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AFFINE STRUCTURES ON COMPLEX MANIFOLDS

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Let M be a complex manifold of complex dimension n and let $C = \{U_i, \varphi_i\}_{i \in I}$ be the maximal atlas defining the complex structure on M . A subset $A = \{U_j, \varphi_j\}_{j \in J}$, $J \subset I$, of C is called an affine atlas of M , if $\varphi_{jk} = \varphi_j \circ \varphi_k^{-1}$ is a complex affine transformation of \mathbb{C}^n whenever $U_j \cap U_k \neq \emptyset$. We can define the notion of a maximal affine atlas of M and we say that each maximal affine atlas of M defines a complex affine structure of the complex manifold M . We shall denote by $A(M)$ the totality of complex affine structures on M .

The aim of this note is to study the structure of $A(M)$ in the case where M is compact and the complex structure of M is homogeneous and we shall prove the following theorems.

Theorem 1. *Let M be a complex torus of complex dimension n . Then there exists a natural one-to-one correspondence between the set $A(M)$ and the set of all commutative associative algebra structure over \mathbb{C} in the complex vector space \mathbb{C}^n . In particular $A(M)$ is a complex affine variety.*

More generally:

Theorem 2. *Let M be a connected compact complex manifold and let $Aut(M)$ be the group of all holomorphic transformations of M . Assume $Aut(M)$ is transitive on M . Then $A(M)$ is a complex affine algebraic variety.*

1. A) Let M be complex manifold and let I be the tensor of the almost complex structure associated with M . For each point $p \in M$, the value I_p of I at p is an endomorphism of the tangent space $T_p(M)$ such that $I_p^2 = -1$. Let $T_p^+(M)$ (resp. $T_p^-(M)$) be the subspace of the complexified tangent space $T_p^{\mathbb{C}}(M)$ consisting of all u such that $I_p u = iu$ (resp. $I_p u = -iu$) with $i = \sqrt{-1}$. Then we have

$$T_p^{\mathbb{C}}(M) = T_p^+(M) \oplus T_p^-(M).$$

If $\{z^1, z^2, \dots, z^n\}$ is a system of complex local coordinates on an open set U , then $\{(\partial/\partial z^i)_p\}_{i=1, \dots, n}$ and $\{(\partial/\partial \bar{z}^i)_p\}_{i=1, \dots, n}$ are bases of $T_p^+(M)$ and $T_p^-(M)$ respectively at each point $p \in U$. The totality of complex tangent vectors belonging to $T_p^+(M)$ ($p \in M$) form a holomorphic vector bundle $T^+(M)$ over M .

Let X be a smooth vector field on M . Then we can write X uniquely in the form

$$X = X^+ + X^-,$$

where $X^+(p) \in T_p^+(M)$ and $X^-(p) \in T_p^-(M)$ at each point $p \in M$ and $X^-(p) = \overline{X^+(p)}$, where $\overline{}$ denotes the conjugation of $T_p^c(M)$.

A complex vector field W on M is, by definition, a smooth section of the vector bundle $T^+(M) \oplus T^-(M)$. Then we can write W uniquely in the form $W = X + iY$, where X and Y are smooth real vector fields on M . A complex vector field W is called holomorphic if W is a holomorphic section of $T^+(M)$. A smooth real vector field X is called holomorphic if X^+ is holomorphic.

Let $\mathfrak{g} = \mathfrak{g}(M)$ be the vector space of all holomorphic real vector fields on M . Then \mathfrak{g} is a complex Lie algebra and if M is compact, \mathfrak{g} is identified with the Lie algebra of the group $\text{Aut}(M)$ of holomorphic transformations of M .

In the following we denote by $\mathfrak{X}(M)$ the real vector space of all smooth vector fields on M . Then a complex vector field on M is identified with an element of $\mathfrak{X}(M)^c$.

B) A linear connection ∇ on M is defined by a bilinear mapping $(X, Y) \rightarrow \nabla_X Y$ of $\mathfrak{X}(M) \times \mathfrak{X}(M)$ into $\mathfrak{X}(M)$ satisfying the following conditions:

- 1) $\nabla_{fY} X = f(\nabla_Y X)$;
- 2) $\nabla_Y fX = f(\nabla_Y X) + Yf \cdot X$.

A linear connection ∇ on M is called a *holomorphic* linear connection if the following two conditions are satisfied:

- a) $\nabla_Y IX = I(\nabla_Y X)$ for all $X, Y \in \mathfrak{X}(M)$;
- b) if X and Y are holomorphic vector fields defined on an open set O of M , then $\nabla_Y X$ is also holomorphic on O .

If ∇ is a linear connection, we can extend ∇ to a complex bilinear mapping of $\mathfrak{X}(M)^c \times \mathfrak{X}(M)^c$ into $\mathfrak{X}(M)^c$. Then the conditions a) and b) are equivalent to the following two conditions a') and b').

- a') $(\nabla_Y X)^+ = \nabla_Y X^+$;
- b') if U and W are complex holomorphic vector fields defined on an open set O , then $\nabla_W U$ is also holomorphic.

C) Let us consider a complex affine structure on M defined by a maximal affine atlas $\{(O, \varphi)\}$. Let $\{z^1, \dots, z^n\}$ be the local coordinates on O defined by the chart (O, φ) . On each of these open sets O we can define uniquely an linear connection ∇^0 on O by the conditions: $\nabla_{Z^i}^0 Z^j = \nabla_{\bar{Z}^i}^0 Z^j = 0$ ($i, j = 1, 2, \dots, n$), where $Z^i = \partial/\partial z^i$ and $\bar{Z}^i = \partial/\partial \bar{z}^i$. Then there exists a unique linear connection ∇ on M such that the restriction of ∇ on each O coincides with ∇^0 . This affine connection ∇ is holomorphic and locally flat, i.e. the torsion and the curvature of ∇ are 0. This means that

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X, Y]; \\ \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) &= \nabla_{[X, Y]} Z \end{aligned}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Thus to each affine structure on M there corresponds a locally flat, holomorphic linear connection on M and to the distinct affine structures there correspond distinct linear connections.

D) Let, conversely, ∇ be any locally flat, holomorphic linear connection on M and let \tilde{M} be the universal covering manifold of M . Then there is defined uniquely a connection $\tilde{\nabla}$ on \tilde{M} such that, for smooth vector fields X and Y on M , we have $\tilde{\nabla}_{Y^*} X^* = (\nabla_Y X)^*$, where, for each vector field X on M , X^* denotes the lift of X . $\tilde{\nabla}$ is also locally flat and holomorphic. Let $\mathfrak{P} = \{P\}$ be the complex vector space of all parallel complex holomorphic vector fields on \tilde{M} . Then the map $P \rightarrow P(\tilde{a})$ ($\tilde{a} \in \tilde{M}$) is a bijection of \mathfrak{P} onto $T_{\tilde{a}}^+(M)$. In particular $\dim_{\mathbb{C}} \mathfrak{P} = n = \dim_{\mathbb{C}} M$. \mathfrak{P} form an abelian Lie algebra, because $[P, P'] = \tilde{\nabla}_P P' - \tilde{\nabla}_{P'} P = 0$.

Let Γ be the fundamental group of M . Then Γ acts from the right on \tilde{M} and the action of each element of Γ is holomorphic and affine.

Fix a point $a \in M$ and let \tilde{a} be a point of \tilde{M} such that $\pi(\tilde{a}) = a$, π denoting the projection of \tilde{M} onto M . For each vector $y \in T_a^+(M)$, there exists one and only one $P \in \mathfrak{P}$ such that $d\pi \cdot P(\tilde{a}) = y$. We denote this vector field P by P_y .

Let $\gamma \in \Gamma$. Then γ is a holomorphic affine transformation and hence $d\gamma \cdot \mathfrak{P} \subset \mathfrak{P}$. Put

$$(1) \quad d\gamma \cdot P_y = P_{f(\gamma)y}.$$

Then $\gamma \rightarrow f(\gamma)$ is a representation of the group Γ in the vector space $T_a^+(M)$.

Now let $\{P_1, \dots, P_n\}$ ($n = \dim_{\mathbb{C}} M$) be a basis of the complex vector space \mathfrak{P} . Then we can define n holomorphic 1-forms $\{\omega^1, \dots, \omega^n\}$ on \tilde{M} by the condition

$$\omega^i(P_j) = \delta_j^i \quad (i, j = 1, \dots, n).$$

These 1-forms are closed. There exists a basis $\{y_1, \dots, y_n\}$ of $T_a^+(M)$ such that $P_i = P_{y_i}$ ($i = 1, 2, \dots, n$). We can define a $T_a^+(M)$ -valued holomorphic 1-form θ on \tilde{M} by

$$\theta = \sum_{i=1}^n \omega^i y_i.$$

Then we have:

$$(2) \quad \theta(P) = d\pi \cdot P(\tilde{a}), \quad P \in \mathfrak{P},$$

and

$$(3) \quad d\theta = 0,$$

Moreover,

$$(4) \quad (d\gamma)^*\theta = f(\gamma)\cdot\theta, \quad \gamma \in \Gamma.$$

In fact, let $P \in \mathfrak{P}$. Then there exists $y \in T_a^+(M)$ such that $P = P_y$. Then

$$((d\gamma)^*\theta)(P) = \theta(d\gamma P_y) = \theta(P_{f(\gamma)y}) = f(\gamma)\cdot y \quad \text{by (2)}.$$

On the other hand, by (2) $y = \theta(P)$ and hence $((d\gamma)^*\cdot\theta)(P) = f(\gamma)\cdot\theta(P)$ and this proves the equality (4). For any $\tilde{x} \in \tilde{M}$, let

$$(5) \quad \varphi(\tilde{x}) = \int_{\tilde{a}}^{\tilde{x}} \theta.$$

Then φ is a holomorphic map of \tilde{M} into $T_a^+(M)$. Put

$$(6) \quad q(\gamma) = \varphi(\gamma\tilde{a})$$

for all $\gamma \in \Gamma$. We have then

$$(7) \quad \varphi(\gamma\tilde{x}) = f(\gamma)\varphi(\tilde{x}) + q(\gamma).$$

In particular for $\tilde{x} = \sigma\tilde{a}$ ($\sigma \in \Gamma$), we have

$$q(\gamma\sigma) = f(\gamma)q(\sigma) + q(\gamma).$$

This shows that q is a 1-cocycle of the group Γ and that, if we denote $a(\gamma)$ the complex affine transformation $x \rightarrow f(\gamma)x + q(\gamma)$ of $T_a^+(M)$, then $\gamma \rightarrow a(\gamma)$ is a homomorphism of Γ into the group of complex affine transformations of $T_a^+(M)$. Moreover (7) shows that

$$(7') \quad \varphi(\gamma\tilde{x}) = a(\gamma)\varphi(\tilde{x}).$$

By the definition of φ , we have

$$(d\varphi)(\tilde{x}) = \theta(\tilde{x})$$

and $\theta(\tilde{x}): T_{\tilde{x}}^+(M) \rightarrow T_a^+(M)$ is bijective. Therefore φ is a holomorphic immersion of \tilde{M} into the n dimensional complex vector space $T_a^+(M)$ which satisfies (7')*. Let U be an open subset of M evenly covered by π such that each connected component of $\pi^{-1}(U)$ is mapped bijectively by φ onto an open set in $\mathbf{C}^n = T_a^+(M)$. Let \tilde{U} be any one of the connected components of $\pi^{-1}(U)$ and let ψ be the holomorphic bijective map of U onto $\varphi(\tilde{U})$ defined by $\psi = \varphi \circ (\pi|_{\tilde{U}})^{-1}$. Then it is easy to check that $\{(U, \psi)\}$ defines a complex affine structure on M and that the locally flat holomorphic linear connection associated with this complex affine structure coincides with ∇ .

* The mapping φ is the "development" of \tilde{M} in \mathbf{C}^n . The present way of defining the development φ is due to J.L. Koszul,

Thus there is a one-to-one correspondence between the set $\mathcal{A}(M)$ of all complex affine structures on a complex manifold M and the set of all locally flat, holomorphic linear connections on M .

2. In the following we shall denote by $\mathcal{A}(M)$ the set of all locally flat, holomorphic linear connections on M . We denote by \mathfrak{g} the complex Lie algebra of all holomorphic vector fields on M . From now on we assume that M is compact. Then \mathfrak{g} is identified with the Lie algebra of the group $\text{Aut}(M)$.

Now let $\nabla \in \mathcal{A}(M)$. Then the map $(X, Y) \rightarrow -\nabla_Y X$ is a bilinear map of $\mathfrak{g} \times \mathfrak{g}$ into \mathfrak{g} . Let

$$(8) \quad X \cdot Y = -\nabla_Y X.$$

Then this multiplication in \mathfrak{g} defines an algebra structure on the complex vector space \mathfrak{g} and we denote this algebra by $\mathfrak{g}(\nabla)$.

DEFINITION. Let A be an algebra over a field k and set $[x, y, z] = x(yz) - (xy)z$ and call it the associator of x, y and z . We call A a pre-Lie algebra, if the relation

$$[x, y, z] = [x, z, y]$$

holds for any x, y and z in A .

For example, an associative algebra A is a pre-Lie algebra. Let A be a pre-Lie algebra and set

$$[x, y] = xy - yx$$

for $x, y \in A$. Then we can show and that the bracket product $[x, y]$ defines a Lie algebra. We call this Lie algebra the Lie algebra associated with A .

REMARK. The notion of pre-Lie algebras has been introduced by M. Gerstenhaber in connection with the deformation of algebras. See [1] and [3].

DEFINITION. Let \mathfrak{g} be a Lie algebra over a field k and A a pre-Lie algebra over k . We call A pre-Lie algebra over \mathfrak{g} if the associated Lie algebra of A is \mathfrak{g} .

Lemma 1. *Let \mathfrak{g} be the algebra of all holomorphic vector fields on a compact complex manifold M and let ∇ be a locally flat holomorphic linear connection on M . Then the algebra $\mathfrak{g}(\nabla)$ is a pre-Lie algebra over \mathfrak{g} .*

This lemma follows easily from the definition of the multiplication in $\mathfrak{g}(\nabla)$ and from the fact that the torsion and the curvature of ∇ are 0.

Let us denote by $\mathcal{A}(\mathfrak{g})$ the set of all pre-Lie algebra structures over \mathfrak{g} . Then the map $\nabla \rightarrow \mathfrak{g}(\nabla)$ defines a map of $\mathcal{A}(M)$ into $\mathcal{A}(\mathfrak{g})$.

Assume now that M is homogeneous. Then for any $p \in M$, the tangent vectors $X(p)$ ($X \in \mathfrak{g}$) span the tangent space $T_p(M)$. Then we see easily that

the map of $A(M)$ into $A(\mathfrak{g})$ is injective.

Now let

$$\mathfrak{g}_p = \{Y \in \mathfrak{g} \mid Y(p) = 0\}, \quad p \in M.$$

If $Y \in \mathfrak{g}_p$, then $(\nabla_Y X)(p) = \nabla_{Y(p)} X = 0$ and hence \mathfrak{g}_p is a left ideal of $\mathfrak{g}(\nabla)$.

Conversely we have the following lemma.

Lemma 2. *Let M be a compact homogeneous complex manifold and let \mathfrak{g} be the Lie algebra of all holomorphic vector fields on M . Let A be a pre-Lie algebra over \mathfrak{g} such that \mathfrak{g}_p is a left ideal of A for every $p \in M$. Then there exists a locally flat holomorphic linear connection ∇ on M such that $A = \mathfrak{g}(\nabla)$.*

Proof. Let $p \in M$ and $u \in T_p(M)$. Then there exists a $Y \in \mathfrak{g}$ such that $Y(p) = u$. For any $X \in \mathfrak{g}$ define $\nabla_u X \in T_p(M)$ by putting

$$\nabla_u X = -(X \cdot Y)(p).$$

This definition does not depend on the choice of $Y \in \mathfrak{g}$ such that $Y(p) = u$, because \mathfrak{g}_p is a left ideal of A . For any vector field Y and any $X \in \mathfrak{g}$, $\nabla_Y X$ will denote the vector field on M such that

$$(\nabla_Y X)(p) = \nabla_{Y(p)} X.$$

Then the following equalities hold:

- 1) $\nabla_{fY} X = f \nabla_Y X$, where f is a smooth function on M ;
- 2) $\nabla_{Y+Y'} X = \nabla_Y X + \nabla_{Y'} X$; $\nabla_Y(X+X') = \nabla_Y X + \nabla_Y X'$,

where Y and Y' are smooth vector fields on M and $X, X' \in \mathfrak{g}$;

3) $\nabla_Y X$ is a smooth vector field; in fact, let $p \in M$. Then in a neighborhood U of p , Y is written uniquely in the form $Y = f^1 Y_1 + \cdots + f^n Y_n$, where Y_1, \dots, Y_n are in \mathfrak{g} and f^1, \dots, f^n are smooth functions on U . At each point $q \in U$, we have $(\nabla_Y X)(q) = \sum_{i=1}^n f^i(q) (\nabla_{Y_i} X)(q) = - \sum_{i=1}^n f^i(q) (X \cdot Y_i)(q)$ and hence $\nabla_Y X$ is smooth on U .

Next let Y be a smooth vector field and $u \in T_p(M)$. Define $\nabla_u Y \in T_p(M)$ by

$$\nabla_u Y = \nabla_{Y(p)} X + [X, Y](p),$$

where X is a vector field in \mathfrak{g} such that $u = X(p)$. We have to show that this definition is consistent. It suffices to show that

$$\nabla_{Y(p)} X + [X, Y](p) = 0$$

whenever $X(p) = 0$ and $X \in \mathfrak{g}$. To see this let $Y = f^1 Y_1 + \cdots + f^n Y_n$ in a neighborhood U of p , where $Y_1, \dots, Y_n \in \mathfrak{g}$. Then

$$\nabla_{Y(p)} X = \sum_{i=1}^n f^i(p) \nabla_{Y_i(p)} X \quad \text{and} \quad [X, Y] = \sum_{i=1}^n f^i [X, Y_i] + \sum_{i=1}^n X f^i \cdot Y_i$$

on U . Since $X(p)=0$, we have $(Xf^i)(p)=0$ and, since X and Y_i are in \mathfrak{g} , $[X, Y_i]=X \cdot Y_i - Y_i \cdot X$. Therefore $[X, Y_i](p)=(X \cdot Y_i)(p) - (Y_i \cdot X)(p) = -\nabla_{Y_i(p)}X + \nabla_{X(p)}Y_i = -\nabla_{Y_i(p)}X$ and hence $\nabla_{Y(p)}X + [X, Y](p) = \sum_{i=1}^n f^i(p) \nabla_{Y_i(p)}X - \sum_{i=1}^n f^i(p) \nabla_{Y_i(p)}X = 0$. Thus we have defined the tangent vector $\nabla_u Y$ for any $u \in T_p(M)$ and any smooth vector field Y . The following conditions hold:

- i) $\nabla_u(Y + Y') = \nabla_u Y + \nabla_u Y'$;
- ii) $\nabla_{u+v} Y = \nabla_u Y + \nabla_v Y$; $\nabla_{\lambda u} Y = \lambda \nabla_u Y$ ($\lambda \in \mathbf{R}$);
- iii) $\nabla_u(fY) = f(p) \nabla_u Y + uf \cdot Y$, where f is a smooth function.

Thus we have defined a linear connection ∇ on M and it is easily seen that the torsion of ∇ is 0 and that $\nabla_Y X = -X \cdot Y$ for $X, Y \in \mathfrak{g}$. Then $\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z = [Z, X, Y] - [Z, Y, X] = 0$ for $X, Y, Z \in \mathfrak{g}$. It follows then that the curvature of ∇ is 0. Moreover it is easily seen that ∇ is holomorphic and $\mathfrak{g}(\nabla) = A$ and the lemma is proved.

Assume now that $\mathfrak{g}_p = \{0\}$ for all $p \in M$. This is the case if and only if M is of the form $M = G/D$, where G is a complex Lie group and D is a discrete subgroup of G . In this case, the map $\nabla \rightarrow \mathfrak{g}(\nabla)$ establishes a one-to-one correspondence between the set of all complex affine structures on M and the set $A(\mathfrak{g})$ of all pre-Lie algebra structures over \mathfrak{g} . In particular, this holds for a complex torus M . In this case the Lie algebra \mathfrak{g} is abelian and Theorem 1 follows from the following lemma and from what we have proved so far.

Lemma 3. *Let A be a pre-Lie algebra over a Lie algebra \mathfrak{g} . Assume \mathfrak{g} is abelian. Then A is a commutative associative algebra.*

In fact, $xy - yx = [x, y] = 0$ and hence $xy = yx$ for $x, y \in A$. Moreover, $x(yz) - (xy)z = x(zx) - (xz)y$ and $yz = zy$ and hence $(xy)z = (xz)y$. But $(xy)z = z(xy)$ and $(xz)y = (zx)y$ and hence $z(xy) = (zx)y$ and this proves that A is associative.

Now each pre-Lie algebra structure over \mathfrak{g} is identified with an element of the vector space $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ and $A(\mathfrak{g})$ is identified with an algebraic subset of $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. For each $p \in M$, Let A_p be the subset of $A(\mathfrak{g})$ consisting of all pre-Lie algebra over \mathfrak{g} such that \mathfrak{g}_p is a left ideal. Then A_p is an algebraic subset of $A(\mathfrak{g})$ and by Lemma 2, $A(M)$ is identified with $\bigcap_{p \in M} A_p$: $A(M) = \bigcap_{p \in M} A_p$. Then there exists a finite number of points p_1, \dots, p_r in M such that $A(M) = \bigcap_{i=1}^r A_{p_i}$ and hence $A(M)$ is a complex affine variety. This proves Theorem 2.

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