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ON TIGHT 6-DESIGNS

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1. Introduction and summary

Let $v$, $k$, $\lambda$ and $t$ be positive integers with $v \geq k \geq t$. Let $X$ be a $v$-set and $\mathcal{A}$ a collection of $k$-subsets of $X$. The elements of $X$ will be called points and the elements of $\mathcal{A}$ will be called blocks. The cardinality of $\mathcal{A}$ will be called $b$.

$\mathcal{D} = \mathcal{D}(X, \mathcal{A})$ is defined to be a $t$-$(v, k, \lambda)$ design (or simply a $t$-design) if each $t$-subset of $X$ is contained in exactly $\lambda$ blocks.

For $0 \leq i \leq t$, let $\lambda_i$ be the number of blocks on a given $i$-subset $S_i$ of $X$. By counting in two ways the number of ordered pairs $(S_i, B)$ where $S_i$ is a $t$-subset of $X$ and $B$ is a block such that $S_i \subseteq S_i \subseteq B$, we get

$$\binom{v-i}{t-i} \lambda = \lambda_i \binom{k-i}{t-i}, \quad 0 \leq i \leq t.$$ 

Thus $\lambda_i$ is independent of the $i$-subset chosen and $\mathcal{D}$ is an $i$-design for $0 \leq i \leq t$.

If there exist $A$ and $B$, distinct elements of $\mathcal{A}$, such that $|A \cap B| = \mu$, then $\mu$ is called an intersection number of $\mathcal{D}$.

A design is called trivial if $\mathcal{A}$ consists of all $k$-subsets of $X$.

Ray-Chaudhuri and Wilson have shown

**Theorem** (Ray-Chaudhuri and Wilson [7], Theorem 1). *Generalized Fisher's Inequality:*

The existence of a $t$-$(v, k, \lambda)$ design with $t$ even, say $t = 2s$, and $v \geq k + s$ implies

$$b \geq \binom{v}{s}.$$ 

If $\mathcal{D}$ is a $2s$-design with $v \geq k + s$ and $b = \binom{v}{s}$, then $\mathcal{D}$ is called a tight $2s$-design. (If $v < k + s$, $\mathcal{D}$ is trivial.)

If $\mathcal{D}$ is a $2s$-design with $v = k + s$, then $X$ has $\binom{v}{v-s}$ $k$-subsets. Since

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1) This is not the most general definition of a design. However, it can be shown that for tight designs, which we will be interested in here, the two definitions are equivalent. See Ray-Chaudhuri and Wilson [7].
The following has also been shown.

**Theorem** (Ray-Chaudhuri and Wilson [7], Theorem 4). *If $\mathcal{D}$ is a 2s-design with $v \geq k + s$ and with $s$ intersection numbers, then $s \geq s$. $s = s$ if and only if $\mathcal{D}$ is a tight 2s-design.*

Tight 2-designs are the symmetric (projective) designs ($v = b$). Ito [4] has shown that the only non-trivial tight 4-designs are the Witt tight designs: a 4-(23, 7, 1) design and its complement, a 4-(23, 16, 52) design.

We will investigate 2s-designs with $s$ intersection numbers (thus tight 2s-designs), for $s \geq 3$.

The following theorems will be proved

**Theorem 4.1.** *Let $\mathcal{D}$ be a tight 2s-design, with $s \geq 3$. If the intersection numbers are "symmetric" about their average ($\beta - a$ is an intersection number whenever $\beta + a$ is an intersection number, where $\beta$ is the average of the intersection numbers), then $\mathcal{D}$ is trivial.*

**Theorem 4.2.** *There are no non-trivial tight 6-designs.*

To prove these theorems, two techniques will be used. One is to establish relationships between certain incidence matrices and adjacency matrices that are associated with tight designs, and in particular, to establish relationships between the eigenvalues of these matrices.

The other technique is to investigate the coefficients of the polynomial whose roots are the intersection numbers, and to investigate the coefficients of the polynomial whose roots are the differences between each intersection number and the average of the intersection numbers.

### 2. Matrices associated with designs

In this section we will define certain incidence matrices and adjacency matrices associated with designs and establish systems of equations relating them. We will compute the eigenvalues of certain of these matrices and show that the matrix equations can be rewritten in terms of eigenvalues.

We begin by examining the structure of adjacency matrices for trivial designs. This structure will then be used to examine the structure of matrices associated with other designs.

Let $X$ be a set of $v$ points and let $P_j, j = 0, \cdots, v$, be the collection of all $j$-subsets of $X$. Define $A_{ij}, i = 0, \cdots, j$, to be the adjacency matrix, corresponding
to $i$, of the trivial design $(X, P_j)$; that is, $A_i^j$ is the matrix with rows and columns indexed by the $j$-subsets of $X$, and with $(\{x_1, \ldots, x_j\}, \{y_1, \ldots, y_j\})$ entry equal to $1$ if $|\{x_1, \ldots, x_j\} \cap \{y_1, \ldots, y_j\}| = i$, and $0$ otherwise.

We will show that, for $j \leq v/2$, the matrices $A_i^j, i=0, \ldots, j$, are simultaneously diagonalizable and we will compute their eigenvalues. To state this precisely, let $W_j = RP_j, j=0, \ldots, v$, be the vector space over the field of real numbers with basis $P_j$. For $j=0, \ldots, v$ and $i=0, \ldots, j$, define $B_{ij}$ to be the matrix associated with the map from $W_i$ to $W_j$ which takes an $i$-subset of $X$ to the formal sum of all $j$-subsets of $X$ containing it. That is, $B_{ij}$ is the matrix with rows indexed by the $i$-subsets of $X$ and columns indexed by the $j$-subsets of $X$, and with $(\{x_1, \ldots, x_i\}, \{y_1, \ldots, y_j\})$ entry equal to $1$ if $\{x_1, \ldots, x_i\} \subseteq \{y_1, \ldots, y_j\}$, and $0$ otherwise.

**Theorem 2.1.** For $j \leq v/2$, there is a decomposition

$$W_j = W_j,0 \oplus \cdots \oplus W_j,i \oplus \cdots \oplus W_j,j,$$

such that for $m=0, \ldots, j$,

i) $W_{j,m} = W_{m,m}B_{mj}$ and\dim W_{j,m} = \binom{v}{m}-\binom{v}{m-1},

ii) $W_{j,m}$ is an eigenspace for $A_i^j, i=0, \ldots, j$,

iii) the eigenvalue of $A_i^j$ on $W_{j,m}$ is $\alpha_{i,j}^m = \sum_{r=0}^{m} (-1)^r \binom{m}{r} (v-j-r)(i-m+r)$.

**Proof.** We argue by induction on $j$. For $j=0$, the statement is trivial.
Suppose that for all $j' < j$,

$$W_{j'} = W_{j',0} \oplus \cdots \oplus W_{j',i} \oplus \cdots \oplus W_{j',j'},$$

is a decomposition of $W_{j'}$, satisfying i), ii) and iii). Define $W_{j,m} = W_{m,m}B_{mj}, m=0, \ldots, j-1$.

It can be shown that the rank of $B_{mj}$ is equal to $\binom{v}{m}$ for $m \leq j \leq v/2$ (see Kantor [5]). Thus, for $m=0, \ldots, j-1$, $\dim W_{j,m} = \dim W_{m,m}$ which, by induction, is equal to $\binom{v}{m}-\binom{v}{m-1}$.

Then

(2) $W_j = W_j,0 \oplus \cdots \oplus W_{j,i} \oplus \cdots \oplus W_{j,j}$

where $W_{j,i} = \{x \in W_j | x(B_{ij})^t = 0, \text{ for all } i=0, \ldots, j-1\}$, and $\dim W_{j,i} = \binom{v}{j} - \binom{v}{j-1}$.

Therefore (2) satisfies i), for all $m=0, \ldots, j$. 
We will verify ii) and iii) first for \( m=0, \ldots, j-1 \), and then for \( m=j \).

For \( i, j, n=0, \ldots, \lfloor v/2 \rfloor \), define \( C^i_j \) to be the matrix with rows indexed by the \( i \)-subsets of \( X \), columns indexed by the \( j \)-subsets of \( X \), and with \((\{x_1, \ldots, x_i\}, \{y_1, \ldots, y_j\}) \) entry equal to 1 if \( |\{x_1, \ldots, x_i\} \cap \{y_1, \ldots, y_j\}| = n \), and 0 otherwise.

We proceed in several steps to investigate \((w_mB_m)A^i_j \) for \( w_m \in W_{m,m} \).

**Step 1.** We show that

\[
B^i_j A^i_j = \sum_{r=0}^{m} \binom{v-j-r}{j-i-r} \binom{j-m+r}{i-m+r} C^m_{m-r}.
\]

\( B^i_j A^i_j \) will be a matrix with rows indexed by the \( m \)-subsets of \( X \) and columns indexed by the \( j \)-subsets of \( X \). If \( S_m \) is an \( m \)-subset of \( X \) and \( S_j \) is a \( j \)-subset of \( X \), then the \((S_m, S_j)\) entry will be equal to the number of \( j \)-subsets of \( X \) which contain \( S_m \) and meet \( S_j \) in exactly \( i \) points. Thus, if \( |S_m \cap S_j| = m-r \), the \((S_m, S_j)\) entry is \( \binom{v-j-r}{j-i-r} \binom{j-m+r}{i-m+r} \). Then

\[
B^i_j A^i_j = \sum_{r=0}^{m} \binom{v-j-r}{j-i-r} \binom{j-m+r}{i-m+r} C^m_{m-r}.
\]

**Step 2.** Similarly, we show that

\[
A^n_m B_m = \sum_{r=0}^{m} \binom{m-r}{i} \binom{j-m+r}{m-i} C^m_{m-r}.
\]

The \((S_m, S_j)\) entry of \( A^n_m B_m \) is equal to the number of \( j \)-subsets which meet \( S_m \) in \( i \) points and are contained in \( S_j \). Thus, if \( |S_m \cap S_j| = m-r \), the \((S_m, S_j)\) entry is \( \binom{m-r}{i} \binom{j-m+r}{m-i} \). Therefore

\[
A^n_m B_m = \sum_{r=0}^{m} \binom{m-r}{i} \binom{j-m+r}{m-i} C^m_{m-r}.
\]

Now in Steps 3 and 4, we proceed to show, by induction on \( j \), that, for \( m=0, \ldots, j-1 \), \( W_{j,m} = W_{m,m} B_m \) is an eigenspace for all \( A^i_j \), \( i=0, \ldots, j \), and in particular, the eigenvalues of \( A^i_j \) on \( W_{j,m} \) is

\[
\alpha^n_{ij} = \sum_{r=0}^{m} (-1)^r \binom{m-r}{i} \binom{v-j-r}{j-i-r} \binom{j-m+r}{i-m+r}.
\]

The above statement is trivial for \( j=0 \). Assume that it holds for \( j^* < j \).

**Step 3.** We show by induction on \( n \) that if \( m < j \) and \( w_m \in W_{m,m} \), then

\[
w_m C^m_{m-j} = (-1)^n \binom{m}{n} (w_mB_m).
\]
If \( n=0 \), (4) gives \( C^m_{mj} = A^m_mB_{mj} = B_{mj} \). Assume that the statement is true for all \( n' < n \). Since \( m < j \), the first induction hypothesis says that the eigenvalue of \( A^m_{m-n} \) on \( W_{m,m} \) is

\[
\alpha^m_{m-n,m} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \binom{v-m-r}{n-r} \binom{r}{j-n} = (-1)^r \binom{m}{n} .
\]

Then

\[
(7) \quad (-1)^r \binom{m}{n} w_mB_{mj} = w_m A^m_{m-n} B_{mj} ,
\]

\[
= w_m \sum_{r=0}^{n} \binom{m-r}{m-n} \binom{j-m+r}{n} C^m_{mj} .
\]

Rearranging (7) and applying the second induction hypothesis gives

\[
\omega_mC^{m-n}_{mj} = \frac{\left( (-1)^r \binom{m}{n} - \sum_{r=0}^{n-1} (-1)^r \binom{m-r}{m-n} \binom{j-m+r}{n} \right)}{\binom{j-m+n}{n}} (w_mB_{mj}) .
\]

Then, since \( \binom{m-r}{m-n} = \binom{m}{n} \binom{n}{r} \),

\[
(8) \quad \omega_mC^{m-n}_{mj} = \frac{(-1)^r \binom{m}{n} \left[ 1 - \sum_{r=0}^{n-1} (-1)^{n-r} \binom{n}{q} \binom{j-m+r}{n} \right]}{\binom{j-m+n}{n}} (w_mB_{mj}) .
\]

Now, using an inductive argument and the fact that \( \binom{n}{q} + \binom{n-1}{q-1} = \binom{n}{q} \), for non-negative integers \( n \) and \( q \), it can be shown that \( \sum_{q=0}^{n} (-1)^q \binom{n}{q} (x+q-n) = 1 \), where \( n \) and \( x \) are non-negative integers.

Applying this to the expression in (8) gives,

\[
1 - \sum_{r=0}^{n-1} (-1)^{n-r} \binom{m-r}{n-r} \binom{j-m+r}{n} = \binom{j-m+n}{n}
\]

and

\[
\omega_mC^{m-n}_{mj} = (-1)^r \binom{m}{n} (w_mB_{mj}) , \quad \text{for} \quad \omega_m \in W_{m,m} .
\]

Step 4. Now combining (3) and (6),

\[
(w_mB_{mj}) A^k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \binom{v-j-r}{n-r} \binom{j-m+r}{i-m+r} (w_mB_{mj}) .
\]

We now have
\[ W_j = W_{j,0} \oplus \cdots \oplus W_{j,m} \oplus \cdots \oplus W_{j,j-1} \oplus W_{j,j}, \]

where, for \( m=0, \ldots , j-1 \), \( W_{j,m}=W_{m,m}B_{m,j} \) is an eigenspace for all \( A_i, i=0, \ldots , j \), (Note \( A_j^j=I \)), with eigenvalue

\[ \alpha_{ij}^m = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \binom{j-r}{j-i-r} \binom{j-m+r}{i-m+r}. \]

It remains to verify ii) and iii) for \( m=j \); that is, we need to show that \( W_{j,j} \)

is an eigenspace for all \( A_i, i=0, \ldots , j \), and to compute the eigenvalues.

For any given \( j, i, \) and \( m \), the matrix \( (B_{m,j})^i A_i ^m B_{m,j} \) can be expressed as a linear combination of the matrices \( A_i, n=0, \ldots , j \), as follows. By (4),

\[ A_i ^m B_{m,j} = \sum_{r=0}^{m-i} \binom{m-r}{i} \binom{j-m+r}{m-i} C_{m-r}^i, \]

and thus,

\[ (B_{m,j})^i A_i ^m B_{m,j} = \sum_{r=0}^{m-i} \binom{m-r}{i} \binom{j-m+r}{m-i} (B_{m,j})^n C_{m-r}^i. \]

Now, \( (B_{m,j})^n C_{m-r}^i \) is a matrix with rows and columns indexed by \( j \)-subsets of \( X \) and \( (S, T) \) entry equal to the number of \( m \)-subsets of \( X \) which are contained in \( S \) and which meet \( T \) in exactly \( m-r \) points. Thus, if \( |S \cap T|=n \), the \( (S, T) \) entry is \( \binom{j-n}{r} \binom{n}{m-r} \).

Therefore,

\[ (B_{m,j})^n C_{m-r}^i = \sum_{n=0}^{j} \binom{j-n}{r} \binom{n}{m-r} A_i^n, \]

and

\[ (B_{m,j})^i A_i ^m B_{m,j} = \sum_{n=0}^{j} \binom{j-n}{r} \binom{n}{m-r} \binom{j-m+r}{m-i} (B_{m,j})^n A_i^n. \]

For \( j=0, \ldots , \left\lfloor \frac{v}{2} \right\rfloor \), let

\[ R_j = \langle A_0^i, \ldots , A_j^i \rangle_\mathbb{R}. \]

Let \( A_i^{-1} = (B_{i-1,j})^i A_i^{-1} B_{i-1,j} \) for \( i=0, \ldots , j-1 \), and let

\[ R_{j-1} = \langle A_i^{-1}, \ldots , A_j^{-1} \rangle_\mathbb{R}. \]

By (9), \( R_{j-1} \subseteq R_j \).

By induction, \( A_i^{-1}, \ldots , A_j^{-1} \) are simultaneously diagonalizable and \( R_{j-1} \) is an algebra of dimension \( j \) over \( \mathbb{R} \). \( R_{j-1} \) has dimension \( j \) over \( \mathbb{R} \) and is commutative and closed under multiplication since \( B_{i-1,j} (B_{i-1,j})^t=(v-j+1)I \). The identity, \( I \), is not contained in \( R_{j-1} \) since \( (B_{i-1,j})^t B_{i-1,j} \) is singular. Therefore \( \langle A_i^{-1}, \ldots , A_j^{-1}, I=\langle A_i^{-1} \rangle_\mathbb{R} \rangle = R_j \).

\( R_j \) is then an algebra of dimension \( j+1 \) over \( \mathbb{R} \), and \( A_i, \ldots , A_j=I \) are simultaneously diagonalizable.

\[ W_{j,j}=\langle w \in W_j \mid w(B_{j-1,j})^t=0 \rangle, \]since \( B_{j-1,j}(B_{j-1,j})^t \) is non-singular. Now,
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since any $A^i_j$, $i=0, \ldots, j$, can be expressed as a linear combination of the matrices $A^{i-1}_j=(B(j-1))A^{i-1}_jB(j-1)_j$, $n=0, \ldots, j-1$, and the identity, $W_{j,j}$ is an eigenspace for all $A^i_j$, $i=0, \ldots, j$.

Setting $m=j-1$ in (9) gives,

$$(10) \quad (B_{(j-1)})^T A^{i-1}_j B_{(j-1)} = \sum_{n=0}^{j} a(i, n) A^i_n$$

where $a(i, n) = \sum_{r=0}^{j-i} \binom{j-1-r}{i} \binom{1+r}{j-1-i} \binom{j-n}{r} \binom{n}{j-1-r}$.

To determine the eigenvalues of $A^i_j$, $i=0, \ldots, j$, on $W^j_j$, consider the matrix $A$ with rows indexed by $A^{i-1}_j$, $i=0, \ldots, j$, (define $A^{-1}_j=I$), and columns indexed by $A^i_j$, $i=0, \ldots, j$, with $(A^{i-1}_j, A^i_i)$ entry equal to $a(i, n)$.

Since $A$ is a non-singular matrix, the equation

$$A \begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_j \end{pmatrix}$$

has a unique solution $\begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix}$.

Let $y_i$, $i=0, \ldots, j$, be equal to the eigenvalues of $A^{i-1}_j$ on $W^j_j$. Then $x_n$, $n=0, \ldots, j$, will be the eigenvalue of $A^i_n$ on $W^j_j$. Since $W^j_j = \langle w \in W_j \mid w(B_{(j-1)})=0 \rangle$,

$$\begin{pmatrix} y_0 \\ \vdots \\ y_{j-1} \\ y_j \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We show that $x_i=(-1)^{i-i}\binom{j}{i}$, $i=0, \ldots, j$, satisfies

$$(11) \quad A \begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using (10) to expand $A \begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix}$, we get a column matrix with $i$-th entry, $i=0, \ldots, j-1$, given by

$$\sum_{m=0}^{j} \sum_{r=0}^{j-i} \binom{j-1-r}{i} \binom{1+r}{j-1-i} \binom{j-n}{r} \binom{n}{j-1-r} (-1)^{j-n} \binom{j}{n},$$

and $j$-th entry equal to 1.

Non-zero entries occur in the above sum when $i \leq j-1-r$, $j-1-i \leq 1+r$, $r \leq j-1$ and $j-1-r \leq n$; that is, when $r=j-i-1$ or $j-1-2$, and $n=j-r$ or $j-r-1$. It is then easily verified that the sum is equal to zero.
Therefore the eigenvalue of $A^i$ on $W_{j, i}$ is $\alpha^j = (-1)^{j-i}(\frac{j}{i})$, $i = 0, \ldots, j$, and ii) and iii) are satisfied.

This completes the proof of Theorem 2.1.

Let $D = \mathcal{D}(X, \mathfrak{A})$ be a $t-(v, k, \lambda)$ design with $s$ intersection numbers $\mu_0, \mu_1, \ldots, \mu_{s-1}$. Define $s$ to be the greatest integer in $t/2$. By Ray-Chaudhuri and Wilson [7], $s \leq s$.

For $j = 0, \ldots, v$, define $I_j$ to be the incidence matrix for the incidence structure $(P_j, \mathfrak{A})$; that is, $I_j$ is the matrix with rows indexed by the $j$-subsets of $X$ and columns indexed by the blocks, with the $\{x_1, \ldots, x_j\}, H_j$ entry equal to 1 if the points $x_1, \ldots, x_j$ are incident with the block $H_j$, and 0 otherwise.

$I_j(I_j)'$ is a matrix with rows and columns indexed by the $j$-subsets of $X$ (where $(I_j)'$ denotes the transpose of $I_j$). If $S$ and $T$ are $j$-subsets of $X$, then the $(S, T)$-entry of $I_j(I_j)'$ is equal to the number of blocks $B$ such that $S \cup T \subseteq B$. If $j \leq s$ and $|S \cap T| = i$, then the $(S, T)$-entry of $I_j(I_j)'$ is $\lambda_{2j-i}$. Therefore, we have

**Proposition 2.1.** $I_j(I_j)' = \sum_{i=0}^{j} \lambda_{2j-i}A_i$, for $j = 0, \ldots, s$.

For $i = 0, \ldots, s-1$, define $N_i$ to be the adjacency matrix, corresponding to $\mu_i$, of $D$; that is, $N_i$ is the matrix with rows and columns indexed by blocks and with $(H, K)$-entry equal to 1 if $|H \cap K| = \mu_i$ and 0 otherwise.

$(I_j)N_i$ is a matrix with rows and columns indexed by blocks, and with $(H, K)$-entry equal to the number of $j$-subsets, $S$, of $X$ such that $S \subseteq H \cap K$. If $|H \cap K| = \mu_i$, then the $(H, K)$-entry is $\binom{\mu_i}{j}$. If $H = K$, then the $(H, K)$-entry is $\binom{k}{j}$. Therefore, if $I$ is the $(b, b)$ identity matrix, we have

**Proposition 2.2.** $(I_j)'I_j = \sum_{i=0}^{j-1} \binom{\mu_i}{j}N_i + \binom{k}{j}I$, for $j = 0, \ldots, v$.

We will compute the eigenvalues of $I_j(I_j)'$ and $(I_j)'I_j$, for $j = 0, \ldots, s$, and show that the equations given in Propositions 2.1 and 2.2 can be rewritten in terms of eigenvalues.

**Lemma 2.1.** $\binom{k-m}{n} \lambda_y = \sum_{i=0}^{n} \binom{v-y}{i} \binom{y-m}{n-i} \lambda_{y+i}$, for $0 \leq y, y+m \leq t$.

**Proof.** From (1), $\binom{v-i}{t-i} \lambda = \lambda_i \binom{k-i}{t-i}$, $0 \leq i \leq t$, it follows that $\binom{v-y}{i} \lambda_{y+i} = \binom{k-y}{i} \lambda_y$. Using the identity $\binom{k-m}{n} = \sum_{i=0}^{n} \binom{k-i}{y} \cdot \binom{y-m}{n-i}$, we obtain $\binom{k-m}{n} \lambda_y = \sum_{i=0}^{n} \binom{k-y}{i} \binom{y-m}{n-i} \lambda_y = \sum_{i=0}^{n} \binom{v-y}{i} \binom{y-m}{n-i} \lambda_{y+i}$.
Applying the definitions and results of Theorem 2.1 to the set of points, \(X\), of \(\mathcal{D}\), gives the decomposition,

\[ W_j = W_{j,0} \oplus \cdots \oplus W_{j,m} \oplus \cdots \oplus W_{j,j}, \]

such that each \(W_{j,m}\) is an eigenspace for all \(A_i\), \(i=0, \ldots, j\).

**Theorem 2.2.**

i) For \(j=0, \ldots, s=[t/2]\), and \(m=0, \ldots, j\), \(W_{j,m}\) is an eigenspace for \(I_j(I_j)'\) with eigenvalue

\[ e_{j,m} = \binom{k-m}{j-m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{j+r}. \]

ii) Let \(V=\mathbb{R}^k\) be the vector space over the field of real numbers with basis \(\mathcal{A}\). For \(m=0, \ldots, s\), define

\[ V_m = W_{m,m} I_m, \quad \text{and} \]

\[ V'_s = \langle v \in V \mid v(I_s)' = 0 \rangle. \]

Then \(V=V_0 \oplus \cdots \oplus V_m \oplus \cdots \oplus V_s \oplus V'_s\) is a decomposition of \(V\) into eigenspaces of \((I_j)'I_j\), for all \(j=0, \ldots, s\). For \(m \leq i\), the eigenvalue of \((I_j)'I_j\) on \(V_m\) is

\[ f_{j,m} = e_{j,m} = \binom{k-m}{j-m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{j+r}, \]

and for \(m > i\), \(f_{j,m}=0\).

\[ |V_m| = \binom{v}{m} - \binom{v}{m-1} \quad \text{for } m=0, \ldots, s, \text{ and } j=0, \ldots, s, \]

and \(|V'_s| = b - \binom{v}{s}\).

**Proof.**

i) Recall that \(W_{j,m}=W_{m,m} B_m\). Let \(w_m \in W_{m,m}\). By Theorem 2.1 (iii) and Proposition 2.1,

\[(w_mB_m)I_j(I_j)' = \sum_{i=0}^{j} \lambda_{2j-i} (w_mB_m) A_i \]

\[= \sum_{r=0}^{m} (-1)^r \binom{m}{r} \left[ \sum_{i=0}^{j} \binom{v-j+r}{i} \binom{j-m+r}{i-m+r} \lambda_{2j-i} \right] (w_mB_m). \]

Then applying Lemma 2.1 to the above expression with \(i\) replaced by \(j-i-r\), \(n\) by \(j-m\) and \(y\) by \(j+r\), gives

\[(w_mB_m)I_j(I_j)' = \binom{k-m}{j-m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{j+r} (w_mB_m). \]

ii) Fix \(j, j=0, \ldots, s\). Let \(m \leq j\). Since \(B_m I_j = \binom{k}{j-m} I_m\), it follows that
\[ V_m = W_{m,m} I_m = W_{m,m} P_j I_j = W_j, m I_j. \] Then \[ V_m(I_j) \gamma I_j = W_{j,m} I_j \gamma I_j = e_{j,m} W_j, m I_j = e_{j,m} V_m. \]

Since the rank of \( I_j \) is \( \binom{q}{j} = |W_j| \), \( V_m(I_j)^i = 0 \) for \( m > i \); so \( V_m(I_j) \gamma I_j = 0 \) for \( m > i \).

The matrix equations

\[ (I_j) \gamma I_j = \sum_{i=0}^{s-1} \binom{\mu_i}{j} N_i + \binom{k}{j} I, \quad j = 0, \ldots, s, \]

can be written in terms of eigenvalues as follows:

**Theorem 2.3.** There is a decomposition

\[ V = V_0 \oplus \cdots \oplus V_m \oplus \cdots V_s \oplus V', \]

of \( V \) such that for \( m = 0, \ldots, s, s', V_m \) is an eigenspace for \( (I_j) \gamma I_j, j = 0, \ldots, s \), and also for \( N_i, i = 0, \ldots, s-1 \), and

\[ f_{j,m} = \sum_{i=0}^{s-1} \binom{\mu_i}{j} g_{i,m} + \binom{k}{j}, \]

where \( f_{j,m} \) is the eigenvalue for \( (I_j) \gamma I_j \) on \( V_m \) and \( g_{i,m} \) is the eigenvalue for \( N_i \) on \( V_m \).

Proof. Take \( V_m = W_{m,m} I_m \) as in the proof of Theorem 2.2. The theorem can then be easily verified by showing that \( N_i \subseteq S = \langle I_j, (I_0) \gamma I_0, \ldots, (I_{s-1}) \gamma I_{s-1} \rangle_R \), for \( i = 0, \ldots, s \) (see Cameron [2]).

### 3. Polynomials associated with designs

Consider the polynomial whose roots are the intersection numbers, \( \mu_0, \ldots, \mu_{s-1} \), for a design, and the polynomial whose roots are the differences, \( \beta - \mu_i, i = 0, \ldots, s-1 \), between each intersection number \( \mu_i \) and the average \( \beta \), of the intersection numbers, for a design, we will compute the coefficients for these polynomials, for a tight design, in terms of the design parameters and the coefficients of the polynomial \( x(x-1) \cdots (x-s+1) \). This will lead to a proof of Theorem 4.1.

Consider the polynomial given by

\[ n! \left( \begin{array}{c} x \\ n \end{array} \right) = x(x-1) \cdots (x-n+1) = \sum_{r=0}^{n} s(n, r) x^r. \]

The coefficients, \( s(n, r) \), \( r = 0, \ldots, n \), are called Stirling numbers of the first kind. It is convenient to define \( s(0, 0) = 1 \). The following facts are known.
**Proposition 3.1.** For \( n \geq 1 \) and \( r \) such that \( n \geq r \geq 0 \),

i) \( s(n, r) = s(n-1, r-1) - (n-1)s(n-1, r) \)

ii) \( \sum_{n=0}^{\infty} s(n, r) = 0 \), for \( n \geq 2 \)

iii) \( s(0, 0) = 1, s(n, n) = 1, s(n, 0) = 0 \)

iv) \( s(n, n-1) = \binom{n}{2} \)

v) \( s(n, n-2) = \frac{1}{4}(3n-1)\binom{n}{3} \)

vi) \( s(n, n-3) = -\binom{n}{2}\binom{n}{4} \)

vii) \( s(n, n-4) = (1/48)(15n^3 - 30n^2 + 5n + 2)\binom{n}{5} \)

viii) \( s(n, n-5) = -\frac{1}{8}(3n^2 - 7n - 2)\binom{n}{6} \)

Proof. i) and iii) follow from the definition of \( s(n, r) \). ii) follows from the fact that 1 is a root of \( x(x-1)(x-n+1) \) for \( n \geq 2 \). iv)-viii) can be proved by inductive arguments using i).

Applying (12) to \( \binom{x-a}{n} \) and rearranging according to powers of \( x \) yields,

**Proposition 3.2.** For \( x-a \geq n \geq 0 \)

\[
\binom{x-a}{n} = \frac{1}{n!} \sum_{j=0}^{n} \left[ \sum_{i=0}^{j} (-1)^{j-i} \binom{r}{i} s(n, r) a^{r-j} \right] x^j.
\]

If we define \( s(-1, -1) = 1 \) and \( s(0, -1) = -1 \), then

**Proposition 3.3.** For \( x \geq n \geq 0 \)

\[
\binom{x}{n} - \binom{x}{n-1} = \frac{1}{n!} \sum_{r=0}^{n} s(n-1, r-1) x^r.
\]

Proof. Follows from (12) using proposition 3.1, i) and ii).

Let \( \mathcal{D} = \mathcal{D}(X, \mathcal{A}) \) be a \( t-(v, k, \lambda) \) design with \( s \) intersection numbers, \( \mu_0, \ldots, \mu_{s-1} \).

Let

\[
\prod_{j=0}^{s-1} (x - \mu_j) = \sum_{i=0}^{s} (-1)^i \sigma_i x^{s-i},
\]

so that \((-1)^i \sigma_i \) is the coefficient of \( x^{s-i} \) in the polynomial whose roots are the intersection numbers of \( \mathcal{D} \).

Let \( y = x - \beta \), where \( \beta = \sigma_1/s \) is the average of the intersection numbers.

Then substituting into (13) yields
\[(14) \quad \prod_{j=0}^{s-1} (y-(\mu_j-\beta)) = \sum_{j=0}^{s} p_j y^{s-j}, \text{ where} \]

\[(15) \quad p_j = \sum_{r=0}^{s} (-1)^r \sigma_r \binom{s-r}{j-r} (\sigma_j s)^{s-r}. \]

Thus \(p_j\) is the coefficient of \(y^{s-j}\) in the polynomial whose roots are the differences \((\mu_j-\beta), j=0, \ldots, s-1.\)

If the intersection numbers of \(\mathcal{D}\) have the property that \(\beta-a\) is an intersection number whenever \(\beta+a\) is an intersection number, where \(\beta\) is the average of the intersection numbers, then we will say that the intersection numbers of \(\mathcal{D}\) are symmetric. It is easy to show the following.

**Proposition 3.4.** If \(\mathcal{D}\) has symmetric intersection numbers, then \(p_j=0\) for odd \(j.\)

We will use the systems of eigenvalue relations of section 2 to compute \(\sigma_i, i=0, \ldots, s,\) in terms of design parameters and Stirling numbers. Then using (15) and Proposition 3.1 we will obtain expressions for \(p_2\) and \(p_3\) in terms of the design parameters. These, along with Proposition 3.4 will be used in proving the main result in section 4.

The following result will be needed in the proof of lemma 3.2.

**Lemma 3.1** (Generalized Vandermonde Determinant).

Let \(D_r, r=0, 1, \ldots, n-1,\) be the \((n, n)\)-matrix with \((i, j)\)-entry defined as follows, for \(0 \leq i,j \leq n-1,\)

\[(x_j)^{i} \text{ for } i<r \text{ and } (x_j)^{i+1} \text{ for } r \leq i.\]

Then \(D_r = \sigma_{n-r}(x_1, \ldots, x_n) \prod_{m>j=1}^{n} (x_m-x_j)\)
where \(\sigma_{n-r}(x_1, \ldots, x_n)\) is defined by

\[\prod_{j=1}^{n} (x-x_j) = \sum_{r=0}^{n} \sigma_{n-r}(x_1, \ldots, x_n)(-1)^{n-r}.\]

In the rest of this section we will assume that \(\mathcal{D}\) is a tight 2s-design. Then \(s=t/2\) and \(s\) is the number of intersection numbers.

If \(h_{jm}=inom{k}{j}\), then Theorem 2.3 becomes

\[(16) \quad \sum_{i=0}^{s-1} \binom{\mu_i}{j} g_{im} = h_{jm}, \quad \text{ for } j=0, \ldots, s, \text{ and } m=0, \ldots, j.\]

**Lemma 3.2.** \(\sum_{i=0}^{s} (-1)^{s-i} S_{s-i} h_{i(s-i)}=0, j=0, \ldots, s,\)
where \(S_{s-i} = \sigma_i,\) and for \(i=0, \ldots, s-1,\)
Proof. For fixed \( m \), \( 0 \leq m \leq s \), the \( s+1 \) equations in the \( s \) variables \( g_{im} \), \( i=0, \ldots, s-1 \), given in (16) are dependent. Therefore, the determinant,

\[
\begin{vmatrix}
\mu_0 & \cdots & \mu_{s-1} & h_{0m} \\
\mu_1 & \cdots & \mu_{s-1} & h_{1m} \\
\vdots & \vdots & \vdots & \vdots \\
\mu_s & \cdots & \mu_{s-1} & h_{sm}
\end{vmatrix} = 0.
\]

(17)

Then by multiplying by constants and performing row operations on (17), we get

\[
\begin{vmatrix}
1 & \cdots & 1 & k_{0m} \\
\mu_0 & \cdots & \mu_{s-1} & k_{1m} \\
\vdots & \vdots & \vdots & \vdots \\
\mu_s & \cdots & \mu_{s-1} & k_{sm}
\end{vmatrix} = 0,
\]

(18)

where

\[
k_{im} = i!h_{im} + \sum_{r=1}^{i-1} r!h_{rm} \sum_{r-l_1<\ldots<l_{n-i}} (-1)^{n-1}s(l_n, l_{n-1}) \cdots s(l_3, l_2)s(l_2, l_1).
\]

(19)

Now, applying Lemma 3.1 in computing the above determinant, we have

\[
\sum_{i=0}^{s} (-1)^j k_{ij} \sigma_{s-i} = 0, \quad \text{for } j=0, \ldots, s.
\]

(20)

The proof is then completed by using (19) to rewrite (20) in terms of the \( h_{ij} \)'s.

Another expression for \( S_{s-i} \) will be given in Lemma 3.5. The following two lemmas will be needed in the proof. The first may be verified by a straightforward manipulation.

**Lemma 3.3.**

\[
\frac{\binom{k}{i+n} \binom{v-k}{q}}{\binom{v}{i+n}} \frac{\binom{k}{i+n+1} \binom{v-k}{q}}{\binom{v}{i+n+1}} = \frac{\binom{k}{i+n} \binom{v-k}{q+1}}{\binom{v}{i+n} \binom{v-(i+n)}{q+1}}.
\]
Lemma 3.4. \( \sum_{n=0}^{\ell} (-1)^n \binom{j}{n} \lambda_{i+n} = b \binom{k}{i} \binom{v-k}{j} \prod_{1 \leq i \leq j} \binom{v-i}{i} \).

Proof. Using the fact that \( \lambda_{i+n} = \binom{i+n}{n} b \), along with Lemma 3.3, we get

\[
\sum_{n=0}^{\ell} (-1)^n \binom{j}{n} \lambda_{i+n} = b \sum_{n=0}^{\ell} (-1)^n \binom{j}{n} \binom{k}{i+n} \frac{(v-k)}{0} \frac{(v-(i+n))}{0} \frac{(v-(i+n))}{1} \frac{(v-(i+n))}{q} \]

\[
= b \sum_{n=0}^{\ell} (-1)^n \binom{j-1}{n} \binom{k}{i+n} \frac{(v-k)}{1} \frac{(v-(i+n))}{1} \frac{(v-(i+n))}{q} \]

\[
= b \sum_{n=0}^{\ell} (-1)^n \binom{j-q}{n} \binom{k}{i+n} \frac{(v-k)}{q} \frac{(v-(i+n))}{q} \]

\[
= b \binom{k}{i} \binom{v-k}{j} \prod_{1 \leq i \leq j} \binom{v-i}{i} \]

Lemma 3.5. \( S_{s-i} = (-1)^{s-i} \frac{s!}{i!} K_{s-i} \), for \( i=0, \cdots, s \),

where \( K_j = \frac{1}{r} \sum_{r=1}^{k-s+r}(k-s+r-1)(v-2s+r) \), for \( j=1, \cdots, s \), and \( K_0 = 1 \).

Proof. Consider the \( s+1 \) equations given in Lemma 3.2,

\[
\sum_{r=0}^{s} (-1)^r S_r h_{tj} = 0, \quad j = 0, \cdots, s.
\]

Subtracting the \( (j=s) \)-equation from the \( (j=s-i) \)-equation gives

\[
\sum_{r=0}^{s} (-1)^r S_{s-n} f_{(s-n)(s-i)} = (-1)^{s-i} f_{s} = 0.
\]

Let \( L_{cd} = \sum_{n=0}^{d} (-1)^n \binom{d}{n} \lambda_{c+n} \). Then \( f_{cd} = \binom{k-d}{c-d} L_{cd} \) by Theorem 2.3.

By Lemma 3.2, \( S_i = (-1)^{s!} \). Using this and rearranging (21), we get
\( S_{s-i} = \frac{1}{L_{s-i}(s-i)} \left[ \sum_{q=0}^{i-1} (-1)^{i-1-q} \binom{k-s+i}{i-q} L_{s-q}(s-q) S_{s-q} \right] \)

for \( i = 0, \ldots, s \).

Now using Lemma 3.4 in (22) and arguing inductively (on \( i \)),

\[ S_{s-i} = (-1)^{i-1} \frac{s!}{i!} K_i, \quad \text{for} \quad i = 1, \ldots, s, \quad \text{and} \quad S_s = (-1)^s s! \]

The values, \( \sigma, \), \( i = 0, \ldots, s \), are the coefficients of the polynomial whose roots are the intersection numbers of \( \mathcal{D} \). Each \( \sigma \) is a function of \( i \) and the design parameters \( v, k, \lambda, \) and \( s \). The next theorem shows that, in fact, \( \sigma \) depends only on \( v, k, \) and \( s \) (in addition to \( i \)).

**Theorem 3.1.**

\[ \sigma_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{s}{j} s(s-j, s-i) K_j, \quad 0 \leq i \leq s, \]

\[ K_j = \sum_{r=1}^{j} \frac{(k-s+r)(k-s+r-1)}{(v-2s+r)}, \quad \text{for} \quad j = 1, \ldots, s, \]

and \( K_0 = 1 \), and \( s(s-j, s-i), j = 0, \ldots, i \), is a Stirling number of the first kind.

Proof. This is trivial for \( i = 0 \). Assume it holds for all \( i' \) such that \( 0 \leq i' \leq i \). Combining Lemmas 3.2 and 3.5, we have

\[ (-1)^{i-i} \frac{s!}{i!} K_i = (s-i)! \left[ (-1)^{s-i} \sigma_i - \sum_{q=0}^{i-1} (-1)^{i-q} Q(q, k) \sigma_q \right], \]

where

\[ Q(q, k) = \sum_{i=0}^{s} \binom{s}{i} s(s-i, s-q) K_i \]

Then, solving for \( \sigma_i \), and applying induction to \( \sigma_q, q = 0, 1, \ldots, i-1 \),

\[ \sigma_i = \left( \binom{s}{i} K_i - \sum_{q=0}^{i-1} (-1)^{i-q} Q(q, k) \sigma_q \right), \]

and

\[ = \left( \binom{s}{i} K_i - \sum_{q=0}^{s} \frac{s(s-j, s-q) s(s-q, l_{i-1}) s(l_2, l_1) K_j}{s(s-j, s-q) s(s-q, l_{i-1}) s(l_2, l_1) K_j} \right) \]

For a given partition of the form, \( s-i = i_1 < \cdots < i_{n-1} = s-q < i_n = s \), we have a pair of terms in the sum which are identical except for opposite sign:

\[ (-1)^{i-j+s-1} \binom{s}{j} s(s-j, s-q) s(s-q, l_{n-2}) s(l_2, l_1) K_j \]
For a partition of the form, \( s - i = l_1 < l_2 = s - j \), there is only one term that occurs in the sum:

\[
(-1)^{i-j+n}\binom{s}{j}s(s-j, s-j)s(s-j, s-q)s(s-q, l_1)\cdots s(l_2, l_1)K_j.
\]

Therefore,

\[
\sigma_i = \sum_{j=0}^{i} (-1)^{i-j}\binom{s}{j}s(s-j, s-i)K_j.
\]

This result can be used to obtain expressions for the coefficients, \( p_j \), of the polynomial whose roots are, \( \beta - \mu_i, i=0, \ldots, s-1 \).

**Theorem 3.2.**

i) \[ p_2 = \frac{-(s+1)s(s-1)}{24}\binom{s}{2}\frac{(k-s+1)(k-s)(v-k-s+1)(v-k-s)}{(v-2s+2)(v-2s+1)^2} \]

ii) \[ p_3 = \binom{s}{3}\frac{(k-s+1)(k-s)(v-k-s+1)(v-k-s)(v-2k+1)(v-2k-1)}{(v-2s+3)(v-2s+2)(v-2s+1)^2} \]

**Proof.** Recall that

\[ p_j = \sum_{r=0}^{i} (-1)^{j-r}\sigma_{j-r}\binom{s}{r}\binom{s-r}{j-r}, \quad j=0, \ldots, s. \]

Using Proposition 3.1, iii)-vi), and Theorem 3.1, we can compute \( \sigma_i, i=0, 1, 2, 3 \), in terms of the design parameters, and then obtain expressions for \( p_2 \) and \( p_3 \).

4. A general result on the existence of tight \( t \)-design, and the non-existence of non-trivial tight 6-designs

**Lemma 4.1.** If \( \mathcal{D} \) is a non-trivial tight 6-design, then \( v \neq 2k \pm 1 \).

**Proof.** Since the complementary design of a non-trivial tight \( t \)-design is also a tight \( t \)-design, it suffices to show that \( v \neq 2k+1 \). If \( s=3 \) and \( v=2k+1 \), then

\[ \sigma_3 = \frac{k(k-1)(k-2)(k-3)}{4(2k-3)} \]

is not an integer, contradicting the definition of \( \sigma_3 \).

The following lemma is due to E. Bannai.
Lemma 4.2 (Bannai [1]). If \( D \) is a tight \( 2s \)-design, with \( s \geq 2 \), then \( v \equiv 2k+1 \).

Proof. The following result of Schur [8, Satz I] is crucial in the proof.

(Schur): Let \( h \) be an integer and let \( m \) be an odd number greater than \( 2h+1 \). Then \( m(m+2) \cdots (m+2h-1) \) is divisible by a prime \( p > 2h+1 \) except for the cases (i) \( h=2, m=25 \) and (ii) \( h=1 \) and \( m=3s \).

Since the complementary design of a non-trivial tight design is also tight, it suffices to show that \( v \equiv 2k+1 \). Assume that \( v=2k+1 \).

First suppose that \( s \geq 4 \). Then the result of Schur shows that there is a prime \( p > 2[\frac{s}{2}]+1 \) (\( \geq s \)) which divides the product of \( \lfloor \frac{s}{2} \rfloor \) consecutive odd numbers,

\[
(2k-2s+2[\frac{s}{2}]+1) \cdots (2k-2s+5)(2k-2s+3),
\]

which appears in the denominator of \( K_s \), unless \( \lfloor \frac{s}{2} \rfloor = 2 \) (i.e., \( s=4 \) or 5) and \( 2k-3s+3=25 \) (i.e., either (i) \( h=15, v=31 \) or (ii) \( h=16, v=33 \)) according as \( s=4 \) or 5). But, these exceptional cases do not occur, because \( \sigma_2 \) is not an integer in these cases.

Therefore, we can choose the smallest even integer \( j \) (less than \( 2[\frac{s}{2}] \)) such that

\[
(2k-2s+3)(2k-2s+5) \cdots (2k-2s+j+1)
\]

has a prime factor \( p > 2[\frac{s}{2}]+1 \). Now, we want to show that \( \sigma_j \) is not an integer. Since the G.C.D. \( (2k-2s+j+1, k-(s-r)) \), with \( 0 \leq r \leq j \), which is equal to G.C.D. \( (2k-2s+j+1, 2r-(j+1)) \), is less than \( p \), we get that \( \binom{s}{j} K_j \) is not a \( p \)-adic integer. On the other hand,

\[
\sigma_j - \left( \begin{array}{c} s \\ j \end{array} \right) K_j = \sum_{l=1}^{j-1} \left( \begin{array}{c} s \\ l \end{array} \right) (-1)^i s(s-l, s-j) K_l,
\]

and the right side is a \( p \)-adic integer. Hence, \( \sigma_j \) is not an integer, which is a contradiction.

The case \( s=2 \) has been shown by Ito [4], and the case \( s=3 \) has been shown in Lemma 4.1.

Theorem 4.1. If \( D \) is a tight \( 2s \)-design (\( s \geq 3 \)) with the coefficient \( p_3=0 \), then \( D \) is trivial. In particular, if \( D \) is a tight \( 2s \)-design (\( s \geq 3 \)) with symmetric intersection numbers, then \( D \) is trivial.

Proof. By Theorem 3.2.

\[
p_3 = - \left( \begin{array}{c} s \\ 3 \end{array} \right) \frac{(k-s+1)(k-s)(v-k-s+1)(v-k-s)(v-2k+1)(v-2k-1)}{(v-2s+3)(v-2s+2)(v-2s+1)^2}
\]
Since $k \geq t=2s+s$ and $k+s-1$. Since $D$ must have $s$ distinct intersection numbers, $v+k+s-1$. By Lemma 4.2, $v+2k\pm 1$. Therefore if $p_3=0$, then $v=k+s$ and $D$ is trivial. If the intersection numbers are symmetric then $p_3=0$ by Proposition 3.4.

Theorem 4.2. If $D$ is a tight 6-design, then $D$ is trivial.

Proof. By Theorem 4.1, it suffices to show that $p_3=0$. Assume $p_3 \neq 0$. By Theorem 3.2,

\[
p_2 = -1 - 3 \frac{(k-2)(k-3)(v-k-2)(v-k-3)}{(v-4)(v-5)^2}
\]

\[
p_3 = -\frac{(k-2)(k-3)(v-k-2)(v-k-3)(v-2k+1)(v-2k-1)}{(v-3)(v-4)(v-5)^3}
\]

Then

\[
0 < -3p_3 = (-p_2-1) \frac{(v-(2k-1))(v-(2k+1))}{(v-3)(v-5)} < -p_2-1.
\]

Let $d_i=\beta-\mu_i$ and $e_i=|d_i|, i=0, 1, 2$. Now, $p_3$ is the constant term of the polynomial whose roots are $d_i, i=0, 1, 2$. Thus

\[
p_3 = d_0d_1d_2 = -e_0e_1e_2.
\]

Similarly, $p_2$ is the coefficient of the linear term of this polynomial. Thus

\[
p_2 = d_0d_1 + d_0d_2 + d_1d_2.
\]

Since $d_0 + d_1 + d_2 = 0$ we get,

\[
p_2 = -\frac{1}{3}(d_0^2 + d_1^2 + d_2^2) = -\frac{1}{3}(e_0^2 + e_1^2 + e_2^2).
\]

Since $0>p_2=d_0d_1d_2$ and $d_0 + d_1 + d_2 = 0$, we may assume that $d_2<0$ and $0<d_0<d_1<|d_2|$. Then since $d_0 + d_1 + d_2 = 0$,

\[
e_0 + e_1 = e_2.
\]

Also,

\[
1/3 < e_0 < e_1 < e_2,
\]

because $0=e_0=\beta-\mu_0=\frac{\mu_0+\mu_1+\mu_2-\mu_0}{3}$

and $\mu_0, \mu_1, \mu_2$ are integers.

Then by (23)-(26),
Now, since $1 < e_0 < e_1$, 

$$e_0 e_1 + 1 < e_0^2 + e_0 e_1, \quad 1 < e_0^2, \quad 1 < e_0.$$ 

Therefore $0 < e_1 < -1/3$. This contradiction proves that $p_3 = 0$.

**References**
