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ON TIGHT 6-DESIGNS

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1. Introduction and summary

Let v, k, λ and t be positive integers with $v \geq k \geq t$. Let X be a v-set and $\mathfrak{A}$ a collection of k-subsets of X. The elements of X will be called *points* and the elements of $\mathfrak{A}$ will be called *blocks*. The cardinality of $\mathfrak{A}$ will be called b.

$\mathfrak{D}=\mathfrak{D}(X, \mathfrak{A})$ is defined to be *a t-(v, k, λ) design* (or simply a *t-design*) if each t-subset of X is contained in exactly λ blocks.[1)]

For $0 \leq i \leq t$, let λ_i be the number of blocks on a given i-subset S_i of X. By counting in two ways the number of ordered pairs (S_t, B) where S_t is a t-subset of X and B is a block such that $S_i \subseteq S_t \subseteq B$, we get

$$(1) \qquad \binom{v-i}{t-i}\lambda = \lambda_i\binom{k-i}{t-i}, \quad 0 \leq i \leq t\,.$$

Thus λ_i is independent of the i-subset chosen and $\mathfrak{D}$ is an i-design for $0 \leq i \leq t$.

If there exist A and B, distinct elements of $\mathfrak{A}$, such that $|A \cap B| = \mu$, then μ is called an *intersection number* of $\mathfrak{D}$.

A design is called *trivial* if $\mathfrak{A}$ consists of all k-subsets of X.

Ray-Chaudhuri and Wilson have shown

Theorem (Ray-Chaudhuri and Wilson [7], Theorem 1). *Generalized Fisher's Inequality:*

The existence of a t-(v, k, λ) design with t even, say $t=2s$, and $v \geq k+s$ implies

$$b \geq \binom{v}{s}.$$

If $\mathfrak{D}$ is a $2s$-design with $v \geq k+s$ and $b=\binom{v}{s}$, then $\mathfrak{D}$ is called a *tight* $2s$-design. (If $v<k+s$, $\mathfrak{D}$ is trivial.)

If $\mathfrak{D}$ is a $2s$-design with $v=k+s$, then X has $\binom{v}{v-s}$ k-subsets. Since

1) This is not the most general definition of a design. However, it can be shown that for tight designs, which we will be interested in here, the two definitions are equivalent. See Ray-Chaudhuri and Wilson [7].

$b\geq\binom{v}{s}=\binom{v}{v-s}$, $\mathfrak{A}$ must consists of all $\binom{v}{s}$ k-subsets of X, and $\mathfrak{D}$ is a tight $2s$-design. It is, however, a trivial $2s$-design.

The following has also been shown.

Theorem (Ray-Chaudhuri and Wilson [7], Theorem 4). *If $\mathfrak{D}$ is a $2s$-design with $v\geq k+s$ and with $\bar{s}$ intersection numbers, then $\bar{s}\geq s$. $\bar{s}=s$ if and only if $\mathfrak{D}$ is a tight $2s$-design.*

Tight 2-designs are the symmetric (projective) designs ($v=b$). Ito [4] has shown that the only non-trivial tight 4-designs are the Witt tight designs: a 4-(23, 7, 1) design and its complement, a 4-(23, 16, 52) design.

We will investigate $2s$-designs with s intersection numbers (thus tight $2s$-designs), for $s\geq3$.

The following theorems will be proved

Theorem 4.1. *Let $\mathfrak{D}$ be a tight $2s$-design, with $s\geq3$. If the intersection numbers are "symmetric" about their average ($\beta-a$ is an intersection number whenever $\beta+a$ is an intersection number, where β is the average of the intersection numbers), then $\mathfrak{D}$ is trivial.*

Theorem 4.2. *There are no non-trivial tight 6-designs.*

To prove these theorems, two techniques will be used. One is to establish relationships between certain incidence matrices and adjacency matrices that are associated with tight designs, and in particular, to establish relationships between the eigenvalues of these matrices.

The other technique is to investigate the coefficients of the polynomial whose roots are the intersection numbers, and to investigate the coefficients of the polynomial whose roots are the differences between each intersection number and the average of the intersection numbers.

2. Matrices associated with designs

In this section we will define certain incidence matrices and adjacency matrices associated with designs and establish systems of equations relating them. We will compute the eigenvalues of certain of these matrices and show that the matrix equations can be rewritten in terms of eigenvalues.

We begin by examining the structure of adjacency matrices for trivial designs. This structure will then be used to examine the structure of matrices associated with other designs.

Let X be a set of v points and let P_j, $j=0, \cdots, v$, be the collection of all j-subsets of X. Define A_i^j, $i=0, \cdots, j$, to be the adjacency matrix, corresponding to i, of the trivial design (X, P_j); that is, A_i^j is the matrix with rows and columns indexed by the j-subsets of X, and with $(\{x_1, \cdots, x_j\}, \{y_1, \cdots, y_j\})$ entry equal to 1 if $|\{x_1, \cdots, x_j\} \cap \{y_1, \cdots, y_j\}|=i$, and 0 otherwise.

We will show that, for $j \leq v/2$, the matrices $A_i^j, i=0, \cdots, j$, are simultaneously diagonalizable and we will compute their eigenvalues. To state this precisely, let $W_j=RP_j, j=0, \cdots, v$, be the vector space over the field of real numbers with basis P_j. For $j=0, \cdots, v$ and $i=0, \cdots, j$, define B_{ij} to be the matrix associated with the map from W_i to W_j which takes an i-subset of X to the formal sum of all j-subsets of X containing it. That is, B_{ij} is the matrix with rows indexed by the i-subsets of X and columns indexed by the j-subsets of X, and with $(\{x_1, \cdots, x_i\}, \{y_1, \cdots, y_j\})$ entry equal to 1 if $\{x_1, \cdots, x_i\} \subseteq \{y_1, \cdots, y_j\}$, and 0 otherwise.

Theorem 2.1. *For $j \leq v/2$, there is a decomposition*

$$W_j = W_{j,0} \oplus \cdots \oplus W_{j,m} \oplus \cdots \oplus W_{j,j},$$

such that for $m=0, \cdots, j$,

i) $W_{j,m} = W_{m,m}B_{mj}$ *and* $\dim W_{j,m} = \binom{v}{m}-\binom{v}{m-1}$,

ii) $W_{j,m}$ *is an eigenspace for A_i^j, $i=0, \cdots, j$,*

iii) *the eigenvalue of A_i^j on $W_{j,m}$ is* $\alpha_{ij}^m = \sum_{r=0}^{m} (-1)^r \binom{m}{r}\binom{v-j-r}{j-i-r}\binom{j-m+r}{i-m+r}$.

Proof. We argue by induction on j. For $j=0$, the statement is trivial. Suppose that for all $j'<j$,

$$W_{j'} = W_{j',0} \oplus \cdots \oplus W_{j',m} \oplus \cdots \oplus W_{j',j'},$$

is a decomposition of $W_{j'}$, satisfying i), ii) and iii). Define $W_{j,m}=W_{m,m}B_{mj}$, $m=0, \cdots, j-1$.

It can be shown that the rank of B_{mj} is equal to $\binom{v}{m}$ *for* $m \leq j \leq v/2$ (see Kantor [5]). Thus, for $m=0, \cdots, j-1$, $\dim W_{j,m}=\dim W_{m,m}$ which, by induction, is equal to $\binom{v}{m}-\binom{v}{m-1}$.

Then

$$W_j = W_{j,0} \oplus \cdots \oplus W_{j,j-1} \oplus W_{j,j} \tag{2}$$

where $W_{j,j}=\{x \in W_j \mid x(B_{ij})^t=0, \text{ for all } i=0, \cdots, j-1\}$, and $\dim W_{j,j}=\binom{v}{j}-\binom{v}{j-1}$.

Therefore (2) satisfies i), for all $m=0, \cdots, j$.

We will verify ii) and iii) first for $m=0, \cdots, j-1$, and then for $m=j$.

For $i, j, n=0, \cdots, [v/2]$, define C^n_{ij} to be the matrix with rows indexed by the i-subsets of X, columns indexed by the j-subsets of X, and with $(\{x_1, \cdots, x_i\}, \{y_1, \cdots, y_j\})$ entry equal to 1 if $|\{x_1, \cdots, x_i\} \cap \{y_1, \cdots, y_j\}|=n$, and 0 otherwise.

We proceed in several steps to investigate $(w_m B_{mj})A^j_i$ for $w_m \in W_{m,m}$.

Step 1. We show that

$$(3) \qquad B_{mj}A^j_i \sum_{r=0}^{m} \binom{v-j-r}{j-i-r}\binom{j-m+r}{i-m+r} C^{m-r}_{mj}\ .$$

$B_{mj}A^j_i$ will be a matrix with rows indexed by the m-subsets of X and columns indexed by the j-subsets of X. If S_m is an m-subset of X and S_j is a j-subset of X, then the (S_m, S_j) entry will be equal to the number of j-subsets of X which contain S_m and meet S_j in exactly i points. Thus, if $|S_m \cap S_j|= m-r$, the (S_m, S_j) entry is $\binom{v-j-r}{j-i-r}\binom{j-m+r}{i-m+r}$. Then

$$B_{mj}A^j_i = \sum_{r=0}^{m} \binom{v-j-r}{j-i-r}\binom{j-m+r}{i-m+r} C^{m-r}_{mj}\ .$$

Step 2. Similarly, we show that

$$(4) \qquad A^m_i B_{mj} = \sum_{r=0}^{m-i} \binom{m-r}{i}\binom{j-m+r}{m-i} C^{m-r}_{mj}\ .$$

The (S_m, S_j) entry of $A^m_i B_{mj}$ is equal to the number of j-subsets which meet S_m in i points and are contained in S_j. Thus, if $|S_m \cap S_j|=m-r$, the (S_m, S_j) entry is $\binom{m-r}{i}\binom{j-m+r}{m-i}$. Therefore

$$A^m_i B_{mj} = \sum_{r=0}^{m-i} \binom{m-r}{i}\binom{j-m+r}{m-i} C^{m-r}_{mj}\ .$$

Now in Steps 3 and 4, we proceed to show, by induction on j, that, for $m=0, \cdots, j-1$, $W_{j,m}=W_{m,m}B_{mj}$ is an eigenspace for all A^j_i, $i=0, \cdots, j$, and in particular, the eigenvalues of A^j_i on $W_{j,m}$ is

$$(5) \qquad \alpha^m_{ij} = \sum_{r=0}^{m} (-1)^r \binom{m}{r}\binom{v-j-r}{j-i-r}\binom{j-m+r}{i-m+r}.$$

The above statement is trivial for $j=0$. Assume that it holds for $j'<j$.

Step 3. We show by induction on n that if $m<j$ and $w_m \in W_{m,m}$, then

$$(6) \qquad w_m C^{m-n}_{mj} = (-1)^n \binom{m}{n} (w_m B_{mj})\ .$$

If $n=0$, (4) gives $C^m_{mj}=A^m_m B_{mj}=B_{mj}$. Assume that the statement is true for all $n'<n$. Since $m<j$, the first induction hypothesis says that the eigenvalue of A^m_{m-n} on $W_{m,m}$ is

$$\begin{aligned}\alpha^m_{(m-n)m} &= \sum_{r=0}^{m}(-1)^r\binom{m}{r}\binom{v-m-r}{n-r}\binom{r}{r-n}\\ &= (-1)^n\binom{m}{n}.\end{aligned}$$

Then

$$\begin{aligned}(7)\qquad (-1)^n\binom{m}{n}w_m B_{mj} &= w_m A^m_{m-n}B_{mj}\\ &= w_m\sum_{r=0}^{n}\binom{m-r}{m-n}\binom{j-m+r}{n}C^{m-r}_{mj}.\end{aligned}$$

Rearranging (7) and applying the second induction hypothesis gives

$$w_m C^{m-n}_{mj} = \frac{\left[(-1)^n\binom{m}{n}-\sum_{r=0}^{n-1}(-1)^r\binom{m}{r}\binom{m-r}{m-n}\binom{j-m+r}{n}\right]}{\binom{j-m+n}{n}}(w_m B_{mj}).$$

Then, since $\binom{m}{r}\binom{m-r}{m-n}=\binom{m}{n}\binom{n}{r}$,

$$(8)\qquad w_m C^{m-n}_{mj} = \frac{(-1)^n\binom{m}{n}\left[1-\sum_{r=0}^{n-1}(-1)^{n-r}\binom{n}{r}\binom{j-m+r}{n}\right]}{\binom{j-m+n}{n}}(w_m B_{mj}).$$

Now, using an inductive argument and the fact that $\binom{n}{q}=\binom{n-1}{q}+\binom{n-1}{q-1}$, for non-negative integers n and q, it can be shown that $\sum_{q=0}^{n}(-1)^q\binom{n}{q}\binom{x+n-q}{n}=1$, where n and x are non-negative integers.

Applying this to the expression in (8) gives,

$$1-\sum_{r=0}^{n-1}(-1)^{n-r}\binom{m-r}{n-r}\binom{j-m+r}{n}=\binom{j-m+n}{n}$$

and

$$w_m C^{m-n}_{mj} = (-1)^n\binom{m}{n}(w_m B_{mj}),\qquad \text{for}\quad w_m\in W_{m,m}.$$

Step 4. Now combining (3) and (6),

$$(w_m B_{mj})A^j_i = \sum_{r=0}^{m}(-1)^r\binom{m}{r}\binom{v-j-r}{j-i-r}\binom{j-m+r}{i-m+r}(w_m B_{mj}).$$

We now have

$$W_j = W_{j,0} \oplus \cdots \oplus W_{j,m} \oplus \cdots \oplus W_{j,j-1} \oplus W_{j,j}\,,$$

where, for $m=0, \cdots, j-1$, $W_{j,m}=W_{m,m}B_{mj}$ is an eigenspace for all A^j_i, $i=0, \cdots, j$, (Note $A^j_j=I$), with eigenvalue

$$\alpha^m_{ij} = \sum_{r=0}^{m}(-1)^r\binom{m}{r}\binom{v-j-r}{j-i-r}\binom{j-m+r}{i-m+r}\,.$$

It remains to verify ii) and iii) for $m=j$; that is, we need to show that $W_{j,j}$ is an eigenspace for all A^j_i, $i=0, \cdots, j$, and to compute the eigenvalues.

For any given j, i, and m, the matrix $(B_{mj})^t A^m_i B_{mj}$ can be expressed as a linear combination of the matrices A^j_n, $n=0, \cdots, j$, as follows. By (4),

$$A^m_j B_{mj} = \sum_{r=0}^{m-i}\binom{m-r}{i}\binom{j-m+r}{m-i} C^{m-r}_{mj}\,,$$

and thus,

$$(B_{mj})^t A^m_i B_{mj} = \sum_{r=0}^{m-i}\binom{m-r}{i}\binom{j-m+r}{m-i}(B_{mj})^t C^{m-r}_{mj}\,.$$

Now, $(B_{mj})^t C^{m-r}_{mj}$ is a matrix with rows and columns indexed by j-subsets of X and (S, T) entry equal to the number of m-subsets of X which are contained in S and which meet T in exactly $m-r$ points. Thus, if $|S \cap T|=n$, the (S, T) entry is $\binom{j-n}{r}\binom{n}{m-r}$.

Therefore, $\quad (B_{mj})^t C^{m-r}_{mj} = \sum_{n=0}^{j}\binom{j-n}{r}\binom{n}{m-r}A^j_n$, and

$$(B_{mj})^t A^m_i B_{mj} = \sum_{n=0}^{j}\sum_{r=0}^{m-i}\binom{m-r}{i}\binom{j-m+r}{m-i}\binom{j-n}{r}\binom{n}{m-r}A^j_n\,. \tag{9}$$

For $j=0, \cdots, [v/2]$, let

$$R_j = \langle A^j_0, \cdots, A^j_j = I\rangle_R\,.$$

Let $\tilde{A}^{j-1}_i = (B_{(j-1)j})^t A^{j-1}_i B_{(j-1)j}$ for $i = 0, \cdots, j-1$, and let

$$\tilde{R}_{j-1} = \langle \tilde{A}^{j-1}_0, \cdots, \tilde{A}^{j-1}_{j-1}\rangle_R\,.$$

By (9), $\tilde{R}_{j-1} \subseteq R_j$.

By induction, $A^{j-1}_0, \cdots, A^{j-1}_{j-1}$ are simultaneously diagonalizable and R_{j-1} is an algebra of dimension j over R. $\tilde{R}_{j-1}$ has dimension j over R and is commutative and closed under multiplication since $B_{(j-1)j}\,(B_{(j-1)j})^t=(v-j+1)I$. The identity, I, is not contained in $\tilde{R}_{j-1}$ since $(B_{(j-1)j})^t B_{(j-1)j}$ is singular. Therefore $\langle \tilde{A}^{j-1}_0, \cdots, \tilde{A}^{j-1}_{j-1}, I \equiv \tilde{A}^{j-1}_j\rangle = R_j$.

R_j is then an algebra of dimension $j+1$ over R, and $A^j_0, \cdots, A^j_j=I$ are simultaneously diagonalizable.

$W_{j,j}=\langle w \in W_j \,|\, w(B_{(j-1)j})^t=0\rangle$, since $B_{(j-1)j}(B_{(j-1)j})^t$ is non-singular. Now, since any A_i^j, $i=0, \cdots, j$, can be expressed as a linear combination of the matrices $\tilde{A}_n^{j-1}=(B_{(j-1)j})^t A_n^{j-1} B_{(j-1)j}$, $n=0, \cdots, j-1$, and the identity, $W_{j,j}$ is an eigenspace for all A_i^j, $i=0, \cdots, j$.

Setting $m=j-1$ in (9) gives,

$$(10) \qquad (B_{(j-1)j})^t A_i^{j-1} B_{(j-1)j} = \sum_{n=0}^{j} a(i, n) A_n^j$$

where $a(i, n)=\sum_{r=0}^{j-1-i} \binom{j-1-r}{i}\binom{1+r}{j-1-i}\binom{j-n}{r}\binom{n}{j-1-r}$.

To determine the eigenvalues of A_i^j, $i=0, \cdots, j$, on $W_{j,j}$, consider the matrix A with rows indexed by $\tilde{A}_i^{j-1}$, $i=0, \cdots, j$, (define $\tilde{A}_j^{j-1}=I$), and columns indexed by A_i^j, $i=0, \cdots, j$, with $(\tilde{A}_i^{j-1}, A_n^j)$ entry equal to $a(i, n)$.

Since A is a non-singular matrix, the equation

$$A\begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix}=\begin{pmatrix} y_0 \\ \vdots \\ y_j \end{pmatrix} \text{ has a unique solution } \begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix}.$$

Let y_i, $i=0, \cdots, j$, be equal to the eigenvalues of $\tilde{A}_i^{j-1}$ on $W_{j,j}$. Then x_n, $n=0, \cdots, j$, will be the eigenvalue of A_n^j on $W_{j,j}$. Since $W_{j,j}=\langle w \in W_j \mid w(B_{(j-1)j})^t=0\rangle$,

$$\begin{pmatrix} y_0 \\ \vdots \\ y_{j-1} \\ y_j \end{pmatrix}=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We show that $x_i=(-1)^{j-i}\binom{j}{i}$, $i=0, \cdots, j$, satisfies

$$(11) \qquad A\begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix}=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

Using (10) to expand $A\begin{pmatrix} x_0 \\ \vdots \\ x_j \end{pmatrix}$, we get a column matrix with *i-th* entry, $i=0, \cdots, j-1$, given by

$$\sum_{m=0}^{j} \sum_{r=0}^{j-1-i} \binom{j-1-r}{i}\binom{1+r}{j-1-i}\binom{j-n}{r}\binom{n}{j-1-r}(-1)^{j-n}\binom{j}{n},$$

and *j-th* entry equal to 1.

Non-zero entries occur in the above sum when $i \leq i-1-r$, $j-1-i \leq 1+r$, $r \leq j-1$ and $j-1-r \leq n$; that is, when $r=j-i-1$ or $j-1-2$, and $n=j-r$ or $j-r-1$. It is then easily verified that the sum is equal to zero.

Therefore the eigenvalue of A_i^j on $W_{j,j}$ is $\alpha_{ij}^j=(-1)^{j-i}\binom{j}{i}$, $i=0, \cdots, j$, and ii) and iii) are satisfied.

This completes the proof of Theorem 2.1.

Let $\mathfrak{D}=\mathfrak{D}(X, \mathfrak{A})$ be a $t-(v, k, \lambda)$ design with s intersection numbers $\mu_0, \mu_1, \cdots, \mu_{s-1}$. Define $\bar{s}$ to be the greatest integer in $t/2$. By Ray-Chaudhuri and Wilson [7], $\bar{s}\leq s$.

For $j=0, \cdots, v$, define I_j to be the incidence matrix for the incidence structure $(P_j, \mathfrak{A})$; that is, I_j is the matrix with rows indexed by the j-subsets of X and columns indexed by the blocks, with the $(\{x_1, \cdots, x_j\}, H_i)$ entry equal to 1 if the points $x_1, \cdots, x_j$ are incident with the block H_i, and 0 otherwise.

$I_j(I_j)^t$ is a matrix with rows and columns indexed by the j-subsets of X (where $(I_j)^t$ denotes the transpose of I_j). If S and T are j-subsets of X, then the (S, T)-entry of $I_j(I_j)^t$ is equal to the number of blocks B such that $S\cup T\subseteq B$. If $j\leq\bar{s}$ and if $|S\cap T|=i$, then the (S, T)-entry of $I_j(I_j)^t$ is λ_{2j-i}. Therefore, we have

Proposition 2.1. $I_j(I_j)^t=\sum_{i=0}^{j}\lambda_{2j-i}A_i^j$, *for* $j=0, \cdots, \bar{s}$.

For $i=0, \cdots, s-1$, define N_i to be the adjacency matrix, corresponding to μ_i, of $\mathfrak{D}$; that is, N_i is the matrix with rows and columns indexed by blocks and with (H, K)-entry equal to 1 if $|H\cap K|=\mu_i$ and 0 otherwise.

$(I_j)^tI_j$ is a matrix with rows and columns indexed by blocks, and with (H, K)-entry equal to the number of j-subsets, S, of X such that $S\subseteq H\cap K$. If $|H\cap K|=\mu_i$, then the (H, K)-entry is $\binom{\mu_i}{j}$. If $H=K$, then the (H, K)-entry is $\binom{k}{j}$. Therefore, if I is the (b, b) identity matrix, we have

Proposition 2.2. $(I_j)^tI_j=\sum_{i=0}^{s-1}\binom{\mu_i}{j}N_i+\binom{k}{j}I$, *for* $j=0, \cdots, v$.

We will compute the eigenvalues of $I_j(I_j)^t$ and $(I_j)^tI_j$, for $j=0, \cdots, \bar{s}$, and show that the equations given in Propositions 2.1 and 2.2 can be rewritten in terms of eigenvalues.

Lemma 2.1. $\binom{k-m}{n}\lambda_y=\sum_{i=0}^{n}\binom{v-y}{i}\binom{y-m}{n-i}\lambda_{y+i}$, *for* $0\leq y, y+m\leq t$.

Proof. From (1), $\binom{v-i}{t-i}\lambda=\lambda_i\binom{k-i}{t-i}$, $0\leq i\leq t$, it follows that $\binom{v-y}{i}\lambda_{y+i}=\binom{k-y}{i}\lambda_y$. Using the identity $\binom{k-m}{n}=\sum_{i=0}^{n}\binom{k-y}{i}\binom{y-m}{n-i}$, we obtain $\binom{k-m}{n}\lambda_y=\sum_{i=0}^{n}\binom{k-y}{i}\binom{y-m}{n-i}\lambda_y=\sum_{i=0}^{n}\binom{v-y}{i}\binom{y-m}{n-i}\lambda_{y+i}$.

Applying the definitions and results of Theorem 2.1 to the set of points, X, of $\mathfrak{D}$, gives the decomposition,

$$W_j = W_{j,0} \oplus \cdots \oplus W_{j,m} \oplus \cdots \oplus W_{j,j}\,,$$

such that each $W_{j,m}$ is an eigenspace for all A_i^j, $i=0, \cdots, j$.

Theorem 2.2.

i) *For* $j=0, \cdots, \bar{s}=[t/2]$, *and* $m=0, \cdots, j$, $W_{j,m}$ *is an eigenspace for* $I_j(I_j)^t$ *with eigenvalue*

$$e_{jm} = \binom{k-m}{j-m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{j+r}\,.$$

ii) *Let* $V=R\mathfrak{A}$ *be the vector space over the field of real numbers with basis* $\mathfrak{A}$. *For* $m=0, \cdots, \bar{s}$, *define*

$$V_m = W_{m,m} I_m, \quad \textit{and}$$
$$V_{\bar{s}'} = \langle v \in V \,|\, v(I_{\bar{s}})^t = 0 \rangle\,.$$

Then $V=V_0 \oplus \cdots \oplus V_m \oplus \cdots \oplus V_{\bar{s}} \oplus V_{\bar{s}'}$ *is a decomposition of* V *into eigenspaces of* $(I_j)^t I_j$, *for all* $j=0, \cdots, \bar{s}$. *For* $m \leq i$, *the eigenvalue of* $(I_j)^t I_j$ *on* V_m *is*

$$f_{j,m} = e_{j,m} = \binom{k-m}{j-m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{j+r}\,,$$

and for $m > i$, $f_{j,m}=0$.

$$|V_m| = \binom{v}{m} - \binom{v}{m-1} \qquad \textit{for } m=0, \cdots, \bar{s}, \textit{ and } j=0, \cdots, \bar{s}\,,$$

and $|V_{\bar{s}'}| = b - \binom{v}{\bar{s}}$.

Proof.

i) Recall that $W_{j,m}=W_{m,m}B_{mj}$. Let $w_m \in W_{m,m}$. By Theorem 2.1 (iii) and Proposition 2.1,

$$\begin{aligned}(w_m B_{mj}) I_j (I_j)^t &= \sum_{i=0}^{j} \lambda_{2j-i} (w_m B_{mj}) A_i^j \\ &= \sum_{r=0}^{m} (-1)^r \binom{m}{r} \left[\sum_{i=0}^{j} \binom{v-j-r}{j-i-r} \binom{j-m+r}{i-m+r} \lambda_{2j-1} \right] (w_m B_{mj}).\end{aligned}$$

Then applying Lemma 2.1 to the above expression with i replaced by $j-i-r$, n by $j-m$ and y by $j+r$, gives

$$(w_m B_{mj}) I_j (I_j)^t = \binom{k-m}{j-m} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_{j+r} (w_m B_{mj})\,.$$

ii) Fix j, $j=0, \cdots, \bar{s}$. Let $m \leq j$. Since $B_{mj} I_j = \binom{k}{j-m} I_m$, it follows that

$V_m = W_{m,m} I_m = W_{m,m} B_{mj} I_j = W_{j,m} I_j$. Then $V_m (I_j)^t I_j = W_{j,m} I_j (I_j)^t I_j = e_{j,m} W_{j,m} I_j = e_{j,m} V_m$.

Since the rank of I_j is $\binom{v}{j} = |W_j|$, $V_m (I_j)^t = 0$ for $m > i$; so $V_m (I_j)^t I_j = 0$ for $m > i$.

The matrix equations

$$(I_j)^t I_j = \sum_{i=0}^{s-1} \binom{\mu_i}{j} N_i + \binom{k}{j} I, \quad j = 0, \cdots. \bar{s},$$

can be written in terms of eigenvalues as follows:

Theorem 2.3. *There is a decomposition*

$$V = V_0 \oplus \cdots \oplus V_m \oplus \cdots \oplus V_{\bar{s}} \oplus V_{\bar{s}'},$$

of V such that for $m = 0, \cdots, \bar{s}, \bar{s}'$, V_m is an eigenspace for $(I_j)^t I_j$, $j = 0, \cdots, \bar{s}$, and also for N_i, $i = 0, \cdots, s-1$, and

$$f_{j,m} = \sum_{i=0}^{s-1} \binom{\mu_i}{j} g_{i,m} + \binom{k}{j}$$

where $f_{j,m}$ is the eigenvalue for $(I_j)^t I_j$ on V_m and $g_{i,m}$ is the eigenvalue for N_i on V_m.

Proof. Take $V_m = W_{m,m} I_m$ as in the proof of Theorem 2.2. The theorem can then be easily verified by showing that $N_i \in S = \langle I, (I_0)^t I_0, \cdots, (I_{s-1})^t I_{s-1} \rangle_R$, for $i = 0, \cdots, s$ (see Cameron [2]).

3. Polynomials associated with designs

Consider the polynomial whose roots are the intersection numbers, $\mu_0, \cdots, \mu_{s-1}$, for a design, and the polynomial whose roots are the differences, $\beta - \mu_i$, $i = 0, \cdots, s-1$, between each intersection number μ_i, and the average β, of the intersection numbers, for a design, we will compute the coefficients for these polynomials, for a tight design, in terms of the design parameters and the coefficients of the polynomial $x(x-1)\cdots(x-s+1)$. This will lead to a proof of Theorem 4.1.

Consider the polynomial given by

$$(12) \qquad n! \binom{x}{n} = x(x-1)\cdots(x-n+1) = \sum_{r=0}^{n} s(n, r) x^r.$$

The coefficients, $s(n, r)$, $r = 0, \cdots, n$, are called Stirling numbers of the first kind. It is convenient to define $s(0, 0) = 1$. The following facts are known.,

Proposition 3.1. *For $n\geq 1$ and r such that $n\geq r\geq 0$,*

i) $s(n, r)=s(n-1, r-1)-(n-1)s(n-1, r)$

ii) $\sum_{r=0}^{n} s(n, r)=0$, *for* $n\geq 2$

iii) $s(0, 0)=1$, $s(n, n)=1$, $s(n, 0)=0$

iv) $s(n, n-1)=-\binom{n}{2}$

v) $s(n, n-2)=\frac{1}{4}(3n-1)\binom{n}{3}$

vi) $s(n, n-3)=-\binom{n}{2}\binom{n}{4}$

vii) $s(n, n-4)=(1/48)(15n^3-30n^2+5n+2)\binom{n}{5}$

viii) $s(n, n-5)=-(1/8)(3n^2-7n-2)\binom{n}{2}\binom{n}{6}$

Proof. i) and iii) follow from the definition of $s(n, r)$. ii) follows from the fact that 1 is a root of $x(x-1)\cdots(x-n+1)$, for $n\geq 2$. iv)-viii) can be proved by inductive arguments using i).

Applying (12) to $\binom{x-a}{n}$ and rearranging according to powers of x yields,

Proposition 3.2. *For $x-a\geq n\geq 0$*

$$\binom{x-a}{n}=\frac{1}{n!}\sum_{j=0}^{n}\left[\sum_{r=j}^{n}(-1)^{r-j}\binom{r}{j}s(n, r)a^{r-j}\right]x^j .$$

If we define $s(-1, -1)=1$ and $s(0, -1)=-1$, then

Proposition 3.3. *For $x\geq n\geq 0$*

$$\binom{x}{n}-\binom{x}{n-1}=\frac{1}{n!}\sum_{r=0}^{n}s(n-1, r-1)x^r .$$

Proof. Follows from (12) using proposition 3.1, i) and ii).

Let $\mathfrak{D}=\mathfrak{D}(X, \mathfrak{A})$ be a $t-(v, k, \lambda)$ design with s intersection numbers, $\mu_0, \cdots, \mu_{s-1}$.

Let

$$(13)\qquad \prod_{j=0}^{s-1}(x-\mu_j)=\sum_{j=0}^{s}(-1)^j\sigma_j x^{s-j} ,$$

so that $(-1)^j\sigma_j$ is the coefficient of x^{s-j} in the polynomial whose roots are the intersection numbers of $\mathfrak{D}$.

Let $y=x-\beta$, where $\beta=\sigma_1/s$ is the average of the intersection numbers. Then substituting into (13) yields

$$(14)\qquad \prod_{j=0}^{s-1}(y-(\mu_j-\beta)) = \sum_{j=0}^{s} p_j y^{s-j}, \text{ where}$$

$$(15)\qquad p_j = \sum_{r=0}^{j}(-1)^r \sigma_r \binom{s-r}{j-r}(\sigma_1/s)^{j-r}.$$

Thus p_j is the coefficient of y^{s-j} in the polynomial whose roots are the differences $(\mu_j-\beta)$, $j=0, \cdots, s-1$.

If the intersection numbers of $\mathfrak{D}$ have the property that $\beta-a$ is an intersection number whenever $\beta+a$ is an intersection number, where β is the average of the intersection numbers, then we will say that the intersection numbers of $\mathfrak{D}$ are *symmetric*. It is easy to show the following.

Proposition 3.4. *If $\mathfrak{D}$ has symmetric intersection numbers, then $p_j=0$ for odd j.*

We will use the systems of eigenvalue relations of section 2 to compute σ_i, $i=0, \cdots, s$, in terms of design parameters and Stirling numbers. Then using (15) and Proposition 3.1 we will obtain expressions for p_2 and p_3 in terms of the design parameters. These, along with Proposition 3.4 will be used in proving the main result in section 4.

The following result will be needed in the proof of lemma 3.2.

Lemma 3.1 (Generalized Vandermonde Determinant).

Let D_r, $r=0, 1, \cdots, n-1$, be the (n, n)-matrix with (i, j)-entry defined as follows, for $0\le i, j\le n-1$,

$$(x_j)^i \text{ for } i<r \quad \text{and} \quad (x_j)^{i+1} \text{ for } r\le i.$$

Then $D_r=\sigma_{n-r}(x_1, \cdots, x_n)\prod_{m>j=1}^{n}(x_m-x_j)$
where $\sigma_{n-r}(x_1, \cdots, x_n)$ is defined by

$$\prod_{j=1}^{n}(x-x_j) = \sum_{r=0}^{n}\sigma_{n-r}(x_1, \cdots, x_n)(-1)^r x^{n-r}.$$

In the rest of this section we will assume that $\mathfrak{D}$ is a tight $2s$-design. Then $\bar{s}=s$, where $\bar{s}=t/2$ and s is the number of intersection numbers.

If $h_{jm}=f_{jm}-\binom{k}{j}$, then Theorem 2.3 becomes

$$(16)\qquad \sum_{i=0}^{s-1}\binom{\mu_i}{j}g_{im} = h_{jm}, \qquad \text{for } j=0, \cdots, \bar{s}, \text{ and } m=0, \cdots, j.$$

Lemma 3.2. $\sum_{i=0}^{s}(-1)^{s-i}S_{s-i}h_{(s-i)j}=0, j=0, \cdots, s,$

where $S_{s-s}=\sigma_s$, and for $i=0, \cdots s-1$,

$$S_{s-i}=(s-i)!\Big[(-1)^{s-i}\sigma_i+\sum_{q=0}^{i-1}(-1)^{s-q}\sum_{\substack{s-i=l_1<\cdots\\ \cdots<l_n=s-q}} s(l_n, l_{n-1})\cdots$$
$$\cdots s(l_3, l_2)s(l_2, l_1)\sigma_q\Big].$$

Proof. For fixed m, $0\leq m\leq s$, the $s+1$ equations in the s variables g_{im}, $i=0, \cdots, s-1$, given in (16) are dependent. Therefore, the determinant,

$$\text{(17)}\qquad \begin{vmatrix} \binom{\mu_0}{0} & \cdots & \binom{\mu_{s-1}}{0} & h_{0m} \\ \binom{\mu_0}{1} & \cdots & \binom{\mu_{s-1}}{1} & h_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{\mu_0}{s} & & \binom{\mu_{s-1}}{s} & h_{sm} \end{vmatrix}=0.$$

Then by multiplying by constants and performing row operations on (17), we get

$$\text{(18)}\qquad \begin{vmatrix} 1 & \cdots & 1 & k_{0m} \\ \mu_0 & \cdots & \mu_{s-1} & k_{1m} \\ (\mu_0)^2 & \cdots & (\mu_{s-1})^2 & k_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_0)^s & & (\mu_{s-1})^s & k_{sm} \end{vmatrix}=0,$$

where

$$\text{(19)}\qquad k_{im}=i!\,h_{im}+\sum_{r=1}^{i-1} r!\,h_{rm}\sum_{r=l_1<\cdots<l_n=i}(-1)^{n-1}s(l_n, l_{n-1})\cdots s(l_3, l_2)s(l_2, l_1).$$

Now, applying Lemma 3.1 in computing the above determinant, we have

$$\text{(20)}\qquad \sum_{i=0}^{s}(-1)^i k_{ij}\sigma_{s-i}=0, \qquad \text{for } j=0, \cdots, s.$$

The proof is then completed by using (19) to rewrite (20) in terms of the h_{ij}'s.

Another expression for S_{s-i} will be given in Lemma 3.5. The following two lemmas will be needed in the proof. The first may be verified by a straightforward manipulation.

Lemma 3.3. $$\frac{\binom{k}{i+n}\binom{v-k}{q}}{\binom{v}{i+n}\binom{v-(i+n)}{q}}-\frac{\binom{k}{i+n+1}\binom{v-k}{q}}{\binom{v}{i+n+1}\binom{v-(i+n+1)}{q}}$$
$$=\frac{\binom{k}{i+n}\binom{v-k}{q+1}}{\binom{v}{i+n}\binom{v-(i+n)}{q+1}}.$$

Lemma 3.4. $\sum_{n=0}^{j}(-1)^n\binom{j}{n}\lambda_{i+n}=b\dfrac{\binom{k}{i}\binom{v-k}{j}}{\binom{v}{i}\binom{v-i}{j}}$.

Proof. Using the fact that $\lambda_{i+n}=\dfrac{\binom{k}{i+n}}{\binom{v}{i+n}}b$, along with Lemma 3.3, we get

$$\begin{aligned}\sum_{n=0}^{j}(-1)^n\binom{j}{n}\lambda_{i+n}&=b\sum_{n=0}^{j}(-1)^n\binom{j}{n}\frac{\binom{k}{i+n}\binom{v-k}{0}}{\binom{v}{i+n}\binom{v-(i+n)}{0}}\\&=b\sum_{n=0}^{j-1}(-1)^n\binom{j-1}{n}\frac{\binom{k}{i+n}\binom{v-k}{1}}{\binom{v}{i+n}\binom{v-(i+n)}{1}}\\&\ \ \vdots\\&=b\sum_{n=0}^{j-q}(-1)^n\binom{j-q}{n}\frac{\binom{k}{i+n}\binom{v-k}{q}}{\binom{v}{i+n}\binom{v-(i+n)}{q}}\\&\ \ \vdots\\&=b\,\frac{\binom{k}{i}\binom{v-k}{j}}{\binom{v}{i}\binom{v-i}{j}}\end{aligned}$$

Lemma 3.5. $S_{s-i}=(-1)^{s-i}\dfrac{s!}{i!}K_{s-i}$, *for* $i=0,\cdots,s$,

where $K_j=\prod_{r=1}^{j}\dfrac{(k-s+r)(k-s+r-1)}{(v-2s+r)}$, *for* $j=1,\cdots,s$, *and* $K_0=1$.

Proof. Consider the $s+1$ equations given in Lemma 3.2,

$$\sum_{i=0}^{s}(-1)^iS_ih_{ij}=0,\quad j=0,\cdots,s\,.$$

Subtracting the $(j=s)$-equation from the $(j=s-i)$ equation gives

$$\sum_{m=0}^{i}(-1)^{s-m}S_{s-m}f_{(s-m)(s-i)}-(-1)^sf_{ss}S_s=0\,. \tag{21}$$

Let $L_{cd}=\sum_{n=0}^{d}(-1)^n\binom{d}{n}\lambda_{c+n}$. Then $f_{cd}=\binom{k-d}{c-d}L_{cd}$, by Theorem 2.3. By Lemma 3.2, $S_s=(-1)^ss!$. Using this and rearranging (21), we get

$$S_{s-i}=\frac{1}{L_{(s-i)(s-i)}}\left[\left(\sum_{q=0}^{i-1}(-1)^{i-1-q}\binom{k-s+i}{i-q}L_{(s-q)(s-q)}S_{s-q}\right)\right.$$
$$\left.-(-1)^{i-1+s}s!L_{ss}\right] \tag{22}$$

for $i=0, \cdots, s$.

Now using Lemma 3.4 in (22) and arguing inductively (on i),

$$S_{s-i}=(-1)^{s-i}\frac{s!}{i!}K_i, \qquad \text{for} \quad i=1, \cdots, s, \quad \text{and} \quad S_s=(-1)^s s!\,.$$

The values, σ_i, $i=0, \cdots, s$, are the coefficients of the polynomial whose roots are the intersection numbers of $\mathfrak{D}$. Each σ_i is a function of i and the design parameters v, k, λ, and s. The next theorem shows that, in fact, σ_i depends only on v, k, and s (in addition to i).

Theorem 3.1.

$$\sigma_i=\sum_{j=0}^{i}(-1)^{i-j}\binom{s}{j}s(s-j,\, s-i)K_j, \quad 0\le i\le s\,,$$
$$K_j=\prod_{r=1}^{j}\frac{(k-s+r)(k-s+r-1)}{(v-2s+r)}, \qquad \textit{for} \quad j=1, \cdots, s\,,$$

and $K_0=1$, *and* $s(s-j,\, s-i)$, $j=0, \cdots, i$, *is a Stirling number of the first kind.*

Proof. This is trivial for $i=0$. Assume it holds for all i' such that $0\le i'\le i$. Combining Lemmas 3.2 and 3.5, we have

$$(-1)^{s-i}\frac{s!}{i!}K_i=(s-i)!\,[(-1)^{s-i}\sigma_i-\sum_{q=0}^{i-1}(-1)^{s-q}Q(q,i)\sigma_q]\,,$$
$$\text{for} \quad i=0, \cdots, s,$$

where

$$Q(q,i)=\sum_{\substack{s-i=l_1<\cdots< \\ l_n=s-q}}(-1)^n s(l_n,\, l_{n-1})\cdots s(l_3,\, l_2)s(l_2,\, l_1)\,.$$

Then, solving for σ_i and applying induction to σ_q, $q=0, 1, \cdots, i-1$,

$$\sigma_i=\binom{s}{i}K_i-\sum_{q=0}^{i-1}(-1)^{i-q}Q(q,i)\sigma_q\,,$$
$$=\binom{s}{i}K_i-\sum_{q=0}^{i-1}\sum_{j=0}^{q}(-1)^{i-j}\binom{s}{j}s(s-j,\, s-q)Q(q,i)K_j\,.$$

For a given partition of the form, $s-i=l_1<\cdots<l_{n-1}=s-q<l_n=s_j$, we have a pair of terms in the sum which are identical except for opposite sign:

$$(-1)^{i-j+n-1}\binom{s}{j}s(s-j,\, s-q)s(s-q,\, l_{n-2})\cdots s(l_2,\, l_1)K_j$$

and

$$(-1)^{i-j+n}\binom{s}{j}s(s-j,\ s-j)s(s-j,\ s-q)s(s-q,\ l_{n-2})\cdots s(l_2,\ l_1)K_j\,.$$

For a partition of the form, $s-i=l_1<l_2=s-j$, there is only one term that occurs in the sum:

$$(-1)^{i-j+2}\binom{s}{j}s(s-j,\ s-j)s(s-j,\ s-i)K_j\,.$$

Therefore,

$$\sigma_i=\sum_{j=0}^{i}(-1)^{i-j}\binom{s}{j}s(s-j,\ s-i)K_j\,.$$

This result can be used to obtain expressions for the coefficients, p_j, of the polynomial whose roots are, $\beta-\mu_i$, $i=0,\cdots,s-1$.

Theorem 3.2.

i) $$p_2=\frac{-(s+1)s(s-1)}{24}-\binom{s}{2}\frac{(k-s+1)(k-s)(v-k-s+1)(v-k-s)}{(v-2s+2)\ (v-2s+1)^2}$$

ii) $$p_3=-\binom{s}{3}\frac{(k-s+1)(k-s)(v-k-s+1)(v-k-s)(v-2k+1)(v-2k-1)}{(v-2s+3)(v-2s+2)(v-2s+1)^3}$$

Proof. Recall that

$$p_j=\sum_{r=0}^{j}(-1)^r\sigma_r\binom{s-r}{j-r}\left(\frac{\sigma_1}{s}\right)^{j-r},\quad j=0,\ \cdots.\ s\,.$$

Using Proposition 3.1, iii)-vi), and Theorem 3.1, we can compute σ_i, $i=0,\ 1,\ 2,\ 3$, in terms of the design parameters, and then obtain expressions for p_2 and p_3.

4. A general result on the existence of tight t-design, and the non-existence of non-trivial tight 6-designs

Lemma 4.1. *If $\mathfrak{D}$ is a non-trivial tight* 6-*design, then* $v\neq 2k\pm 1$.

Proof. Since the complementary design of a non-trivial tight t-design is also a tight t-design, it suffices to show that $v\neq 2k+1$. If $s=3$ and $v=2k+1$, then

$$\sigma_3=\frac{k(k-1)(k-2)(k-3)}{4(2k-3)}$$

is not an interger, contradicting the definition of σ_3.

The following lemma is due to E. Bannai.

Lemma 4.2 (Bannai [1]). *If $\mathfrak{D}$ is a tight 2s-design, with $s\geq 2$, then $v\neq 2k\pm 1$.*

Proof. The following result of Schur [8, Satz I] is crucial in the proof.

(Schur): Let h be an integer and let m be an odd number greater than $2h+1$. Then $m(m+2)\ (m+4)\cdots(m+2(h-1))$ is divisible by a prime $p>2h+1$ except for the cases (i) $h=2$, $m=25$ and (ii) $h=1$ and $m=3^a$.

Since the complementary design of a non-trivial tight design is also tight, it suffices to show that $v\neq 2k+1$. Assume that $v=2k+1$.

First suppose that $s\geq 4$. Then the result of Schur shows that there is a prime $p>2[s/2]+1$ ($\geq s$) which divides the product of $[s/2]$ consecutive odd numbers,

$$(2k-2s+2[s/2]+1)\cdots(2k-2s+5)(2k-2s+3)\,,$$

which appears in the denominator of K_s, unless $[s/2]=2$ (*i.e.*, $s=4$ or 5) and $2k-3s+3=25$ (*i.e.*, either (i) $h=15$, $v=31$ or (ii) $h=16$, $v=33$ according as $s=4$ or 5). But, these exceptional cases do not occur, because σ_2 is not an integer in these cases.

Therefore, we can choose the smallest even integer j (less than $2[s/2]$) such that

$$(2k-2s+3)(2k-2s+5)\cdots(2k-2s+j+1)$$

has a prime factor $p>2[s/2]+1$. Now, we want to show that σ_j is not an integer. Since the *G.C.D.* $(2k-2s+j+1,\ k-(s-r))$, with $0\leq r\leq j$, which is equal to *G.C.D.* $(2k-2s+j+1,\ 2r-(j+1))$, is less than p, we get that $\binom{s}{j}K_j$ is not a p-adic interger. On the other hand,

$$\sigma_j-\binom{s}{j}K_j=\sum_{l=1}^{j-1}\binom{s}{l}(-1)^{j-1}s(s-l,\ s-j)K_l\,,$$

and the right side is a p-adic integer. Hence, σ_j is not an integer, which is a contradiction.

The case $s=2$ has been shown by Ito [4], and the case $s=3$ has been shown in Lemma 4.1.

Theorem 4.1. *If $\mathfrak{D}$ is a tight 2s-design ($s\geq 3$) with the coefficient $p_3=0$, then $\mathfrak{D}$ is trivial. In particular, if $\mathfrak{D}$ is a tight 2s-design ($s\geq 3$) with symmetric intersection numbers, then $\mathfrak{D}$ is trivial.*

Proof. By Theorem 3.2.

$$p_3=-\binom{s}{3}\frac{(k-s+1)(k-s)(v-k-s+1)(v-k-s)(v-2k+1)(v-2k-1)}{(v-2s+3)(v-2s+2)(v-2s+1)^3}$$

Since $k \geq t=2s>s$, $k \neq s$ and $k \neq s-1$. Since $\mathfrak{D}$ must have s distinct intersection numbers, $v \neq k+s-1$. By Lemma 4.2, $v \neq 2k \pm 1$. Therefore if $p_3=0$, then $v=k+s$ and $\mathfrak{D}$ is trivial. If the intersection numbers are symmetric then $p_3=0$ by Proposition 3.4.

Theorem 4.2. *If $\mathfrak{D}$ is a tight 6-design, then $\mathfrak{D}$ is trivial.*

Proof. By Theorem 4.1, it suffices to show that $p_3=0$.
Assume $p_3 \neq 0$. By Theorem 3.2,

$$p_2 = -1-3\frac{(k-2)(k-3)(v-k-2)(v-k-3)}{(v-4)(v-5)^2}$$

$$p_3 = \frac{-(k-2)(k-3)(v-k-2)(v-k-3)(v-2k+1)(v-2k-1)}{(v-3)(v-4)(v-5)^3}.$$

Then

$$0<-3p_3 = (-p_2-1)\frac{(v-(2k-1))(v-(2k+1))}{(v-3)(v-5)}<-p_2-1\,.$$

Let $d_i=\beta-\mu_i$ and $e_i=|d_i|$, $i=0, 1, 2$.

Now, p_3 is the constant term of the polynomial whose roots are d_i, $i=0, 1, 2$. Thus

$$(23) \qquad p_3 = d_0d_1d_2 = -e_0e_1e_2\,.$$

Similarly, p_2 is the coefficient of the linear term of this polynomial. Thus

$$p_2 = d_0d_1+d_0d_2+d_1d_2\,.$$

Since $d_0+d_1+d_2=0$ we get,

$$(24) \qquad p_2 = -\tfrac{1}{2}(d_0^2+d_1^2+d_2^2) = -\tfrac{1}{2}(e_0^2+e_1^2+e_2^2)\,.$$

Since $0>p_3=d_0d_1d_0$ and $d_0+d_1+d_2=0$, we may assume that $d_2<0$ and $0<d_0<d_1<|d_2|$. Then since $d_0+d_1+d_2=0$,

$$(25) \qquad e_0+e_1 = e_2\,.$$

Also,

$$(26) \qquad 1/3<e_0<e_1<e_2\,,$$

because $0 \neq e_0=\beta-\mu_0=\dfrac{\mu_0+\mu_1+\mu_2}{3}-\mu_0$

and μ_0, μ_1, μ_2 are integers.

Then by (23)-(26),

$$e_1(e_0+e_1)\leq 3e_0e_1e_2=-3p_3<-p_2-1=\frac{e_0^2+e_1^2+(e_0+e_1)^2}{2}-1$$
$$=(e_0+e_1)^2-e_0e_1-1\,.$$

Then $e_1<e_0+e_1-\left(\frac{e_0e_1+1}{e_0+e_1}\right)$

$$\frac{e_0e_1+1}{e_0+e_1}<e_0\,,\quad e_0e_1+1<e_0^2+e_0e_1\,,\quad 1<e_0^2\,,\quad 1<e_0\,.$$

Now, since $1<e_0<e_1$,

$$3e_1^2+3e_1<3e_0e_1(e_0+e_1)<\frac{e_0^2+e_1^2+(e_0+e_1)^2}{2}-1$$
$$=e_0^2+e_1^2+e_0e_1-1$$
$$<3e_1^2-1\,.$$

Therefore $0<e_1<-1/3$. This contradiction proves that $p_3=0$.

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