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MINIMAL IMMERSIONS OF 3-DIMENSIONAL SPHERE INTO SPHERES

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Introduction

Let S_c^n be the n-dimensional sphere with constant curvature c. Let Δ be the Laplace-Beltrami operator on S_1^n. The spectre and eigen-functions of Δ are well-known [2]. Let V^d be the eigen-space of Δ corresponding to the d-th eigen-value $\lambda_d = d(d+n-1)$. Let $f_0, f_1, \cdots, f_{m(d)}$ be an orthonormal basis of V^d with respect to the inner product. Then

$$\begin{aligned}\psi_{n,d}\colon\ & S^n_{k(d)} \to S_1^{m(d)} (\subset \boldsymbol{R}^{m(d)+1}) \\ & ;\ p \to 1/(m(d)+1)(f_0(p), f_1(p), \cdots, f_{m(d)}(p)),\end{aligned}$$

is an isometric minimal immersion, where $k(d)$ and $m(d)$ are as follows [6];

$$\begin{aligned} k(d) &= n/d(d+n-1)\,, \\ m(d) &= (2d+n-1)(d+n-2)!/d!(n-1)!-1\,. \end{aligned}$$

It is proved that any isometric minimal immersion of S_c^2 into S_1^N is equivalent to $\psi_{2,d}$ for some d, [3], [6]. But it is not true if the dimension n is greater than 3. In fact do Carmo and Wallach proved the following

Theorem 0.1 (do Carmo and Wallach, [7]). *Let $f\colon S_c^n \to S_1^N$ be an isometric minimal immersion. Then*
(i) *there exists an integer d such that $c = k(d)$.*
(ii) *There exists a positive semi-definite matrix A of size $(m(d)+1)\times(m(d)+1)$ such that f is equivalent to $A\circ\psi_{n,d}$.*
(iii) *If $n=2$ or $d\leqq 3$, then A is the identity matrix.*
(iv) *If $n\geqq 3$ and $d\geqq 4$, then A is parametrized by a compact convex body L in some finite dimensional vector space,* $\dim L \geqq 18$. *If A is an interior point of L then $N=m(d)$, and if A is a boundary point of L then $N<m(d)$.*

There are some problems concerning (iv) of the above Theorem.

Problem 0.2 (Chern, [4]). *Let $S_c^3 \to S_1^7$ be an isometric minimal immersion. Is it totally geodesic?*

In [5], do Carmo posed a more general

Problem 0.3. *Determine the lower bound* $1(d)$ *of the dimension* N *of the sphere* S_1^N *into which a given* $S_{k(d)}^n$ *can be isometrically and minimally immersed.*

Recently Problem 0.2 was negatively answered by N. Ejiri [8]. In fact he proved that there exists an isometric minimal immersion $S_{1/16}^3 \to S_1^6$.

As for the Problem 0.3, scarcely anything is known.

In this paper we confine our consideration to the case $n=3$. In this case S^3 has a structure of a Lie group, $S^3=SU(2)$. We investigate whether there exists an orbit in a representation space V of $SU(2)$, which is a minimal submanifold in the unit sphere in V. And we give an estimate for $1(d)$ (of the Problem 0.3 in the case $n=3$). The following will be proved.

Theorem A. *Let* d *be an integer,* $d \geqq 4$. *Then there exists an isometric minimal immersion of* $S_{3/d(d+2)}^3$ *into* S_1^{2d+1}.

Theorem B. *Let* d *be an even integer,* $d \geqq 6$. *Then there exists an isometric minimal immersion of* $S_{3/d(d+2)}^3$ *into* S_1^d.

1. Complex linear representations of $SU(2)$

In this section we give a brief review on the complex linear representation of $SU(2)$.

The special unitary group $SU(2)$ is the group of matrices which acts on $\boldsymbol{C}^2$ and leaves invariant the usual Hermitian inner product on $\boldsymbol{C}$. We can identify $SU(2)$ with the 3-dimensional unit sphere S_1^3 ($\subset \boldsymbol{C}^2$) by

$$SU(2) \to S_1^3:\ g \to g\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ g \in SU(2)\,.$$

Then the induced metric on $SU(2)$ by the above diffeomorphism is the bi-invariant metric on $SU(2)$.

A homogeneous polynomial on $\boldsymbol{C}^2$ is called of degree d if it satisfies

$$P(\lambda z, \lambda w) = \lambda^d P(z, w),\ \lambda \in \boldsymbol{C},\ z,\ w \in \boldsymbol{C}\,.$$

For each positive integer d, let $V(d)$ be the space of homogeneous polynomials of type $(d, 0)$ on $\boldsymbol{C}^2$. Then $SU(2)$ acts on $V(d)$ as follows

$$(\rho(g)(P))(z, w) = P({}^t(g^{-1}\cdot{}^t(z, w))),\ g \in SU(2),\ z,\ w \in \boldsymbol{C},\ P \in V(d)\,.$$

Then $(V(d), \rho)$ is a complex irreducible representation and each complex irreducible representation of $SU(2)$ is equivalent to $(V(d), \rho)$ for some d [12].

Define a Hermitian inner product in $V(d)$ by

$$(1.1) \qquad (P, Q) = (d+1)\int_{g\in SU(2)} P({}^t(g\cdot{}^t(1.0)))\overline{Q({}^t(g\cdot{}^t(1.0)))}\,dg$$

where dg is the normalized Haar measure on $SU(2)$. Let P_i be the polynomial in $V(d)$ defined by

$$P_i(z, w) = ({}_dC_i)^{1/2} z^{d-i} w^i,\ z,\ w\in \boldsymbol{C}.$$

Then $P_0, P_1, \cdots, P_d$ is an orthonormal basis of $V(d)$.

Let $\mathfrak{su}(2)$ be the Lie algebra of $SU(2)$. Take the following basis of $\mathfrak{su}(2)$ and fix them once for all.

$$X_1 = \begin{bmatrix} (-1)^{1/2} & 0 \\ 0 & -(-1)^{1/2} \end{bmatrix},\quad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad X_3 = \begin{bmatrix} 0 & (-1)^{1/2} \\ (-1)^{1/2} & 0 \end{bmatrix}.$$

Then the bracket relations of X_1, X_2 and X_3 are

$$[X_1, X_2] = 2X_3,\quad [X_2, X_3] = 2X_1,\quad [X_3, X_1] = 2X_2.$$

We denote also by ρ the representation of $\mathfrak{su}(2)$ induced by the representation of $SU(2)$, i.e.,

$$\rho(A)(P) = d/dt\,|_{t=0}\,\rho(\exp tA)(P),\ A\in\mathfrak{su}(2).$$

Then by a direct calculation we get

$$(1.2)_1 \qquad \rho(X_1)(P_j) = (-1)^{1/2}(2j-d)P_j, \qquad 0\leqq j\leqq d,$$

$$(1.2)_2 \qquad \rho(X_2)(P_j) = -((d-j)(j+1))^{1/2}P_{j+1} + (j(d-j+1))^{1/2}P_{j-1}, \qquad 0\leqq j\leqq d,$$

$$(1.2)_3 \qquad \rho(X_3)(P_j) = -(-(d-j)(j+1))^{1/2}P_{j+1} - (-j(d-j+1))^{1/2}P_{j-1}, \qquad 0\leqq j\leqq d,$$

where we put $P_{-1}=P_{d+1}=0$.

2. Real irreducible representations of $SU(2)$

In this section we give a brief review on real irreducible representations of $SU(2)$.

Let G be a compact Lie group and (V, ρ) be a complex irreducible representation of G. Then (V, ρ) is said to be self-conjugate if V has a structure map j, i.e., a conjugate linear map on V such that

$$j(\rho(g)v) = \rho(g)j(v),\qquad g\in G,\ v\in V,$$
$$j(\alpha v+\beta w) = \bar{\alpha} j(v) + \bar{\beta} j(w),\qquad \alpha,\ \beta\in\boldsymbol{C},\ v,\ w\in V,$$
$$j^2 = \pm 1.$$

A self-conjugate representation (V, ρ) is said to be of index 1 (resp. -1) if $j^2=1$ (resp. $j^2=-1$). For simple Lie groups self-conjugate representations and their indices are known [13]. We denote by $(V_{\boldsymbol{R}}, \rho)$ the representation of G over $\boldsymbol{R}$ obtained by the restriction of the coefficient field from $\boldsymbol{C}$ to $\boldsymbol{R}$.

Let (V, ρ) be a self-conjugate representation of G of index -1. Then $(V_{\boldsymbol{R}}, \rho)$ is also irreducible. But $(V_{\boldsymbol{R}}, \rho)$ is reducible if (V, ρ) is a self-conjugate representation of G of index 1. Namely $(1+j)V_{\boldsymbol{R}}$ and $(1-j)V_{\boldsymbol{R}}$ are mutually equivalent real irreducible representation of G and

$$V_{\boldsymbol{R}} = (1+j)V_{\boldsymbol{R}}+(1-j)V_{\boldsymbol{R}}\,, \text{ (direct sum)}\,.$$

For these facts we refer, for instance, to [1].

Now we confine our attention to the case $G=SU(2)$.

Let j be a conjugate-linear automorphism on $\boldsymbol{C}^2$ defined by

$$j(z, w) = (-\bar{w}, \bar{z})\,, \qquad z, w\in \boldsymbol{C}\,.$$

Extend j to an automorphism on $V(d)$ by

$$(jP)(z, w) = \overline{P(j(z, w))}\,, \qquad z, w\in \boldsymbol{C}\,.$$

Then j is a structure map on $V(d)$ with $j^2=(-1)^d 1$. So $(V(d)_{\boldsymbol{R}}, \rho)$ is a self-conjugate representation of index $(-1)^d$. Let d be an even integer $d=2d'$ and put $Q_i=(-1)^{1/2}P_i$, $0\leqq i\leqq d$. Then

$$jP_i = (-1)^i P_{d-i}\,, \quad jQ_i = -(-1)^i Q_{d-i}\,, \quad 0\leqq i\leqq d\,.$$

Since $P_0, P_1, \cdots, P_d, Q_0, Q_1, \cdots, Q_d$ are basis of $V(d)_{\boldsymbol{R}}$, $(1+j)P_i$, $(1+j)Q_i$, $0\leqq i\leqq d$, are generators of $(1+j)V(d)_{\boldsymbol{R}}$. It is easily seen that $(1+j)P_i$, $(1-j)Q_i$, $0\leqq i\leqq d-1$ and $(1+j)P_{d'}$ [resp. $(1+j)Q_{d'}$] are basis of $(1+j)V(d)_{\boldsymbol{R}}$ if a' is an even [resp. odd] integer. We denote $(1+j)V(d)_{\boldsymbol{R}}$ by $V_0(d)$.

Lemma 2.1. *Let d be an even integer, $d=2d'$. Then $\sum_{i=0}^d z_iP_i$ is contained in $V_0(d)$ if and only if*

$$z_i = (-1)^i \bar{z}_{d-i}\,, \qquad 0\leqq i\leqq d'\,.$$

Proof.

$$\begin{aligned}\textstyle\sum_{i=0}^d z_iP_i = {} & (\mathrm{Re}\, z_0P_0+\mathrm{Re}\, z_dP_d)+(\mathrm{Im}\, z_0Q_0+\mathrm{Im}\, z_dQ_d)\\ & +(\mathrm{Re}\, z_1P_1+\mathrm{Re}\, z_{d-1}P_{d-1})+(\mathrm{Im}\, z_1Q_1+\mathrm{Im}\, z_{d-1}Q_{d-1})\\ & +\cdots\cdots\cdots\\ & +z_{d'}P_{d'}\,.\end{aligned}$$

Remember that $P_j+(-1)^jP_{d-j}$, $Q_j-(-1)^jQ_{d-j}$, $0\leqq j\leqq d'-1$ and $P_{d'}$ [resp. $Q_{d'}$] are basis of $V_0(d)$ if d' is an even [resp. odd] integer. So $\sum_{i=0}^d z_iP_i$ is contained in $V_0(d)$ if and only if

$\operatorname{Re} z_i = (-1)^i \operatorname{Re} z_{d-i}$, $\operatorname{Im} z_i = -(-1)^i \operatorname{Im} z_{d-i}$, $0 \leqq i \leqq d'-1$.
$\operatorname{Im} z_{d'} = 0$ [resp. $\operatorname{Re} z_{d'} = 0$] if d' is even [resp. odd].

So we get the Lemma. Q.E.D.

3. Orbits in a sphere

Let G be a Lie subgroup in $SO(N+1)$. Then G acts on the unit sphere S_1^N in $\boldsymbol{R}^{N+1}$ centered at the origin in a natural manner. Take a point p_0 in S_1^N and let M be the orbit of the action of G through p_0.

Let $\mathfrak{g}$ be the Lie algebra of G. We denote by A^* the vector field on S_1^N defined by

$$A^*{}_{|p} = d/dt_{|t=0} \exp(tA)(p), \qquad p \in S_1^N. \tag{3.1}$$

We consider elements of $\mathfrak{g}$ as skew symmetric $(N+1)\times(N+1)$-matrices in a natural manner. Then we get from (3.1) the following

$$A^*{}_{|p} = A(p), \qquad A \in \mathfrak{g}, \; p \in S_1^N.$$

So the tangent space of M at p is

$$T_p(M) = \{A(p) \,|\, A \in \mathfrak{g}\}.$$

Let $N_p(M)$ be the normal space at p in S_1^N. Consider the tangent space $T_p(M)$ and the normal space $N_p(M)$ as a subspace in $\boldsymbol{R}^{N+1}$. Then $\boldsymbol{R}^{N+1}$ is decomposed into the direct sum

$$\boldsymbol{R}^{N+1} = \boldsymbol{R}p + T_p(M) + N_p(M). \tag{3.2}$$

For a vector A in $\boldsymbol{R}^{N+1}$, we denote A^T and A^N the $T_p(M)$-component and $N_p(M)$-component of A in the decomposition (3.2) respectively.

Lemma 3.1. *Let G be a Lie subgroup in $SO(N+1)$. Let α be the second fundamental form of the orbit $G\cdot p$ in S_1^N. Then*

$$\alpha(A^*, B^*)_{|p} = (A(B(p)))^N, \tag{3.3}$$

$$\nabla_{B^*} A^*{}_{|p} = (A(B(p)))^T, \qquad A, B \in \mathfrak{g}. \tag{3.4}$$

where ∇ is the Riemannian connection on M.

Proof. Let D be the Riemannian connection in $\boldsymbol{R}^{N+1}$. Then

$$\begin{aligned} D_{B^*} A^*{}_{|p} &= d/dt_{|t=0} A^*{}_{|\exp(tB)(p)} \\ &= d/dt_{|t=0} A(\exp(tB)(p)) \\ &= A(B(p)). \end{aligned}$$

Since $\alpha(A^*, B^*)_{|p}=(D_{B^*}A^*_{|p})^N$ and $\nabla_{B^*}A^*_{|p}=(D_{B^*}A^*_{|p})^T$, we get the Lemma.
Q.E.D.

4. Left invariant metrics on $SU(2)$ and $SO(3)$

In this section we denote by G the Lie group $SU(2)$ or $SO(3)$. The Lie algebras of $SU(2)$ and $SO(3)$ are mutually isomorphic. We denote them by $\mathfrak{su}(2)$.

Let B be the Killing form of $\mathfrak{su}(2)$. Then X_1, X_2, X_3 defined in § 1 are orthonormal with respect to $-B/8$. Let g_0 be the Riemannian metric on G which is the bi-invariant extension of $-B/8$.

Lemma 4.1. [11]. *Let g be an inner product on $\mathfrak{su}(2)$. Then there exists an element σ in G such that*
(i) $X'_i=Ad(\sigma)(X_i)$, $i=1, 2, 3$, *are mutually orthogonal with respect to g.*
(ii) $g=\lambda_1\omega_1^2+\lambda_2\omega_2^2+\lambda_3\omega_3^2$, *where λ_i are positive constants and $\omega_i(\cdot)=g_0(X'_i, \cdot)$, $i=1, 2, 3$.*

Let g be the Riemannian metric on G which is the left invariant extension of the inner product g on $\mathfrak{su}(2)$. Extend $X'_i/(\lambda_i)^{1/2}$, $1\leqq i\leqq 3$, to the left invariant vector fields Y_i, $1\leqq i\leqq 3$. Let θ_i, $1\leqq i\leqq 3$, be the dual coframe fields on G to Y_i, $1\leqq i\leqq 3$. Let θ_{ij} (resp. Ω_{ij}) be the connection (resp. curvature) form of (G, g) with respect to the orthonormal frame fields Y_1, Y_2, Y_3. Then we get easily

$$\begin{aligned}
\theta_{12} &= -(\lambda_1+\lambda_2-\lambda_3)/(\lambda_1\lambda_2\lambda_3)^{1/2}\theta_3\,,\\
\theta_{23} &= -(\lambda_2+\lambda_3-\lambda_1)/(\lambda_1\lambda_2\lambda_3)^{1/2}\theta_1\,,\\
\theta_{31} &= -(\lambda_3+\lambda_1-\lambda_2)/(\lambda_1\lambda_2\lambda_3)^{1/2}\theta_2\,,\\
\Omega_{12} &= (((\lambda_1-\lambda_2)^2-3\lambda_3^2+2\lambda_3(\lambda_1+\lambda_2))/\lambda_1\lambda_2\lambda_3)\theta_1\Lambda\theta_2\,,\\
\Omega_{23} &= (((\lambda_2-\lambda_3)^2-3\lambda_1^2+2\lambda_1(\lambda_2+\lambda_3))/\lambda_1\lambda_2\lambda_3)\theta_2\Lambda\theta_3\,,\\
\Omega_{31} &= (((\lambda_3-\lambda_1)^2-3\lambda_2^2+2\lambda_2(\lambda_3+\lambda_1))/\lambda_1\lambda_2\lambda_3)\theta_3\Lambda\theta_1\,.
\end{aligned}$$

So (G, g) is a space of constant curvature k if and only if $\lambda_1=\lambda_2=\lambda_3=1/k$, i.e., $g=(1/k)g_0$.

Let (V, ρ) be a real representation of G and $\langle\,,\rangle$ be a G-invariant inner product on V. Then an orbit M of G through a unit vector $p\in V$ is contained in the unit sphere S_1 (in V centered at the origin).

Lemma 4.2. (i) *The orbit M is a 3-dimensional space of constant curvature k if and only if*

$$\langle\rho(X_i)(p), \rho(X_j)(p)\rangle = \delta_{ij}/k\,, \qquad 1\leqq i, j\leqq 3\,.$$

(ii) *Assume that the orbit M is a 3-dimensional space of constant curvature k.*

Then M is a minimal submanifold in S_1 if and only if

$$\sum_{j=1}^{3} \rho(X_j)^2(p) = -3kp .$$

Proof. Define a map $f: G \to S_1$ by

$$f(\sigma) = \rho(\sigma)(p) , \qquad \sigma \in S_1 .$$

Then

$$f_*(X_i) = \rho(X_i)(p) .$$

Let g be the induced metric on G of f_*. Then g is a left invariant metric. So (G, g) is a 3-dimensional space of constant curvature k if and only if $g=(1/k)g_0$. By definition of g

$$\begin{aligned} g(X_i, X_j) &= \langle \rho(X_i)(p), \rho(X_j)(p) \rangle \\ &= g_0(X_i, X_j)/k \\ &= \delta_{ij}/k , \qquad 1 \leqq i, j \leqq 3 , \end{aligned}$$

if and only if $g=(1/k)g_0$.

(ii) Since (G, g) is a space of constant curvature, exp tX_i are geodesics in (G, g). By Lemma 3.1, $(\rho(X_i))^2(p)$ is normal to M. Consider the vector $\sum_{i=1}^{3} (\rho(X_i))^2(p)$ in V, which is normal to M. Then its $N_p(M)$-components in the decomposition (3.2) is the mean curvature vector of M in S_1 at p. Since M is an orbit of a representation of G, M is a minimal submanifold in S_1 if and only if the mean curvature vector of M in S_1 at one point is 0. So M is a minimal submanifold if and only if

$$\sum_{i=1}^{3} (\rho(X_i))^2(p) = cp , \tag{4.1}$$

for some constant c. Assume that (4.1) holds, then

$$\begin{aligned} c &= \langle \sum_{i=1}^{3} (\rho(X_i))^2(p), p \rangle \\ &= -\sum_{i=1}^{3} \langle \rho(X_i)(p), \rho(X_i)(p) \rangle \\ &= -3k . \end{aligned}$$

Q.E.D.

5. Proof of Theorems

For each integer d, there exists a (complex) irreducible linear representation of $SU(2)$. We denote by $(V(d)_{\mathbf{R}}, \rho)$ the real representation of $SU(2)$ obtained by the restriction of the coefficient field. Then $(V(d)_{\mathbf{R}}, \rho)$ is irreducible if d is odd. $(V(d)_{\mathbf{R}}, \rho)$ is reducible if d is even and we denote by $V_0(d)$ one of the irreducible component of $V(d)_{\mathbf{R}}$. In this section we study whether there exists an orbit of constant curvature which is a minimal submanifold in the unit sphere in $V(d)_{\mathbf{R}}$ or $V_0(d)$.

Let $\langle\ ,\ \rangle$ be the real part of the $SU(2)$-invariant Hermitian inner product $(\ ,\ )$ on $V(d)$ defined in (1.1). Then $\langle\ ,\ \rangle$ is an $SU(2)$-invariant inner product on $V(d)_{\boldsymbol{R}}$.

Let $p=\sum_{i=0}^{d} z_i P_i \in S_1^{2d+1}$, i.e.,

$$\sum_{i=0}^{d} z_i \bar{z}_i = 1 . \tag{5.1}$$

By a formula of Freudenthal [14], we have

$$\rho(X_1)^2+\rho(X_2)^2+\rho(X_3)^2 = -d(d+2)1 . \tag{5.2}$$

Then the following is an immediate consequence of Lemma 4.2.

Lemma 5.1. *If an orbit $M=\rho(SU(2))\ (p)$ is a space of constant curvature k, then*
(i) $k=3/d(d+2)$,
(ii) *M is a minimal submanifold in S_1^{2d+1}.*

By virtue of the above Lemma, we have only to verify the existence of an orbit of constant curvature in S_1^{2d+1} to prove Theorem A.

Extend ρ: $\mathfrak{su}(2)\to\mathfrak{gl}(d+1, \boldsymbol{C})$ to $\mathfrak{sl}(2, \boldsymbol{C})=(\mathfrak{su}(2))^{\boldsymbol{C}}$ and put

$$H = -(-1)^{1/2}X_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = X_2-(-1)^{1/2}X_3 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},$$
$$Y = -X_2-(-1)^{1/2}X_3 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

Then from (1.2), we get

$$\rho(H)(P_j) = (2j-d)P_j , \qquad 0\leqq j\leqq d , \tag{5.3$_1$}$$
$$\rho(X)(P_j) = -2((d-j)(j+1))^{1/2}P_{j+1} , \qquad 0\leqq j\leqq d , \tag{5.3$_2$}$$
$$\rho(Y)(P_j) = -2(j(d-j+1))^{1/2}P_{j-1} , \qquad 0\leqq j\leqq d . \tag{5.3$_3$}$$

where we put $P_{-1}=P_{d+1}=0$.

Lemma 5.2. *An orbit $M=\rho(SU(2))\ (p)$ is a space of constant curvature $3/d(d+2)$ if and only if*

$$(\rho(H)(p), \rho(X)(p))+\overline{(\rho(H)(p), \rho(Y)(p))} = 0 , \tag{5.4$_1$}$$
$$(\rho(X)(p), \rho(Y)(p)) = 0 , \tag{5.4$_2$}$$
$$(\rho(H)(p), \rho(H)(p)) = d(d+2)/3 . \tag{5.4$_3$}$$

Proof. By definition of H, X and Y

$$X_1 = (-1)^{1/2}H , \quad X_2 = X-Y , \quad X_3 = (-1)^{1/2}(X+Y) .$$

A simple computation shows

$$\begin{aligned}&\langle\rho(X_1)(p),\ \rho(X_2)(p)\rangle\\&=\langle(-1)^{1/2}\rho(H)(p),\ \rho(X)(p)-\rho(Y)(p)\rangle\\&=-\mathrm{Im}\,(\rho(H)(p),\ \rho(X)(p))+\mathrm{Im}\,(\rho(H)(p),\ \rho(Y)(p))\,.\end{aligned}$$

Similarly

$$\begin{aligned}&\langle\rho(X_1)(p),\ \rho(X_3)(p)\rangle\\&=\mathrm{Re}\,(\rho(H)(p),\ \rho(X)(p))+\mathrm{Re}\,(\rho(H)(p),\ \rho(Y)(p))\,,\\&\langle\rho(X_2)(p),\ \rho(X_3)(p)\rangle\\&=2\,\mathrm{Im}\,(\rho(X)(p),\ \rho(Y)(p))\,,\\&\langle\rho(X_1)(p),\ \rho(X_1)(p)\rangle\\&=(\rho(H)(p),\ \rho(H)(p))\,,\\&\langle\rho(X_2)(p),\ \rho(X_2)(p)\rangle\\&=(\rho(X)(p),\ \rho(X)(p))+(\rho(Y)(p),\ \rho(Y)(p))-2\mathrm{Re}\,(\rho(X)(p),\ \rho(Y)(p))\,,\\&\langle\rho(X_3)(p),\ \rho(X_3)(p)\rangle\\&=(\rho(X)(p),\ \rho(X)(p))+(\rho(Y)(p),\ \rho(Y)(p))+2\,\mathrm{Re}\,(\rho(X)(p),\ \rho(Y)(p))\,.\end{aligned}$$

An orbit $M=\rho(SU(2))(p)$ is a space of constant curvature $3/d(d+2)$ if and only if

$$\langle\rho(X_i)(p),\ \rho(X_j)(p)\rangle=d(d+2)/3\ \delta_{ij}\,,\qquad 1\leqq i,\ i\leqq 3\,,$$

by Lemma 4.2. Taking (5.2) into account, the Lemma is an immediate consequence. Q.E.D.

Proof of Theorems. Let $p=\sum_{j=0}^{d} z_jP_j$ be a point in S_1^{2d+1}, i.e.,

(5.1) $$\sum_{j=0}^{d} z_j\bar{z}_j=1\,.$$

From $(5.3)_1$, $(5.3)_2$ and $(5.3)_3$, we get

$$\begin{aligned}\rho(H)(p)&=\sum_{j=0}^{d}(2j-d)z_jP_j\,,\\\rho(X)(p)&=-2\sum_{j=0}^{d-1}((d-j)(j+1))^{1/2}z_jP_{j+1}\,,\\\rho(Y)(p)&=-2\sum_{j=1}^{d}(j(d-j+1))^{1/2}z_jP_{j-1}\,.\end{aligned}$$

Then

$$\begin{aligned}&\rho((H)(p),\ \rho(X)(p))+\overline{(\rho(H)(p),\ \rho(Y)(p))}\\&=-2\sum_{j=1}^{d}(2j-d)(j(d-j+1))^{1/2}z_j\bar{z}_{j-1}-2\sum_{j=0}^{d-1}(2j-d)((j+1)(d-j))^{1/2}z_j\bar{z}_{j+1}\,,\\&(\rho(X)(p),\ \rho(Y)(p))\\&=4\sum_{j=0}^{d-1}(j(j+1)(d-j+1)(d-j))^{1/2}z_{j-1}\bar{z}_{j+1}\,,\\&(\rho(H)(p),\ \rho(H)(p))\\&=\sum_{j=0}^{d}(d^2-4dj+4j^2)z_j\bar{z}_j\,.\end{aligned}$$

So $(5.4)_1$ and $(5.4)_2$ is equivalent to the following

$$(5.5)_1 \quad \sum_{j=1}^{d} (2j-d)(j(d-j+1))^{1/2} z_j \bar{z}_{j-1} + \sum_{j=0}^{d-1} (2j-d)((j+1)(d-j))^{1/2} z_j \bar{z}_{j+1} = 0,$$

$$(5.5)_2 \quad \sum_{j=1}^{d-1} (j(j+1)(d-j+1)(d-j))^{1/2} z_{j-1} \bar{z}_{j+1} = 0.$$

Taking (5.1) into account, $(5.4)_3$ is equivalent to

$$(5.5)_3 \quad \sum_{j=0}^{d} (6j^2 - 6dj + d^2 - d)\, z_j \bar{z}_j = 0.$$

Now we prove the system of equations $(5.5)_1$, $(5.5)_2$ and $(5.5)_3$ has a solution under the condition (5.1)

When $d=4$ we put

$$z_i = \begin{cases} 1/2 \quad , & \text{if } i = 0, 4, \\ (-2)^{1/2}/2, & \text{if } i = 2, \\ 0 \quad , & \text{if } i = 1, 3. \end{cases}$$

When d is an even ınteger $d=2d'$ and $d \geqq 6$, we put

$$z_i = \begin{cases} ((d'+1)/6d')^{1/2} \quad , & \text{if } i = 0, d, \\ (-1)^{d'/2}((2d'-1)/3d')^{1/2}, & \text{if } j = d', \\ 0 \quad , & \text{if otherwise}. \end{cases}$$

When d is an odd integer $d=2d'+1$, $d' \geqq 2$, we put

$$z_i = \begin{cases} ((d'+2)/(3d'+3))^{1/2} \;, & \text{if } i = 0, \\ ((2d'+1)/(3d'+3))^{1/2}, & \text{if } i = d'+1, \\ 0 \quad , & \text{if otherwise}. \end{cases}$$

Then it is easily verified that $(z_0, z_1, \cdots, z_d)$ is a solution of the equation. So Theorem A is proved.

When d is an even integer, $d \geqq 6$, $\sum_{i=0}^{d} z_i P_i$ is contained in $V_0(d)$ by Lemma 2.1. So the orbit passing this point must be contained in the unit sphere in $V_0(d)$. So we get Theorem B. Q.E.D.

In Theorem B the case $d=4$ is excluded. But this is a natural consequence of the following

Theorem 5.7 (J.D. Moore, [10]). *Let M be a connected n-dimensional Riemannian manifold of constant curvature k isometrically and minimally immersed in a simply connected $(2n-1)$-dimensional Riemannian manifold N of constant curvature K. Then either M is totally geodesic or it is flat.*

Recently Li [9] proved the following

Theorem. *If* Φ: $S^m \to S_1^n$ *is an isometric minimal immersion, then* $\Phi(S^m)$ *is either an embedded sphere or an embedded projective space.*

But this is not true if the codimension is not maximal. Let M be the orbit passing $(2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3$ in $V_0(6)$. As we proved, M is a space of constant curvature $1/16$ and is a minimal submanifold in S_1^6. But the orbit is neither an embedded sphere nor an embedded projective space in S_1^6. Namely we have the following

Proposition 5.8. *Let* π *be the covering map*

$$\pi\colon SU(2) \to M;\ g \to \rho(g)((2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3)\,.$$

Then π *is at least 6-fold.*

Proof. Put $g=\begin{bmatrix}\alpha & \\ & \alpha^{-1}\end{bmatrix}$, $\alpha=e^{(-1)^{1/2}k\pi/3}$ $(0\leqq k\leqq 5)$. Then

$$\begin{aligned}&\rho(g)((2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3)\\ &=(2^{1/2}\alpha^{-6}P_0-(-5)^{1/2}\alpha^{-3}\alpha^3P_3+2^{1/2}\alpha^6P_6)/3\\ &=(2^{1/2}P_0-(-5)^{1/2}P_3+2^{1/2}P_6)/3\end{aligned}$$

So the covering π is at least 6-fold. Q.E.D.

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