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<th>Spatial-graph isotopy and the rearrangement theorem</th>
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A 1-dimensional finite CW-complex is called a graph. The set of all (piecewise linear) embeddings $\Gamma : G \rightarrow \mathbb{R}^3$ of $G$ is denoted by $S(G)$. In this paper, we will study spatial-graph isotopy and cobordism, equivalence relations on $S(G)$ introduced by Taniyama [6], and obtain interaction between them. The subset of $S(G)$ consisting of all elements isotopic to (resp. cobordant to) $\Gamma \in S(G)$ is denoted by $[\Gamma]_{\text{isotopy}}$ (resp. by $[\Gamma]_{\text{cobord}}$), and called the isotopy class (resp. the cobordism class) of $\Gamma$. Here, we note that any isotopy between two embeddings $\Gamma, \Gamma' \in S(G)$ is realized by a finite sequence of blowing-downs \( \backslash \) and ups \( / \). In Soma [5] and Inaba-Soma [2], we saw that it is useful for the study of spatial-graph isotopy to rearrange the order of blowing-ups and downs, and presented a rearrangement theorem valid for trivalent graphs, [5, Theorem 2], and that for connected graphs without cut vertices, [2, Theorem 3]. The following shows that such a rearrangement theorem holds for any graphs.

**Theorem 1** (The Rearrangement Theorem on Spatial-Graph Isotopy). For any graph $G$, let $\Gamma_1, \Gamma_2 : G \rightarrow \mathbb{R}^3$ be embeddings isotopic to each other. Then, there exists an embedding $\Gamma_3 : G \rightarrow \mathbb{R}^3$ and a sequence of blowing-downs followed by blowing-ups such that $\Gamma_1 \backslash \cdots \backslash \Gamma_{(i-1)} / \cdots / \Gamma_3 / \cdots / \Gamma_2$.

Our proof of Theorem 1 is based on arguments in [2]. However, for the completion of the proof, we must clear the hurdle which the author could not there.

An element $\Gamma_{\text{red}} \in S(G)$ is said to be isotopically reduced if the ambient-isotopy type of $\Gamma_{\text{red}}$ can not be changed by any blowing-down of $\Gamma_{\text{red}}$. We note that the isotopy class $[\Gamma]_{\text{isotopy}}$ of any $\Gamma \in S(G)$ contains an isotopically reduced element, see [5, §3, Proposition 1]. Corollary 1 is proved by the argument quite similar to that in [5, Corollary 1] which was effective only for trivalent graphs.

**Corollary 1.** Let $\Gamma_1, \Gamma_2 : G \rightarrow \mathbb{R}^3$ be embeddings of any graph $G$. Suppose that $\Gamma_{i_{\text{red}}}$ is any isotopically reduced element in $[\Gamma_i]_{\text{isotopy}}$ for $i = 1, 2$. Then, $\Gamma_1$ is isotopic to $\Gamma_2$ if and only if $\Gamma_{1_{\text{red}}}$ is ambient isotopic to $\Gamma_{2_{\text{red}}}$. 
The following corollary is a restatement of Corollary 1.

**Corollary 2.** For any embedding \( \Gamma : G \rightarrow \mathbb{R}^3 \) of a graph \( G \), the isotopy class \([\Gamma]_{\text{isotopy}}\) contains a unique isotopically reduced element up to ambient isotopy.

This corollary suggests the other question whether the cobordism class \([\Gamma]_{\text{cobord}}\) contains an isotopically reduced element. A graph \( G \) is called a *generalized bouquet* if \( G \) contains a vertex \( v \) such that \( G - \{v\} \) is acyclic. According to Taniyama [6, Theorem A], if \( G \) is a generalized bouquet, then any embedding \( \Gamma : G \rightarrow \mathbb{R}^3 \) is isotopic to a planar embedding \( \Gamma_0 : G \rightarrow \mathbb{R}^2 \subseteq \mathbb{R}^3 \), so the quotient set \( S(G)/\text{isotopy} \) consists of a single element. If the graph \( G \) is non-acyclic, then \( S(G) \) has infinitely many cobordism classes. However, except the unknotted class \([\Gamma_0]_{\text{cobord}}\), any other classes \([\Gamma]_{\text{cobord}}\) contain no isotopically reduced elements. For non-generalized-bouquet graphs, we have the following theorem in contrast to Corollary 2.

**Theorem 2.** Suppose that \( G \) is any graph other than a generalized bouquet. Then, for any embedding \( \Gamma \in S(G) \), the cobordism class \([\Gamma]_{\text{cobord}}\) contains infinitely many isotopically reduced elements which are not ambient isotopic to each other.

Note that an embedding \( \Gamma' \in S(G) \) obtained by blowing-downs of \( \Gamma \) is, in general, not cobordant to \( \Gamma \). Thus, the blowing-down method is not applicable to construct isotopically reduced elements in \([\Gamma]_{\text{cobord}}\). In §3, we will construct such embeddings by replacing mutually disjoint, trivial tangles \((B_1, B_1 \cap \Gamma(G)), \ldots, (B_m, B_m \cap \Gamma(G))\) in \((S^3, \Gamma(G))\) by certain simple tangles.

Corollary 3 follows immediately from Theorems 1 and 2.

**Corollary 3.** For any graph \( G \), let \( \phi : S(G) \rightarrow S(G)/\text{isotopy} \) be the natural quotient map. If \( G \) is not a generalized bouquet, then for any element \( \Gamma \in S(G) \), the image \( \phi([\Gamma]_{\text{cobord}}) \) is an infinite subset of \( S(G)/\text{isotopy} \).

The referee suggested that it is not hard to prove the following proposition where the positions of isotopy and cobordism in Corollary 3 are exchanged.

**Proposition 1.** For any graph \( G \), let \( \psi : S(G) \rightarrow S(G)/\text{cobord} \) be the natural quotient map. If \( G \) is not acyclic, then for any element \( \Gamma \in S(G) \), the image \( \psi([\Gamma]_{\text{isotopy}}) \) is an infinite subset of \( S(G)/\text{cobord} \).

1. **Preliminaries**

Let \( G \) be a graph, and \( I \) the closed interval \([0,1]\). Consider a pair of elements \( \Gamma, \Gamma' \in S(G) \) admitting a PL-embedding \( \Phi : G \times I \rightarrow \mathbb{R}^3 \times I \) such that, for
Some $0 < \varepsilon < 1/2$, $\Phi(x,t) = (\Gamma(x),t)$ if $(x,t) \in G \times [0,\varepsilon]$, $\Phi(x,t) = (\Gamma'(x),t)$ if $(x,t) \in G \times [1 - \varepsilon,1]$, and $\Phi(G \times [\varepsilon, 1 - \varepsilon]) \subseteq \mathbb{R}^3 \times [\varepsilon, 1 - \varepsilon]$. We say that (i) $\Gamma$ is ambient isotopic to $\Gamma'$ if $\Phi$ is locally flat and level-preserving, (ii) $\Gamma$ is cobordant to $\Gamma'$ if $\Phi$ is locally flat, and (iii) $\Gamma$ is isotopic to $\Gamma'$ if $\Phi$ is level-preserving.

A graph $H$ is a star of degree $n \in \mathbb{N}$ and centered at $v$ if $H$ is a tree consisting of $n$ edges which have $v$ as a common vertex. For a given 3-ball $B$ in $\mathbb{R}^3$, we fix a point $v \in \text{int}B$, called the center of $B$. For an element $\Gamma \in S(G)$, the pair $(B, B \cap \Gamma(G))$ is a star.

A ball-star pair $(B, B \cap \Gamma(G))$ is standard if there exists a properly embedded disk $D$ in $B$ with $D \supset B \cap \Gamma(G)$. For an embedding $\Gamma : G \to \mathbb{R}^3$ with a ball-star pair $(B, B \cap \Gamma(G))$, set $J = G - \Gamma^{-1}(\text{int}B)$. Then, we say that $\Gamma' : G \to \mathbb{R}^3$ is obtained from $\Gamma$ by a blowing-down in $B$ and denote it by $\Gamma' \prec_B \Gamma$ (or shortly $\Gamma' \prec \Gamma$). As was pointed out in [6, §2], for two elements $\Gamma, \Gamma' \in S(G)$, $\Gamma$ is isotopic to $\Gamma'$ if and only if $\Gamma'$ is obtained from $\Gamma$ by a finite sequence of blowing-downs and ups. Consider double blowing-ups $\Gamma' \prec_B \Gamma, \Gamma' \prec_B \Gamma''$ for $\Gamma \in S(G)$. Since $(B_2, B_2 \cap \Gamma'')$ is a standard pair, one can shrink $B_2$ by an ambient isotopy of $\mathbb{R}^3$ fixing $\Gamma(G)$ as a set so that either $B_1 \cap B_2 = \emptyset$ or $B_2 \subset \text{int}B_1$. If $B_2 \subset \text{int}B_1$, then the double blowing-ups can be replaced by a single blowing-up $\Gamma \prec_B \Gamma''$, see Fig. 3 in [5].

First of all, we will give the proof of Proposition 1.

**Proof of Proposition 1.** Any non-acyclic graph $G$ contains a cycle $l$. For any embedding $\Gamma \in S(G)$ and any $n \in \mathbb{N}$, let $B_n = B_1 \cup \cdots \cup B_n$ be a disjoint union of 3-balls in $\mathbb{R}^3$ such that each $B_i \cap \Gamma(G)$ is an unknotted, proper arc in $B_i$ with $\alpha_i = \Gamma^{-1}(B_i) \subseteq l$. Consider an embedding $\Gamma_n \in S(G)$ such that each $\Gamma_n(\alpha_i)$ is a left-handed trefoil in $B_i$ and $\Gamma_n|_{H_n} = \Gamma|_{H_n}$ for $H_n = G - \text{int}(\alpha_1 \cup \cdots \cup \alpha_n)$. Since $\Gamma_n \prec \cdots \prec B_n \Gamma, \Gamma_n$ is contained in $[\Gamma]_{\text{isotope}}$. Since $\text{sign}(\Gamma_n(l)) = \text{sign}(\Gamma(l)) + 2n$ and since the knot signature is well known to be a cobordism invariant, $\psi(\Gamma_n)$ ($n = 1, 2, \ldots$) are mutually distinct points of $S(G)/\text{cobar}$. This completes the proof.

We identify the 3-sphere $S^3$ with $\mathbb{R}^3 \cup \{\infty\}$. So, any element $\Gamma \in S(G)$ can be regarded as an embedding of $G$ into $S^3$. For any subset $X$ of $S^3$, an ambient isotopy of $(S^3, X)$ means an ambient isotopy of $S^3$ fixing $X$ as a set.
2. Proof of the rearrangement theorem

Throughout this section, fix a graph $G$ and a pair of blowing-up and down $\Gamma_1 \searrow_B \Gamma_2 \swarrow_C \Gamma_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are elements of $S(G)$ and $B, C$ are 3-balls with centers $v_B, v_C$. Note that isolated vertices and free edges of a graph $G$ do not affect equivalence relations on $S(G)$ such as ambient isotopy, isotopy and cobordism. Thus, we may always assume without loss of generality that $G$ contains no isolated vertices and free edges, that is, the degree of each vertex of $G$ is at least two. If necessary adding extra vertices to $G$, we may also assume that any cycle in $G$ contains at least two vertices of $G$. In particular, $G$ satisfies the condition $(**)$ in [2, §2].

It is easily seen that the following proposition implies Theorem 1.

**Proposition 2.** With the notation as above, there exist embeddings $\Gamma'_2, \Gamma'_3 \in S(G)$ and a sequence $\Gamma_1 \searrow_{C'} \Gamma'_2 \searrow_{C'} \Gamma'_3 \searrow_{C''} \Gamma_3$, where $B', C', C''$ are 3-balls with centers $v_B, v_C, v_C$ respectively.

Note that, in the case of $v_B = v_C$, the double blowing-ups $\Gamma_2 \searrow_B \Gamma_3 \searrow_C \Gamma_3$ in Proposition 2 are replaced by a single blowing up $\Gamma'_2 \searrow_{B'} \Gamma'_3 \searrow_{C''} \Gamma_3$. From now on, for any proper subset $X$ of $S^3$, we set $X^0 = X - X \cap \Gamma_2(G)$. By [2, Lemma 3], we may assume that each component of $\partial B^o \cap \partial C^o$ is a loop non-contractible both in $\partial B^o$ and $\partial C^o$ (even in the case where $\partial B^o, \partial C^o$ are compressible in $S^3 - \Gamma_2(G)$). For each component $R$ of $B \cap \partial C$, let $W_R$ denote the closure in $B$ of a component of $B - R$ disjoint from $v_B$. A closure $W_R$ is said to be innermost among these closures if $\text{int} W_R \cap \partial C = \emptyset$. According to [2, Lemma 2], if $F_R = W_R \cap \partial B$ is connected for an innermost closure $W_R$, then we have a sequence $\Gamma_1 \searrow_B \Gamma'_2 \searrow_{C'} \Gamma'_3 \searrow_{C''} \Gamma_3$ with $|\partial B \cap \partial C'| < |\partial B \cap \partial C|$, where $|Y|$ denotes the number of connected components of a compact set $Y$. In fact, when $v_B \neq v_C$, we showed in [2, Lemma 4] that, for any component $R$ of $B \cap \partial C$, $F_R$ is connected (even if $W_R$ is not innermost), and hence Proposition 2 was proved inductively. So, it suffices to consider the case of $v_B = v_C = v$. Remark that, in this case, the result corresponding to [2, Lemma 4] does not hold in general. We will complete the proof of Proposition 2 by showing that either $F_R$ is connected for at least one innermost $W_R$ or each component of $S^3 - \text{int}(B \cup C)$ is a 3-ball.

For unoriented loops $l, l'$ in $S^3$ with $l \cap l' = \emptyset$, $\text{lk}(l, l')$ is the absolute linking number of $l$ and $l'$ in $S^3$. For a loop $l$ in the punctured surface $\partial(B \cup C)^o$, $l^+$ represents a loop in $S^3 - \Gamma_2(G) \cup B \cup C$ isotopic to $l$ in $S^3 - \Gamma_2(G) \cup \text{int}(B \cup C)$. Intuitively, $l^+$ is obtained by pushing $l$ outside of $B \cup C$ slightly.

**Lemma 1.** With the notation and assumptions as above, suppose that $X$ is a connected component of $S^3 - \text{int}(B \cup C)$. Then, one of the following (i) and (ii) holds.

(i) $X$ is homeomorphic to a 3-ball.
(ii) There exists a simple proper arc $\alpha$ in $Q = X \cap \partial C$ connecting distinct components $l, l'$ of $\partial Q$ such that $\text{lk}(\alpha \cup \beta_1 \cup \beta_2, l^+) = 1$, where $\beta_1, \beta_2$ are simple arcs in $B$ connecting the end points of $\alpha$ with $v$ and satisfying $\beta_1 \cap \beta_2 = \{v\}$.

Proof. We assume that the conclusion (ii) does not hold and will show then that the conclusion (i) holds. Let $Y_1, \ldots, Y_m$ be the components of $Q$. For each $Y_i$, there exist mutually disjoint disks $D_1(i), \ldots, D_r(i)$ in $\partial B$ such that $\partial D(i) \subset \partial Y_i$ and $X \cap \partial B \subset \mathcal{D}(i)$, where $\mathcal{D}(i)$ is the union $D_1(i) \cup \cdots \cup D_r(i)$. When $\partial Y_i \cap \text{int} D_j(i) \neq \emptyset$, consider a component $l$ of $\partial Y_i \cap \text{int} D_j(i)$ which is not disconnected from $\partial D_j(i)$ by any other components of $\partial Y_i \cap \text{int} D_j(i)$. Then the triad of $l, l' = \partial D_j(i)$ and any simple arc $\alpha$ in $Y_i$ connecting $l$ with $l'$ would satisfy (ii), a contradiction. Thus, we have $\partial Y_i \cap \text{int} D_j(i) = \emptyset$. Then, the union $S_i = Y_i \cup \mathcal{D}(i)$ is a 2-sphere bounding a 3-ball $B_i$ in $S^3 - \text{int} B$ with $B_i \supset X$. Our $X$ coincides with the intersection $B_1 \cap \cdots \cap B_m$.

We set $W_i = S^3 - \text{int}(B \cup B_i)$ and $Z_i = \partial W_i - \text{int} Y_i$. Note that $Z_i$ is a connected surface in $\partial B$ homeomorphic to $Y_i$. For any distinct $i, j \in \{1, \ldots, m\}$, since $Y_i \subset X$ is disjoint from $\text{int} W_j$, $W_i$ is either contained in $W_j$ or disjoint from $W_j$. If $W_i \subset W_j$, then $X$ would meet $\text{int} W_j$ non-trivially, a contradiction. It follows that $W_i \cap W_j = \emptyset$. Thus, the boundary $\partial X = (\partial B - Z_1 \cup \cdots \cup Z_m) \cup (Y_1 \cup \cdots \cup Y_m)$ is homeomorphic to the 2-sphere $\partial B = (\partial X - Y_1 \cup \cdots \cup Y_m) \cup (Z_1 \cup \cdots \cup Z_m)$. This shows that $X$ is homeomorphic to a 3-ball.

Proof of Proposition 2 (and Theorem 1). As was seen above, we may assume that $v_B = v_C = v$.

First, we consider the case where all components $X_1, \ldots, X_m$ of $N_0 = S^3 - \text{int}(B \cup C)$ are 3-balls. Note that $N_0 \cap \Gamma_1(G) = N_0 \cap \Gamma_2(G) = N_0 \cap \Gamma_3(G)$ and the graph $(B \cup C) \cap \Gamma_i(G)$ is a star centered at $v$ for $i = 1, 2, 3$. Take mutually disjoint, simple proper arcs $\alpha_1, \ldots, \alpha_m$ in $B \cup C$ such that each $\alpha_j$ connects $\partial X_j$ with $\partial X_{j+1}$ and

$$(\alpha_1 \cup \cdots \cup \alpha_m) \cap (\Gamma_1(G) \cup \Gamma_2(G) \cup \Gamma_3(G)) = \emptyset.$$ 

The union $N_1$ of a small regular neighborhood of $\alpha_1 \cup \cdots \cup \alpha_m$ in $B \cup C$ and $N_0$ is a 3-ball with $N_1 \cap \Gamma_1(G) = N_1 \cap \Gamma_2(G) = N_1 \cap \Gamma_3(G)$ and, for the 3-ball $\hat{B} = S^3 - \text{int} N_1$ and $i = 1, 2, 3$, $(\hat{B}, \hat{B} \cap \Gamma_i(G)) = (\hat{B}, (B \cup C) \cap \Gamma_i(G))$ is a ball-star pair. This shows that there exists a (common) embedding $\Gamma'_2 \in S(G)$ admitting blowing-downs $\Gamma_1 \searrow \Gamma'_2 \Gamma'_2 \searrow \Gamma_2$ and $\Gamma_3 \searrow \Gamma'_2 \Gamma'_2$. Thus, we have the pair of blowing-down and up $\Gamma_1 \searrow \Gamma'_2 \Gamma'_2 \nearrow \Gamma_3$ from $\Gamma_1$ to $\Gamma_3$.

Next, we suppose that $S^3 - \text{int}(B \cup C)$ contains a component $X$ not homeomorphic to a 3-ball. By Lemma 1, there exists a simple proper arc $\alpha$ in $Q = X \cap \partial C$, simple arcs $\beta_1, \beta_2$ in $B$ as in Lemma 1 (ii) and a component $l$ of $\partial Q$ with $\text{lk}(\alpha \cup \beta_1 \cup \beta_2, l^+) = 1$. Consider the 2-fold branched covering $p : S^3 \rightarrow S^3$ branched over $l^+$, and set $p^{-1}(v) = \{v_1, v_2\}$. The preimage $p^{-1}(B)$ (resp. $p^{-1}(C)$)
is a union of mutually disjoint 3-balls $\tilde{B}_1$, $\tilde{B}_2$ with $v_1 \in \tilde{B}_1$, $v_2 \in \tilde{B}_2$ (resp. $\tilde{C}_1$, $\tilde{C}_2$ with $v_1 \in \tilde{C}_1$, $v_2 \in \tilde{C}_2$). Let $\tilde{\alpha}$ be the lift of $\alpha$ contained in $\partial \tilde{C}_2$. Since $\tilde{\alpha}$ connects $\tilde{B}_1$ with $\tilde{B}_2$, $\tilde{C}_2$ meets both $\tilde{B}_1$ and $\tilde{B}_2$. Note that $\tilde{\Gamma} = p^{-1}(\Gamma_2(G))$ is a spatial graph, and $(\tilde{B}_j, \tilde{B}_j \cap \tilde{\Gamma})$, $(\tilde{C}_j, \tilde{C}_j \cap \tilde{\Gamma})$ are ball-star pairs for $j = 1, 2$. Since $\tilde{v}_1 \neq \tilde{v}_2$, Lemma 4 in [2] implies that, for any component $\tilde{R}_2$ of $\tilde{B}_1 \cap \partial \tilde{C}_2$, $\tilde{F}_2 = \tilde{W}_2 \cap \partial \tilde{B}_1$ is connected and hence homeomorphic to $\tilde{R}_2$, where $\tilde{W}_2$ is the closure in $\tilde{B}_1$ of a component of $\tilde{B}_1 - \tilde{R}_2$ disjoint from $\tilde{v}_1$. When $\tilde{W}_2 \cap \partial \tilde{C}_1 = \emptyset$ for a closure $\tilde{W}_2$ with $\text{int} \tilde{W}_2 \cap \partial \tilde{C}_2 = \emptyset$, we set $W = p(\tilde{W}_2)$. Otherwise, consider the closure $\tilde{W}_1$ in $\tilde{W}_2$ of a component $\tilde{W}_2 - \partial \tilde{C}_1$ with $\text{int} \tilde{W}_1 \cap \partial \tilde{C}_1 = \emptyset$ and $\tilde{W}_1 \cap \tilde{R}_2 = \emptyset$. Note that $\tilde{R}_1 = \tilde{W}_1 \cap \partial \tilde{C}_1$ is a connected surface, see Fig. 1. If $\tilde{C}_2 \cap \text{int} \tilde{W}_2 \neq \emptyset$, then $\tilde{C}_2$ would contain $\tilde{W}_2 \supset \tilde{R}_1$, and hence $\tilde{C}_1 \cap \tilde{C}_2 \neq \emptyset$, a contradiction. This implies that $\tilde{C}_2' = \tilde{C}_2 \cup \tilde{W}_2$ is a 3-ball. If $\tilde{W}_2 \cap \tilde{\Gamma} \neq \emptyset$, then any edge $e$ of the star $\tilde{\Gamma} \cap \tilde{B}_1$ connecting a point of $\tilde{F}_2 \cap \tilde{\Gamma}$ with $\tilde{v}_1$ would meet $\tilde{R}_2$, so $e$ would tend toward $\tilde{v}_2$. This contradicts that $\tilde{v}_1 \neq \tilde{v}_2$. It follows that $(\tilde{C}_2', \tilde{C}_2 \cap \tilde{\Gamma}) = (\tilde{C}_2', \tilde{C}_2 \cap \tilde{\Gamma})$ is a ball-star pair centered at $\tilde{v}_2$. By applying Lemma 4 in [2] to the pair of the 3-balls $\tilde{C}_1$, $\tilde{C}_2$ with distinct centers, one can show that $\tilde{F}_1 = \tilde{W}_1 \cap \partial \tilde{C}_1 = \tilde{W}_1 \cap \partial \tilde{B}_1$ is connected. Then, we set $W = p(\tilde{W}_1)$. In either case, $W$ is a compact 3-manifold in $B$ bounded by the union of the connected surfaces $R = W \cap \partial C$, $F = W \cap \partial B$ and satisfying $\text{int} W \cap \partial C = \emptyset$. Then, by [2, Lemma 2], we have a sequence $\Gamma_1 \to_B \Gamma^{(1)}_2 \to_{\partial C(1)} \Gamma^{(1)}_3 \to_{\partial C(1)} \Gamma_3$ with $|\partial B \cap \partial C(1)| < |\partial B \cap \partial C|$. Repeating the same process finitely many times, we have a sequence $\Gamma_1 \to_B \Gamma^{(r)}_2 \to_{\partial C(r)} \Gamma^{(r)}_3 \to_{\partial C(r)} \Gamma_3$ such that each component of
$S^3 - \text{int}(B' \cup C')$ is a 3-ball. As was seen in the previous case, one can then exchange the blowing-up and down of $\Gamma_1 \setminus_{B'} \Gamma_2^{(r)} \setminus_{C'} \Gamma_3^{(r)}$ and obtain our desired sequence.

3. Construction of isotopically reduced embeddings

In this section, we will prove that, if a graph $G$ is not a generalized bouquet, then for any embedding $\Gamma \in S(G)$, the cobordism class $[\Gamma]_{\text{cobor}}$ contains infinitely many isotopically reduced elements which are not ambient isotopic to each other.

Our proof here is based on arguments in Soma [3] and [4], where the author constructed simple links cobordant to given links in $S^3$ and closed 3-manifolds by using certain simple tangles. Here, a (2-string) tangle $(B, t_1 \cup t_2)$ is a pair of a 3-ball $B$ and a disjoint union $t_1 \cup t_2$ of two simple proper arcs in $B$. A tangle $(B, t_1 \cup t_2)$ is trivial if there exists a properly embedded disk in $B$ containing $t_1 \cup t_2$. A tangle $(B, t_1 \cup t_2)$ is simple if $\partial B - \partial t_1 - \partial t_2$ is incompressible in $B - t_1 \cup t_2$ and if $B - t_1 \cup t_2$ contains no incompressible tori. We refer to [3, §2] for examples of simple tangles.

In particular, a clasp tangle $(B, t_1 \cup t_2)$ as in Fig. 2 is simple. Let $A$ be a properly embedded annulus in the complement $B - t_1 \cup t_2$ of a simple tangle such that $\partial A$ bounds an annulus $A'$ in $\partial B - \partial t_1 - \partial t_2$. If $A$ is incompressible in $B - t_1 \cup t_2$, then any compressing disk $\Delta$ for the torus $T = A \cup A'$ is contained in the compact 3-manifold $V$ in $B - t_1 \cup t_2$ bounded by $T$. Since $V$ is a solid torus and each component of $\partial A$ is contractible in $B - \text{int} V$, $A$ is parallel to $A'$ in $V \subset B - t_1 \cup t_2$. Here, we say that a compact surface $F$ properly embedded in a 3-manifold $X$ is parallel in $X$ to a surface $F'$ if there exists an embedding $h : F \times I \rightarrow X$ with $h(F \times \{0\}) = F$ and $h(F \times \{1\}) \cup \partial F \times I) = F'$.

A compact, connected surface homeomorphic to a closed region in $\mathbb{R}^2$ with $n$ boundary components is called an $n$-ply connected disk. In particular, a doubly connected disk is an annulus.

Lemma 2. Let $(B, t_1 \cup t_2)$ be a clasp tangle, and let $R$ be an incompressible, triply connected disk properly embedded in $B - t_1 \cup t_2$. Suppose that there exist mutually disjoint disks $D_1, D_2, D_3$ in $\partial B$ satisfying $\partial(D_1 \cup D_2 \cup D_3) = \partial R$, $D_1 \cap t_1 \neq \emptyset$, $D_2 \cap t_1 \neq \emptyset$ and $(D_1 \cup D_2) \cap t_2 = \emptyset$. Then, $R$ is parallel in $B - t_1 \cup t_2$ to a surface in $\partial B - \partial t_1 - \partial t_2$.

Proof. By the assumptions as above, both $D_1 \cap t_1$ and $D_2 \cap t_1$ consist of single points. Since $R$ is incompressible in $B - t_1 \cup t_2$ and $(D_1 \cup D_2) \cap t_2 = \emptyset$, $\partial t_2$ is contained in $D_3$. The 2-sphere $R \cup D_1 \cup D_2 \cup D_3$ bounds a 3-ball $C$ in $B$ containing $t_1 \cup t_2$. Consider the closure $W$ of $B - C$ in $B$. Note that $F = W \cap \partial B$ is a triply connected disk in $\partial B^o = \partial B - \partial t_1 - \partial t_2$ with $\partial F = \partial R$. Let $\Delta$ be an embedded disk in $B$ as illustrated in Fig. 2 such that $\partial \Delta \subset t_1$ and $\Delta \cap t_2$ is two points in $\text{int} \Delta$. It is easily seen that $\Delta^o = \Delta - \Delta \cap (t_1 \cup t_2)$ is incompressible in $B - t_1 \cup t_2$. We
may assume that $\Delta$ meets $R$ transversely, and each loop component of $\Delta \cap R$ is non-contractible both in $\Delta^o$ and $R$. Since the arc $\Delta \cap \partial B$ connects the end points of $t_1$, the union $J$ of all arc components of $\Delta \cap R$ are non-empty. Let $\Delta_1, \ldots, \Delta_n$ ($n \geq 2$) be the closures in $\Delta$ of all components of $\Delta - J$ such that $\Delta_n \supset t_1$ and $\Delta_1$ is innermost, that is, $\gamma = \Delta_1 \cap J$ is a single arc. We need to consider the following three cases, though the reader will see that Cases 2 and 3 do not occur really.

**CASE 1.** $\Delta_1 \cap t_2$ is empty.

In this case, $\text{int} \Delta_1 \cap R$ contains no loop components, so $\text{int} \Delta_1 \cap R = \emptyset$. If $\Delta_1 \subset C$, then for some $j \in \{1, 2, 3\}$, $D_j \cap \Delta_1$ is an arc separating $D_j$ into two disks $D_{j1}$ and $D_{j2}$ such that $D_{j1} \cap (t_1 \cup t_2) = \emptyset$. The union $\Delta_1 \cup D_{j1}$ is a disk in $C - t_1 \cup t_2$ with $\partial(\Delta_1 \cup D_{j1}) \subset R$. Since $R$ is incompressible in $C - t_1 \cup t_2$, $\Delta_1$ excises a 3-ball $C_1$ from $C - t_1 \cup t_2$. Deforming $\Delta$ in a small neighborhood of $C_1$ by an ambient isotopy of $B$ rel. $t_1 \cup t_2$, one can reduce the number $|\Delta \cap R|$. Thus, we may assume that $\Delta_1 \cap \text{int} C = \emptyset$. If $\gamma$ is inessential in $R$, that is, $\gamma$ excises a disk from $R$, then one can reduce $|\Delta \cap R|$ as above by invoking the incompressibility of $R$ in $W$. In the case where $\gamma$ is essential in $R$, consider the surface $R'$ obtained by surgery on $R$ along $\Delta_1$. The surface $R'$ consists of at most two annuli which are incompressible in $W$. Since the boundary of each component $A'$ of $R'$ bounds an annulus in $F$, $A'$ is parallel to the annulus in $W$. This implies that $R$ is parallel in $B - t_1 \cup t_2$ to $F$.

**CASE 2.** $\Delta_1 \cap t_2$ consists of a single point.

If $\text{int} \Delta_1 \cap R \neq \emptyset$, then there would exist a disk $\Delta_0$ in $\Delta_1$ with $\partial \Delta_0 \subset \text{int} \Delta_1 \cap R$, $\text{int} \Delta_0 \cap R = \emptyset$ and such that $\text{int} \Delta_0 \cap t_2$ is a single point. Since $\Delta_0 \cap D_3 \subset \Delta_0 \cap \partial B = \emptyset$ and $\partial t_2 \subset D_3$, the algebraic intersection number of $\Delta_0$ with $t_2$ in the 3-ball $C$ would be zero, a contradiction. Thus, $\text{int} \Delta_1 \cap R$ is empty. A similar argument implies that
Δ₁ ∩ (D₁ ∪ D₂) = ∅, and so Δ₁ ∩ D₃ is an arc. Since Δ₁ ∩ t₁ = ∅, γ is inessential in R, and hence Δ₁ excises a 3-ball C₀ from C such that t₀ = C₀ ∩ t₂ = C₀ ∩ (t₁ ∪ t₂) is a proper arc in C. Since t₂ is unknotted in B, t₀ is also unknotted in C₀. Thus, there exists a disk E₀ in C₀ bounded by the union of t₀ and an arc u₀ in the disk Δ₁ ∪ (C₀ ∩ D₃) with ∂u₀ = ∂t₀ and such that u₀ ∩ Δ₁ ∩ D₃ is a single point. The tangle (B, t₁ ∪ t₂) admits an orientation reversing involution h which is the reflection with respect to the horizontal plane containing the barycenter of the 3-ball B in Fig. 2. By using an elementary cut-and-paste argument, one can take E₀ so that E₀ ∩ h(E₀) = ∅. Move t₂ in a small neighborhood of E₀ ∩ h(E₀) in B by an ambient isotopy of B rel. t₁ so that t₂ ∩ Δ = ∅. Then, for a regular neighborhood N of Δ in B − t₂, ∂N − int(N ∩ ∂B) is a proper disk in B separating t₁ and t₂. This implies that ∂B° is compressible in B − t₁ ∪ t₂, a contradiction. Thus, Case 2 cannot occur.

CASE 3. Δ₁ ∩ t₂ consists of two points.

Since Δ₁ ∩ t₂ = ∅ for i = 2, ..., n, Δ₁ ∩ J consists of two arcs for i = 2, ..., n − 1 and Δₙ ∩ J consists of a single arc. Moreover, Δₙ is a disk in C with Δₙ ∩ ∂C = ∂Δₙ − intt₁. For the 3-ball C' obtained by cutting C open along Δₙ, ∂C' − intD₃ is a proper disk in B separating t₁ from t₂. This contradiction implies that Case 3 cannot occur.

Let Γ : G → R³ ⊂ S³ be any embedding of a graph G other than a generalized bouquet. As in §2, G can be assumed to contain no isolated vertices and free edges. We denote by V = {v₁, ..., vₙ} the set of all vertices of G. Consider the projection p : R³ → R²(⊂ R³) defined by p(x, y, z) = (x, y, 0). Slightly deforming Γ by an ambient isotopy, we may assume that p ∩ Γ is a regular projection, that is, (i) the restriction p ∩ Γ|V is an embedding, (ii) p(Γ(V)) ∩ p(Γ(G − V)) = ∅, and (iii) each singular value of p ∩ Γ is a transversal double point. We regard that the image Γ = p(Γ(G)) is a plane graph, where each double point of p ∩ Γ\G is considered to be a vertex of Γ of degree four. Let D₁, ..., Dₙ be mutually disjoint disks in R² such that Dᵢ ∩ Γ is a star centered at vi = p(Γ(vᵢ)) for i = 1, ..., n, and let D = D₁ ∪ ⋯ ∪ Dₙ. Since G is not a generalized bouquet, for each vᵢ ∈ V, G contains a cycle lᵢ disjoint from vᵢ. We note that lᵢ may be equal to lⱼ even if i ≠ j. Let α₁, ..., αₙ be mutually disjoint arcs in Γ − D ∩ Γ disjoint from the set of vertices of Γ and with αᵢ ⊂ p(Γ(lᵢ)). We set \(\tilde{\alpha}_i = (p ∩ Γ)^{-1}(α_i)\) and \(\tilde{A} = \tilde{\alpha}_1 ∪ ⋯ ∪ \tilde{\alpha}_n\). Consider simple arcs β₁, ..., βₙ in R² − intD meeting each other and Γ transversely and such that each βᵢ connects a point in intαᵢ with a point xᵢ in ∂Dᵢ − ∂Dᵢ ∩ Γ. Let Γ₁ ∈ S(G) be an embedding ambient isotopic to Γ rel. G − A such that p ∩ Γ₁(\(\tilde{\alpha}_i\)) is an arc which tends toward ∂Dᵢ along βᵢ, and meets βᵢ at a point yᵢ near xi, and then goes round a neighborhood of ∂Dᵢ until meeting yᵢ again, and finally returns to αⱼ along βᵢ as illustrated in Fig. 3. If necessary deforming Γ₁ by an ambient
isotopy, the plane graph $\hat{\Gamma}_1 = p \circ \Gamma_1(G)$ can be assumed to satisfy the following (2.1) and (2.2).

(2.1) $\hat{\Gamma}_1$ is connected.
(2.2) $\hat{\Gamma}_1$ contains no cut vertices.

Here, a cut vertex $v$ of a graph $H$ means a vertex disconnecting the component of $H$ containing $v$. Let $\{\hat{w}_1, \ldots, \hat{w}_m\}$ be the set of the vertices of $\hat{\Gamma}_1$ corresponding to the double points of $p \circ \Gamma_1$, and let $C_1, \ldots, C_m$ be small regular neighborhoods of $\hat{w}_1, \ldots, \hat{w}_m$ in $S^3$. Note that each $(C_j, C_j \cap \hat{\Gamma}_1)$ is a standard ball-star pair of degree four centered at $\hat{w}_j$. Let $\hat{\Gamma}_1$ be the regular diagram for $\Gamma_1$ obtained by replacing each $C_j \cap \hat{\Gamma}_1$ by a suitable 2-string trivial tangle in $C_j$. We set $C = C_1 \cup \cdots \cup C_m$. Let $\Gamma_2 : G \to S^3$ be an embedding such that $\Gamma_2(G) \cap \text{int}C = \hat{\Gamma}_1 \cap \text{int}C$, and for each $j = 1, \ldots, m$, $(C_j, C_j \cap \Gamma_2(G))$ is obtained by exchanging each trivial tangle $(C_j, C_j \cap \hat{\Gamma}_1)$ by a clasp tangle so that $\Gamma_2$ is cobordant to $\Gamma_1$ and hence to $\Gamma$.

Now, we will prove the following lemma which is crucial in the proof of Theorem 2.

**Lemma 3.** With the notation as above, any ball-star pair $(B, B \cap \Gamma_2(G))$ in $(S^3, \Gamma_2(G))$ is standard. In particular, $\Gamma_2$ is isotopically reduced.

**Proof.** The argument quite similar to that in Assertion 1 of [3, Theorem 3] implies that $\Gamma_2(G)$ is "prime", that is, any 2-sphere in $S^3$ meeting $\Gamma_2(G)$ transversely in two points bounds a 3-ball $B_0$ in $S^3$ such that $B_0 \cap \Gamma_2(G)$ is an unknotted arc in $B_0$. In particular, any ball-arc pair in $(S^3, \Gamma_2(G))$ is standard. Thus, we may assume that $B$ contains a vertex of $\Gamma_2(G)$, say $\hat{v}_1$. As in §2, for a proper subset $X$
of $S^3$, we set $X^o = X - X \cap \Gamma_2(G)$. By (2.1), $S^3 - C \cup \Gamma_2(G) = S^3 - C \cup \hat{\Gamma}_1$ is irreducible. Since a clasp tangle is simple, $\partial C_j^o$ is incompressible in $C_j^o$. By (2.2), each $\partial C_j^o$ is also incompressible in $S^3 - \text{int}C \cup \Gamma_2(G)$. This shows that $\partial C_j^o$ is incompressible in $S^3 - \Gamma_2(G)$ and $S^3 - \Gamma_2(G)$ is irreducible. Set $B = \hat{B}$ if $\partial B^o$ is incompressible in $S^3 - \Gamma_2(G)$. If $\partial B^o$ is compressible in $S^3 - \Gamma_2(G)$, then we consider mutually disjoint, compressing disks $\Delta_1, \ldots, \Delta_r$ for $\partial B^o$ in $S^3 - \text{int}B \cup \Gamma_2(G)$ and 3-balls $B_1, \ldots, B_r$ in $S^3$ with $\partial B_i \subset \Delta_i \cup \partial B$ and $B_i \cap \Gamma_2(l_1) = \emptyset$. Then, the union $\hat{B} = B \cup B_1 \cup \cdots \cup B_r$ is a 3-ball disjoint from $\Gamma_2(l_1)$ and such that $\partial \hat{B}^o$ is incompressible in $S^3 - \Gamma_2(G)$. Note that $\partial \hat{B} \cap \Gamma_2(G) \subset \partial B \cap \Gamma_2(G)$, and $\partial \hat{B} \cap \Gamma_2(G) = \partial B \cap \Gamma_2(G)$ if and only if $\partial B^o$ is incompressible in $S^3 - \Gamma_2(G)$. Since $S^3 - \Gamma_2(G)$ is irreducible, $\partial \hat{B} \cap \Gamma_2(G)$ is non-empty. If necessary deforming $\partial B$ by an ambient isotopy of $(S^3, \Gamma_2(G))$, one can assume that $\partial \hat{B}$ meets $\partial C$ transversely and each component of $\partial \hat{B} \cap \partial C$ is non-contractible both in $\partial \hat{B}^o$ and $\partial C^o$. Renumber $\tilde{\omega}_j$'s so that the subset $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_k\}$ of $\{\omega_1, \ldots, \omega_m\}$ consists of the double points of $\tilde{\Gamma}_1$ surrounding $\tilde{v}_1$, and $\tilde{\omega}_k$ corresponds to the double point $y_1$ of $p \circ \Gamma_1(\tilde{\alpha}_1)$. Let $\epsilon_j$ ($j = 1, \ldots, k - 1$) be the edge of $\Gamma_2(G)$ meeting both $\hat{v}_1$ and $C_j$, see Fig. 4. Since $C_j$ meets $\Gamma_2(l_1)$ non-trivially for any $j = 1, \ldots, k$, $C_j$ is not contained in $\hat{B}$. If there existed a disk $\Delta$ in $\partial C_j$ with $\partial \Delta \subset \partial \hat{B} \cap \partial C_j$, $\text{int} \Delta \cap \partial \hat{B} = \emptyset$ and such that $\Delta \cap \Gamma_2(l_1)$ is a single point, then $\Delta$ would be a non-separating proper disk in the 3-ball $S^3 - \text{int}\hat{B}$, a contradiction. Thus, in the case of $\partial \hat{B} \cap \partial C_j = \emptyset$, the closure $F$ in $\partial C_j$ of any connected component of $\partial C_j - \partial \hat{B} \cap \partial C_j$ is either a disk with $1 \leq \#(F \cap \Gamma_2(G)) \leq 3$, or an annulus with $0 \leq \#(F \cap \Gamma_2(G)) \leq 2$, or a triply connected disk with $F \cap \Gamma_2(G) = \emptyset$, where $\#(X)$ denotes the number of elements of a finite set $X$. If $F$ is either a disk with $\#(F \cap \Gamma_2(G)) = 3$ or an annulus with $\#(F \cap \Gamma_2(G)) = 2$, then $F \cap \Gamma_2(l_2) \neq \emptyset$, and hence $F$ is not contained in $\hat{B}$. 

Fig. 4.
We set $C(k) = C_1 \cup \cdots \cup C_k \subset C$. If $\partial C(k) \cap \hat{B}$ contains a disk component $F$ with $\#(F \cap \Gamma_2(G)) = 1$, then $\partial F$ bounds a disk $F'$ in $\partial \hat{B}$ with $\#(F' \cap \Gamma_2(G)) = 1$. Since $\Gamma_2(G)$ is prime, the 2-sphere $F \cup F'$ bounds a 3-ball $B'$ in $\hat{B}$ such that $B' \cap \Gamma_2(G)$ is an unknotted arc in $B'$. This enables us to reduce the number $|\partial C(k) \cap \partial \hat{B}|$ by deforming $\partial C(k)$ in a small neighborhood of $B'$. Similarly, if $\partial C(k) \cap \hat{B}$ contains an annulus component $F$ with $F \cap \Gamma_2(G) = \emptyset$, then one can reduce the number $|\partial C(k) \cap \partial \hat{B}|$, for example see Assertion 2 in the proof of [3, Theorem 3]. Thus, we may assume that, for each $C_j$ $(j = 1, \ldots, k)$ with $\partial \hat{B} \cap \partial C_j \neq \emptyset$, $F_j = \partial C_j \cap \hat{B}$ is a connected surface which is either a disk with $\#(F_j \cap \Gamma_2(G)) = 2$, or an annulus with $\#(F_j \cap \Gamma_2(G)) = 1$, or a triply connected disk with $F_j \cap \Gamma_2(G) = \emptyset$. One can reduce the former two cases to the latter case, by pushing a small neighborhood of $F_j \cap \Gamma_2(G)$ toward the outside of $\hat{B}$ along the edges of $\Gamma_2(G)$ meeting $F_j$. So, it suffices to consider the case where $F_j$ is a triply connected disk disjoint from $\Gamma_2(G)$. Let $W_j$ be the closure in $\hat{B}$ of a component of $\hat{B} - F_j$ disjoint from $\hat{v}_1$. It is easy to see that $R_j = \partial W_j \cap \partial \hat{B}$ is also a triply connected disk. We assume that $W_1$ is innermost among all $W_j$'s, that is, $\text{int} W_1 \cap \partial C(k) = \emptyset$. If $W_1$ were not contained in $C_1$, then $C_1$ would contain $\hat{v}_1$, a contradiction. It follows that $W_1$ is contained in $C_1$, and hence Lemma 2 shows that $R_1$ is parallel to $F_1$ in $C_1^\circ$. This implies that one can reduce the number $|\partial \hat{B} \cap \partial C(k)|$, and finally get the situation of $\hat{B} \cap C(k) = \emptyset$.

Since $\partial B \cap \Gamma_2(G) \cap \partial \hat{B} \cap \Gamma_2(G) \neq \emptyset$, at least one of $\varepsilon_1, \ldots, \varepsilon_{k-1}$, say $\varepsilon_1$, meets $\partial \hat{B}$ non-trivially. Let $\alpha$ be the subarc of $\varepsilon_1$ connecting $\hat{v}_1$ with $\varepsilon_1 \cap \partial \hat{B}$. If $\alpha \cap C_1 \neq \emptyset$,
then $C_1$ would meet $\tilde{B} \supset \alpha$ non-trivially. This contradiction implies $\alpha \cap C_1 = \emptyset$.

Since $\partial \tilde{B} \cap (\Gamma_2(l_1) \cup \mathcal{C}(k)) = \emptyset$ and since $\partial \tilde{B}$ meets any $\varepsilon_j (j = 1, \ldots, k - 1)$ at most one point, the component $l_0$ of $\partial \tilde{B} \cap (\mathbb{R}^2 - \text{int} \mathcal{C} \cap \mathbb{R}^2)$ containing $\varepsilon_1 \cap \partial \tilde{B}$ is a loop in $\mathbb{R}^2 - \mathcal{C} \cap \mathbb{R}^2$ bounding a disk $\Delta_0$ such that $\Delta_0 \cap \Gamma_2(G)$ is a star of degree $k - 1$ centered at $\tilde{v}_1$, see Fig. 5. Since $\#(\partial \tilde{B} \cap \Gamma_2(G))$ is equal to the degree of $\tilde{v}_1$ in $\Gamma_2(G)$, we have $B = \tilde{B}$ or equivalently that $\partial B^o$ is incompressible in $S^3 - \Gamma_2(G)$. Since each component of $\partial B - l_0$ is an open disk disjoint from $\Gamma_2(G)$, one can deform $\partial B$ by an ambient isotopy of $(S^3, \Gamma_2(G))$ rel. $l_0$ so that $\partial B \cap (\mathbb{R}^2 \cup \mathcal{C}) = l_0$. In particular, $(B, B \cap \Gamma_2(G))$ is a standard ball-star pair. This shows that $\Gamma_2$ is isotopically reduced.

Proof of Theorem 2. For any positive integer $m$, choose the regular projection $\hat{\Gamma}_1 = p(\Gamma_1(G))$ as above so that $\hat{\Gamma}_1$ has at least $m$ double points. Then, for the isotopically reduced embedding $\Gamma_2 \in [\Gamma]_{\text{cobor}}$ given in Lemma 3, the complement $S^3 - \Gamma_2(G)$ contains at least $m$ mutually disjoint and non-parallel, incompressible, four-punctured 2-spheres. On the other hand, by Haken’s Finiteness Theorem [1], there exists a positive integer $n(\Gamma_2)$ depending only on the ambient isotopy type of $\Gamma_2$ so that the number of such four-punctured 2-spheres in $S^3 - \Gamma_2(G)$ is not greater than $n(\Gamma_2)$. This observation implies that one can construct infinitely many isotopically reduced elements of $[\Gamma]_{\text{cobor}}$ which are not ambient isotopic to each other.

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References

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