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# PRIMARY DECOMPOSITION OF ELEMENTS IN COMPACTLY GENERATED INTEGRAL MULTIPLICATIVE LATTICES

Dedicated to Professor Keizo Asano on his 60th birthday

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## 1. Introduction

A complete lattice $L$ is said to be compactly generated, when it has a subset $\Sigma$ which satisfies that (1) if $x \leq \sup N$ for an element $x$ of $\Sigma$ and a subset $N$ of $\Sigma$, there exists a finite number of elements $x_1, \cdots, x_n$ of $N$ satisfying $x_1 \cup \cdots \cup x_n \geq x$, and (2) every element of $L$ is expressible as a join (supremum) of a subset of $\Sigma$. $\Sigma$ is called a compact generator system of $L$[1]. The purpose of the present paper is to investigate primary decompositions of elements in compactly generated integral multiplicative lattices[2].

Throughout this paper, we let $L$ be a compactly generated integral multiplicative lattice with a compact generator system $\Sigma$. In Section 2 we define a $\mu$-system as a suitable subset of $\Sigma$, which is somewhat different from the one introduced in [13]. By using the $\mu$-systems, we define radicals of elements in $L$ and consider meet decompositions of radicals by prime elements. In Section 3 right primary elements are defined by using radicals defined in Section 2. The result in this section is a uniqueness theorem of short decomposition for elements having right primary decompositions. Section 4 deals with right upper $M$-components of elements, where $M$ is a $\mu$-system. A right upper $M$-component is defined by using the concept of $M$-$\nu$-systems, which are also somewhat different from the one introduced in [13]. It will be shown in this section that the right upper $M$-component of an element has two different representations (Theorem 3). Section 5 is mainly concerned with minimal primes of elements and decompositions of upper isolated $p$-components of elements. The results in this section are obtained under two conditions. The

1) In [12, §9] $\Sigma$ is called an $aj$-system of $L$. It can be proved that a lattice is compactly generated in the sense of Dilworth and Crawley ([5], [6]), if and only if it has an $aj$-system, that is, it is compactly generated in our sense.

2) Cf. [3, CHAP. XIII].

one is the ascending chain condition for elements, and the other is the condition (N), which is concerned with weight and type of product-forms. Under some modified semi-modularity for $L$, it can be proved, in Section 6, that every element of $L$ has right primary decomposition if and only if $L$ has right weak Artin-Rees property. The proofs of the results obtained in this section are similar as in [7], [9] and [10]. But in order to make this paper self-contained we include proofs of the results. Section 7 lays two applications. The ideal theory in non-associative rings has been developed in [2] and [8]. The results obtained in the first half of this section are generalizations of the classical primary decompositions of ideals in commutative rings to ideals in (N)-rings (non-associative and non-commutative), and which are concerned with [2], [8], [10] and [15]. In [3] Birkhoff has pointed out that the lattice of normal subgroups of a group is a commutative integral residuated *cm*-lattice under the commutator-product and the set-inclusion. It is easy to see that the set of the normal subgroups with single generators is a compact generator system of the lattice. In the latter half of this section, primary decompositions of normal subgroups of (N)-groups are obtained as an application of the results in the preceding sections, where (N)-groups are regarded as a generalization of nilpotent groups. Recently primary decomposition theory has been studied in various algebraic systems ([1], [17], [18], etc.). In particular, the theory in groupoids is obtained, among others, in [1]. We shall note here that the results in Sections 2∼6 are applicable to subsystems of some sorts of groupoids, but which is not collected in this paper.

Elements of $L$ will be denoted, throughout this paper, by $a, b, c, \cdots$, and those of $\Sigma$, in particular, by $x, y, z, \cdots$ with or without suffices. The greatest element of $L$ will be denoted by $e$, which is not necessarily multiplicative unit of $L$ ([3, CHAP. XIII]). $ab \leq a$ and $ab \leq b$ are assumed for two elements $a, b$ of $L$. An element $a$ is said to be less than $b$ if $a \leq b$. The symbols $\vee$ and $\wedge$ will denote the set-theoretic union and the intersection respectively. By $\{a \in A \mid a \text{ has property } P\}$ we mean the set of all elements $a$ in $A$, each of which has property $P$.

## 2. Radicals of elements

Let $a, b$ be any two elements of $L$. The set of the elements $x$ of $\Sigma$ such that $xb \leq a$ is not void. The join (supremum) of such elements $x$ will be denoted by $a/b$, and called a (*right*) *quotient* of $a$ by $b$. It is easily verified that $a/b$ is not necessarily the join of the elements $c$ of $L$ such that $cb \leq a$. The quotient has the following properties: (1) $a \leq a/b$, (2) $(a/b)b \leq a$, (3) $b \leq a$ implies $a/b = e$, (4) $b \leq a$ implies $b/c \leq a/c$, (5) $c \leq b$ implies $a/b \leq a/c$, (6) $\inf_\lambda (a/b_\lambda) = a/(\sup_\lambda b_\lambda)$ and (7) $\inf_\lambda (a_\lambda/b) = (\inf_\lambda a_\lambda)/b$.

From now on, the symbols $P(a)$ and $\Sigma(a)$ will mean the sets $\{x\in\Sigma \mid a/x=a\}$ and $\{x\in\Sigma \mid x\leq a\}$, respectively. The complements of $P(a)$ and $\Sigma(a)$ in $\Sigma$ will be denoted by $P'(a)$ and $\Sigma'(a)$ respectively. It is then easy to see that $P(a)$ is contained in $\Sigma'(a)$ for every element $a\neq e$.

DEFINITION 1. A subset $M$ of $\Sigma$ is called a *$\mu$-system*, if there exists an $z$ of $M$ such that $z\leq xy$ for any two elements $x$, $y$ of $M$. The void set is to be element considered as a $\mu$-system.

An element $p$ of $L$ is said to be prime if whenever a product of two elements of $L$ is less than $p$, then at least one of the factors is less than $p$.

**Lemma 1.** *The following conditions are equivalent to one another.*

1) *$p$ is prime,*
2) *$xy\leq p$ $(x, y\in\Sigma)$ implies $x\leq p$ or $y\leq p$,*
3) *$\Sigma'(p)$ is a $\mu$-system.*

**Lemma 2.** *An element $p$ $(\neq e)$ is prime if and only if $P(p)=\Sigma'(p)$.*

Proofs. These two lemmas are immediate.

The following lemma is somewhat different from Lemma 1 in [14].

**Lemma 3.** *Let $a$ be an element of $L$, and let $M$ be a $\mu$-system which does not meet $\Sigma(a)$. Then there exists an element $p$ which is maximal in the set consisting of the elements $b$ such that $b\geq a$ and $\Sigma(b)$ does not meet $M$. $p$ is necessarily a prime element.*

Proof. Since $L$ is compactly generated, we can show, by Zorn's lemma, the existence of $p$ mentioned in the first part of the lemma. To prove the last part of the lemma, we suppose that $xy\leq p$, $x\not\leq p$ and $y\not\leq p$ for $x$, $y$ in $\Sigma$. Then there exist $x'$ and $y'$ in $M$ such that $x'\leq p\cup x$ and $y'\leq p\cup y$. Since there exists an element $u$ of $M$ such that $u\leq x'y'$, we obtain that $u\leq(p\cup x)(p\cup y)\leq p\cup xy=p$. This is a contradiction.

DEFINITION 2. Let $a$ be an element of $L$. A ***radical*** of $a$, denoted by rad$(a)$, is the join of all elements $x$ of $\Sigma$ having the property that every $\mu$-system which contains $x$ meets $\Sigma(a)$.

**Theorem 1.** *For every element $a$ of $L$, rad$(a)$ is the meet (infimum) of the primes $p_\lambda$ such that $p_\lambda\geq a$.*

Proof. First we shall show that rad$(a)\leq p_\lambda$ for every prime $p_\lambda$ such that $p_\lambda\geq a$. If we suppose that there exists $p$ such that $p\geq a$ and $p\not\geq$ rad$(a)$, we can take an element $x$ of $\Sigma$ such that $x\not\leq p$ and $x\leq$ rad$(a)$. Then there exists a finite number of elements $x_1, \cdots, x_n$ such that $x\leq x_1\cup\cdots\cup x_n$ and each $x_i$ has the property that every $\mu$-system which contains $x_i$ meets $\Sigma(a)$. Now, since there exists $x_j$ such that $x_j \not\leq p$, $\Sigma'(p)$ meets $\Sigma(a)$, which is a contradiction. We have therefore $\mathrm{rad}(a) \leq \inf_{\lambda} p_{\lambda}$. Next, let $x$ be any element of $\Sigma$ such that it is not less than rad$(a)$. Then there exists a $\mu$-system $M$ which does not meet $\Sigma(a)$ and contains $x$. Hence by using Lemma 3 we can take a prime element $p$ such that $p \geq a$ and $\Sigma(p)$ does not meet $M$. Then evidently $x$ is not less than $p$. Therefore $x$ is not less than $\inf_{\lambda} p_{\lambda}$. This completes the proof.

DEFINITION 3. Let $a$ be an element of $L$. A prime element $p$ of $L$ is said to be a *minimal prime belonging to* $a$, if (1) $p \geq a$ and (2) there exists no prime element $p'$ such that $a \leq p' < p$.

Let $p$ be a prime element such that $p \geq a$. Then it is proved that the set of the primes which are in the closed interval $[a, p]$ is inductive for downwards; that is, for every descending chain $C$ consisting of primes in $[a, p]$, inf $C$ is a prime in $[a, p]$. Hence Zorn's lemma assures the existence of a minimal prime belonging to $a$ which is less than $p$. Therefore we obtain the following

**Corollary.** *For every element $a$ of $L$, rad$(a)$ is the meet of the minimal primes belonging to $a$.*

For radicals we can prove the following

**Lemma 4.** (1°) $a \leq \mathrm{rad}(a)$, (2°) $a \leq b$ *implies* $\mathrm{rad}(a) \leq \mathrm{rad}(b)$, (3°) $\mathrm{rad}(\mathrm{rad}(a)) = \mathrm{rad}(a)$, (4°) $\mathrm{rad}(a \cap b) = \mathrm{rad}(a) \cap \mathrm{rad}(b) = \mathrm{rad}(ab)$.

## 3. Elements with right primary decompositions

DEFINITION 4. An element $q$ of $L$ is said to be (*right*) *primary*, if whenever $xy \leq q$ and $y \not\leq \mathrm{rad}(q)$ for $x$, $y$ in $\Sigma$, then $x \leq q$.

It is easy to see that $q$ is primary if and only if $ab \leq q$ and $b \not\leq \mathrm{rad}(q)$ imply $a \leq q$ for $a$, $b$ in $L$.

**Lemma 5.** *An element $q$ of $L$ is primary if and only if $\Sigma(\mathrm{rad}(q))$ contains $P'(q)$.*

Proof. This is immediate.

**Lemma 6.** *If $q_1, \cdots, q_n$ is a finite number of primary elements with the same radicals, say $\mathrm{rad}(q_i) = c\,(i=1, \cdots, n)$, then $q = q_1 \cap \cdots \cap q_n$ is primary and has the radical $c$.*

Proof. It is eivdent that $\mathrm{rad}(q) = c$ by the property (4°) in Lemma 4. In order to prove that $q$ is primary, we suppose that $xy \leq q$ and $y \not\leq \mathrm{rad}(q) = c$. Then $xy \leq q_i$ and $y \not\leq \mathrm{rad}(q_i)$; hence $x \leq q_i$ for $i=1, \cdots, n$. We obtain therefore $x \leq q_1 \cap \cdots \cap q_n = q$, completing the proof.

**Lemma 7.** *Let $a = q_1 \cap \cdots \cap q_n$ be an irredundant decomposition of $a$ into* *a finite number of primary elements $q_i$. If $rad(q_i)\neq rad(q_k)$ for some $i$ and $k$, $a$ is not primary.*

Proof. Put $t_j=q_1\cap\cdots\cap_{j-1}\cap q_{j+1}\cap\cdots\cap q_n$. Then $t_jq_j\leq a$. Since $t_j\nleq a$, we have $q_j\leq\mathrm{rad}(a)=\cap_{i=1}^n\mathrm{rad}(q_i)$. Hence $\mathrm{rad}(q_j)\leq\cap_{i=1}^n\mathrm{rad}(q_i)$ for $j=1,\cdots,n$. We obtain therefore $\mathrm{rad}(q_1)=\cdots=\mathrm{rad}(q_n)$, a contradiction.

DEFINITION 5. An irredudant decomposition

$$a=q_1\cap\cdots\cap q_n \tag{$*$}$$

of $a$ into primary elements $q_i$ is called a *short decomposition* of $a$, if none of the meets of two (or more) of $q_1,\cdots,q_n$ are primary.

**Theorem 2.** *If an element $a$ of $L$ can be decomposed as a meet of a finite number of primary elements, $a$ has a short decomposition. In any two short decompositions of $a$, the number of primary components as well as their radicals are necessarily the same.*

Proof. By Lemmas 6 and 7, $a$ has a short decomposition. Now, let $(*)$ and $a=q_1^*\cap\cdots\cap q_m^*$ be any two short decompositions of $a$. Take a maximal element in the *po*-set $\{\mathrm{rad}(q_1),\cdots,\mathrm{rad}(q_n),\ \mathrm{rad}(q_1^*),\cdots,\mathrm{rad}(q_m^*)\}$. We may suppose, without loss of generality, that the maximal element is $\mathrm{rad}(q_1)$. We now show that $\mathrm{rad}(q_1)$ occurs among $\mathrm{rad}(q_k^*)$, $k=1,\cdots,m$. Assume that $\mathrm{rad}(q_1)\neq\mathrm{rad}(q_k^*)$ for all $k$. Then we have that $q_1\nleq\mathrm{rad}(q_k^*)$ for all $k$. Because, if cotrary, we have a contradiction by using (2°), (3°) in Lemma 4 and the maximality of $\mathrm{rad}(q_1)$. On the other hand it is easily verified that $q_i/q_1=q_i$ for $i\neq 1$, and $q_k^*/q_1=q_k^*$ for $k=1,\cdots,m$. Hence we obtain that $a=q_1^*\cap\cdots\cap q_m^*=(q_1^*/q_1)\cap\cdots\cap(q_m^*/q_1)=(q_1/q_1)\cap(q_2/q_1)\cap\cdots\cap(q_n/q_1)=e\cap q_2\cap\cdots\cap q_n=q_2\cap\cdots\cap q_n$, which is a contradiction. We can now suppose, without loss of generality, that $\mathrm{rad}(q_1)=\mathrm{rad}(q_1^*)$, and make

$$(q_2/q_1)\cap\cdots\cap(q_n/q_1)=(q_1^*/q_1)\cap\cdots\cap(q_m^*/q_1) \tag{$\alpha$}$$

Then since $q_1\nleq\mathrm{rad}(q_i)$ for $i\neq 1$, and $q_1\nleq\mathrm{rad}(q_k^*)$ for $k\neq 1$, we have $q_i/q_1=q_i$ $(i\neq 1)$, and $q_k^*/q_1=q_k^*$ $(k\neq 1)$. Hence by $(\alpha)$ we have $q_2\cap\cdots\cap q_n=(q_1^*/q_1)\cap q_2^*\cap\cdots\cap q_m^*$, and have

$$(q_2/q_1^*)\cap\cdots\cap(q_n/q_1^*)=((q_1^*/q_1)/q_1^*)\cap(q_2^*/q_1^*)\cap\cdots\cap(q_m^*/q_1^*) \tag{$\beta$}$$

Since it is easily verified that $q_1^*\nleq\mathrm{rad}(q_i)$ for $i\neq 1$, and $q_1^*\nleq\mathrm{rad}(q_k^*)$ for $k\neq 1$, and since $q_1^*/q_1\geq q_1^*$, we have $q_i/q_1^*=q_i$ for $i\neq 1$; $q_k^*/q_1^*=q_k^*$ for $k\neq 1$ and $(q_1^*/q_1)/q_1^*=e$. Hence by $(\beta)$, we have

$$q_2\cap\cdots\cap q_n=q_2^*\cap\cdots\cap q_m^* \tag{$\gamma$}$$

Continuing an exactly similar argument for ($\gamma$), we atain after a finite number of steps that $m=n$, and $\mathrm{rad}(q_i)=\mathrm{rad}(q_i^*)$ for $i=1, \cdots, m=n$.

## 4. Isolated components of elements

DEFINITION 6. A subset $N$ of $\Sigma$ is called a (right) *$M$-$\nu$-system*, if (1) $N$ contains a $\mu$-system $M$ and (2) for every element $u$ of $N$ and every element $x$ of $M$ there exists an element $z$ of $N$ such that $z \leq ux$. If $M$ is void, the only $M$-$\nu$-system is, by definition, the void set itself.

Let $a$ be an element of $L$ and $M$ a $\mu$-system which does not meet $\Sigma(a)$. Then it is easily verified that the set-union $N^*$ of all $M$-$\nu$-systems, each of which does not meet $\Sigma(a)$, is the unique maximal $M$-$\nu$-system which does not meet $\Sigma(a)$. $N^*$ is uniquely determined by $a$ and $M$.

**Lemma 8.** *Let $a$ ($\neq e$) be an element of $L$, $M$ a $\mu$-system, and $N$ an $M$-$\nu$-system. If $\Sigma(a)$ does not meet $N$, there exists an element $q$ which is maximal in the set consisting of the elements $c$ such that $c \geq a$ and $\Sigma(c)$ does not meet $N$, and $P(q)$ contains $M$.*

Proof. Since $L$ is compactly generated, we can prove, by using Zorn's lemma, the existence of $q$ mentioned in the first part of the lemma. In order to prove the last part of the lemma, it is sufficient to show that $q/x > q$ implies $x \notin M$. Take an element $y$ of $\Sigma$ such that $y \leq q/x$ and $y \not\leq q$. Then, since $q < q \cup y$, we can take an element $v$ of $N$ such that $v \leq q \cup y$. Hence we have that $vx \leq (q \cup y)x = qx \cup yx \leq q$. If we suppose that $x \in M$, we can choose an element $z$ of $N$ such that $z \leq vx$. Hence $z \leq q$, that is, $\Sigma(q)$ meets $N$, which is a contradiction.

**Lemma 9.** *Suppose that $M \neq \phi$, $a \neq e$. Then $\Sigma'(a)$ forms an $M$-$\nu$-system if and only if $P(a)$ contains $M$.*

Proof. First we suppose that $\Sigma'(a)$ is an $M$-$\nu$-system. If $P(a)$ does not contain $M$, we can take an element $y$ such that $y \in M$ and $y \in P'(a)$. Since $a < a/y$, there exists an element $x$ of $\Sigma$ such that $x \leq a/y$ and $x \not\leq a$. Then we have that $z \leq a$ for every element $z$ of $\Sigma$ satisfying $z \leq xy$. On the other hand, since $\Sigma'(a)$ is an $M$-$\nu$-system, there exists an element $u$ of $\Sigma$ such that $u \leq xy$ and $u \not\leq a$. This is a contradiction. Next, we suppose that $a \neq e$ and $P(a)$ contains $M$. Then, since $P(a)$ is contained in $\Sigma'(a)$, we have $u \not\leq a = a/x$ for any $u$ of $\Sigma'(a)$ and $x$ of $M$. Hence $ux$ is not less than $a$. Therefore we can take an element $z$ of $\Sigma$ such that $z \leq ux$ and $z \not\leq a$. This shows that $\Sigma'(a)$ is an $M$-$\nu$-system.

**Lemma 10.** *Let $a$ ($\neq e$) be an element of $L$, let $M$ ($\neq \phi$) be a $\mu$-system such that it does not meet $\Sigma(a)$, let $N^*$ be the unique maximal $M$-$\nu$-system which does* *not meet* $\Sigma(a)$, *and let* $S(a, M)$ *be the set of the elements* $s$ *of* $L$ *having the properties that* $s \geq a$ *and* $P(s)$ *contains* $M$. *Then the join of the complement of* $N^*$ *in* $\Sigma$, *say* $sup(\Sigma \backslash N^*)$, *is a minimal element in* $S(a, M)$.

Proof. By Lemma 8, there exists a maximal element $q$ such that $q \geq a$ and $\Sigma(q)$ does not meet $N^*$, and $P(q)$ contains $M$. Since $\Sigma'(q)$ contains $P(q)$, it forms an $M$-$\nu$-system by the "if part" of Lemma 9. Obviously $\Sigma'(q)$ contains $N^*$. Hence $\Sigma'(q)=N^*$ by the maximality of $N^*$. Hence $\Sigma(q)=\Sigma \backslash N^*$. Therefore we have that $q=\sup \Sigma(q)=\sup(\Sigma \backslash N^*)$. It remains to prove that $P(c)$ does not contain $M$ for every $c$ such that $q > c \geq a$. If we suppose that $P(c)$ contains $M$, then $\Sigma'(c)$ is an $M$-$\nu$-system by Lemma 9, and meets $\Sigma(a)$. Hence we can find an element $u$ of $\Sigma$ such that $u$ is less than $a$ and not less than $c$, a contradiction.

**Lemma 11.** *Suppose that* $a$, $M$, $N^*$ *and* $S(a, M)$ *are the same as in Lemma* 10. *If* $q$ *is a minimal element in* $S(a, M)$ *and* $q \neq e$ *then* $q=sup(\Sigma \backslash N^*)$.

Proof. By Lemma 9, $\Sigma'(q)$ is an $M$-$\nu$-system, and it is evident that $\Sigma'(q)$ does not meet $\Sigma(a)$. By using Lemma 10, we have that $a \leq \sup(\Sigma^* \backslash N) \equiv q'$, and $q'$ is a minimal element such that $\Sigma(q')$ contains $M$. Then, since $\Sigma'(q)$ is contained in $N^*$, we have that $q=\sup \Sigma(q) \geq \sup(\Sigma \backslash N^*)=q'$. Therefore we obtain $q=q'$ by the minimality of $q$.

DEFINITION 7. Let $a$ be an element of $L$, and let $M$ be a $\mu$-system which does not meet $\Sigma(a)$. A (*right*) *upper* $M$-*component* of $a$ is the join of all elements $x$ of $\Sigma$ such that every $M$-$\nu$-system which contains $x$ meets $\Sigma(a)$. The upper $M$-component of $a$ will be denoted by $u(a, M)$.

**Theorem 3.** *Let* $a$ *be an element of* $L$, $M$ $(\neq \phi)$ *a* $\mu$-*system which does not meet* $\Sigma(a)$, *and* $N^*$ *the unique maximal* $M$-$\nu$-*system which does not meet* $\Sigma(a)$. *If* $S(a, M)$ *contains an element* $\neq e$, *then*

$$u(a, M) = inf(S(a, M)) = sup(\Sigma \backslash N^*) .$$

Proof. For simplisity, we put $q=\inf(S(a, M))$. First, we shall prove that $q=\sup(\Sigma \backslash N^*)$. Since $q/x=\inf_{s \in S(a,M)}\{s/x\}=\inf_{s \in S(a,M)}\{s\}=\inf(S(a, M))=q$ for every element $x$ of $M$, $P(q)$ contains $M$. Hence, by Lemma 11 we obtain $q=\sup(\Sigma \backslash N^*)$. Next we prove that $u(a, M)=q$. Evidently every element of $\Sigma(q)$ is not contained in $N^*$. Since $N^*$ is the unique maximal $M$-$\nu$-system which does not meet $\Sigma(a)$, every $M$-$\nu$-system which contains $x$ of $\Sigma(q)$ meets $\Sigma(a)$, that is, $x$ is less than $u(a, M)$. This implies that $q \leq u(a, M)$. Let $x$ be any element of $\{x \in \Sigma \mid x \in N(M\text{-}\nu\text{-system}) \Rightarrow N \cap \Sigma(a) \text{ is not void}\}$. Then evidently $x$ is not contained in $N^*$. Hence $x$ is less than $\sup(\Sigma \backslash N^*)=q$. Therefore we have $u(a, M) \leq q$, completing the proof.

**Corollary 1.** *Let $a$, $b$ be two elements of $L$ such that $a \geq b$, and let $M$ be a $\mu$-system which does not meet $\Sigma(a)$. Then $u(a, M) \geq u(b, M)$.*

Proof. Since $S(b, M)$ contains $S(a, M)$, this is immediate by Theorem 3.

**Corollary 2.** *Let $a$ be an element of $L$, and let $M_1$, $M_2$ be two $\mu$-systems such that $M_1$ contains $M_2$, and $M_1$ does not meet $\Sigma(a)$. Then $u(a, M_1) \geq u(a, M_2)$.*

Proof. Let $N_i^*$ be the maximal $M_i$-$\nu$-systems ($i=1, 2$), each of which does not meet $\Sigma(a)$. Then it is easy to see that $N_1^*$ is contained in $N_2^*$. Therefore we obtain that $u(a, M_1)=\sup(\Sigma \backslash N_1^*) \geq \sup(\Sigma \backslash N_2^*)=u(a, M_2)$.

DEFINITION 8. Let $p$ be a prime element such that $p \geq a$, and let $M=\Sigma'(p)$. $u(a, M)$ is called a *(right) upper isolated $p$-component* of $a$, and denoted by $u(a, p)$.

Suppose that $(*)$ in §3 is a decomposition of $a$ into primary elements $q_i$, and suppose that each $\Sigma'(\text{rad}(q_i))$ contains the unique maximal $\mu$-system $M_i$, $i=1, \cdots, n$. If $p$ is a prime element such that $M_1 \supseteq \Sigma'(p), \cdots, M_s \supseteq \Sigma'(p)$, $M_{s+1} \not\supseteq \Sigma'(p), \cdots, M_n \not\supseteq \Sigma'(p)$, then $u(a, p)=q_1 \cap \cdots \cap q_s$. Because, for $i=1, \cdots, s$, we have $u(a, p) \leq u(a, M_i)$ by Corollary 2 to Theorem 3. Now by Lemma 5, we have $P(q_i) \supseteq \Sigma'(\text{rad}\,(q_i))$. Hence $P(q_i)$ contains $M_i$. Hence we have, by Theorem 3, $u(a, M_i) \leq q_i$. Therefore $u(a, p) \leq q_1 \cap \cdots \cap q_s$. If $s=n$, we obtain $a \leq u(a, p) \leq q_1 \cap \cdots \cap q_n = a$, $u(a, p)=q_1 \cap \cdots \cap q_n$. If $s<n$, then, since $\Sigma'(p)$ is not contained in $\Sigma'(\text{rad}(q_j))$, we have $\text{rad}(q_j) \not\leq p$, and have $q_j \not\leq p$ for $j>s$ (by Theorem 1). Hence we can take elements $x_j$ such that $x_j \leq q_j$ and $x_j \in \Sigma'(p)$, $j=s+1, \cdots, n$. Since $\Sigma'(p)$ is a $\mu$-system, there exists a finite number of elements $y_j$ in $\Sigma'(p)$ such that $y_{s+1} \leq x_{s+1} \cdot x_{s+2}$, $y_{s+2} \leq y_{s+1} \cdot x_{s+3}, \cdots, y_{n-1} \leq y_{n-2} x_n$. Then we have $y_{n-1} \leq (\cdots((x_{s+1} \cdot x_{s+2}) x_{s+3}) \cdots) x_n \leq q_{s+1} \cap \cdots \cap q_n$. Let $z$ be an arbitrary element of $\Sigma$ such that $z \leq q_1 \cap \cdots \cap q_s$. Then we obtain $z y_{n-1} \leq (q_1 \cap \cdots \cap q_s) \cap (q_{s+1} \cap \cdots \cap q_n) = a$. Now, take any $\Sigma'(p)$-$\nu$-system $N$ containing $z$. Then there exists an element $v$ of $N$ such that $v \leq z y_{n-1}$. Since $v \leq a$, $N$ meets $\Sigma(a)$. Hence we have $z \leq u(a, p)$, $q_1 \cap \cdots \cap q_s \leq u(a, p)$. Therefore we obtain $q_1 \cap \cdots \cap q_s = u(a, p)$, completing the proof.

## 5. Ascending chain condition, Condition (N)

We shall assume, throughout this section, that the ascending chain condition (a. c. c.) holds for elements of $L$.

**Lemma 12.** *Every element of $L$ has a finite number of minimal primes belonging to it.*

Proof. Let $c$ be an element of $L$. If $c$ is prime, the lemma is trivially evident. Suppose now that $c$ is not prime. If there exists an infinite number of minimal primes $p_\lambda$ belonging to $c$, then, since $a_1 \not\leq c$, $b_1 \not\leq c$ and $a_1 b_1 \leq c$ for suitable elements $a_1$, $b_1$ of $L$, $a_1$ or $b_1$ is less than $p_\lambda$ for an infinite number of $p_\lambda$. Suppose that it is $a_1$, and put $c_1 = c \cup a_1$. Then evidently $c < c_1$ and $c_1 \leq p_\lambda$. $c_1$ is not prime. Hence $c_1$ has the same property as that of $c$. Continuing in this way, we obtain an ascending chain $c < c_1 < c_2 < \cdots$, which is a contradiction.

**Lemma 13.** *Let $p_1, \cdots, p_n$ be the minimal primes belonging to an element $c$ of $L$. Then there exists a product $\mathfrak{P}(p_{i_1}, \cdots, p_{i_m})$ which is less than $c$, where $\mathfrak{P}$ denote a product-form of some type of weight $m$, and $i_1, \cdots, i_m$ is some finite permutation of $1, \cdots, n$ with repetitions allowed.*

Proof. The lemma is evident if $c$ is prime. Suppose that $c$ is not prime. Then there exist two elements $x$ and $y$ of $\Sigma$ such that $x \not\leq c$, $y \not\leq c$ and $xy \leq c$. Put $a_1 = c \cup x$ and $b_1 = c \cup y$. Then $c < a_1$ and $c < b_1$. Now, let $p_1', \cdots, p_r'$ and $p_1'', \cdots, p_s''$ be the minimal primes belonging to $a_1$ and $b_1$ respectively. If we suppose that both $a_1$ and $b_1$ have the same property that we wish to prove of $c$, so that $\mathfrak{P}'(p'_{j_1}, \cdots, p'_{j_\lambda}) \leq a_1$ and $\mathfrak{P}''(p''_{k_1}, \cdots, p''_{k_\mu}) \leq b_1$, then, since $a_1 b_1 = (c \cup x) \cdot (c \cup y) \leq c \cup xy = c$, we have $\mathfrak{P}'(p'_{j_1}, \cdots, p'_{j_\lambda}) \cdot \mathfrak{P}''(p''_{k_1}), \cdots, p''_{k_\mu}) \leq c$. The interval $[c, p'_{j_\rho}]$ contains a minimal prime belonging to $c$, $\rho = 1, \cdots, \lambda$, and similarly for $[c, p''_{k_\sigma}]$, $\sigma = 1, \cdots, \mu$. Hence we have $\mathfrak{P}(p_{i_1}, \cdots, p_{i_m}) \leq c$, where $p_{i_1}, \cdots, p_{i_m}$ are minimal primes belonging to $c$, and $\mathfrak{P} = \mathfrak{P}' \cdot \mathfrak{P}''$. Hence, if the lemma is false for $c$, it is false for $a_1$ or for $b_1$. Continuing in this way, we atain a contradiction of the a. c. c.

DEFINITION 9. A product-form $\mathfrak{Q}(X_1, \cdots, X_m) = (\cdots((X_1 X_2) X_3) \cdots) X_m$ is called that it has a (*right*) *nested type* of weight $m$, where $X_i$ are indeterminates over $L$.

We now consider the following condition:

(N) *For every product-form $\mathfrak{P}$ of weight $n$, and for every elements $c_1, \cdots, c_n$ (repetitions allowed) of $L$, there exists a product-form $\mathfrak{Q}$ with nested type of weight $m$ such that*

$$\mathfrak{Q}(c_{i_1}, \cdots, c_{i_m}) \leq \mathfrak{P}(c_1, \cdots, c_n),$$

*where $i_1 \leq \cdots \leq i_m$.*

If $L$ is associative, the condition (N) is satisfied trivially. But there are important examples which are compactly generated non-associative multiplicative lattices satisfying the condition (N), which will be shown in the last section of this paper.

**Lemma 14.** *Suppose that the condition (N) holds for $L$. Then, for every element $a$ $(\neq e)$ of $L$, there exists a minimal prime $p$ of $a$ such that $a/p > a$.*

Proof. If $a$ is prime, the lemma is trivially evident. Suppose that $a$ is not prime. Then by Lemma 13 and the condition (N), there exist minimal primes $p_1, \cdots, p_m$ (not necessarily distinct) belonging to $a$ and a product form $\mathfrak{Q}$ of nested type such that $\mathfrak{Q}(p_1, \cdots, p_m) \leq a$. It is then easy to see that $m>1$, and that there exists $p_i$ such that $a/p_i > a$. This completes the proof.

Behrens showed in [2] that the radicals of primary ideals in non-associative rings are not necessarily prime. He gave two examples in that paper. Each of those examples is a commutative algebra with some finite base over a field. It is now easily verified that the ideals in each of the algebras is a compactly generated multiplicative lattice. Accordingly, those examples assure the existence of the lattices in which the radicals of primary elements are not prime. Now we have the following

**Theorem 4.** *Suppose that the condition (N) holds for L. Then the radical of every primary element is prime.*

Proof. Let $q$ be a primary element of $L$. If $q=e$, the theorem is evident. We suppose that $q<e$. Then by Lemma 14, we can find a minimal prime $p$ belonging to $q$ such that $x_z \cdot z \leq q$ and $x_z \not\leq q$ for an arbitrary element $z$ of $\Sigma(p)$ and a suitable element $x_z$ of $\Sigma$. Hence $z \leq \text{rad}(q)$, and hence $p \leq \text{rad}(q)$. On the other hand, since $\text{rad}(q) \leq p$ by Theorem 1, we obtain $\text{rad}(q)=p$, as desired.

REMARK. Under the condition (N) for $L$, we can show that if $\text{rad}(c)=p$ is prime for an element $c(\neq e)$, then $c/p>c$. Because, the assertion is trivially evident if $p=c$. Hence we can suppose that $p>c$. Then by Lemma 13 and the condition (N), there exists a nested product $\mathfrak{Q}$ of $p$ such that $\mathfrak{Q} \equiv \mathfrak{Q}'p \leq c$. If we suppose that $c/p=c$, then $\mathfrak{Q}' \equiv \mathfrak{Q}''p \leq c$. Continuing in this way, we obtain $p=c$, which is a contradiction.

**Theorem 5.** *Suppose that* (*) *(in §3) is a decomposition of a into primary elements $q_i$ with prime radicals $p_i$. Then the minimal primes belonging to a coincide with the minimal elements in the po-set $\{p_1, \cdots, p_n\}$.*

Proof. By Lemma 13, we have that $\mathfrak{P}^{(i)}(p_i) \equiv \mathfrak{P}^{(i)}(p_i, \cdots, p_i) \leq q_i$ for suitable product-forms $\mathfrak{P}^{(i)}$, $i=1, \cdots, n$. Hence, for any product-form of $n$-th weight, we obtain $\mathfrak{P}(\mathfrak{P}^{(1)}(p_1), \cdots, \mathfrak{P}^{(n)}(p_n)) \leq \mathfrak{P}(q_1, \cdots, q_n) \leq q_1 \cap \cdots \cap q_n = a$. This implies the existence of $p_i$ such that $p_i \leq p$ for any prime $p$ satisfying $p \geq a$. In particular, any minimal prime belonging to $a$ coincides with some $p_i$, and there is no $p_j$ such that $p_j < p_i$. Conversely, let $p_i$ be any minimal element in the *po*-set $\{p_i, \cdots, p_n\}$. If $p$ is a prime element contained in $[a, p_i]$, we can show, similarly as above, the existence of a prime element $p_k$ such that $p_k \leq p$. We obtain therefore $p_k \leq p_i$, $p_k = p_i$, completing the proof.

The following theorem have been established by the last part of §4.

**Theorem 6.** *Suppose that* (*) *is a decomposition of a into primary elements*

*$q_i$ with prime radicals $p_i$. If $p(\neq e)$ is a prime element such that $p_1 \leq p, \cdots, p_s \leq p$, $p_{s+1} \not\leq p, \cdots, p_n \not\leq p$, then*

$$u(a, p) = q_1 \cap \cdots \cap q_s .$$

**Theorem 7.** *Suppose that (*) is a short decomposition of a into primary elements $q_i$ with prime radicals $p_i$. If $p$ is any minimal prime element belonging to a then $u(a, p) = q_i$ for some $i$, and $u(a, p)$ is primary.*

Proof. By Theorem 5, we have $p = p_i$ for some $i$. Since there exists no $j$ such that $p_j \leq p$ $(j \neq i)$, we obtain $u(a, p) = q_i$ by Theorem 6.

REMARK. If $p = e$ in Theorem 7, $a$ is primary such as $\text{rad}(a) = e$.

**Corollary 1.** *Suppose that (*) is a short decomposition of a, and let $p_1, \cdots, p_s$ be the minimal primes belonging to a $(i = 1, \cdots, s)$. Then*

$$a = u(a, p_1) \cap \cdots \cap u(a, p_s) \cap q_{s+1} \cap \cdots \cap q_n .$$

**Corollary 2.** *Suppose that a has a decomposition into primary elements with prime radicals. If $p_1, \cdots, p_s$ are the minimal primes belonging to a, then $u(a, p_1), \cdots, u(a, p_s)$ are primary.*

Proof. This is immediate by Theorems 2, 5 and 7.

Now let $V$ be a compactly generated lattice with compact generator system $\Sigma$. If $\Sigma$ is a join-semi-lattice, $\Sigma$ is said to be join-closed. Let $\Sigma$ be any compact generator system of $V$. Then it can be proved that the join-semi-lattice $\Sigma'$ generated by $\Sigma$ satisfies the conditions (1) and (2) in §1. Hence $\Sigma'$ is a join-closed compact generator system of $V$.

In the rest of this section we suppose that $\Sigma$ is join-closed. Then it is easy to see by the a.c.c. that $\Sigma$ coincides with $L$. But it is convenient to remain the symbol $\Sigma$.

**Lemma 15.** *Let $p_1, \cdots, p_n$ be a finite number of prime elements of a compactly generated multiplicative lattice with a join-closed compact generator system. If $\Sigma(a)$ is contained in the set-union $\vee_{i=1}^{n} \Sigma(p_i)$, there exists $p_i$ such that $p_i \geq a$.*

Proof. If $n = 1$, the lemma is trivially evident. If $n = 2$, then $\Sigma(a)$ is contained in $\Sigma(p_1) \vee \Sigma(p_2)$. Suppose that $a \not\leq p_1$ and $a \not\leq p_2$. Then we can take $z_i$ of $\Sigma$ such that $z_i \leq a$, $z_i \leq p_i$ $(i = 1, 2)$, $z_1 \not\leq p_2$ and $z_2 \not\leq p_1$. Since $z_1 \cup z_2$ is less than $a$, $\Sigma(z_1 \cup z_2)$ is contained in $\Sigma(p_1)$ or $\Sigma(p_2)$. This implies $z_2 \leq p_1$ or $z_1 \leq p_2$, which is a contradiction. If $n \geq 3$, we can assume, no loss of generality, that $\Sigma(a)$ is contained in $\vee_{i=1}^{m} \Sigma(p_i)$ $(m \leq n)$, and not contained in $\vee_{i=1}^{k-1} \Sigma(p_i) \vee \vee_{i=k+1}^{m} \Sigma(p_i)$ for every $k = 2, \cdots, m-1$. Then we can take elements $z_k$ of $\Sigma$ such that $z_k \leq a$, $z_k \leq p_k$ and $z_k \not\leq p_i$ for $i \neq k$; $i, k = 1, \cdots, m$. Since $z_2, \cdots, z_m$ are contained in a $\mu$-system $\Sigma'(p_1)$, we can find a finite number of elements

$v_1, \cdots, v_{m-2}$ of $\Sigma'(p_1)$ such that $v_1 \leq z_2 z_3$, $v_2 \leq v_1 z_4, \cdots, v_{m-2} \leq v_{m-3} z_m$. Then we have $v^{(1)} \equiv v_{m-2} \leq ((\cdots((z_2 \cdot z_3)z_4)\cdots)z_{m-1})z_m \leq p_j$ for $j=2, \cdots, m$, and $v^{(1)} \not\leq p_1$. Similarly, we can find $v^{(i)}$ of $\Sigma'(p_i)$ such that $v^{(i)} \leq p_j$ $(j \neq i)$ and $v^{(i)} \not\leq p_i$ for $i=2, \cdots, m$. Now let $v \equiv v^{(1)} \cup \cdots \cup v^{(m)}$. Then, since $\Sigma$ is closed under finite join operation, $v$ is contained in $\Sigma(a)$. Hence we have $v \leq p_i$ for a suitable prime $p_i$. This implies $v^{(i)} \leq p_i$, which is a contradiction. This completes the proof.

**Theorem 8.** *Suppose that* (*) *is a short decomposition of a with prime radicals, and let p be a prime element such that* $a \leq p \neq e$. *Then* $p = rad(q_i)$ *for some* $q_i$, *if and only if* $u(a, p)/p > u(a, p)$.

Proof. We have, by Theorem 6, $u(a, p) = q_1 \cap \cdots \cap q_k$, where $q_1, \cdots, q_k$ are those whose radicals $p_i$ are less than $p$. This is a short decomposition of $u(a, p)$, and $p$ is one of $p_1, \cdots, p_k$. Since an element $x$ of $\Sigma = L$ is contained in $P'(u(a, p))$ if and only if $x \leq p$, we have $u(a, p)/p > u(a, p)$. Conversely, let $u(a, p)/p > u(a, p)$. Then the minimal primes of $a$ are the minimal elements in the *po*-set $\{p_1, \cdots, p_n\}$. Hence $p_i \leq p$ for some $p_i$. We let $p_1, \cdots, p_k$ be the primes such that $p_i \leq p$ $(i=1, \cdots, k)$. Then $u(a, p) = q_1 \cap \cdots \cap q_k$, $\text{rad}(q_i) = p_i$, and that is a short decomposition of $u(a, p)$. Now by the assumption $\Sigma(p)$ is contained in $P'(u(a, p)) = \bigvee_{i=1}^{k} \Sigma(p_i)$. Hence, we have by Lemma 15 $p \leq p_i$ for a suitable $p_i (1 \leq i \leq k)$. We obtain therefore $p = p_i$.

## 6. Artin-Rees property

In this section, we let $L$ be a compactly generated integral multiplicative lattice with the compact generator system $\Sigma$.

DEFINITION 10. $L$ is said to have the (*right*) *weak Artin-Rees property*, if for any $a$ in $L$ and any $x$ in $\Sigma$, there exists a product $\mathfrak{P}$ of $x$ such that $a \cap \mathfrak{P} \leq ax$.

**Theorem 9.** *Suppose that the a.c.c. holds for elements of L. If every element of L may be decomposed into a meet of a finite number of primary elements, then the weak Artin-Rees property holds for L.*

Proof. Let $a \in L$, and $x \in \Sigma$, and suppose that $ax = q_1 \cap \cdots \cap q_n$ is a primary decomposition of $ax$. If $a \leq q_i$ for every $i=1, \cdots, n$, we have that $a \cap x \leq a = ax$. Hence we can suppose that $a \not\leq q_i$ for $i=1, \cdots, m$, where $1 \leq m \leq n$. Then $ax = a \cap q_1 \cap \cdots \cap q_m$. Since there exists an element $u$ of $\Sigma$ such that $ux \leq q_i$ and $u \not\leq q_i$ $(1 \leq i \leq m)$, we obtain $x \leq \text{rad}(q_i)$ $(1 \leq i \leq m)$. Hence we have that $\mathfrak{P}_i \equiv \mathfrak{P}_i(x, \cdots, x) \leq q_i$ for suitable product-forms $\mathfrak{P}_i$ $(1 \leq i \leq m)$. Hence $\mathfrak{P}' \equiv (\cdots((\mathfrak{P}_1 \cdot \mathfrak{P}_2)\mathfrak{P}_3 \cdots)\mathfrak{P}_m \leq q_1 \cap \cdots \cap q_m$. Therefore we obtain $a \cap \mathfrak{P}' \leq a \cap q_1 \cap \cdots \cap q_m = ax$, completing the proof.

Let $\Sigma^*$ be the multiplicative monoid generated by $\Sigma$.

DEFINITION 11. $L$ is called a *strictly upper semi-modular lattice related to* $\Sigma^*$, if the relations $a \cap u < b < a < a \cup u$ hold for $a, b \in L$, and $u \in \Sigma^*$, then there exists an element $c$ of $L$ such that $a \cap u < c \leq u$ and $(c \cup b) \cap a = b$.

This is a modification of the semi-modular lattice defined in [19, §45].

**Lemma 16.** *Let $L$ be a strictly upper semi-modular lattice related to $\Sigma^*$, and let $q$ be an irreducible element of $L$. If $q \cap u = a \cap u$ and $q < a$ for $a \in L$, $u \in \Sigma^*$, then $u \leq q$.*

Proof. Put $b = (q \cup u) \cap a$. Then $q \leq b$. If $b = q$, then since $q = (q \cup u) \cap a$ and $q < a$, we have $q = q \cup u$, $u \leq q$. Next we suppose that $q < b$. Now we have that $a \cap u \leq q < a \leq a \cup u$. If $a = a \cup u$, then $u \leq a$, $u = a \cap u = q \cap u$. This implies $u \leq q$. If $a \cap u = q$, then $q \cap u = q$, $q \leq u$. This implies $q \cup u = u$. Hence we have $b = a \cap u = q \cap u = q$, a contradiction. Now it remains to consider the case of $a \cap u < q < a < a \cup u$. Then there exists an element $c$ of $L$ such that $a \cap u < c \leq u$ and $q = (q \cup c) \cap a$. Since $q$ is irredicible, we have $q = q \cup c$. Hence $c \leq q < a$, $c \leq a \cap u$. This contradicts $a \cap u < c$.

**Lemma 17.** *A non-void $\mu$-system $M$ meets $\Sigma(\mathfrak{P}(x, \cdots, x))$ for every element $x \in M$ and every product-form $\mathfrak{P}$.*

Proof. The proof will be given by induction with respect to the weight $m$ of $\mathfrak{P}$. If $m = 1$, the lemma is evident. We suppose that the lemma has been proved for $\mathfrak{P}'$ with any weight $m' < m$. Now $\mathfrak{P}$ is expressible as $\mathfrak{P} = \mathfrak{P}_1 \cdot \mathfrak{P}_2$. Of course the weight of $\mathfrak{P}_i$ is strictly less than that of $\mathfrak{P}$. Hence by the induction hypothesis $M$ meets $\Sigma(\mathfrak{P}_i)$; accordingly there exists $u_i$ such that $u_i \in M$ and $u_i \leq \mathfrak{P}_i$ $(i = 1, 2)$. Since there exists an element $u$ of $M$ such that $u \leq u_1 u_2$, $M$ meets $\Sigma(\mathfrak{P})$, as desired.

**Theorem 10.** *Let $L$ be a strictly upper semi-modular lattice related to $\Sigma^*$, and suppose that the a. c. c. holds for elements of $L$. If the weak Artin-Rees property holds for $L$, every element of $L$ is decomposed into a meet of a finite number of primary elements.*

Proof. Since $L$ satisfies the a. c. c., it is sufficient to show that every irreducible element of $L$ is primary. Suppose that $q$ is irreducible, and let $xy \leq q$ but $x \not\leq q$ for two elements $x$, $y$ in $\Sigma$. Put $a = x \cup q$. Then $a > q$ and $ay = (x \cup q)y = xy \cup qy \leq q$. Now let $\mathfrak{P} \equiv \mathfrak{P}(y, \cdots, y)$ be a product of $y$ such that $a \cap \mathfrak{P} \leq ay \leq q$. Then we have $a \cap \mathfrak{P} \leq q \cap \mathfrak{P}$. Hence $a \cap \mathfrak{P} = q \cap \mathfrak{P}$. Since $q < a$, we have by Lemma 16 $\mathfrak{P} \leq q$. Next, we let $M$ be an arbitrary $\mu$-system containing $y$. Then by Lemma 17 $M \wedge \Sigma(\mathfrak{P})$ is not void. Since $\mathfrak{P} \leq q$, we have that $M \wedge \Sigma(\mathfrak{P}) \subseteq M \wedge \Sigma(q)$. Therefore $M$ meets $\Sigma(q)$, that is, $y \leq \operatorname{rad}(q)$, as desired.

## 7. Applications

[**1**] Let $R$ be a non-associative (not necessarily) ring with or without unity quantity. The word "ideals" will mean always "two-sided ideals" of $R$. Ideals of $R$ will be denoted by $A, B, P, Q, \cdots$. For an element $x$ of $R$, $(x)$ will denote the principal ideal generated by $x$. $(x)$ consists of the elements $u$ such of that $u=\Sigma\mathfrak{P}(\cdots, x, \cdots)$, where $\mathfrak{P}(\cdots, x, \cdots)$ is a product with $x$ as its factor, and $\Sigma$ is a finite sum.

Now it can be proved that the set of all ideals of $R$ forms a compactly generated integral multiplicative lattice with the compact generator system consisting of the principal ideals. The results in the preceding sections are accordingly applicable to the ideals of $R$.

Throughout [**1**], there is a complete parallelism between the theory of right-side and that of left-side. We shall therefore state the results for right-side only.

For any two ideals $A$ and $B$, the (right) quotient $A$ by $B$, denoted by $A/B$, is the set of the elements $u$ in $R$ such that $(u)B\subseteq A$ (Cf. [2], [8]). Then $A/B$ is an ideal of $R$, and it can be proved easily that $A/B$ coincides with the set-union of all the principal ideals $(u)$ such that $(u)B\subseteq A$. An element $x$ of $R$ is said to be *(right) related* an ideal $A$, if and only if $A/(x)$ contains $A$ properly. Otherwise $x$ is said to be *(right) unrelated* to $A$. It is then easily seen that if $x$ is related to $A$, every element in $(x)$ is also related to $A$.

A family $\mathfrak{M}$ of principal ideals of $R$ is called a *$\mu$-system*, if there exists $(z)$ of $\mathfrak{M}$ such that $(z)\subseteq(x)(y)$ for any two principal ideals $(x)$ and $(y)$ in $\mathfrak{M}$. The void set is also defined to be a $\mu$-system. Let $P$ be a prime ideal of $R$ (Cf. [2]). It is then easily verified that the family of principal ideals $\mathfrak{M}_P=\{(x)\mid(x)$ is not contained in $P\}$ forms a $\mu$-system. Conversely, if $\mathfrak{M}_P$ is a $\mu$-system for an ideal $P$, then $P$ is prime. Let $A$ be an ideal of $R$, and let $\mathfrak{M}$ be a $\mu$-system which does not contain any ideal $(x)\subseteq A$. Then we can show that the existence of the (maximal) prime ideal $P$ such that $P$ contains $A$ and every principal ideal in $P$ does not contained in $\mathfrak{M}$ (Cf. [16, §14]).

Let $M$ be an $M$-system in the sense of Behrens [2]. If we make the family $\mathfrak{M}=\{(x)\mid x\in M\}$ of principal ideals, it is easily verified that $\mathfrak{M}$ is a $\mu$-system. But, for any $\mu$-system $\mathfrak{M}$, it can not be proved in general, that the set $\{x\mid(x)\in\mathfrak{M}\}$ is an $M$-system in the sense of Behrens. By Definition 2, we define the radical of an ideal $A$, which is denoted by rad$(A)$, is the ideal generated by the set-union of principal ideals $(x)$ with the property that every $\mu$-system which contains $(x)$ contains a principal ideal in $A$. Definition of a minimal prime ideal of an ideal is the same as in the case of an associative ring (Cf. [11]). Then by Corollary to Theorem 1, we obtain that the radical of an ideal $A$ is the intersection of all the minimal prime ideals of $A$. Therefore we obtain that rad$(A)$ coincides with the Behrens' radical $\mathfrak{r}(A)$.

In order that an ideal $Q$ of $R$ is (right) primary (Cf. [2]) it is necessary and sufficient that every element which is (right) related to $Q$ is contained in $\mathfrak{r}(Q)$. Irredundant decomposition of an ideal of $R$ is defined as usual. Let

$$A = Q_1 \cap \cdots \cap Q_n \qquad (*)$$

be an irredundant decomposition of an ideal $A$ into primary components $Q_i$.

The representation $(*)$ of $A$ is called a short decomposition of $A$, if none of the meets of two (or more) of $Q_1, \cdots, Q_n$ are primary. By Theorem 2, we obtain the following statement.

1) *If an ideal $A$ of $R$ can be decomposed as an intersection of a finite number of primary ideals, $A$ has a short decomposition. In any two short decompositions of $A$, the number of primary components as well as their radicals are necessarily the same.*

Let $\mathfrak{M}$ be a non-void $\mu$-system. A family $\mathfrak{N}$ of principal ideals of $R$ is called a *(right) $\mathfrak{M}$-$\nu$-system* of $R$, if $\mathfrak{N}$ contains $\mathfrak{M}$ and if for every $(u)$ in $\mathfrak{N}$ and every $(x)$ in $\mathfrak{M}$, there exists an ideal $(z)$ in $\mathfrak{N}$ such that $(z) \subseteq (u)(x)$. If $\mathfrak{M}$ is void, the $\mathfrak{M}$-$\nu$-system is also void. Let $\mathfrak{M}$ be a $\mu$-system such that every ideal in $\mathfrak{M}$ is not contained in an ideal $A$. A *(right) upper $\mathfrak{M}$-component* of $A$ is defined to be the ideal generated by the set-union of all the principal ideals $(x)$ having the property that every $\mathfrak{M}$-$\nu$-system which contains $(x)$ has an ideal in $A$. The upper $\mathfrak{M}$-component of $A$ will be denoted by $U(A, \mathfrak{M})$. Let $P$ be a prime ideal containing $A$. Then the (right) upper isolated $P$-component of $A$, which is denoted by $U(A, P)$, means $U(A, \mathfrak{M})$, where $\mathfrak{M} = \{(x) \mid (x)$ is not contained in $P\}$. If $P$ is a minimal prime of $A$, $U(A, P)$ is called an isolated (right) primary component of $A$. Now let $\mathfrak{M}$ $(\neq \phi)$ be a $\mu$-system which does not contain any ideal in $A$, and let $\mathfrak{N}^*$ be the (unique) maximal $\mathfrak{M}$-$\nu$-system such that every ideal in $\mathfrak{N}^*$ is not contained in $A$. Then by Theorem 3 $U(A, \mathfrak{M})$ is the intersection of all the ideals $B$ having the property that (1) $B$ contains $A$ and (2) $\{(x) \mid B/(x) = B\}$ contains $\mathfrak{M}$. Moreover $U(A, \mathfrak{M})$ is the ideal generated by the set-union of all the principal ideals, each of which is not in $\mathfrak{N}^*$.

A product-form $\mathfrak{Q}(X_1, \cdots, X_m) = (\cdots((X_1X_2)X_3)\cdots)X_m$ is called that it has a (right) *nested* type of weight $m$, where $X_i$ are indeterminates over the ideal-$m$-lattice of $R$.

A non-associative ring $R$ is called here an (N)-*ring* if it satisfies the following condition:

(N) *For every product-form $\mathfrak{P}$ of weight $n$, and for every ideals $A_1, \cdots, A_n$ (repetitions allowed) of $R$, there exists a product-form $\mathfrak{Q}$ with nested type of weight $m$ such that*

$$\mathfrak{Q}(A_{i_1}, \cdots, A_{i_m}) \subseteq \mathfrak{P}(A_1, \cdots, A_n),$$

*where $i_1 \leq \cdots \leq i_m$.*

Any associative ring is evidently an (N)-ring. Any nilpotent Lie ring is also an (N)-ring. Now we have the following statement.

2) *Suppose that the a. c. c. holds for ideals of an (N)-ring R. Then the radical of every primary ideal of R is prime.* (by Theorem 4).

We now suppose that (*) in this section is an irredundant decomposition of an ideal $A$ of a ring $R$ into primary ideals $Q_i$ with prime radicals $P_i$. If the a. c. c. holds for ideals of $R$, the minimal primes belonging to $A$ coincide with the minimal elements in the *po*-set $\{P_1, \cdots, P_n\}$. This is the immediate consequence of Theorem 5. In particular, we obtain the following:

3) *Assume that the a. c. c. holds for ideals of an (N)-ring R. If* (*) *is a decomposition of an ideal A into primary ideals $Q_i$, the minimal primes belonging to A coincide with the minimal elements in the po-set consisting of the radicals of $Q_i$* (by Theorem 5).

In the rest of this paragraph, we let $R$ be an (N)-ring with the a. c. c. for ideals of $R$. Then by Theorems 6, 7, 8, 9 and 10 we have the followings 4)～8).

4) *Suppose that* (*) *is a decomposition of an ideal A of R into primary ideals $Q_i$ with prime radicals $P_i$. If $P(\neq R)$ is a prime ideal such that $P_1 \subseteq P, \cdots, P_s \subseteq P$, $P_{s+1} \not\subseteq P, \cdots, P_n \not\subseteq P$, then*

$$U(A, P) = Q_1 \cap \cdots \cap Q_s .$$

5) *Suppose that* (*) *is a short decomposition of A with (prime) radicals $P_i = rad(Q_i)$. If P is any minimal prime ideal belonging to A and $P \neq R$, then $U(A, P) = Q_i$ for some i, and $U(A, P)$ is primary.*

6) *Suppose that* (*) *is a short decomposition of A, and let $P_1, \cdots, P_s$ be the minimal primes belonging to A. Then*

$$A = U(A, P_1) \cap \cdots \cap U(A, P_s) \cap Q_{s+1} \cap \cdots \cap Q_n .$$

7) *Suppose that* (*) *is a short decomposition of A, and let P be a prime ideal such that $A \subseteq P \neq R$. Then $P = rad(Q_i)$ for some $Q_i$, if and only if $U(A, P)/P \supsetneqq (A, P)$.*

$R$ is said to have the (*right*) *weak Artin-Rees property*, if for any ideal $A$ and any principal ideal $(x)$ of $R$, there exists a product $\mathfrak{P}$ of $(x)$ such that $A \cap \mathfrak{P} \subseteq A(x)$. (Cf. [8]). Then we have

8) *In order that every ideal of R is decomposed into a meet of a finite number of primary ideals, it is necessary and sufficient that the weak Artin-Rees property holds for R.*

[2] Let $G$ be a group. The set of all normal subgroups $A, B, N, \cdots$ of $G$ is a commutative residuated *cm*-lattice under commutator-product $[A, B]$ and the set-inclusion relation. The *residual* of $A$ by $B$, which is denoted by $A:B$, is defined as the set-union of the elements $u \in G$ such that $[(u), B] \subseteq A$, where $(u)$ is the normal subgroup generated by $u \in G$, that is, $(u) = \{\Pi x_\rho^{-1} u^\rho x_\rho \mid x \in G,\ \rho \in Z$

(the integers)}. Then it can be proved that $A:B$ is a normal subgroup of $G$. It is easily be seen that the *cm*-lattice has the *zero element* 1 (the group identity) (Cf. [3]). Now we can show that the set of the normal subgroups of $G$ is a compactly generated multiplicative lattice with the compact generator system consisting of normal subgroups, each of which is generated by a single element.

An element $x$ of $G$ is said to be *unrelated* to a normal subgroup $N$, if $N:(x)=N$. Otherwise, $x$ is *related* to $N$. A family $\boldsymbol{M}$ consisting of normal subgroups with single generators is called a *$\mu$-system,* if there exists $(z)$ of $\boldsymbol{M}$ such that $(z)\subseteq[(x), (y)]$ for any two $(x)$ and $(y)$ or $\boldsymbol{M}$. The void set is also defined to be a $\mu$-system. A normal subgroup $P$ of $G$ is said to be prime, if $[A, B]\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$. Then it can be proved that $P$ is prime if and only if $[(x), (y)]\subseteq P$ implies $(x)\subseteq P$ or $(y)\subseteq P$. If $P$ is prime, the family $\{(x)\mid x\notin P\}$ forms a $\mu$-system. Moreover a normal subgroup $P(\neq G)$ of $G$ is prime if and only if $\{(x)\mid x$ is related to $P\}$ is a $\mu$-system.

Let $\boldsymbol{M}$ be a $\mu$-system which does not contain $(x)$ such that $(x)\subseteq A$. Then there exists a normal subgroup $P$ which is maximal in the family of normal subgroups $B$ such that $B\supseteq A$ and $(b)\notin\boldsymbol{M}$ for every $b\in B$. $P$ is necessariry prime.

A *radical* of normal subgroup $N$ of $G$ is the normal subgroup generated by the set-union of $(x)$ with the property that the every $\mu$-system containing $(x)$ contains a subgroup in $N$. In symbol: rad$(N)$. Minimal primes of a normal subgroup is defined in the obvious way. Then by Corollary to Theorem 1 we obtain that rad$(N)$ is the intersection of all minimal primes of $N$.

A normal subgroup $Q$ of $G$ is called *primary,* if $[(x), (y)]\subseteq Q$ and $(y)\nsubseteq$ rad$(Q)$ imply that $(x)\subseteq Q$.

Let

$$N=Q_1\cap\cdots\cap Q_n \qquad (**)$$

be an irredundant decomposition of a normal subgroup $N$ into primary normal subgroups $Q_i$. The representation $(**)$ of $N$ is called a short decomposition of $N$, if none of the meets of two (or more) of $Q_1,\cdots, Q_n$ are primary. By Theorem 2, we obtain the following statement.

1) *If a normal subgroup $N$ of $G$ can be decomposed as an intersection of a finite number of primary normal subgroups, then $N$ has a short decomposition. In any two short decompositions of $N$, the number of primary components as well as their radicals are necessarily the same.*

Let $\boldsymbol{M}$ be a non-void $\mu$-system. A family $\boldsymbol{N}$ of principal normal subgroups of $G$ is called an *$\boldsymbol{M}$-$\nu$-system,* if $\boldsymbol{N}$ contains $\boldsymbol{M}$ and if for every $(u)$ in $N$ and every $(x)$ in $\boldsymbol{M}$ there exists $(z)$ in $\boldsymbol{N}$ such that $(z)\subseteq[(u), (x)]$. If $\boldsymbol{M}$ is void, the $\boldsymbol{M}$-$\nu$-system is also void. By using $\boldsymbol{M}$-$\nu$-system, the *upper $\boldsymbol{M}$-component* $U(N, \boldsymbol{M})$ of $N$ is defined in an obvious way. In particular, *upper isolated $P$-component* $U(N, P)$ of $N$ is defined for any minimal prime of $N$. Now let $\boldsymbol{M}$

be a $\mu$-system which does not contain a normal subgroup (of $G$) in $N$, and let $\boldsymbol{N}^*$ be the (unique) maximal $\boldsymbol{M}$-$\nu$-system such that every normal subgroup in $\boldsymbol{N}^*$ does not contained in $N$. Then by Theorem 3 $U(N, \boldsymbol{M})$ is the intersection of all the normal subgroups $H$ having the property that (1) $H \supseteq N$ and (2) $\{(a) \mid H:(a)=H\} \supseteq \boldsymbol{M}$. Moreover $U(N, \boldsymbol{M})$ is the normal subgroup generated by the set-union of all the normal subgroups such that each of which has a single generator and is not contained in $\boldsymbol{N}^*$.

A product-form $\mathfrak{Q}(X_1, \cdots, X_m) = (\cdots((X_1X_2)X_3))\cdots)X_m$ is called here that it has a *nested* type of weight $m$, where $X_i$ are the indeterminates over the $m$-lattice of the normal subgroups of $G$.

A group $G$ is called an (N)-*group* if it satisfies the following condition:

(N) *For every product-form $\mathfrak{P}$ of weight $n$, and for every normal subgroup $N_1, \cdots, N_n$ (repetitions allowed) of $G$, there exists a product-form $\mathfrak{Q}$ with nested type of weight $m$ such that*

$$\mathfrak{Q}(N_{i_1}, \cdots, N_{i_m}) \subseteq \mathfrak{P}(N_1, \cdots, N_n),$$

*where $i_1 \leq \cdots \leq i_m$.*

Nilpotent groups are evidently (N)-groups.

Now we let $G$ be an ($N$)-group with the a. c. c. for normal subgroups. Then by Theorems 4, 5, 6, 7, 8, 9 and 10 we obtain the following statements:

2) *The radical of any normal subgroup of $G$ is prime.*

3) *If* (**) *is an irredundant decomposition of a normal subgroup $N$ of $G$ into primary normal subgroups $Q_i$, the minimal primes belonging to $N$ coincide with the minimal elements in the po-set consisting of the rad$(Q_i)$.*

4) *Suppose that* (**) *is a decomposition of a normal subgroup $N$ of $G$ into primary normal subgroups $Q_i$ with prime radicals $P_i$. If $P(\neq G)$ is a prime normal subgroup such that $P_1 \subseteq P, \cdots, P_s \subseteq P$, $P_{s+1} \nsubseteq P, \cdots, P_n \nsubseteq P$, then*

$$U(N, P) = Q_1 \cap \cdots \cap Q_s .$$

5) *Suppose that* (**) *is a short decomposition of $N$ with (prime) radicals $P_i = rad(Q_i)$. If $P$ is any minimal prime belonging to $N$ and $P \neq G$, then $U(N, P) = Q_i$ for some $i$, and $U(N, P)$ is primary.*

6) *Suppose that* (**) *is a short decompolition of $N$, and let $P_1, \cdots, P_s$ be the minimal primes belonging to $N$ such that $P_i \neq G$ $(i=1, \cdots, s)$. Then*

$$N = U(N, P_1) \cap \cdots \cap U(N, P_s) \cap Q_{s+1} \cap \cdots \cap Q_n .$$

7) *Suppose that* (**) *is a short decomposition of $N$, and let $P$ be a prime normal subgroup such that $N \subseteq P \neq G$. Then $P = rad(Q_i)$ for some $Q_i$, if and only if $U(N, P):P \supsetneqq U(N, P)$.*

$G$ is said to have *the weak Artin-Rees property*, if for any normal subgroup $N$ of $G$ and for any normal subgroup $(x)$ with single generator $x$, there exists a commutator-product $\mathfrak{P}$ of $(x)$ such that $N \cap \mathfrak{P} \subseteq [N, (x)]$. Then we obtain

8) *In order that every normal subgroup of G is decomposed into a finite number of primary normal subgroups, it is necessary and sufficient that the weak Artin-Rees property holds for G.*

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