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ON BLOCKS OF FINITE GROUPS WITH RADICAL CUBE ZERO

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Let G be a finite group and k be an algebraically closed field of characteristic p, a prime number. Let B be a block algebra of the group algebra kG with defect group D and let $J(B)$ denote the Jacobson radical of B. It is well known that $J(B)=0$ if and only if $D=1$. Furthermore it is true that $J(B)^2=$ if and only if $p=2$ and $|D|=2$.

In this paper we shall prove the following theorem.

Theorem 1. *$J(B)^3=0$ (but $J(B)^2\neq 0$) if and only if one of the following conditions holds;*

(1) $p=2$, D is a four group and B is isomorphic to the matrix ring over kD or is Morita equivalent to kA_4 where A_4 is the alternating group of degree 4,

(2) p is odd, $|D|=p$, the number of simple kG-modules in B is $p-1$ or $p-1/2$ and the Brauer tree of B is a straight line segment such that the exceptional vertex is in an end point (if it exists).

For the prime 2 we have the following.

Theorem 2. *Assume $p=2$. Let U be the projective indecomposable kG-module with $U/Rad(U)=k_G$, the trivial kG-module. If Loewy length of U is 3, then a 2-Sylow subgroup of G is dihedral.*

EXAMPLE.

(1) The principal p-block of the following groups satisfies the conditions in Theorem 1.

(a) G is a four group or A_4 and $p=2$.

(b) G is the symmetric group or the alternating group of degree p and p is odd.

(2) Erdmann [6] shows that for each prime power q with $q\equiv 3$ (mod. 4) the group PSL $(2, q)$ satisfies the assumption in Theorem 2.

1. Preliminaries

In this section we shall prove some lemmas which will be used to prove

Theorem 1. Throughout this section, B is an arbitrary block algebra of a finite group G. Let D be a defect group of B. For a positive integer n let n_p denote the p-part of n.

Lemma 1. *There exists a simple kG-module S in B such that a vertex of S is D and a source of S is p'-dimensional.*

Proof. There exists a simple kG-module S in B such that $(\dim_k S)_p = |G:D|_p$ (Theorem 4.5, Chap. IV [9]). This module S satisfies the conditions in the lemma.

Let Ω denote the Heller's syzygy functor. Then the following lemma follows from the fact that kG is a symmetric algebra.

Lemma 2. *Let X be a kG-module with no nonzero projective direct summand. Then $Soc(\Omega^1(X)) \cong X/rad(X)$.*

Lemma 3. *Let P be a nontrivial cyclic subgroup of D. The there exists a kG-module X in B such that*

(1) *a vertex of each indecomposable direct summand of X is P and* $(\dim_k X)_p = |G:P|_p$ *and*

(2) $\Omega^1(X) \cong \Omega^{-1}(X)$.

Proof. Let $N = N_G(P)$ and $C = C_G(P)$. Then there exists a block b of C with $b^G = B$. Put $B_1 = b^N$. There exists an indecomposable kC-module Y in b such that $\operatorname{Ker} Y \supset P$, Y is projective as a kC/P-module and $(\dim_k Y)_p = |C:P|_p$. We claim that $\Omega^1(Y) = \Omega^{-1}(Y)$. Let U be a projective cover of Y. Then $Y \cong U/UJ(kP)$ and $\Omega^1(Y) \cong UJ(kP)$, where $J(kP)$ denotes the Jacobson radical of kP. Furthermore U is an injective hull of Y, $Y \cong \operatorname{Inv}_P(U)$ and $\Omega^{-1}(Y) = U/\operatorname{Inv}_P(U)$. Since P is cyclic and central in C, we have $UJ(kP) \cong U/\operatorname{Inv}_P(U)$ and therefore $\Omega^1(Y) = \Omega^{-1}(Y)$. Thus our claim follows. Let $Y^N = Y_1 \oplus \cdots \oplus Y_n$, where each Y_i is an indecomposable kN-module. Then Y_i is in B_1 and has P as a vertex for each i. Put $X_i = f^{-1}(Y_i)$, where f denotes the Green correspondence with respect to (G, N, P) and set $X = X_1 \oplus \cdots \oplus X_n$. By the properties of the Green correspondence (Theorem 7.8 [9], [11], [12]) X_i is in B, $\dim_k X_i \equiv \dim_k Y_i^G$ (mod. $p|G:P|_p$) and $\Omega^m(X_i) \cong f^{-1}(\Omega^m(Y_i))$ for every integer m. Thus $(\dim_k X)_p = |G:P|_p$ and $\Omega^1(X) = \Omega^{-1}(X)$.

2. Proof of Theorem 1

If a block B satisfies one of the conditions (1) and (2) in Theorem 1, then it is easy to show that $J(B)^3 = 0$ and $J(B)^2 \neq 0$. In the rest of this section we assume that $J(B)^3 = 0$, $J(B)^2 \neq 0$ and we shall prove that B satisfies one of the conditions (1) and (2).

Step 1. *If X is a nonsimple nonprojective indecomposable kG-module in B, then $Soc(X)=Rad(X)$.*

Proof. Since X is nonprojective, $Rad(X)\subset Soc(X)$ ([14]). Then it follows that $Rad(X)=Soc(X)$ as X is nonsimple.

Step 2. *If p is odd, then $|D|=p$.*

Proof. Suppose $|D|\neq p$. Let P be a subgroup of D of order p and let X be a kG-module in B which satisfies the conditions in Lemma 3. Then by a result of Erdmann [5] X and $\Omega^1(X)$ have no simple direct summand. By Lemma 2 $Soc(\Omega^1(X))\cong X/Rad(X)$ and $Soc(X)\cong\Omega^{-1}(X)/Rad\Omega^{-1}((X))$. As $\Omega^1(X)\cong\Omega^{-1}(X)$ it follows that $Soc(X)\cong\Omega^1(X)/Rad\Omega^1((X))$. Then by Step 1 we have $\dim_k X=\dim_k\Omega^1(X)$. On the other hand $\dim_k X+\dim_k\Omega^1(X)$ is divisible by the order of a Sylow p-subgroup of G. Thus we have a contradiction as p is odd and $(\dim_k X)_p=|G\colon P|_p$.

Step 3. *If $p=2$, then D is elementary abelian.*

Proof. By Proposition (6G) [2] and [15] D is not cyclic. Suppose that there exists a cyclic subgroup P of D of order 4. Then by a similar argument as in the proof of Step 2 it follows that there exists a kG-module X in B such that $(\dim_k X)_2=|G\colon P|_2$ and $\dim_k X=\dim_k\Omega^1(X)$. Since $\dim_k X+\dim_k\Omega^1(X)$ is divisible by the order of a Sylow 2-subgroup of G, this is a contradiction. Thus every nontrivial element in D is of order 2 and therefore D is elementary abelian.

Step 4. *If $p=2$, then D is a four group.*

Proof. Suppose that $|D|>4$ and let P be a four group contained in D. Then by a result of Knörr [13] and Step 3 any simple kG-module in B is not P-projective. Let $I=\{i\in\boldsymbol{Z};\Omega^i(k_P)$ is a direct summand of $S_{|P}$ for some simple kG-module S in $B\}$, where k_P denotes the trivial kP-module and $\boldsymbol{Z}$ denotes the set of all integers. By a result of Conlon [3] each indecomposable kP-module of odd dimension is isomorphic to $\Omega^i(k_P)$ for some integer i. Thus by Lemma 1 we can conclude that I is not empty. Let i be the largest integer in I and choose a simple kG-module S in B such that $\Omega^i(k_P)$ is a direct summand of $S_{|P}$. Let U be a projective cover of S. By the assumption that $J(B)^3=0$ and $J(B)^2\neq 0$, $Rad(U)/Soc(U)$ is nonzero and completely reducible. $Rad(U)/Soc(U)$ appears in the Asulander-Reiten sequence; $0\to\Omega^1(S)\to Rad(U)/Soc(U)\oplus U\to\Omega^{-1}(S)\to 0$ (Proposition 4.11 [1]). Then by the result of Roggenkamp (Proposition 2.10 [17]) $\Omega^{i+1}(k_P)$ is a direct summand of $(Rad(U)/Soc(U))_{|P}$ which contradicts the maximality of i.

Step 5. *Conclusion.*

First assume $p=2$. Then by Step 4 D is a four group. By results of Erdmann [8] we have two cases (i) and (ii) in Theorem 4, [8]. In the case (i), it follows easily that the basic ring of B is isomorphic to kD. In the case (ii), B has three simple modules S_1, S_2 and S_3. Let e_i ($i=1, 2$ and 3) be pairwise orthogonal primitive idempotents with $e_i kG/e_i J(kG)=S_i$ and put $e=e_1+e_2+e_3$. By Theorem 4, [8] $\dim_k ekGe=12$ and $\dim_k e_i kGe_j=1+\delta_{ij}$. Then we can show that $ekGe$ is isomorphic to kA_4. Next assume that p is odd. By the thoery of Brauer-Dade [4] and a result of Peacock [16], it follows that $Rad(U)/Soc(U)$ is simple or a sum of two non-isomorphic simple modules for every projective indecomposable kG-module U in B. Then the result follows easily.

3. Proof of Theorem 2

Put $S=Rad(U)/Soc(U)$. Suppose that a Sylow 2-subgroup of G is not dihedral. Then by the result of Webb (Theorem E [18]) and our assumption S is simple and self dual. Let V be a projective cover of S. Since V is also self dual, for any simple kG-module T the multiplicity of T in the composition factors of V is equal to that of its dual. By a result of Fong [10] the dimension of a nontrivial self dual simple kG-module is even. Thus we have a contradiction as the multiplicity of the trivial kG-module in the composition factors of V is 1 and $\dim_k V$ is even.

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