Okuyama, T.
Osaka J. Math.
23 (1986), 461-465

# ON BLOCKS OF FINITE GROUPS WITH RADICAL CUBE ZERO

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(Received January 24, 1985)

Let $G$ be a finite group and $k$ be an algebraically closed field of characteristic $p$, a prime number. Let $B$ be a block algebra of the group algebra $kG$ with defect group $D$ and let $J(B)$ denote the Jacobson radical of $B$. It is well known that $J(B)=0$ if and only if $D=1$. Furthermore it is true that $J(B)^2=$ if and only if $p=2$ and $|D|=2$.

In this paper we shall prove the following theorem.

**Theorem 1.** *$J(B)^3=0$ (but $J(B)^2\neq 0$) if and only if one of the following conditions holds;*

(1) *$p=2$, $D$ is a four group and $B$ is isomorphic to the matrix ring over $kD$ or is Morita equivalent to $kA_4$ where $A_4$ is the alternating group of degree 4,*

(2) *$p$ is odd, $|D|=p$, the number of simple $kG$-modules in $B$ is $p-1$ or $p-1/2$ and the Brauer tree of $B$ is a straight line segment such that the exceptional vertex is in an end point (if it exists).*

For the prime 2 we have the following.

**Theorem 2.** *Assume $p=2$. Let $U$ be the projective indecomposable $kG$-module with $U/Rad(U)=k_G$, the trivial $kG$-module. If Loewy length of $U$ is 3, then a 2-Sylow subgroup of $G$ is dihedral.*

EXAMPLE.

(1) The principal $p$-block of the following groups satisfies the conditions in Theorem 1.

(a) $G$ is a four group or $A_4$ and $p=2$.

(b) $G$ is the symmetric group or the alternating group of degree $p$ and $p$ is odd.

(2) Erdmann [6] shows that for each prime power $q$ with $q\equiv 3$ (mod. 4) the group PSL $(2, q)$ satisfies the assumption in Theorem 2.

## 1. Preliminaries

In this section we shall prove some lemmas which will be used to prove

Theorem 1. Throughout this section, $B$ is an arbitrary block algebra of a finite group $G$. Let $D$ be a defect group of $B$. For a positive integer $n$ let $n_p$ denote the $p$-part of $n$.

**Lemma 1.** *There exists a simple $kG$-module $S$ in $B$ such that a vertex of $S$ is $D$ and a source of $S$ is $p'$-dimensional.*

Proof. There exists a simple $kG$-module $S$ in $B$ such that $(\dim_k S)_p = |G:D|_p$ (Theorem 4.5, Chap. IV [9]). This module $S$ satisfies the conditions in the lemma.

Let $\Omega$ denote the Heller's syzygy functor. Then the following lemma follows from the fact that $kG$ is a symmetric algebra.

**Lemma 2.** *Let $X$ be a $kG$-module with no nonzero projective direct summand. Then $Soc(\Omega^1(X)) \cong X/rad(X)$.*

**Lemma 3.** *Let $P$ be a nontrivial cyclic subgroup of $D$. The there exists a $kG$-module $X$ in $B$ such that*

(1) *a vertex of each indecomposable direct summand of $X$ is $P$ and $(\dim_k X)_p = |G:P|_p$ and*

(2) $\Omega^1(X) \cong \Omega^{-1}(X)$.

Proof. Let $N = N_G(P)$ and $C = C_G(P)$. Then there exists a block $b$ of $C$ with $b^G = B$. Put $B_1 = b^N$. There exists an indecomposable $kC$-module $Y$ in $b$ such that $\operatorname{Ker} Y \supset P$, $Y$ is projective as a $kC/P$-module and $(\dim_k Y)_p = |C:P|_p$. We claim that $\Omega^1(Y) = \Omega^{-1}(Y)$. Let $U$ be a projective cover of $Y$. Then $Y \cong U/UJ(kP)$ and $\Omega^1(Y) \cong UJ(kP)$, where $J(kP)$ denotes the Jacobson radical of $kP$. Furthermore $U$ is an injective hull of $Y$, $Y \cong \operatorname{Inv}_P(U)$ and $\Omega^{-1}(Y) = U/\operatorname{Inv}_P(U)$. Since $P$ is cyclic and central in $C$, we have $UJ(kP) \cong U/\operatorname{Inv}_P(U)$ and therefore $\Omega^1(Y) = \Omega^{-1}(Y)$. Thus our claim follows. Let $Y^N = Y_1 \oplus \cdots \oplus Y_n$, where each $Y_i$ is an indecomposable $kN$-module. Then $Y_i$ is in $B_1$ and has $P$ as a vertex for each $i$. Put $X_i = f^{-1}(Y_i)$, where $f$ denotes the Green correspondence with respect to $(G, N, P)$ and set $X = X_1 \oplus \cdots \oplus X_n$. By the properties of the Green correspondence (Theorem 7.8 [9], [11], [12]) $X_i$ is in $B$, $\dim_k X_i \equiv \dim_k Y_i^G$ (mod. $p|G:P|_p$) and $\Omega^m(X_i) \cong f^{-1}(\Omega^m(Y_i))$ for every integer $m$. Thus $(\dim_k X)_p = |G:P|_p$ and $\Omega^1(X) = \Omega^{-1}(X)$.

## 2. Proof of Theorem 1

If a block $B$ satisfies one of the conditions (1) and (2) in Theorem 1, then it is easy to show that $J(B)^3 = 0$ and $J(B)^2 \neq 0$. In the rest of this section we assume that $J(B)^3 = 0$, $J(B)^2 \neq 0$ and we shall prove that $B$ satisfies one of the conditions (1) and (2).

Step 1. *If $X$ is a nonsimple nonprojective indecomposable $kG$-module in $B$, then $Soc(X)=Rad(X)$.*

Proof. Since $X$ is nonprojective, $Rad(X)\subset Soc(X)$ ([14]). Then it follows that $Rad(X)=Soc(X)$ as $X$ is nonsimple.

Step 2. *If $p$ is odd, then $|D|=p$.*

Proof. Suppose $|D|\neq p$. Let $P$ be a subgroup of $D$ of order $p$ and let $X$ be a $kG$-module in $B$ which satisfies the conditions in Lemma 3. Then by a result of Erdmann [5] $X$ and $\Omega^1(X)$ have no simple direct summand. By Lemma 2 $Soc(\Omega^1(X))\cong X/Rad(X)$ and $Soc(X)\cong\Omega^{-1}(X)/Rad\Omega^{-1}((X))$. As $\Omega^1(X)\cong\Omega^{-1}(X)$ it follows that $Soc(X)\cong\Omega^1(X)/Rad\Omega^1((X))$. Then by Step 1 we have $\dim_k X=\dim_k\Omega^1(X)$. On the other hand $\dim_k X+\dim_k\Omega^1(X)$ is divisible by the order of a Sylow $p$-subgroup of $G$. Thus we have a contradiction as $p$ is odd and $(\dim_k X)_p=|G:P|_p$.

Step 3. *If $p=2$, then $D$ is elementary abelian.*

Proof. By Proposition (6G) [2] and [15] $D$ is not cyclic. Suppose that there exists a cyclic subgroup $P$ of $D$ of order 4. Then by a similar argument as in the proof of Step 2 it follows that there exists a $kG$-module $X$ in $B$ such that $(\dim_k X)_2=|G:P|_2$ and $\dim_k X=\dim_k\Omega^1(X)$. Since $\dim_k X+\dim_k\Omega^1(X)$ is divisible by the order of a Sylow 2-subgroup of $G$, this is a contradiction. Thus every nontrivial element in $D$ is of order 2 and therefore $D$ is elementary abelian.

Step 4. *If $p=2$, then $D$ is a four group.*

Proof. Suppose that $|D|>4$ and let $P$ be a four group contained in $D$. Then by a result of Knörr [13] and Step 3 any simple $kG$-module in $B$ is not $P$-projective. Let $I=\{i\in\boldsymbol{Z};\ \Omega^i(k_P)$ is a direct summand of $S_{|P}$ for some simple $kG$-module $S$ in $B\}$, where $k_P$ denotes the trivial $kP$-module and $\boldsymbol{Z}$ denotes the set of all integers. By a result of Conlon [3] each indecomposable $kP$-module of odd dimension is isomorphic to $\Omega^i(k_P)$ for some integer $i$. Thus by Lemma 1 we can conclude that $I$ is not empty. Let $i$ be the largest integer in $I$ and choose a simple $kG$-module $S$ in $B$ such that $\Omega^i(k_P)$ is a direct summand of $S_{|P}$. Let $U$ be a projective cover of $S$. By the assumption that $J(B)^3=0$ and $J(B)^2\neq 0$, $Rad(U)/Soc(U)$ is nonzero and completely reducible. $Rad(U)/Soc(U)$ appears in the Asulander-Reiten sequence; $0\to\Omega^1(S)\to Rad(U)/Soc(U)\oplus U\to\Omega^{-1}(S)\to 0$ (Proposition 4.11 [1]). Then by the result of Roggenkamp (Proposition 2.10 [17]) $\Omega^{i+1}(k_P)$ is a direct summand of $(Rad(U)/Soc(U))_{|P}$ which contradicts the maximality of $i$.

Step 5. *Conclusion.*

First assume $p=2$. Then by Step 4 $D$ is a four group. By results of Erdmann [8] we have two cases (i) and (ii) in Theorem 4, [8]. In the case (i), it follows easily that the basic ring of $B$ is isomorphic to $kD$. In the case (ii), $B$ has three simple modules $S_1$, $S_2$ and $S_3$. Let $e_i$ ($i=1, 2$ and 3) be pairwise orthogonal primitive idempotents with $e_ikG/e_iJ(kG)=S_i$ and put $e=e_1+e_2+e_3$. By Theorem 4, [8] $\dim_k ekGe=12$ and $\dim_k e_ikGe_j=1+\delta_{ij}$. Then we can show that $ekGe$ is isomorphic to $kA_4$. Next assume that $p$ is odd. By the thoery of Brauer-Dade [4] and a result of Peacock [16], it follows that $Rad(U)/Soc(U)$ is simple or a sum of two non-isomorphic simple modules for every projective indecomposable $kG$-module $U$ in $B$. Then the result follows easily.

## 3. Proof of Theorem 2

Put $S=Rad(U)/Soc(U)$. Suppose that a Sylow 2-subgroup of $G$ is not dihedral. Then by the result of Webb (Theorem E [18]) and our assumption $S$ is simple and self dual. Let $V$ be a projective cover of $S$. Since $V$ is also self dual, for any simple $kG$-module $T$ the multiplicity of $T$ in the composition factors of $V$ is equal to that of its dual. By a result of Fong [10] the dimension of a nontrivial self dual simple $kG$-module is even. Thus we have a contradiction as the multiplicity of the trivial $kG$-module in the composition factors of $V$ is 1 and $\dim_k V$ is even.

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