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<th>Spatial-graph isotopy and the rearrangement theorem</th>
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A 1-dimensional finite CW-complex is called a graph. The set of all (piecewise linear) embeddings $\Gamma : G \to \mathbb{R}^3$ of $G$ is denoted by $S(G)$. In this paper, we will study spatial-graph isotopy and cobordism, equivalence relations on $S(G)$ introduced by Taniyama [6], and obtain interaction between them. The subset of $S(G)$ consisting of all elements isotopic to (resp. cobordant to) $\Gamma \in S(G)$ is denoted by $[\Gamma]_{\text{isotopy}}$ (resp. by $[\Gamma]_{\text{cobor}}$), and called the isotopy class (resp. the cobordism class) of $\Gamma$. Here, we note that any isotopy between two embeddings $\Gamma, \Gamma' \in S(G)$ is realized by a finite sequence of blowing-downs $\backslash$ and ups $\slash$. In Soma [5] and Inaba-Soma [2], we saw that it is useful for the study of spatial-graph isotopy to rearrange the order of blowing-ups and downs, and presented a rearrangement theorem valid for trivalent graphs, [5, Theorem 2], and that for connected graphs without cut vertices, [2, Theorem 3]. The following shows that such a rearrangement theorem holds for any graphs.

**Theorem 1** (The Rearrangement Theorem on Spatial-Graph Isotopy). For any graph $G$, let $\Gamma_1, \Gamma_2 : G \to \mathbb{R}^3$ be embeddings isotopic to each other. Then, there exists an embedding $\Gamma_3 : G \to \mathbb{R}^3$ and a sequence of blowing-downs followed by blowing-ups such that $\Gamma_1 \backslash \cdots \backslash \Gamma_3 \slash \cdots \slash \Gamma_2$.

Our proof of Theorem 1 is based on arguments in [2]. However, for the completion of the proof, we must clear the hurdle which the author could not there.

An element $\Gamma^\text{red} \in S(G)$ is said to be isotopically reduced if the ambient-isotopy type of $\Gamma^\text{red}$ can not be changed by any blowing-down of $\Gamma^\text{red}$. We note that the isotopy class $[\Gamma]_{\text{isotopy}}$ of any $\Gamma \in S(G)$ contains an isotopically reduced element, see [5, §3, Proposition 1]. Corollary 1 is proved by the argument quite similar to that in [5, Corollary 1] which was effective only for trivalent graphs.

**Corollary 1.** Let $\Gamma_1, \Gamma_2 : G \to \mathbb{R}^3$ be embeddings of any graph $G$. Suppose that $\Gamma_i^\text{red}$ is any isotopically reduced element in $[\Gamma_i]_{\text{isotopy}}$ for $i = 1, 2$. Then, $\Gamma_1$ is isotopic to $\Gamma_2$ if and only if $\Gamma_1^\text{red}$ is ambient isotopic to $\Gamma_2^\text{red}$. 
The following corollary is a restatement of Corollary 1.

**Corollary 2.** For any embedding \( \Gamma : G \to \mathbb{R}^3 \) of a graph \( G \), the isotopy class \([\Gamma]_{\text{isotopy}}\) contains a unique isotopically reduced element up to ambient isotopy.

This corollary suggests the other question whether the cobordism class \([\Gamma]_{\text{cobor}}\) contains an isotopically reduced element. A graph \( G \) is called a generalized bouquet if \( G \) contains a vertex \( v \) such that \( G - \{v\} \) is acyclic. According to Taniyama [6, Theorem A], if \( G \) is a generalized bouquet, then any embedding \( \Gamma : G \to \mathbb{R}^3 \) is isotopic to a planar embedding \( \Gamma_0 : G \to \mathbb{R}^2 \subset \mathbb{R}^3 \), so the quotient set \( S(G)/\text{isotopy} \) consists of a single element. If the graph \( G \) is non-acyclic, then \( S(G) \) has infinitely many cobordism classes. However, except the unknotted class \([\Gamma_0]_{\text{cobor}}\), any other classes \([\Gamma]_{\text{cobor}}\) contain no isotopically reduced elements. For non-generalized-bouquet graphs, we have the following theorem in contrast to Corollary 2.

**Theorem 2.** Suppose that \( G \) is any graph other than a generalized bouquet. Then, for any embedding \( \Gamma \in S(G) \), the cobordism class \([\Gamma]_{\text{cobor}}\) contains infinitely many isotopically reduced elements which are not ambient isotopic to each other.

Note that an embedding \( \Gamma' \in S(G) \) obtained by blowing-downs of \( \Gamma \) is, in general, not cobordant to \( \Gamma \). Thus, the blowing-down method is not applicable to construct isotopically reduced elements in \([\Gamma]_{\text{cobor}}\). In §3, we will construct such embeddings by replacing mutually disjoint, trivial tangles \((B_1, B_1 \cap \Gamma(G)), \ldots, (B_m, B_m \cap \Gamma(G))\) in \( (S^3, \Gamma(G)) \) by certain simple tangles.

Corollary 3 follows immediately from Theorems 1 and 2.

**Corollary 3.** For any graph \( G \), let \( \phi : S(G) \to S(G)/\text{isotopy} \) be the natural quotient map. If \( G \) is not a generalized bouquet, then for any element \( \Gamma \in S(G) \), the image \( \phi([\Gamma]_{\text{cobor}}) \) is an infinite subset of \( S(G)/\text{isotopy} \).

The referee suggested that it is not hard to prove the following proposition where the positions of isotopy and cobordism in Corollary 3 are exchanged.

**Proposition 1.** For any graph \( G \), let \( \psi : S(G) \to S(G)/\text{cobor} \) be the natural quotient map. If \( G \) is not acyclic, then for any element \( \Gamma \in S(G) \), the image \( \psi([\Gamma]_{\text{isotopy}}) \) is an infinite subset of \( S(G)/\text{cobor} \).

1. Preliminaries

Let \( G \) be a graph, and \( I \) the closed interval \([0,1]\). Consider a pair of elements \( \Gamma, \Gamma' \in S(G) \) admitting a PL-embedding \( \Phi : G \times I \to \mathbb{R}^3 \times I \) such that, for
some \(0 < \varepsilon < 1/2\), \(\Phi(x,t) = (\Gamma(x),t)\) if \((x,t) \in G \times [0,\varepsilon]\), \(\Phi(x,t) = (\Gamma'(x),t)\) if \((x,t) \in G \times [1-\varepsilon,1]\), and \(\Phi(G \times [\varepsilon,1-\varepsilon]) \subset \mathbb{R}^3 \times [\varepsilon,1-\varepsilon]\). We say that (i) \(\Gamma\) is \textit{ambient isotopic} to \(\Gamma'\) if \(\Phi\) is locally flat and level-preserving, (ii) \(\Gamma\) is \textit{cobordant} to \(\Gamma'\) if \(\Phi\) is locally flat, and (iii) \(\Gamma\) is \textit{isotopic} to \(\Gamma'\) if \(\Phi\) is level-preserving.

A graph \(H\) is a \textit{star} of degree \(n \in \mathbb{N}\) and centered at \(v\) if \(H\) is a tree consisting of \(n\) edges which have \(v\) as a common vertex. For a given 3-ball \(B\) in \(\mathbb{R}^3\), we fix a point \(v \in \text{int}B\), called the \textit{center} of \(B\). For an element \(\Gamma \in S(G)\), the pair \((B, B \cap \Gamma(G))\) is called a \textit{ball-star} pair if \(\Gamma_\alpha\) is a star centered at \(v\) and with \(\partial \varepsilon \subset \partial B \cup \{v\}\) for each edge \(\varepsilon\) of \(B \cap \Gamma(G)\). When \(\alpha = B \cap \Gamma(G)\) is a proper arc in \(B\), \((B, \alpha)\) is regarded as a ball-star pair of degree two even if \(\alpha\) contains no vertices of \(\Gamma(G)\). A ball-star pair \((B, B \cap \Gamma(G))\) is \textit{standard} if there exists a properly embedded disk \(D\) in \(B\) with \(D \supset B \cap \Gamma(G)\). For an embedding \(\Gamma : G \rightarrow \mathbb{R}^3\) with a ball-star pair \((B, B \cap \Gamma(G))\), set \(J = G - \Gamma\) and \((J_\alpha, J_\cap \Gamma(G))\) is a standard ball-star pair. Conversely, \(\Gamma\) is said to be obtained from \(\Gamma'\) by a \textit{blowing-up} occurring in \(B\) and denote it by \(\Gamma' / B\). As was pointed out in [6, §2], for two elements \(\Gamma, \Gamma' \in S(G)\), \(\Gamma\) is isotopic to \(\Gamma'\) if and only if \(\Gamma'\) is obtained from \(\Gamma\) by a finite sequence of blowing-downs and ups. Consider double blowing-ups \(\Gamma' / B_1, \Gamma' / B_2, \Gamma''\) for \(\Gamma \in S(G)\). Since \((B_2, B_2 \cap \Gamma'(G))\) is a standard pair, one can shrink \(B_2\) by an ambient isotopy of \(\mathbb{R}^3\) fixing \(\Gamma'(G)\) as a set so that either \(B_1 \cap B_2 = \emptyset\) or \(B_2 \subset \text{int}B_1\). If \(B_2 \subset \text{int}B_1\), then the double blowing-ups can be replaced by a single blowing-up \(\Gamma / B_1, \Gamma''\), see Fig. 3 in [5].

First of all, we will give the proof of Proposition 1.

Proof of Proposition 1. Any non-acyclic graph \(G\) contains a cycle \(l\). For any embedding \(\Gamma \in S(G)\) and any \(n \in \mathbb{N}\), let \(B_n = B_1 \cup \cdots \cup B_n\) be a disjoint union of 3-balls in \(\mathbb{R}^3\) such that each \(B_i \cap \Gamma(G)\) is an unknotted, proper arc in \(B_i\) with \(\alpha_i = \Gamma^{-1}(B_i) \subset l\). Consider an embedding \(\Gamma_n \in S(G)\) such that each \(\Gamma_n(\alpha_i)\) is a left-handed trefoil in \(B_i\) and \(\Gamma_n|_{H_n} = \Gamma|_{H_n}\) for \(H_n = G - \text{int}(\alpha_1 \cup \cdots \cup \alpha_n)\). Since \(\Gamma_n \cap \cdots \cap \Gamma_n\) is contained in \(\Gamma_{\text{isotopy}}\). Since \(\text{sign}(\Gamma_n(l)) = \text{sign}(\Gamma(l)) + 2n\) and since the knot signature is well known to be a cobordism invariant, \(\psi(\Gamma_n)\) \((n = 1,2,\ldots)\) are mutually distinct points of \(S(G)/\text{cobor}\). This completes the proof. \[\square\]

We identify the 3-sphere \(S^3\) with \(\mathbb{R}^3 \cup \{\infty\}\). So, any element \(\Gamma \in S(G)\) can be regarded as an embedding of \(G\) into \(S^3\). For any subset \(X\) of \(S^3\), an \textit{ambient isotopy} of \((S^3, X)\) means an ambient isotopy of \(S^3\) fixing \(X\) as a set.
2. Proof of the rearrangement theorem

Throughout this section, fix a graph \( G \) and a pair of blowing-up and down \( \Gamma_1 / B \Gamma_2 \setminus C \Gamma_3 \), where \( \Gamma_1, \Gamma_2, \Gamma_3 \) are elements of \( S(G) \) and \( B, C \) are 3-balls with centers \( v_B, v_C \). Note that isolated vertices and free edges of a graph \( G \) do not affect equivalence relations on \( S(G) \) such as ambient isotopy, isotopy and cobordism. Thus, we may always assume without loss of generality that \( G \) contains no isolated vertices and free edges, that is, the degree of each vertex of \( G \) is at least two. If necessary adding extra vertices to \( G \), we may also assume that any cycle in \( G \) contains at least two vertices of \( G \). In particular, \( G \) satisfies the condition \((**) \) in [2, §2].

It is easily seen that the following proposition implies Theorem 1.

**Proposition 2.** With the notation as above, there exist embeddings \( \Gamma'_2, \Gamma'_3 \in S(G) \) and a sequence \( \Gamma_1 \setminus C. \Gamma'_2 / B. \Gamma'_3 / C''. \Gamma'_3 \), where \( B', C', C'' \) are 3-balls with centers \( v_B, v_C, v_C \) respectively.

Note that, in the case of \( v_B = v_C \), the double blowing-ups \( \Gamma'_2 / B. \Gamma'_3 / C''. \Gamma'_3 \) in Proposition 2 are replaced by a single blowing up \( \Gamma'_2 / B'. \Gamma'_3 / C'' \). From now on, for any proper subset \( X \) of \( S^3 \), we set \( X^0 = X - X \cap \Gamma_2(G) \). By [2, Lemma 3], we may assume that each component of \( \partial B^o \cap \partial C^o \) is a loop non-contractible both in \( \partial B^o \) and \( \partial C^o \) (even in the case where \( \partial B^o, \partial C^o \) are compressible in \( S^3 - \Gamma_2(G) \)). For each component \( R \) of \( B \cap \partial C \), let \( W_R \) denote the closure in \( B \) of a component of \( B - R \) disjoint from \( v_B \). A closure \( W_R \) is said to be innermost among these closures if \( \int W_R \cap \partial C = \emptyset \). According to [2, Lemma 2], if \( F_R = W_R \cap \partial B \) is connected for an innermost closure \( W_R \), then we have a sequence \( \Gamma_1 / B \Gamma'_2 \setminus C'. \Gamma'_3 / C'. \Gamma'_3 \) with \( |\partial B \cap \partial C'| < |\partial B \cap \partial C| \), where \( |Y| \) denotes the number of connected components of a compact set \( Y \). In fact, when \( v_B \neq v_C \), we showed in [2, Lemma 4] that, for any component \( R \) of \( B \cap \partial C \), \( F_R \) is connected (even if \( W_R \) is not innermost), and hence Proposition 2 was proved inductively. So, it suffices to consider the case of \( v_B = v_C = v \). Remark that, in this case, the result corresponding to [2, Lemma 4] does not hold in general. We will complete the proof of Proposition 2 by showing that either \( F_R \) is connected for at least one innermost \( W_R \) or each component of \( S^3 - \int (B \cup C) \) is a 3-ball.

For unoriented loops \( l, l' \) in \( S^3 \) with \( l \cap l' = \emptyset \), \( \text{lk}(l, l') \) is the absolute linking number of \( l \) and \( l' \) in \( S^3 \). For a loop \( l \) in the punctured surface \( \partial (B \cup C)^o \), \( l^+ \) represents a loop in \( S^3 - \Gamma_2(G) \cup B \cup C \) isotopic to \( l \) in \( S^3 - \Gamma_2(G) \cup \int (B \cup C) \). Intuitively, \( l^+ \) is obtained by pushing \( l \) outside of \( B \cup C \) slightly.

**Lemma 1.** With the notation and assumptions as above, suppose that \( X \) is a connected component of \( S^3 - \int (B \cup C) \). Then, one of the following \((i) \) and \((ii) \) holds.

(i) \( X \) is homeomorphic to a 3-ball.
(ii) \textbf{There exists a simple proper arc }\alpha \textbf{ in } Q = X \cap \partial C \textbf{ connecting distinct components } l, l' \textbf{ of } \partial Q \textbf{ such that } lk(\alpha \cup \beta_1 \cup \beta_2, l^+) = 1, \textbf{ where } \beta_1, \beta_2 \textbf{ are simple arcs in } B \textbf{ connecting the end points of } \alpha \textbf{ with } v \textbf{ and satisfying } \beta_1 \cap \beta_2 = \{v\}.\]

Proof. We assume that the conclusion (ii) does not hold and will show then that the conclusion (i) holds. Let \( Y_1, \ldots, Y_m \) be the components of \( Q \). For each \( Y_i \), there exist mutually disjoint disks \( D_1, \ldots, D_i \) in \( \partial B \) such that \( \partial D(i) \subset \partial Y_i \) and \( X \cap \partial B \subset D(i) \), where \( D(i) \) is the union \( D_1 \cup \cdots \cup D_i \). When \( \partial Y_i \cap \text{int}D_j(i) \neq \emptyset \), consider a component \( l \) of \( \partial Y_i \cap \text{int}D_j(i) \) which is not disconnected from \( \partial D_j(i) \) by any other components of \( \partial Y_i \cap \text{int}D_j(i) \). Then the triad of \( l, l' = \partial D(i) \) and any simple arc \( \alpha \) in \( Y_i \) connecting \( \partial Y_i \cup \alpha \) with \( l \) would satisfy (ii), a contradiction. Thus, we have \( \partial Y_i \cap \text{int}D(i) = \emptyset \). Then, the union \( S_i = Y_i \cup D(i) \) is a 2-sphere bounding a 3-ball \( B_i \) in \( S^3 - \text{int}B \) with \( B_i \subset X \). Our \( X \) coincides with the intersection \( B_1 \cap \cdots \cap B_m \).

We set \( W_i = S^3 - \text{int}(B_1 \cup B_i) \) and \( Z_i = \partial W_i - \text{int}Y_i \). Note that \( Z_i \) is a connected surface in \( \partial B \) homeomorphic to \( Y_i \). For any distinct \( i, j \in \{1, \ldots, m\} \), since \( Y_i \subset X \) is disjoint from \( \text{int}W_j \), \( W_i \) is either contained in \( W_j \) or disjoint from \( W_j \). If \( W_i \subset W_j \), then \( X \) would meet \( \text{int}W_j \) non-trivially, a contradiction. It follows that \( W_i \cap W_j = \emptyset \).

Thus, the boundary \( \partial X = (\partial B - Z_1 \cup \cdots \cup Z_m) \cup (Y_1 \cup \cdots \cup Y_m) \) is homeomorphic to the 2-sphere \( \partial B = (\partial X - Y_1 \cup \cdots \cup Y_m) \cup (Z_1 \cup \cdots \cup Z_m) \). This shows that \( X \) is homeomorphic to a 3-ball. \( \square \)

Proof of Proposition 2 (and Theorem 1). As was seen above, we may assume that \( v_B = v_C = v \).

First, we consider the case where all components \( X_1, \ldots, X_m \) of \( N_0 = S^3 - \text{int}(B \cup C) \) are 3-balls. Note that \( N_0 \cap \Gamma_i(G) = N_0 \cap \Gamma_2(G) = N_0 \cap \Gamma_3(G) \) and the graph \( (B \cup C) \cap \Gamma_i(G) \) is a star centered at \( v \) for \( i = 1, 2, 3 \). Take mutually disjoint, simple proper arcs \( \alpha_1, \ldots, \alpha_m-1 \) in \( B \cup C \) such that each \( \alpha_j \) connects \( \partial X_j \) with \( \partial X_{j+1} \) and

\[
(\alpha_1 \cup \cdots \cup \alpha_m-1) \cap (\Gamma_1(G) \cup \Gamma_2(G) \cup \Gamma_3(G)) = \emptyset.
\]

The union \( N_1 \) of a small regular neighborhood of \( \alpha_1 \cup \cdots \cup \alpha_m-1 \) in \( B \cup C \) and \( N_0 \) is a 3-ball with \( N_1 \cap \Gamma_i(G) = N_1 \cap \Gamma_2(G) = N_1 \cap \Gamma_3(G) \) and, for the 3-ball \( \hat{B} = S^3 - \text{int}N_1 \) and \( i = 1, 2, 3 \), \((\hat{B}, \hat{B} \cap \Gamma_i(G)) = (\hat{B}, (B \cup C) \cap \Gamma_i(G)) \) is a ball-star pair. This shows that there exists a (common) embedding \( \Gamma_2' \subset S(G) \) admitting blowing-downs \( \Gamma_1 \setminus \hat{B} \Gamma_2' \) and \( \Gamma_3 \setminus \hat{B} \Gamma_2' \). Thus, we have the pair of blowing-down and up \( \Gamma_1 \setminus \hat{B} \Gamma_2' \Gamma_3 \setminus \hat{B} \Gamma_2' \Gamma_3 \) from \( \Gamma_1 \) to \( \Gamma_3 \).

Next, we suppose that \( S^3 - \text{int}(B \cup C) \) contains a component \( X \) not homeomorphic to a 3-ball. By Lemma 1, there exists a simple proper arc \( \alpha \) in \( Q = X \cap \partial C \), simple arcs \( \beta_1, \beta_2 \) in \( B \) as in Lemma 1 (ii) and a component \( l \) of \( \partial Q \) with \( lk(\alpha \cup \beta_1 \cup \beta_2, l^+) = 1 \). Consider the 2-fold branched covering \( p : S^3 \rightarrow S^3 \) branched over \( l^+ \), and set \( p^{-1}(v) = \{\tilde{v}_1, \tilde{v}_2\} \). The preimage \( p^{-1}(B) \) (resp. \( p^{-1}(C) \))
is a union of mutually disjoint 3-balls $\tilde{B}_1$, $\tilde{B}_2$ with $\tilde{v}_1 \in \tilde{B}_1$, $\tilde{v}_2 \in \tilde{B}_2$ (resp. $\tilde{C}_1$, $\tilde{C}_2$ with $\tilde{v}_1 \in \tilde{C}_1$, $\tilde{v}_2 \in \tilde{C}_2$). Let $\tilde{\alpha}$ be the lift of $\alpha$ contained in $\partial \tilde{C}_2$. Since $\tilde{\alpha}$ connects $\tilde{B}_1$ with $\tilde{B}_2$, $\tilde{C}_2$ meets both $\tilde{B}_1$ and $\tilde{B}_2$. Note that $\tilde{\Gamma} = p^{-1}(\Gamma_2(G))$ is a spatial graph, and $(\tilde{B}_j, \tilde{B}_j \cap \tilde{\Gamma})$, $(\tilde{C}_j, \tilde{C}_j \cap \tilde{\Gamma})$ are ball-star pairs for $j = 1, 2$. Since $\tilde{v}_1 \neq \tilde{v}_2$, Lemma 4 in [2] implies that, for any component $\tilde{R}_2$ of $\tilde{B}_1 \cap \partial \tilde{C}_2$, $\tilde{R}_2 = \tilde{W}_2 \cap \partial \tilde{B}_1$ is connected and hence homeomorphic to $\tilde{R}_2$, where $\tilde{W}_2$ is the closure in $\tilde{B}_1$ of a component of $\tilde{B}_1 - \tilde{R}_2$ disjoint from $\tilde{v}_1$. When $\tilde{W}_2 \cap \partial \tilde{C}_1 = \emptyset$ for a closure $\tilde{W}_2$ with $\text{int} \tilde{W}_2 \cap \partial \tilde{C}_2 = \emptyset$, we set $W = p(\tilde{W}_2)$. Otherwise, consider the closure $\tilde{W}_1$ in $\tilde{W}_2$ of a component $\tilde{W}_2 - \partial \tilde{C}_1$ with $\text{int} \tilde{W}_1 \cap \partial \tilde{C}_1 = \emptyset$ and $\tilde{W}_1 \cap \tilde{R}_2 = \emptyset$. Note that $\tilde{R}_1 = \tilde{W}_1 \cap \partial \tilde{C}_1$ is a connected surface, see Fig. 1. If $\tilde{C}_2 \cap \text{int} \tilde{W}_2 = \emptyset$, then $\tilde{C}_2$ would contain $\tilde{W}_2 \supset \tilde{R}_1$, and hence $\tilde{C}_1 \cap \tilde{C}_2 = \emptyset$, a contradiction. This implies that $\tilde{C}_2 = \tilde{C}_2 \cup \tilde{W}_2$ is a 3-ball. If $\tilde{W}_2 \cap \tilde{\Gamma} \neq \emptyset$, then any edge $e$ of the star $\tilde{\Gamma} \cap \tilde{B}_1$ connecting a point of $\tilde{F}_2 \cap \tilde{\Gamma}$ with $\tilde{v}_1$ would meet $\tilde{R}_2$, so $e$ would tend toward $\tilde{v}_2$. This contradicts that $\tilde{v}_1 \neq \tilde{v}_2$. It follows that $(\tilde{C}_2', \tilde{C}_2' \cap \tilde{\Gamma}) = (\tilde{C}_2', \tilde{C}_2' \cap \tilde{\Gamma})$ is a ball-star pair centered at $\tilde{v}_2$. By applying Lemma 4 in [2] to the pair of the 3-balls $\tilde{C}_1$, $\tilde{C}_2'$ with distinct centers, one can show that $\tilde{F}_1 = \tilde{W}_1 \cap \partial \tilde{C}_1$ is connected. Then, we set $W = p(\tilde{W}_1)$. In either case, $W$ is a compact 3-manifold in $B$ bounded by the union of the connected surfaces $R = W \cap \partial C$, $F = W \cap \partial B$ and satisfying $\text{int} W \cap \partial C = \emptyset$. Then, by [2, Lemma 2], we have a sequence $\Gamma_1 \not\sim_B \Gamma_2 \not\sim_{C^{(1)}} \Gamma_3$ with $|\partial B \cap \partial C^{(1)}| < |\partial B \cap \partial C|$. Repeating the same process finitely many times, we have a sequence $\Gamma_1 \not\sim_{B'} \Gamma_2 \not\sim_{C'} \Gamma_3$ such that each component of
\(S^3 - \text{int}(B' \cup C')\) is a 3-ball. As was seen in the previous case, one can then exchange the blowing-up and down of \(\Gamma_1 / B' \Gamma_2^{(r)} \backslash C' \Gamma_3^{(r)}\) and obtain our desired sequence.

3. Construction of isotopically reduced embeddings

In this section, we will prove that, if a graph \(G\) is not a generalized bouquet, then for any embedding \(\Gamma \in S(G)\), the cobordism class \([\Gamma]\) contains infinitely many isotopically reduced elements which are not ambient isotopic to each other.

Our proof here is based on arguments in Soma [3] and [4], where the author constructed simple links cobordant to given links in \(S^3\) and closed 3-manifolds by using certain simple tangles. Here, a (2-string) tangle \((B, t_1 \cup t_2)\) is a pair of a 3-ball \(B\) and a disjoint union \(t_1 \cup t_2\) of two simple proper arcs in \(B\). A tangle \((B, t_1 \cup t_2)\) is trivial if there exists a properly embedded disk in \(B\) containing \(t_1 \cup t_2\). A tangle \((B, t_1 \cup t_2)\) is simple if \(\partial B - \partial t_1 \cup \partial t_2\) is incompressible in \(B - t_1 \cup t_2\) and if \(B - t_1 \cup t_2\) contains no incompressible tori. We refer to [3, §2] for examples of simple tangles.

In particular, a clasp tangle \((B, t_1 \cup t_2)\) as in Fig. 2 is simple. Let \(A\) be a properly embedded annulus in the complement \(B - t_1 \cup t_2\) of a simple tangle such that \(\partial A\) bounds an annulus \(A'\) in \(\partial B - \partial t_1 \cup \partial t_2\). If \(A\) is incompressible in \(B - t_1 \cup t_2\), then any compressing disk \(\Delta\) for the torus \(T = A \cup A'\) is contained in the compact 3-manifold \(V\) in \(B - t_1 \cup t_2\) bounded by \(T\). Since \(V\) is a solid torus and each component of \(\partial A\) is contractible in \(B - \text{int}V\), \(A\) is parallel to \(A'\) in \(V \subset B - t_1 \cup t_2\). Here, we say that a compact surface \(F\) properly embedded in a 3-manifold \(X\) is parallel in \(X\) to a surface \(F'\) in \(\partial X\) if there exists an embedding \(h : F \times I \to X\) with \(h(F \times \{0\}) = F\) and \(h(F \times \{1\} \cup \partial F \times I) = F'\).

A compact, connected surface homeomorphic to a closed region in \(\mathbb{R}^2\) with \(n\) boundary components is called an \(n\)-ply connected disk. In particular, a doubly connected disk is an annulus.

**Lemma 2.** Let \((B, t_1 \cup t_2)\) be a clasp tangle, and let \(R\) be an incompressible, triply connected disk properly embedded in \(B - t_1 \cup t_2\). Suppose that there exist mutually disjoint disks \(D_1, D_2, D_3\) in \(\partial B\) satisfying \(\partial(D_1 \cup D_2 \cup D_3) = \partial R\), \(D_1 \cap t_1 \neq \emptyset\), \(D_2 \cap t_1 \neq \emptyset\) and \((D_1 \cup D_2) \cap t_2 = \emptyset\). Then, \(R\) is parallel in \(B - t_1 \cup t_2\) to a surface in \(\partial B - \partial t_1 \cup \partial t_2\).

**Proof.** By the assumptions as above, both \(D_1 \cap t_1\) and \(D_2 \cap t_1\) consist of single points. Since \(R\) is incompressible in \(B - t_1 \cup t_2\) and \((D_1 \cup D_2) \cap t_2 = \emptyset\), \(\partial t_2\) is contained in \(D_3\). The 2-sphere \(R \cup D_1 \cup D_2 \cup D_3\) bounds a 3-ball \(C\) in \(B\) containing \(t_1 \cup t_2\). Consider the closure \(W\) of \(B - C\) in \(B\). Note that \(F = W \cap \partial B\) is a triply connected disk in \(\partial B^o = \partial B - \partial t_1 \cup \partial t_2\) with \(\partial F = \partial R\). Let \(\Delta\) be an embedded disk in \(B\) as illustrated in Fig. 2 such that \(\partial \Delta \supset t_1\) and \(\Delta \cap t_2\) is two points in \(\text{int} \Delta\). It is easily seen that \(\Delta^o = \Delta - \Delta \cap (t_1 \cup t_2)\) is incompressible in \(B - t_1 \cup t_2\). We
may assume that $\Delta$ meets $R$ transversely, and each loop component of $\Delta \cap R$ is non-contractible both in $\Delta^\circ$ and $R$. Since the arc $\Delta \cap \partial B$ connects the end points of $t_1$, the union $J$ of all arc components of $\Delta \cap R$ are non-empty. Let $\Delta_1, \ldots, \Delta_n$ ($n \geq 2$) be the closures in $\Delta$ of all components of $\Delta \cap R$ that are non-empty. Let $\Delta_1, \ldots, \Delta_n$ ($n \geq 2$) be the closures in $\Delta$ of all components of $\Delta \cap R$ that are non-empty. Let $\Delta_1, \ldots, \Delta_n$ ($n \geq 2$) be the closures in $\Delta$ of all components of $\Delta \cap R$ that are non-empty. Let $\Delta_1, \ldots, \Delta_n$ ($n \geq 2$) be the closures in $\Delta$ of all components of $\Delta \cap R$ that are non-empty.

**CASE 1.** $\Delta_1 \cap t_2$ is empty.

In this case, $\text{int}\Delta_1 \cap R$ contains no loop components, so $\text{int}\Delta_1 \cap R = \emptyset$. If $\Delta_1 \subseteq C$, then for some $j \in \{1, 2, 3\}$, $D_j \cap \Delta_1$ is an arc separating $D_j$ into two disks $D_{j1}$ and $D_{j2}$ such that $D_{j1} \cap (t_1 \cup t_2) = \emptyset$. The union $\Delta_1 \cup D_{j1}$ is a disk in $C - t_1 \cup t_2$ with $\partial(\Delta_1 \cup D_{j1}) \subset R$. Since $R$ is incompressible in $C - t_1 \cup t_2$, $\Delta_1$ excises a 3-ball $C_1$ from $C - t_1 \cup t_2$. Deforming $\Delta$ in a small neighborhood of $C_1$ by an ambient isotopy of $B$ rel. $t_1 \cup t_2$, one can reduce the number $|\Delta \cap R|$. Thus, we may assume that $\Delta_1 \cap \text{int}C = \emptyset$. If $\gamma$ is *inessential* in $R$, that is, $\gamma$ excises a disk from $R$, then one can reduce $|\Delta \cap R|$ as above by invoking the incompressibility of $R$ in $W$. In the case where $\gamma$ is essential in $R$, consider the surface $R'$ obtained by surgery on $R$ along $\Delta_1$. The surface $R'$ consists of at most two annuli which are incompressible in $W$. Since the boundary of each component $A'$ of $R'$ bounds an annulus in $F$, $A'$ is parallel to the annulus in $W$. This implies that $R$ is parallel in $B - t_1 \cup t_2$ to $F$.

**CASE 2.** $\Delta_1 \cap t_2$ consists of a single point.

If $\text{int}\Delta_1 \cap R \neq \emptyset$, then there would exist a disk $\Delta_0$ in $\Delta_1$ with $\partial \Delta_0 \subset \text{int}\Delta_1 \cap R$, $\text{int}\Delta_0 \cap R = \emptyset$ and such that $\text{int}\Delta_0 \cap t_2$ is a single point. Since $\Delta_0 \cap D_3 \subset \Delta_0 \cap \partial B = \emptyset$ and $\partial t_2 \subset D_3$, the algebraic intersection number of $\Delta_0$ with $t_2$ in the 3-ball $C$ would be zero, a contradiction. Thus, $\text{int}\Delta_1 \cap R$ is empty. A similar argument implies that...
Δ₁ \cap (D₁ \cup D₂) = \emptyset, and so Δ₁ \cap D₃ is an arc. Since Δ₁ \cap t₁ = \emptyset, γ is inessential in R, and hence Δ₁ excises a 3-ball C₀ from C such that t₀ = C₀ \cap t₂ = C₀ \cap (t₁ \cup t₂) is a proper arc in C. Since t₂ is unknotted in B, t₀ is also unknotted in C₀. Thus, there exists a disk E₀ in C₀ bounded by the union of t₀ and an arc u₀ in the disk Δ₁ \cup (C₀ \cap D₃) with ∂u₀ = ∂t₀ and such that u₀ \cap Δ₁ \cap D₃ is a single point.

The tangle (B, t₁ \cup t₂) admits an orientation reversing involution f which is the reflection with respect to the horizontal plane containing the barycenter of the 3-ball B in Fig. 2. By using an elementary cut-and-past argument, one can take E₀ so that E₀ \cap h(E₀) = \emptyset. Move t₂ in a small neighborhood of E₀ \cup h(E₀) in B by an ambient isotopy of B rel. t₁ so that t₂ \cap Δ = \emptyset. Then, for a regular neighborhood N of Δ in B – t₂, ∂N – int(N \cap ∂B) is a proper disk in B separating t₁ and t₂. This implies that ∂B° is compressible in B – t₁ \cup t₂, a contradiction. Thus, Case 2 cannot occur.

**Case 3.** Δ₁ \cap t₂ consists of two points.

Since Δᵢ \cap t₂ = \emptyset for i = 2, \ldots, n, Δᵢ \cap J consists of two arcs for i = 2, \ldots, n – 1 and Δₙ \cap J consists of a single arc. Moreover, Δₙ is a disk in C with Δₙ \cap ∂C = ∂Δₙ – intt₁. For the 3-ball C' obtained by cutting C open along Δₙ, ∂C' – intD₃ is a proper disk in B separating t₁ from t₂. This contradiction implies that Case 3 cannot occur.

Let Γ : G → \mathbb{R}^3 ⊂ S^3 be any embedding of a graph G other than a generalized bouquet. As in §2, G can be assumed to contain no isolated vertices and free edges. We denote by V = \{v₁, \ldots, vₙ\} the set of all vertices of G. Consider the projection p : \mathbb{R}^3 → \mathbb{R}^2(\subset \mathbb{R}^3) defined by p(x, y, z) = (x, y, 0). Slightly deforming Γ by an ambient isotopy, we may assume that p \circ Γ is a regular projection, that is, (i) the restriction p \circ Γ|V is an embedding, (ii) p(Γ(V)) \cap p(Γ(G – V)) = \emptyset, and (iii) each singular value of p \circ Γ is a transversal double point. We regard that the image Γ = p(Γ(G)) is a plane graph, where each double point of p \circ Γ|G is considered to be a vertex of Γ of degree four. Let D₁, \ldots, Dₙ be mutually disjoint disks in \mathbb{R}^2 such that Dᵢ \cap \tilde{Γ} is a star centered at \tilde{v}_i = p(Γ(vᵢ)) for i = 1, \ldots, n, and let D = D₁ \cup \cdots \cup Dₙ. Since G is not a generalized bouquet, for each vᵢ ∈ V, G contains a cycle lᵢ disjoint from vᵢ. We note that lᵢ may be equal to lⱼ even if i ≠ j.

Let α₁, \ldots, αₙ be mutually disjoint arcs in \tilde{Γ} – D \cap \tilde{Γ} disjoint from the set of vertices of \tilde{Γ} and with αᵢ ⊂ p(Γ(lᵢ)). We set \tilde{α}_i = (p \circ Γ)^{-1}(αᵢ) and \tilde{A} = \tilde{α}_1 \cup \cdots \cup \tilde{α}_n. Consider simple arcs β₁, \ldots, βₙ in \mathbb{R}^2 – intD meeting each other and \tilde{Γ} transversely and such that each βᵢ connects a point in intαᵢ with a point xᵢ in ∂Dᵢ – ∂Dᵢ ∩ \tilde{Γ}. Let Γ₁ ∈ S(G) be an embedding ambient isotopic to Γ rel. G – \tilde{A} such that p \circ Γ₁(\tilde{α}_i) is an arc which tends toward ∂Dᵢ along βᵢ, and meets βᵢ at a point yᵢ near xᵢ, and then goes round a neighborhood of ∂Dᵢ until meeting yᵢ again, and finally returns to αᵢ along βᵢ as illustrated in Fig. 3. If necessary deforming Γ₁ by an ambient
isotopy, the plane graph $\hat{\Gamma}_1 = p \circ \Gamma_1(G)$ can be assumed to satisfy the following (2.1) and (2.2).

(2.1) $\hat{\Gamma}_1$ is connected.

(2.2) $\hat{\Gamma}_1$ contains no cut vertices.

Here, a cut vertex $v$ of a graph $H$ means a vertex disconnecting the component of $H$ containing $v$. Let $\{\hat{w}_1, \ldots, \hat{w}_m\}$ be the set of the vertices of $\hat{\Gamma}_1$ corresponding to the double points of $p \circ \Gamma_1$, and let $C_1, \ldots, C_m$ be small regular neighborhoods of $\hat{w}_1, \ldots, \hat{w}_m$ in $S^3$. Note that each $(C_j, C_j \cap \hat{\Gamma}_1)$ is a standard ball-star pair of degree four centered at $\hat{w}_j$. Let $\hat{\Gamma}_1$ be the regular diagram for $\Gamma_1$ obtained by replacing each $C_j \cap \hat{\Gamma}_1$ by a suitable 2-string trivial tangle in $C_j$. We set $C = C_1 \cup \cdots \cup C_m$.

Let $\Gamma_2 : G \to S^3$ be an embedding such that $\Gamma_2(G) - \text{int}C = \hat{\Gamma}_1 - \text{int}C$, and for each $j = 1, \ldots, m$, $(C_j, C_j \cap \Gamma_2(G))$ is obtained by exchanging each trivial tangle $(C_j, C_j \cap \hat{\Gamma}_1)$ by a clasp tangle so that $\Gamma_2$ is cobordant to $\Gamma_1$ and hence to $\Gamma$.

Now, we will prove the following lemma which is crucial in the proof of Theorem 2.

**Lemma 3.** With the notation as above, any ball-star pair $(B, B \cap \Gamma_2(G))$ in $(S^3, \Gamma_2(G))$ is standard. In particular, $\Gamma_2$ is isotopically reduced.

**Proof.** The argument quite similar to that in Assertion 1 of [3, Theorem 3] implies that $\Gamma_2(G)$ is "prime", that is, any 2-sphere in $S^3$ meeting $\Gamma_2(G)$ transversely in two points bounds a 3-ball $B_0$ in $S^3$ such that $B_0 \cap \Gamma_2(G)$ is an unknotted arc in $B_0$. In particular, any ball-arc pair in $(S^3, \Gamma_2(G))$ is standard. Thus, we may assume that $B$ contains a vertex of $\Gamma_2(G)$, say $\hat{v}_1$. As in §2, for a proper subset $X$
of \( S^3 \), we set \( X^o = X - X \cap \Gamma_2(G) \). By (2.1), \( S^3 - C \cup \Gamma_2(G) = S^3 - C \cup \tilde{\Gamma}_1 \) is irreducible. Since a clasp tangle is simple, \( \partial C_j^o \) is incompressible in \( C_j^o \). By (2.2), each \( \partial C_j^o \) is also incompressible in \( S^3 - \text{int} C \cup \Gamma_2(G) \). This shows that \( \partial C_j^o \) is incompressible in \( S^3 - \Gamma_2(G) \) and \( S^3 - \Gamma_2(G) \) is irreducible. Set \( B = \hat{B} \) if \( \partial B^o \) is incompressible in \( S^3 - \Gamma_2(G) \). If \( \partial B^o \) is compressible in \( S^3 - \Gamma_2(G) \), then we consider mutually disjoint, compressing disks \( \Delta_1, \ldots, \Delta_r \) for \( \partial B^o \) in \( S^3 - \text{int} B \cup \Gamma_2(G) \) and 3-balls \( B_1, \ldots, B_r \) in \( S^3 \) with \( \partial B_i \subset \Delta_i \cup \partial B \) and \( B_i \cap \Gamma_2(l_1) = \emptyset \). Then, the union \( \hat{B} = B \cup B_1 \cup \cdots \cup B_r \) is a 3-ball disjoint from \( \Gamma_2(l_1) \) and such that \( \partial \hat{B}^o \) is incompressible in \( S^3 - \Gamma_2(G) \). Note that \( \partial \hat{B} \cap \Gamma_2(G) \subset \partial B \cap \Gamma_2(G) \), and \( \partial \hat{B} \cap \Gamma_2(G) = \partial B \cap \Gamma_2(G) \) if and only if \( \partial B^o \) is incompressible in \( S^3 - \Gamma_2(G) \). Since \( S^3 - \Gamma_2(G) \) is irreducible, \( \partial \hat{B} \cap \Gamma_2(G) \) is non-empty. If necessary deforming \( \partial B \) by an ambient isotopy of \( (S^3, \Gamma_2(G)) \), one can assume that \( \partial \hat{B} \) meets \( \partial C \) transversely and each component of \( \partial \hat{B} \cap \partial C \) is non-contractible both in \( \partial \hat{B}^o \) and \( \partial C^o \). Renumber \( \tilde{w}_j \)'s so that the subset \( \{\tilde{w}_1, \ldots, \tilde{w}_k\} \) of \( \{\tilde{w}_1, \ldots, \tilde{w}_m\} \) consists of the double points of \( \tilde{\Gamma}_1 \) surrounding \( \tilde{v}_1 \), and \( \tilde{w}_k \) corresponds to the double point \( y_1 \) of \( p \circ \Gamma_1(\tilde{\alpha}_1) \). Let \( \varepsilon_j \) \( (j = 1, \ldots, k - 1) \) be the edge of \( \Gamma_2(G) \) meeting both \( \tilde{v}_1 \) and \( C_j \), see Fig. 4. Since \( C_j \) meets \( \Gamma_2(l_1) \) non-trivially for any \( j = 1, \ldots, k \), \( C_j \) is not contained in \( \hat{B} \). If there existed a disk \( \Delta \) in \( \partial C_j \) with \( \partial \Delta \subset \partial \hat{B} \cap \partial C_j \), \( \text{int} \Delta \cap \partial \hat{B} = \emptyset \) and such that \( \Delta \cap \Gamma_2(l_1) \) is a single point, then \( \Delta \) would be a non-separating proper disk in the 3-ball \( S^3 - \text{int} \hat{B} \), a contradiction. Thus, in the case of \( \partial \hat{B} \cap \partial C_j \neq \emptyset \), the closure \( F \) in \( \partial C_j \) of any connected component of \( \partial C_j - \partial \hat{B} \cap \partial C_j \) is either a disk with \( 1 \leq \#(F \cap \Gamma_2(G)) \leq 3 \), or an annulus with \( 0 \leq \#(F \cap \Gamma_2(G)) \leq 2 \), or a triply connected disk with \( F \cap \Gamma_2(G) = \emptyset \), where \( \#(X) \) denotes the number of elements of a finite set \( X \). If \( F \) is either a disk with \( \#(F \cap \Gamma_2(G)) = 3 \) or an annulus with \( \#(F \cap \Gamma_2(G)) = 2 \), then \( F \cap \Gamma_2(l_2) \neq \emptyset \), and hence \( F \) is not contained in \( \hat{B} \).
We set \( C(k) = C_1 \cup \cdots \cup C_k \subset C \). If \( \partial C(k) \cap \hat{B} \) contains a disk component \( F \) with \( \#(F \cap \Gamma_2(G)) = 1 \), then \( \partial F \) bounds a disk \( F' \) in \( \partial \hat{B} \) with \( \#(F' \cap \Gamma_2(G)) = 1 \). Since \( \Gamma_2(G) \) is prime, the 2-sphere \( F \cup F' \) bounds a 3-ball \( B' \) in \( \hat{B} \) such that \( B' \cap \Gamma_2(G) \) is an unknotted arc in \( B' \). This enables us to reduce the number \( |\partial C(k) \cap \partial \hat{B}| \) by deforming \( \partial C(k) \) in a small neighborhood of \( B' \). Similarly, if \( \partial C(k) \cap \hat{B} \) contains an annulus component \( F \) with \( F \cap \Gamma_2(G) = 0 \), then one can reduce the number \( |\partial C(k) \cap \partial \hat{B}| \), for example see Assertion 2 in the proof of [3, Theorem 3]. Thus, we may assume that, for each \( C_j \ (j = 1, \ldots, k) \) with \( \partial \hat{B} \cap \partial C_j \neq \emptyset \), \( F_j = \partial C_j \cap \hat{B} \) is a connected surface which is either a disk with \( \#(F_j \cap \Gamma_2(G)) = 2 \), or an annulus with \( \#(F_j \cap \Gamma_2(G)) = 1 \), or a triply connected disk with \( F_j \cap \Gamma_2(G) = 0 \). One can reduce the former two cases to the latter case, by pushing a small neighborhood of \( F_j \cap \Gamma_2(G) \) toward the outside of \( \hat{B} \) along the edges of \( \Gamma_2(G) \) meeting \( F_j \). So, it suffices to consider the case where \( F_j \) is a triply connected disk disjoint from \( \Gamma_2(G) \).

Let \( W_j \) be the closure in \( \hat{B} \) of a component of \( \hat{B} - F_j \) disjoint from \( \hat{v}_1 \). It is easy to see that \( R_j = \partial W_j \cap \partial \hat{B} \) is also a triply connected disk. We assume that \( W_1 \) is innermost among all \( W_j \)'s, that is, \( \text{int} W_1 \cap \partial C(k) = \emptyset \). If \( W_1 \) were not contained in \( C_1 \), then \( C_1 \) would contain \( \hat{v}_1 \), a contradiction. It follows that \( W_1 \) is contained in \( C_1 \), and hence Lemma 2 shows that \( R_1 \) is parallel to \( F_1 \) in \( C_1 \). This implies that one can reduce the number \( |\partial \hat{B} \cap \partial C(k)| \), and finally get the situation of \( \hat{B} \cap C(k) = \emptyset \).

Since \( \partial B \cap \Gamma_2(G) \supset \partial \hat{B} \cap \Gamma_2(G) \neq \emptyset \), at least one of \( \varepsilon_1, \ldots, \varepsilon_{k-1} \), say \( \varepsilon_1 \), meets \( \partial \hat{B} \) non-trivially. Let \( \alpha \) be the subarc of \( \varepsilon_1 \) connecting \( \hat{v}_1 \) with \( \varepsilon_1 \cap \partial \hat{B} \). If \( \alpha \cap C_1 \neq \emptyset \),

\[ \begin{align*}
\Gamma_2(l_t) \cup C(k) \\
\Delta_0 \\
\hat{v}_1 \\
\alpha \\
\end{align*} \]

Fig. 5.
then $C_1$ would meet $\tilde{B} \supset \alpha$ non-trivially. This contradiction implies $\alpha \cap C_1 = \emptyset$. Since $\partial \tilde{B} \cap (\Gamma_2(l_1) \cup C(k)) = \emptyset$ and since $\partial \tilde{B}$ meets any $\varepsilon_j$ $(j = 1, \ldots, k - 1)$ at most one point, the component $l_0$ of $\partial \tilde{B} \cap (\mathbb{R}^2 - \text{int} C \cap \mathbb{R}^2)$ containing $\varepsilon_1 \cap \partial \tilde{B}$ is a loop in $\mathbb{R}^2 - C \cap \mathbb{R}^2$ bounding a disk $\Delta_0$ such that $\Delta_0 \cap \Gamma_2(G)$ is a star of degree $k - 1$ centered at $\tilde{v}_1$, see Fig. 5. Since $\#(\partial \tilde{B} \cap \Gamma_2(G))$ is equal to the degree of $\tilde{v}_1$ in $\Gamma_2(G)$, we have $B = \tilde{B}$ or equivalently that $\partial B^\circ$ is incompressible in $S^3 - \Gamma_2(G)$. Since each component of $\partial B - l_0$ is an open disk disjoint from $\Gamma_2(G)$, one can deform $\partial B$ by an ambient isotopy of $(S^3, \Gamma_2(G))$ rel. $l_0$ so that $\partial B \cap (\mathbb{R}^2 \cup C) = l_0$. In particular, $(B, B \cap \Gamma_2(G))$ is a standard ball-star pair. This shows that $\Gamma_2$ is isotopically reduced.

Proof of Theorem 2. For any positive integer $m$, choose the regular projection $\hat{\Gamma}_1 = p(\Gamma_1(G))$ as above so that $\hat{\Gamma}_1$ has at least $m$ double points. Then, for the isotopically reduced embedding $\Gamma_2 \in [\Gamma]_{\text{cobor}}$ given in Lemma 3, the complement $S^3 - \Gamma_2(G)$ contains at least $m$ mutually disjoint and non-parallel, incompressible, four-punctured 2-spheres. On the other hand, by Haken's Finiteness Theorem [1], there exists a positive integer $n(\Gamma_2)$ depending only on the ambient isotopy type of $\Gamma_2$ so that the number of such four-punctured 2-spheres in $S^3 - \Gamma_2(G)$ is not greater than $n(\Gamma_2)$. This observation implies that one can construct infinitely many isotopically reduced elements of $[\Gamma]_{\text{cobor}}$ which are not ambient isotopic to each other.

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