for \( E_{YM} \), and the obstruction space

\[
\text{Ker} \delta^v/\text{Im} E_{YM}' \cong YMEID(\nabla)
\]

by equation (1.5.6).

**Proposition 4.1.** Let \( \nabla \) be a Yang-Mills connection. The obstruction space for \( E_{YM} \)-deformation of \( \nabla \) with respect to \( \delta^v \) is isomorphic with the space \( YMEID(\nabla) \) of essential infinitesimal deformations.

Hence we apply Lemma 3.3 to the situation that we deform the metric \( g \) on \( M \) and Yang-Mills connection follows it. Denote by \( \mathcal{M}' \) the space of all \( H^\prime \)-riemannian metrics on \( M \) and define maps \( E, I \) by
Then $I$ is an identity for $E$.

**Theorem 4.2.** Let $\nabla$ be a Yang-Mills connection over $(M,g)$, i.e., $E(g, \nabla) = 0$. If $YMEID(\nabla) = 0$, then for any deformation $g_t$ of $g$ there exists a 1-parameter family of connections $\nabla_t$ so that each $\nabla_t$ is a Yang-Mills connection with respect to $g_t$, provided that $|t|$ is small.

Proof. The obstruction space for $E$-deformation of $(g, \nabla)$ with respect to $I$ coincides with the space

$$\text{Ker } \delta^*_E / \text{Im } E'_{(g, \nabla)},$$

which is a quotient space of the space $YMEID(\nabla)$ by equation (1.5.6), and vanishes by assumption. Therefore by Lemma 3.3 the set $E^{-1}(0)$ around $(g, \nabla)$ forms a manifold whose tangent space at $(g, \nabla)$ coincides with $\text{Ker } E'_{(g, \nabla)}$. But here the projection map from $\text{Ker } E'_{(g, \nabla)}$ to $T_g \mathcal{M}^{t+1}$ is surjective, which completes the proof by the implicit function theorem. In fact, for any $h \in T_g \mathcal{M}^{t+1}$, we get $E'_{(g, \nabla)}(h, 0) \in \text{Ker } \delta^*_E$, therefore by equation (1.5.6) and assumption, there is $A \in T_v C^{t+2}$ such that $\text{Ker } E'_v(h, 0) = E_{YM}(A)$, i.e., $E'_{(g, \nabla)}(h, -A) = 0$. Q.E.D.

5. **Holomorphic structures**

Let $M$ be a compact complex manifold and $P^c$ a principal $G^c$-bundle, where $G^c$ is a complexification of $G$. A $G^c$-invariant almost complex structure on $P^c$ is called an almost holomorphic structure of $P^c$. If it is integrable, then it is called a holomorphic structure. An almost holomorphic structure can be regarded as a first order differential operator

$$\bar{\partial}: C^\infty(g^c) \to C^\infty(\Lambda^{0,1} \otimes g^c),$$

(5.0.1)

and it is a holomorphic structure if and only if the torsion $T(\bar{\partial})$ of $\bar{\partial}$ vanishes, where $T(\bar{\partial}) \in C^\infty(\Lambda^{0,2} \otimes g^c)$ is defined by

$$T(\bar{\partial})(X, Y)v = \bar{\partial}_X \bar{\partial}_Y v - \bar{\partial}_Y \bar{\partial}_X v - \bar{\partial}_{[X,Y]} v.$$

(5.0.2)

An almost holomorphic structure $\bar{\partial}$ extends to the operators

$$\bar{\partial}^\beta: C^\infty(\Lambda^{0,\beta} \otimes g^c) \to C^\infty(\Lambda^{0,\beta+1} \otimes g^c),$$

(5.0.3)

and if $\bar{\partial}$ is a holomorphic structure, then they define an elliptic complex and the cohomology groups

$$H^{0,\beta}(g^c) = \text{Ker } \bar{\partial}^\beta / \text{Im } \bar{\partial}^{\beta+1}$$

(5.0.4)
are defined.

We can study deformations of holomorphic structures by a similar way as deformations of complex structures on manifold (c.f. [13, pp 172-176]), but we use here notations similar with [6].

The space $\mathcal{A}$ of all almost holomorphic structures forms an affine space with standard vector space $C^\infty(\Lambda^{0,1}\otimes g^e)$, and the complex gauge group $G^C = C^\infty(G^C)$ acts on it. Let $\mathcal{A}_G$ be the group of isotropy, i.e., $\mathcal{A}_G = \{ \gamma \in G^C | \gamma \ast = \delta \}$. Then the $H^{c+1}$-gauge group $G^{c+1}$ acts on $\mathcal{A}$ holomorphically and the coset space $\mathcal{A}/G^{c+1}$ forms a complex analytic manifold. The following proposition is proved by a similar manner as Proposition 2.1.

**Proposition 5.1.** Let $\delta \in \mathcal{A}$. There exist a neighbourhood $S_{\delta}^c$ of $\delta$ in $\delta$ $+$ Ker $(\delta^* | H^c)$, a neighbourhood $U^{c+1}$ of [id] in $\mathcal{A}_G \setminus G^{c+1}$ and a neighbourhood $W^{c+1}$ of $\delta$ in $\mathcal{A}$ so that the action

$$\mathcal{A}^{c+1} : U^{c+1} \times S_{\delta}^c \to W^{c+1}$$

becomes a complex analytic diffeomorphism. Here, the formal adjoint $\delta^*$ of $\delta$ is defined by some (and fixed) hermitian inner product of $g^e$.

Let $\delta$ be a holomorphic structure of $P^c$. For $\alpha \in C^\infty(\Lambda^{0,1}\otimes g^e)$ we see that $T(\delta + A) = T(\delta) + [\delta, A] + [A, \delta]$. Therefore the equation of infinitesimal deformation of holomorphic structure of $\delta$ is given by

$$\delta^A = 0 .$$

The space of all essential infinitesimal deformations of $\delta$ is given by

$$\text{EHID}(\delta) = \text{Ker} \delta^i \cap \text{Ker} \delta^* .$$

By a similar way as Theorem 2.7, we have

**Theorem 5.2.** Let $\delta$ be a holomorphic structure. There are a neighbourhood $U^{c+1}$ of $\delta$ in $\mathcal{A}_G$ and a complex analytic submanifold $Z^c$ of $U^{c+1}$ so that the set of all $H^c$-holomorphic structures in $U^{c+1}$ forms a complex analytic set of $Z^c$.

**Definition 5.3.** The set of all $H^c$-holomorphic structures in $U^{c+1}$ is called the local pre-moduli space of holomorphic structures around $\delta$ and denoted by $\text{HLPM}(\delta)$. The manifold $Z^c$ is called its support manifold.

Equalities (5.1.2) and (5.0.4) mean that the space $\text{HEID}(\delta)$ is canonically isomorphic to $H^{0,1}(g^e)$. Moreover, for any $\delta \in \mathcal{A}$ we have

$$\delta^T(\delta) = 0 ,$$

which means that $\delta^\theta$ is an identity for $T$. Therefore, by Lemma 3.3, we get
Theorem 5.4. Let $\delta$ be a holomorphic structure. If $H^{2,2}(g\delta)=0$, then the space $HLPM(\delta)$ forms a (complex) manifold whose tangent space at $\delta$ coincides with the space $HEID(\delta)$.

Remark 5.5. Since the action of $G^{c,+1}$ on $\mathcal{J}\mathcal{M}$ is complex analytic, the complex structure of the above space $HLPM(\delta)$ is canonical. I.e., if $\delta_i \in HLPM(\delta)$, then the “projection map”: $HLPM(\delta) \rightarrow HLPM(\delta)$ defined by Proposition 5.1 is complex analytic.

Remark 5.6. The space $HLPM(\delta)$ has similar properties as $YMLPM(V)$ in Theorem 2.9. But property (c) does not hold for $HLPM(\delta)$, because $G^c$ is not compact. Therefore the quotient space $G^c/HLPM(\delta)$ is not necessarily identified with an open set of global moduli space of holomorphic structures.

6. Einstein holomorphic connections

Let $(M,g)$ be a compact Kähler manifold, $\omega$ its Kähler form. Then the $(0,1)$ component of a connection $\nabla$ on $P$ is a almost holomorphic structure $\delta$ of $P^c$. Since $T(\delta)$ coincides with the $(0,2)$ component of $R^{\omega}$, $\delta$ is a holomorphic structure if and only if $R^\omega$ is of type $(1,1)$.

Definition 6.1. A connection $\nabla$ of $P$ is said to be holomorphic if the $(0,1)$ component of $\nabla$ is a holomorphic structure, or equivalently, if $R^\omega$ is of type $(1,1)$. (Remark that this definition is not exactly the same with [6].)

Denote by $R^o_\gamma$ (resp. $R^s_\gamma$) the hermitian (resp. skew-hermitian) part of $R^\omega$. Elements of Lie algebra $\mathfrak{z}$ of the center $Z(G)$ of $G$ define parallel sections of $C^\infty(V)$, and are denoted also by $\mathfrak{z}$.

Definition 6.2. A holomorphic connection $\nabla$ is called an Einstein holomorphic connection if $(\omega, R^\omega) \in \mathfrak{z}$ as section.

For example, if $G=U(r)$, a connection $\nabla$ is an Einstein holomorphic connection if and only if $\nabla$ is an Einstein hermitian connection for some holomorphic structure.

Lemma 6.3 (Itoh, Personal communication). An Einstein holomorphic connection takes the minimum value of the Yang-Mills functional $F_{YM}$ on C. Conversely, a connection which takes the value is an Einstein holomorphic connection.

Proof. Let $\nabla$ be a connection of $P$ and consider the characteristic classes of $P$. For each $c \in \mathfrak{z}$, the classes represented by $(c, R^\omega)$ and $\text{Tr}(R^\omega \wedge R^\omega)$ do not depend on $\nabla$, and so the values $\int_M (c, R^\omega) \wedge \omega^{n-1}$ and $\int_M \text{Tr}(R^\omega \wedge R^\omega) \wedge \omega^{n-2}$ are constant for $\nabla$. Therefore there are $c_0 \in \mathfrak{z}$ and a real number $C$ such that equalities
and

\[ \langle \omega \otimes e, R^\psi \rangle = \langle c_0, e \rangle \]

hold for all \( \nabla \). Let \( (\omega, R^\psi) = c_1 + v \) where \( c_1 \in \mathcal{H} \) and \( v \) is orthogonal to \( \mathcal{H} \) with respect to the global inner product. Then

\[ \langle c_0, e \rangle = \langle c, (\omega, R^\psi) \rangle = \langle c, c_1 \rangle \]

for any \( c \in \mathcal{H} \). Therefore \( c_1 = c_0 \) and

\[
||R^\psi||^2 = ||R^\psi||^2 + ||R^\psi||^2 \\
= C + 2||R^\psi||^2 + ||(\omega, R^\psi)||^2 \\
\geq C + ||(\omega, R^\psi)||^2 \\
= C + ||c_0||^2 + ||v||^2 \\
\geq C + ||c_0||^2.
\]

The equality \( ||R^\psi||^2 = C + ||c_0||^2 \) holds if and only if \( R^\psi = 0 \) and \( v = 0 \), i.e., \( (\omega, R^\psi) \in \mathcal{H} \). Q.E.D.

**Remark 6.4.** We saw that if \( \nabla \) is an Einstein holomorphic connection then \( (\omega, R^\psi) = c_0 \).

**Corollary 6.5.** All Einstein holomorphic connections are Yang-Mills connections. Conversely all Yang-Mills connections which are sufficiently close to an Einstein holomorphic connection are Einstein holomorphic connections.

**Proof.** Easy to see by Corollary 2.11 (3). Q.E.D.

Next we consider infinitesimal deformations of Einstein holomorphic connections. Define a map \( E_{EH}: C^1 \rightarrow \mathcal{H}^{\ast} \) by

\[
\nabla \rightarrow (p^0 \cdot R^\psi, (\omega, R^\psi) - c_0),
\]

where \( p^0 \) is the projection map from \( \Lambda^r \) to \( \Lambda^{0,r} \) and \( c_0 \in \mathcal{H} \) is defined in Proof of Lemma 6.3. By Remark 6.4, a connection \( \nabla \) is an Einstein holomorphic connection if and only if \( E_{EH}(\nabla) = 0 \).

**Definition 6.6.** Let \( \nabla \) be an Einstein holomorphic connection. An element \( A \) of \( H^1(\Lambda^1 \otimes \mathfrak{g}_R^0) \) is called an Einstein holomorphic infinitesimal deformation of \( \nabla \) if \( E_{EH}(A) = 0 \). An Einstein holomorphic infinitesimal deformation is said to be essential if it is orthogonal to all trivial infinitesimal deformations of \( \nabla \), and the space of all Einstein holomorphic essential infinitesimal deformations is denoted by \( E_{HEID}(\nabla) \).
By a similar way as Theorem 2.7, we get

**Theorem 6.7.** Let \( \nabla \) be an Einstein holomorphic connection. There are a neighbourhood \( U^{\prime} \) of \( \nabla \) in \( S_{\nu} \) and a closed \( C^{\infty} \)-submanifold \( Z \) of \( U^{\prime} \) whose tangent space at \( \nabla \) coincides with \( EHEID(\nabla) \) such that the set \( EHLPM(\nabla) \) of all Einstein holomorphic connections in \( U^{\prime} \) is a real analytic set of \( Z \).

Moreover, the combination of an obvious inclusion: \( EHLPM(\nabla) \subseteq YMLPM(\nabla) \) and the converse inclusion \( YMLPM(\nabla) \subseteq EHLPM(\nabla) \) by Corollary 6.5 means that \( EHLPM(\nabla) = YMLPM(\nabla) \). Let \( \nabla \) be a connection. Define a map \( I_{\nabla} : H^{r-1}(\Lambda^{0,3} \otimes g_{P}) \oplus H^{r-1}(g_{P}) \rightarrow H^{r-2}(\Lambda^{0,3} \otimes g_{P}) \oplus g_{P} \) by

\[
I_{\nabla}(P, \eta) = \left(p^{0,3}(d^{P}P), \text{ 3-part of } \eta \right).
\]

**Lemma 6.8.** The map \( I \) is an identity for \( E_{EH} \).

**Proof.** For any \( \nabla \), we see

\[
p^{0,3}(d^{P}R^{P}) = 0,
\]

and (6.3.1) means that \( \text{3-part of } (\omega, R^{P}) - c_{0} \) vanishes. Q.E.D.

Therefore if \( \nabla \) is an Einstein holomorphic connection and \( \text{Ker } I_{\nabla}/\text{Im } E_{EH}^{\prime}_{\nabla} \) vanishes, then the local pre-moduli space \( EHLPM(\nabla) \) of Einstein holomorphic connections forms a manifold with tangent space \( EHEID(\nabla) \) at \( \nabla \).

**Theorem 6.9.** Let \( \nabla \) be an Einstein holomorphic connection. In general, the space \( EHLPM(\nabla) \) forms a real analytic set of the support manifold \( Z \) whose tangent space at \( \nabla \) is isomorphic with the cohomology group \( H^{0,1}(M, g_{P}) \). If \( H^{0,2}(M, g_{P}) = 0 \) and \( H^{0}(M, g_{P}) = 3^{c} \), then the space \( EHLPM(\nabla) \) coincides with the support manifold.

**Proof.** We must show that \( EHEID(\nabla) \cong H^{0,1}(M, g_{P}) \) and \( \text{Ker } I_{\nabla}/\text{Im } E_{EH}^{\prime}_{\nabla} \cong H^{0,2}(M, g_{P}) \oplus H^{0}(M, g_{P})/3^{c} \), where \( H^{0}(M, g_{P}) \) denotes the vector space of all parallel sections of \( g_{P} \). First we see that the sequence

\[
C^{\infty}(g_{P}) \xrightarrow{\nabla} C^{\infty}(\Lambda^{1} \otimes g_{P}) \xrightarrow{E_{EH}^{\prime}_{\nabla}} C^{\infty}(\Lambda^{0,3} \otimes g_{P}) \oplus C^{\infty}(g_{P}) \xrightarrow{\text{pr} \circ I_{\nabla}} C^{\infty}(\Lambda^{0,3} \otimes g_{P})
\]

is an elliptic complex. Therefore

\[
(\text{Ker } I_{\nabla}/\text{Im } E_{EH}^{\prime}_{\nabla})/3^{c} \cong \text{Ker } (\text{pr} \circ I_{\nabla})/\text{Im } E_{EH}^{\prime}_{\nabla}.
\]

Let \((P, \eta) \in \text{Ker } (\text{pr} \circ I_{\nabla}) \cap \text{Ker } (E_{EH}^{\prime}_{\nabla})^{\ast} \). We easily see that

\[
(\omega, d^{P}A) = 4 \text{Re}(\sqrt{-1} \nabla^{a}A_{a}).
\]

Thus \((P, \eta) \in \text{Ker } (E_{EH}^{\prime}_{\nabla})^{\ast} \) means that
for all $A \in C^\infty(\Lambda^1 \otimes \mathfrak{g}_\mathfrak{P})$, from which we have
\begin{equation}
-\nabla^\mathfrak{P}_a + 2\sqrt{-1} \nabla_a \eta = 0.
\end{equation}
Here we know that $\nabla^\mathfrak{P}_a \nabla^\mathfrak{P}_a = 0$ since $\nabla$ is Einstein. Therefore we see that
\begin{equation}
-\nabla^\mathfrak{P}_a = 0 \quad \text{and} \quad \nabla_\eta = 0.
\end{equation}
Combining with the assumption that $(P, \eta) \in \text{Ker}(\text{pr}_0 I_\mathfrak{V})$, we see that $P$ is harmonic and $\eta$ is parallel. The converse is obvious, and we get
\begin{equation}
\text{Ker}(\text{pr}_0 I_\mathfrak{V}) \cap \text{Ker}(E_{EH})^* \approx H^{p, q}(M, \mathfrak{g}_\mathfrak{P}) \oplus H^0(M, \mathfrak{g}_\mathfrak{P}).
\end{equation}
Let $A \in E_{EH}/\mathfrak{V}$. Then by definition and equality (6.9.3) we get
\begin{equation}
\nabla_\mathfrak{P}_a A - \nabla_a A = 0,
\end{equation}
\begin{equation}
\nabla^\mathfrak{P}_a A \in C^\infty(\mathfrak{g}_\mathfrak{P})
\end{equation}
and
\begin{equation}
\nabla^\mathfrak{P}_a A + \nabla^\mathfrak{P}_a A = 0.
\end{equation}
Thus we see that $P^0 A$ is harmonic and so the first isomorphism holds. Q.E.D.

The above results are resumed as follows.

**Theorem 6.10.** Let $\nabla$ be an Einstein holomorphic connection. The space $E_{EH}/\mathfrak{V}(\nabla)$ coincides with the space $Y_{MLPM}(\nabla)$ around $\nabla$, which is a real analytic set of the support manifold $Z$ whose tangent space at $\nabla$ coincides with the space $E_{EH}(\nabla)$. If $H^{p, q}(M, \mathfrak{g}_\mathfrak{P}) = 0$ and $H^0(M, \mathfrak{g}_\mathfrak{P}) = \mathfrak{g}$, then the space $E_{EH}/\mathfrak{V}(\nabla)$ coincides with the support manifold $Z$.

**Remark 6.11.** The above statement suggests the equality $Y_{MLPM}(\nabla) = E_{EH}(\nabla)$, which in fact holds.

7. The deformation of Einstein holomorphic connections caused by a deformation of complex structure of the base manifold

In this section we discuss on deformations of Einstein holomorphic connections in a similar situation as in section 4. Let $(M, \mathfrak{f}, g)$ be the base Kahler manifold and $(\mathfrak{f}_t, \mathfrak{g}_t)$ be a one-parameter family of Kahler structure such that $(\mathfrak{f}_0, \mathfrak{g}_0) = (\mathfrak{f}, \mathfrak{g})$. Define maps $E$, $I$
\begin{align*}
E: (-\varepsilon, \varepsilon) & \times C^\infty \to H^{-1}(\Lambda^{0, 2} \otimes \mathfrak{g}_\mathfrak{P}) \oplus H^{1, 1}(\mathfrak{g}_\mathfrak{P}), \\
I_{(\mathfrak{P}, \mathfrak{Q})}: H^{-1}(\Lambda^{0, 2} \otimes \mathfrak{g}_\mathfrak{P}) \oplus H^{1, 1}(\mathfrak{g}_\mathfrak{P}) & \to H^{-2}(\Lambda^{0, 3} \otimes \mathfrak{g}_\mathfrak{P}) \oplus \mathfrak{g}.
\end{align*}
by

\[ E(t, \nabla) = (p^0, \omega, R_{\omega}) - c_t \]

and

\[ I_{(t, \omega)}(P, \eta) = (p^0, d^0 \omega, \text{5-part of } \eta) \]

where all operators and \( c_t \) depending on base Kahler structure are defined by \((J_t, g_t)\). Then we know that \( I \) is an identity for \( E \).

**Theorem 7.1.** Let \( \nabla \) be an Einstein holomorphic connection on \((M, J, g)\). If \( H^0(M, \mathfrak{g}_\mathcal{P}) = 0 \) and \( H^0(M, \mathfrak{g}_\mathcal{P}) = 5 \), then for any deformation \((J, g)\) of Kahler structures of \((J, g)\) there exists a one-parameter family of connections \( \nabla_t \) of \( \mathcal{P} \) so that each \( \nabla_t \) is an Einstein holomorphic connection over \((M, J_t, g_t)\), provided that \( |t| \) is sufficiently small. Moreover, each local pre-moduli space \( \text{EHLM} \nabla \) over \((M, J_t, g_t)\) forms a manifold of the same dimension.

**Proof.** The obstruction space for \( E \)-deformation of \((0, \nabla)\) with respect to \( I \) coincides with the space

\[ \text{Ker } I_{\omega}/\text{Im } E_{(0, \omega)} \]

where \( I_{\omega} \) is introduced before Lemma 6.8. It is a quotient space of the space \( \text{Ker } I_{\omega}/\text{Im } E_{\mathcal{H}} \) and vanishes by assumption. Therefore by Lemma 3.3 the set \( E^{-1}(0) \) around \((0, \nabla)\) forms a manifold whose tangent space at \((0, \nabla)\) is given by \( \text{Ker } E_{(0, \omega)} \). But here the projection map from \( \text{Ker } E_{(0, \omega)} \) to \( T_0(-\varepsilon, \varepsilon) \) is surjective, which completes the proof by the implicit function theorem. In fact, for \( u \in T_0(-\varepsilon, \varepsilon) \), we get \( E_{(0, \omega)}(u, 0) \in \text{Ker } I_{\omega} \), therefore by assumption there is \( A \in T_\varepsilon \mathcal{C}^* \) such that \( E_{(0, \omega)}(u, 0) = E_{\mathcal{H}}(A) \), i.e., \( E_{(0, \omega)}(u, -A) = 0 \). Q.E.D.

8. Einstein holomorphic connections and holomorphic structures

For a connection \( \nabla \) of \( \mathcal{P} \) we denote by \( \Psi(\nabla) \) the \((0, 1)\)-part of \( \nabla \), which is an almost holomorphic structure of \( \mathcal{P}^c \). Remark that the map \( \Psi \) commutes with the action of the gauge group \( G \). Therefore \( \Psi \) induces a map from the moduli space of Einstein holomorphic connections to the moduli space of holomorphic structures. This map locally corresponds to a map \( \phi: \text{EHLM} \nabla \rightarrow \text{HLPM}(\delta) \), where \( \nabla \) is an Einstein holomorphic connection and \( \delta = \Psi(\nabla) \).

**Theorem 8.1.** Let \( \nabla \) be an Einstein holomorphic connection. If \( H^0(M, \mathfrak{g}_\mathcal{P}) \approx \mathbb{C} \), then the map \( p^0 \Psi \) gives a bijection between \( \text{EHLM} \nabla \) and \( \text{HLPM}(\Psi(\delta)) \) around \( \nabla \), where the map \( p: W^c \rightarrow S_{\Psi(\nabla)}^c \) is defined by Proposition 5.1.

**Proof.** Set \( \mathcal{E} = \{ \nabla_1 \in S_{\nabla}^c; (\omega, R_{\omega}) - c_0 = 0 \} \). The derivative of the map \( f: \nabla_1 \rightarrow (\omega, R_{\omega}) - c_0 \) at \( \nabla \) is given by

\[ f'_{\nabla}(u, \omega, R_{\omega}) = (d^0 \omega, d^0 R_{\omega}) \]
\[ A \to 2\sqrt{-1} (\nabla^g A_\sigma - \nabla^g A_\omega) \] 

Set \( A_\omega = \sqrt{-1} \nabla_\omega \psi \) for \( \psi \in H^{r+1}(g_\rho) \). Then 
\[ A \in T_\omega S^i_\psi \]

and 
\[ 2\sqrt{-1} (\nabla^g A_\sigma - \nabla^g A_\omega) = 2\nabla^* \nabla \psi. \]

Therefore the image of the derivative of the map \( f \) from \( S^i_\psi \) is closed in \( H^{r+1}(g_\rho) \), and coincides with the orthogonal complement of \( H^p(M, g_\rho) \). Therefore by assumption and Lemma 6.8, the map \( f \) from \( S^i_\psi \) to the orthogonal complement of \( \mathfrak{g} \) has surjective derivative, from which we see that \( \mathcal{E} \) is a manifold whose tangent space at \( \nabla \) coincides with the space 
\[ \{ A \in H^p(A \otimes g_\rho); \ -\nabla^g A_\sigma = 0 \} \]

Since the derivative of the map \( p \circ \Psi \) from \( \mathcal{E} \) is nothing but the correspondence: \( A \to (0, 1) \)-part of \( A \), \( p \circ \Psi \) gives a local diffeomorphism from \( \mathcal{E} \) to \( S^i_\psi \). If \( \nabla_1 \in EHLPM(\nabla) \) then \( p \circ \Psi(\nabla_1) \in HLPM(\Psi(\nabla)) \), conversely, if \( \mathfrak{g}_1 \in HLPM(\Psi(\nabla)) \) then \( (p \circ \Psi | \mathcal{E})^{-1}(\mathfrak{g}_1) \) is Einstein holomorphic by definition of \( \mathcal{E} \). Q.E.D.

**Remark 8.2.** Theorem 8.1 and Theorem 5.4 give another proof of Theorem 6.10.

Combining with Theorem 6.10, we get the following

**Theorem 8.3.** Let \( \nabla \) be an Einstein holomorphic connection and set \( \mathfrak{g} = \Psi(\nabla) \). Then there exists a natural correspondence 
\[ YMLPM(\nabla) = EHLPM(\nabla) \to HLPM(\mathfrak{g}) \]
where \( \to \) is an injection, and becomes a bijection if \( \text{Ker } \nabla = \mathfrak{g} \).

**9. A structure on the moduli space**

Let \( \nabla \) be an Einstein holomorphic connection and set \( \mathfrak{g} = \Psi(\nabla) \). Assume that \( H^0(\mathfrak{g}) = 3^c \) and \( H^2(\mathfrak{g}) = 0 \). Then the manifolds \( EHLPM(\nabla) \) and \( HLPM(\mathfrak{g}) \) are isomorphic by Theorem 8.1, and become complex manifolds by Theorem 5.4. The complex structures are realized by the almost complex structures given by multiplying \( \sqrt{-1} \) on \( T_\mathfrak{g} EHLPM(\mathfrak{g}) \) and \( J \) on \( T_\mathfrak{g} EHLPM(\nabla) \), where \( J \) is defined by \( (JA)_i = -A_j J^i_j \). In fact, we see that 
\[ \Psi(JA) = (JA)^{(0,1)} = -A_\beta J^i_\beta = \sqrt{-1} A_\beta = \sqrt{-1} \Psi(A). \]

On the other hand, the space \( \mathcal{C} \) has the riemannian metric \( \langle \cdot, \cdot \rangle \), which is
invariant under the action of \( G^{*+1} \). Therefore the manifold \( EHLPM(\nabla) \) has a canonical riemannian metric, which is given as follows. Let \( \nabla_1 \in EHLPM(\nabla) \) and \( A, B \in T_{\nabla_1} EHLPM(\nabla) \). The elements \( A \) and \( B \) are Einstein holomorphic infinitesimal deformations of \( \nabla_1 \), and are decomposed into the essential parts \( A_e, B_e \) and trivial parts \( A_t, B_t \) (see (1.5.2)). We define the inner product of \( A \) and \( B \) by \( \langle A_e, B_e \rangle \). From Lemma 13.1, we see that this inner product becomes a \( C^\infty \)-riemannian metric.

**Definition 9.1.** The above riemannian metric on \( EHLPM(\nabla) \) is called the natural riemannian metric.

**Remark 9.2.** Let \( \nabla_1 \) and \( \nabla_2 \) be Einstein holomorphic connections and assume that there are \( \nabla_0 \in EHLPM(\nabla_1) \) and \( \gamma \in G^{*+1} \) such that \( \gamma^* \nabla_0 \in EHLPM(\nabla_2) \). Then for each \( \nabla \in EHLPM(\nabla_1) \) sufficiently close to \( \nabla_0 \) there is \( \gamma^* \nabla \in EHLPM(\nabla_2) \), and this correspondence: \( \nabla \mapsto \gamma^* \nabla \) becomes an isometry. Therefore we may say that the canonical riemannian metric is independent of \( \nabla \).

**Theorem 9.3.** Let \( \nabla \) be an Einstein holomorphic connection and set \( \overline{\partial} = \Psi(\nabla) \). If \( H^0(\overline{\partial}^c)^* = 0^c \) and \( H^2(\overline{\partial}^c)^* = 0 \), then the canonical riemannian metric on \( EHLPM(\nabla) \) is a Kähler metric with respect to the complex structure on \( HLPM(\overline{\partial}) \).

**Proof.** We easily see that the canonical riemannian metric is a hermitian metric. We have to show that the Kähler form is closed. We replace \( \nabla \) by \( \nabla_0 \) and denote by \( \nabla \) elements of \( HLPM(\nabla_0) \) regarded as variable. Consider the fiber bundle \( p: P \times EHLPM \to EHLPM \). In general, a diffeomorphism from a fiber to another fiber which commutes with the action of \( G \) and fixes \( M \) pull backs a \( G \)-invariant structure, and so if a vector field \( v \) on \( P \times EHLPM \) is \( p \)-projectable, \( G \)-invariant and \( \pi^* v = 0 \), where \( \pi \) is the projection to \( M \), then the Lie derivation \( \mathcal{L}_v \) on a family of \( G \)-invariant structures is defined. For example,

\[
\mathcal{L}_v \nabla \equiv \frac{d}{ds} \bigg|_0 (\exp sv)^* \nabla .
\]

If we decompose \( v \) into the \( P \)-part \( v_P \) and the \( EHLPM \)-part \( v_M \), we see that

\[
\mathcal{L}_v \nabla = v_M [\nabla] + L_{v_P} \nabla .
\]

Now, we denote the almost complex structure on \( EHLPM \) by \( J^\overline{\partial} \), the canonical riemannian metric by \( g^\overline{\partial} \) and the Kähler form by \( \omega^\overline{\partial} \). Decompose \( v \in T(EHLPM) \) into \( v_E \) and \( v_T \) so that \( \mathcal{L}_{v_E} \nabla \) is essential and \( \mathcal{L}_{v_T} \nabla \) is trivial. This decomposition is not unique, but we may assume that it depends \( C^\infty \)-ly on \( v \) by Lemma 13.1. Then we see that

\[
\mathcal{L}_{(J^\overline{\partial} v)_E} \nabla = J \mathcal{L}_{v_E} \nabla ,
\]

\[
g^\overline{\partial}(v, w) = \langle \mathcal{L}_{v_E} \nabla, \mathcal{L}_{w_E} \nabla \rangle ,
\]

\[
\omega^\overline{\partial}(v, w) = g^\overline{\partial}(J^\overline{\partial} v, w) = \langle J \mathcal{L}_{v_E} \nabla, \mathcal{L}_{w_E} \nabla \rangle ,
\]
where $J$ is defined in the first paragraph of this section. We may assume that $[v,w]=[w,z]=[z,v]=0$ without loss of generality, and see that

$$(d\omega^F)(v,w,z) = v \cdot \omega^F(w,z) + \text{alternating terms}$$

$$= v \cdot \langle J\mathcal{L}_{w^B} \nabla, \mathcal{L}_{z^B} \nabla \rangle + \text{alt}.$$  

$$= \langle J\mathcal{L}_{w^B} \mathcal{L}_{w^B} \nabla, \mathcal{L}_{z^B} \nabla \rangle + \langle J\mathcal{L}_{w^B} \mathcal{L}_{z^B} \nabla, \mathcal{L}_{w^B} \nabla \rangle + \text{alt}.$$ 

$$= -\langle \mathcal{L}_{w^B} \mathcal{L}_{w^B} \nabla, J\mathcal{L}_{z^B} \nabla \rangle + \langle \mathcal{L}_{w^B} \mathcal{L}_{z^B} \nabla, J\mathcal{L}_{w^B} \nabla \rangle + \text{alt}.$$ 

$$= -\langle \mathcal{L}_{w^B} \mathcal{L}_{w^B} \nabla, J\mathcal{L}_{z^B} \nabla \rangle + \text{alt}.$$ 

But here $p^*([v_E, w_E])=[v, w]=0$ and so $[v_E, w_E]$ is vertical, which implies that $\mathcal{L}_{w^B} \mathcal{L}_{w^B} \nabla$ is trivial. Q.E.D.

10. Example I

Let $M$ be a flat torus $T^2$, $P$ the trivial principal $U(2)$-bundle and $\nabla_0$ the canonical connection of $P$. $\nabla_0$ is a flat connection, and so an Einstein holomorphic connection. Therefore, by Lemma 6.3, all Einstein holomorphic connections of $P$ are flat. Fix a point $x$ in $M$ and an element $p$ in $P_x$. Any closed curve $c$ ($c(0)=c(1)=x$) in $M$ is horizontally lifted to a curve $\tilde{c}$ in $P$ so that $\tilde{c}(0)=p$, and we get an element $\tilde{c}(1)$ in $P_x$. Let $a$ be an element of $U(2)$ such that $\tilde{c}(1)=p \cdot a$. Since $\nabla$ is flat, this mapping: $c \mapsto a$ induces a homomorphism: $\pi_1(M) \rightarrow U(2)$, defined by $[c] \mapsto a$. Taking generators $\{[c_1], [c_2]\}$ of $\pi_1(M)$, we get corresponding elements $\{a_1, a_2\}$ in $U(2)$ such that $a_1^{-1} \cdot a_2^{-1} \cdot a_1 \cdot a_2 = \text{id}$. Denote by $f(\nabla)$ this pair $(a_1, a_2)$. We see that by a gauge transformation $\eta$ of $P$, $f(\nabla)=(a_1, a_2)$ is transformed as

$$(10.1) \quad f(\eta^* \nabla) = (b^{-1} \cdot a_1 \cdot b, b^{-1} \cdot a_2 \cdot b),$$

where $b \in U(2)$ is defined by $\eta(x) \cdot p = p \cdot b$.

Thus the global moduli space of Einstein holomorphic connections is identified with the quotient space (commuting pair in $U(2) \times U(2)$) $/{\sim}$, where $\sim$ is defined by $(b^{-1} \cdot a_1 \cdot b, b^{-1} \cdot a_2 \cdot b) \sim (a_1, a_2)$ for $b \in U(2)$. By diagonalization, this space becomes the space $T^2 \times T^2/{\sim}$, where

$$\left( \begin{array}{cc} (\alpha & 0) \\ 0 & \beta \end{array} \right), \left( \begin{array}{cc} (\gamma & 0) \\ 0 & \delta \end{array} \right) \sim \left( \begin{array}{cc} (\alpha' & 0) \\ 0 & \beta' \end{array} \right), \left( \begin{array}{cc} (\gamma' & 0) \\ 0 & \delta' \end{array} \right)$$

if and only if they coincide or $\beta'=\alpha$, $\alpha'=\beta$, $\delta'=\gamma$ and $\gamma'=\delta$.

On the other hand, the space $EHEID(\nabla_0)$ is the space of harmonic sections of $\Lambda^1 \otimes u(2)$, and is isomorphic with $R^2 \otimes u(2)$. Let $A \in EHEID(\nabla_0)$ and consider the connection $\nabla_0 + A$. Since $\nabla_0 A = 0$, we see that
\[ E_{EH}(\nabla_0 + A) = (0, 2 \{ [A^{(0,1)}, A^{(0,1)}] \} , \]

and

\[ (\omega, d^s B) = 2\sqrt{-1} (\nabla_0^s B_a - \nabla_0^s B_a) , \]

which implies that \( \nabla_0 + A \) is an element of the support manifold of \( EHLPM(\nabla_0) \). Thus we see that the support manifold is locally isomorphic with \( R^2 \otimes \mathfrak{u}(2) \). Moreover \( \nabla_0 + A \) belongs to \( EHLPM(\nabla_0) \) if and only if

\[ [A^{(0,1)}, A^{(0,1)}] = 0 . \]

Therefore the space \( EHLPM(\nabla_0) \) is a proper subset of the support manifold. Moreover, the group \( \mathcal{G}_{\nabla_0} \cong U(2) \) acts on the space \( EHLPM(\nabla_0) \) analogously as (10.1), and we see that

\[ \mathcal{G}_{\nabla_0} \backslash EHLPM(\nabla_0) \cong R^2 \times R^2 / \sim . \]

By a similar way we see that the space \( HEID(\nabla_0) \) is canonically isomorphic with the space \( C \otimes \mathfrak{gl}(2, C) \), and \( \nabla_0 + HEID(\nabla_0) \) is the support manifold of \( HLPM(\nabla_0) \). In this case, the space \( HLPM(\nabla_0) \) is an open set of the support manifold. We can see more details as follows. The group \( \mathcal{G}_{\nabla_0} \) acts on the space \( HLPM(\nabla_0) \), and

\[ \mathcal{G}_{\nabla_0} \backslash HLPM(\nabla_0) \cong GL(2, C) \backslash \mathfrak{gl}(2, C) , \]

whose elements are classified using Jordan's normal form. An element of \( \mathfrak{gl}(2, C) \) corresponds to an Einstein holomorphic connection if and only if it is diagonalizable. Thus

\[ \mathcal{G}_{\nabla_0} \backslash EHLPM(\nabla_0) \cong \mathcal{G}_{\nabla_0} \backslash HLPM(\nabla_0) . \]

Remark that the space \( \mathcal{G}_{\nabla_0} \backslash HLPM(\nabla_0) \) is not a Hausdorff space. In fact, any neighbourhood of the element \( (\lambda 0)

\( 0 \lambda) \) in \( U(2) \) implies some \( (\lambda t)

\( t=0) \), which is conjugate with \( (\lambda 1)

\( 0 \lambda) \).

11. Example II

Let \( (M, g) \) be an Einstein-Kähler manifold with Ricci tensor \( = e \cdot g \), \( \nabla \) an Einstein holomorphic connection and \( \nabla = \Psi(\nabla) \). Then we can see that

\[ (11.0.1) \quad \{(\overline{\delta} \delta + 2\overline{\delta} \delta) A\}_a = ([\nabla^s \nabla A]_a + e A_a + 2[R^{\gamma \lambda}_a, A]_a) \]

\[ = 2 \{- [\nabla^\delta \nabla A]_a + e A_a + [R^{\gamma \lambda}_a, A]_a \} \]

for \( gl^c \)-valued \( (0,1) \)-form \( A \),
for $g^\xi$-valued $(0,2)$-form $A$. Therefore, to see whether $H^1_\xi$ and $H^2_\xi$ vanish, we have to get eigenvalues of these operators.

Let $M$ be a homogeneous space $K/H$ and $P$ the principal $G$-bundle $K \times \rho G$, where $\rho$ is a homomorphism: $H \to G$. Then we have

$$G_P = K \times \text{Ad}_\rho G, \quad g_P = K \times \text{Ad}_\rho g.$$ 

As usual, we identify $C^\infty(g_P)$ with $C^\infty(K, Q)$. Let $\mathfrak{f} = \mathfrak{h} + \mathfrak{m}$ be an $H$-invariant decomposition and define a differential operator $D: C^\infty(K, Q) \to C^\infty(K, \pi^*\otimes \xi)$ by

$$(D\xi)(X) = (X\xi).$$

Then this operator $D$ gives a covariant derivative of $g_P$, which is identified with the standard connection $\nabla$ of $P$. Let $C_K$ (resp. $C_H$) be the Cassimir operator of the $K$-module (resp. $H$-module) $C^\infty(K, \mathfrak{g})$. We can check that

$$\nabla^*\nabla = C_K - C_H$$

and

$$R^\xi(X, Y) = -\rho[X, Y] \quad \text{for } X, Y \in \mathfrak{m}.$$ 

(See e.g., [10, Proposition 5.3].)

Therefore the eigenvalues of operators (11.0.1) and (11.0.2) are calculated explicitly by the representation theory. The calculation is easy but complicated, and we omit the detail. See e.g. [10, §7].

Let $M = P^*(\mathcal{C}) = SU(n+1)/S(U(n) \times U(1))$ and $P$ the unitary frame bundle of $T^*M$. Then $\mathfrak{g} = \mathfrak{m}^- \otimes \mathfrak{m}^+$, and the operator (11.0.1) has only positive eigenvalues. Thus $H^1_{\xi}(M, \mathfrak{g}^\xi) = 0$.

**Proposition 11.1.** The standard connection of the unitary frame bundle of $T^*P^*(\mathcal{C})$ is isolated in the moduli space.

Next, let $P$ be the unitary frame bundle of the symmetric tensor product $S^2T^*M$ of $T^*M$. Then $\mathfrak{g} = (S^2\mathfrak{m}^-) \otimes (S^2\mathfrak{m}^+)$. In this case the operator (11.0.1) has 0 as an eigenvalue, and all eigenvalues of the operator (11.0.2) are positive. Moreover, we can easily check that $H^1_\xi(M, \mathfrak{g}^\xi) = 0$. Thus by Theorem 6.10, we get the following

**Proposition 11.2.** The local pre-moduli space around the standard connection of the unitary frame bundle of $S^2T^*P^*(\mathcal{C})$ $(n \geq 2)$ forms a non-trivial manifold.
12. Regularity of Yang-Mills connections

In this section we consider not a family of connections but one connection. Let $V$ be a Yang-Mills $C^{2+\alpha}$-connection of $P$ ($0<\alpha<1$). I.e., if we represent $V$ by a local frame $\{\xi^p\}$ of $\mathcal{G}_P$ as

$$\nabla_{\xi^q} \xi^p = \Gamma^p_{iqr} \xi^r,$$

then $\Gamma^p_{iqr}$ are $C^{2+\alpha}$. A local section $\xi$ of $\mathcal{G}_P$ is said to be harmonic if $\nabla^* \nabla \xi = 0$. The defining equation of harmonic section is a linear elliptic differential equation with $C^{2+\alpha}$-coefficients. Therefore we can take a local frame by harmonic sections, which are $C^{3+\alpha}$ ([2, p. 228 Theorem 1]). The coefficients $\Gamma^p_{iqr}$ with respect to the frame are $C^{2+\alpha}$. But we know that $\{\Gamma^p_{iqr}\}$ satisfies Yang-Mills equation:

$$g^{kl} \partial_k (\partial_l \Gamma^p_{iqr} - \partial_q \Gamma^p_{lir}) + \text{lower terms} = 0,$$

and harmonic equation

$$g^{kl} \partial_k \Gamma^p_{lir} + \text{lower terms} = 0,$$

which is quasi-linear elliptic system with $C^\infty$-coefficients. Thus $\Gamma^p_{iqr}$ are $C^\infty$ ([11, Theorem 6.8.1]). If $(M, g)$ is a $C^\infty$-riemannian manifold, then $\Gamma^p_{iqr}$ are $C^\infty$ ([11, Theorem 6.7.6]).

**Theorem 12.1.** Let $(M, g)$ be a $C^\infty$ (resp. $C^\omega$) riemannian manifold and $\nabla$ a Yang-Mills $C^3$-connection. Then there exists a $C^3$-gauge transformation $\gamma$ so that $\gamma^* V$ is $C^\infty$ (resp. $C^\omega$).

**Corollary 12.2.** Let $(M, g)$ be a simply connected $C^\omega$-riemannian manifold. Let $\nabla_1$ and $\nabla_2$ be Yang-Mills connections on $M$. Assume that there is an open set $U$ of $M$ and a gauge transformation $\gamma$ on $U$ such that $\gamma^* \nabla_1 = \nabla_2$. Then $\gamma$ extends to a global gauge transformation $\tilde{\gamma}$ so that $\tilde{\gamma}^* \nabla_1 = \nabla_2$ on $M$.

**Proof.** We may assume that $\gamma = \text{id}$ on $U$ and $\nabla_1$ is $C^\omega$. For $x \in U$ and $y \in M$, take a joining geodesic $c: [0, 1] \to M$ and a $C^\omega$-tubular neighbourhood $V \cong (-\varepsilon, 1+\varepsilon) \times D^{n-1}$ of $c[0, 1]$. Take a $C^\omega$-frame of $\mathcal{G}_P$ on $\{0\} \times D^{n-1}$ and take the parallel extension $\{\xi^p\}$ (resp. $\{\xi^p\}$) for the direction $(-\varepsilon, 1+\varepsilon)$ with respect to $\nabla_1$ (resp. $\nabla_2$). Let $\tilde{\gamma}$ be the gauge transformation on $V$ which transforms $\{\xi^p\}$ to $\{\xi^p\}$. Since $\nabla_2$ is $C^\omega$ with respect to $\{\xi^p\}$, $\tilde{\gamma}^{-1} \nabla_2$ is $C^\omega$ with respect to $\{\xi^p\}$. But here $\tilde{\gamma} = \text{id}$ on $U$, which implies that $\tilde{\gamma}^{-1} \nabla_2 = \nabla_1$ on $V$ by analyticity. Moreover the extension of $\gamma$ to $\tilde{\gamma}$ is unique and well-defined since $M$ is simply connected. Q.E.D.

**Remark 12.3.** This is an analogy of the unique extension theorem of Einstein metrics in [3, Section 5].
13. Some basic lemmas

**Lemma 13.1** ([8, Lemma 4.3]). Let \( v_t \) be a family of volume elements on \( M \), \( E_t, F_t \) families of vector bundles over \( M \) with fiber metrics \( g_t^E, g_t^F \) and \( Q_t: C^\infty(E_t) \to C^\infty(F_t) \) a family of differential operators of order \( k \) with injective symbol. Assume that \( v_t, E_t, F_t, g_t^E, g_t^F \) and \( Q_t \) depend \( C^\infty \)-ly (resp. real analytically) on \( t \). That is, there are bundle isomorphism \( e_t: E_0 \to E_t \) and \( f_t: F_0 \to F_t \) such that the coefficients of \( e_t^* g_t^E, f_t^* g_t^F \) and \( (f_t^{-1})^* Q_t (e_t)_* \) depend \( C^\infty \)-ly (resp. real analytically) on \( t \). Then the dimension of the space \( \text{Ker} \ Q_t \) is upper semicontinuous. If the dimension of the space \( \text{Ker} \ Q_t \) is constant, then the decompositions

\[
H^r(E_t) = Q_t^*(H^{r+k}(F_t)) \oplus \text{Ker} \ Q_t,
\]

\[
H^r(F_t) = Q_t(H^{r+k}(E_t)) \oplus \text{Ker} \ Q_t^*
\]
depend \( C^\infty \)-ly (resp. real analytically) on \( t \), where \( Q_t^* \) is the formal adjoint operator of \( Q_t \) with respect to \( g_t^E, g_t^F \) and \( v_t \). Moreover the isomorphisms

\[
Q_t^* + 1: Q_t(H^{r+k}(E_t)) \oplus \text{Ker} \ Q_t \to H^r(E_t),
\]

\[
Q_t + 1: Q_t^*(H^{r+k}(F_t)) \oplus \text{Ker} \ Q_t^* \to H^r(F_t)
\]
also depend \( C^\infty \)-ly (resp. real analytically) on \( t \).

**Lemma 13.2** ([4, Theorem 3.12]). In the real analytic category in Banach spaces, the implicit function theorem holds.

**Lemma 13.3** ([8, Lemma 13.7]). Let \( E \) and \( F \) be vector bundles over \( M \) and \( E^C, F^C \) their complexifications. Let \( f \) be a \( C^\infty \)-cross section of \( E \) and \( \varphi: E \to F \) a fiber preserving \( C^\infty \)-map defined on an open set of \( E \) which contains the image of \( f \). Assume that \( \varphi \) has an extension to a fiber preserving map \( \varphi^C: E^C \to F^C \) defined on an open set of \( E^C \) such that the restriction \( \varphi^C \) to each fiber \( E^C \) is holomorphic. Then the map \( \Psi: H^s(E) \to H^s(F) \) defined by

\[
\Psi(u) = \varphi \cdot u,
\]
defined on an open neighbourhood of \( f \), is real analytic provided that \( s > [n/2] + 1 \).

**References**


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