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MINIMAL IMMERSIONS OF 3-DIMENSIONAL SPHERE INTO SPHERES

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Introduction

Let S_c^n be the n -dimensional sphere with constant curvature c . Let Δ be the Laplace-Beltrami operator on S_1^n . The spectre and eigen-functions of Δ are well-known [2]. Let V^d be the eigen-space of Δ corresponding to the d -th eigen-value $\lambda_d = d(d+n-1)$. Let $f_0, f_1, \dots, f_{m(d)}$ be an orthonormal basis of V^d with respect to the inner product. Then

$$\begin{aligned} \psi_{n,d}: S_{k(d)}^n &\rightarrow S_1^{m(d)}(\subset \mathbf{R}^{m(d)+1}) \\ &; p \rightarrow 1/(m(d)+1)(f_0(p), f_1(p), \dots, f_{m(d)}(p)), \end{aligned}$$

is an isometric minimal immersion, where $k(d)$ and $m(d)$ are as follows [6];

$$\begin{aligned} k(d) &= n/d(d+n-1), \\ m(d) &= (2d+n-1)(d+n-2)!/d!(n-1)!-1. \end{aligned}$$

It is proved that any isometric minimal immersion of S_c^2 into S_1^N is equivalent to $\psi_{2,d}$ for some d , [3], [6]. But it is not true if the dimension n is greater than 3. In fact do Carmo and Wallach proved the following

Theorem 0.1 (do Carmo and Wallach, [7]). *Let $f: S_c^n \rightarrow S_1^N$ be an isometric minimal immersion. Then*

- (i) *there exists an integer d such that $c=k(d)$.*
- (ii) *There exists a positive semi-definite matrix A of size $(m(d)+1) \times (m(d)+1)$ such that f is equivalent to $A \circ \psi_{n,d}$.*
- (iii) *If $n=2$ or $d \leq 3$, then A is the identity matrix.*
- (iv) *If $n \geq 3$ and $d \geq 4$, then A is parametrized by a compact convex body L in some finite dimensional vector space, $\dim L \geq 18$. If A is an interior point of L then $N=m(d)$, and if A is a boundary point of L then $N < m(d)$.*

There are some problems concerning (iv) of the above Theorem.

Problem 0.2 (Chern, [4]). *Let $S_c^3 \rightarrow S_1^7$ be an isometric minimal immersion. Is it totally geodesic?*

In [5], do Carmo posed a more general

Problem 0.3. *Determine the lower bound $1(d)$ of the dimension N of the sphere S_1^N into which a given $S_{k(d)}^n$ can be isometrically and minimally immersed.*

Recently Problem 0.2 was negatively answered by N. Ejiri [8]. In fact he proved that there exists an isometric minimal immersion $S_{1/16}^3 \rightarrow S_1^6$.

As for the Problem 0.3, scarcely anything is known.

In this paper we confine our consideration to the case $n=3$. In this case S^3 has a structure of a Lie group, $S^3 = SU(2)$. We investigate whether there exists an orbit in a representation space V of $SU(2)$, which is a minimal submanifold in the unit sphere in V . And we give an estimate for $1(d)$ (of the Problem 0.3 in the case $n=3$). The following will be proved.

Theorem A. *Let d be an integer, $d \geq 4$. Then there exists an isometric minimal immersion of $S_{3/4(d+2)}^3$ into S_1^{2d+1} .*

Theorem B. *Let d be an even integer, $d \geq 6$. Then there exists an isometric minimal immersion of $S_{3/4(d+2)}^3$ into S_1^d .*

1. Complex linear representations of $SU(2)$

In this section we give a brief review on the complex linear representation of $SU(2)$.

The special unitary group $SU(2)$ is the group of matrices which acts on \mathbf{C}^2 and leaves invariant the usual Hermitian inner product on \mathbf{C} . We can identify $SU(2)$ with the 3-dimensional unit sphere $S_1^3 (\subset \mathbf{C}^2)$ by

$$SU(2) \rightarrow S_1^3: g \rightarrow g \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g \in SU(2).$$

Then the induced metric on $SU(2)$ by the above diffeomorphism is the bi-invariant metric on $SU(2)$.

A homogeneous polynomial on \mathbf{C}^2 is called of degree d if it satisfies

$$P(\lambda z, \lambda w) = \lambda^d P(z, w), \lambda \in \mathbf{C}, z, w \in \mathbf{C}.$$

For each positive integer d , let $V(d)$ be the space of homogeneous polynomials of type $(d, 0)$ on \mathbf{C}^2 . Then $SU(2)$ acts on $V(d)$ as follows

$$(\rho(g)(P))(z, w) = P({}^t(g^{-1} \cdot {}^t(z, w))), g \in SU(2), z, w \in \mathbf{C}, P \in V(d).$$

Then $(V(d), \rho)$ is a complex irreducible representation and each complex irreducible representation of $SU(2)$ is equivalent to $(V(d), \rho)$ for some d [12].

Define a Hermitian inner product in $V(d)$ by

$$(1.1) \quad (P, Q) = (d+1) \int_{g \in SU(2)} P({}^t(g \cdot {}^t(1.0))) \overline{Q({}^t(g \cdot {}^t(1.0)))} dg$$

where dg is the normalized Haar measure on $SU(2)$. Let P_i be the polynomial in $V(d)$ defined by

$$P_i(z, w) = ({}_d C_i)^{1/2} z^{d-i} w^i, \quad z, w \in \mathbf{C}.$$

Then P_0, P_1, \dots, P_d is an orthonormal basis of $V(d)$.

Let $\mathfrak{su}(2)$ be the Lie algebra of $SU(2)$. Take the following basis of $\mathfrak{su}(2)$ and fix them once for all.

$$X_1 = \begin{bmatrix} (-1)^{1/2} & 0 \\ 0 & -(-1)^{1/2} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & (-1)^{1/2} \\ (-1)^{1/2} & 0 \end{bmatrix}.$$

Then the bracket relations of X_1, X_2 and X_3 are

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

We denote also by ρ the representation of $\mathfrak{su}(2)$ induced by the representation of $SU(2)$, i.e.,

$$\rho(A)(P) = d/dt|_{t=0} \rho(\exp tA)(P), \quad A \in \mathfrak{su}(2).$$

Then by a direct calculation we get

$$\begin{aligned} (1.2)_1 \quad \rho(X_1)(P_j) &= (-1)^{1/2}(2j-d)P_j, & 0 \leq j \leq d, \\ (1.2)_2 \quad \rho(X_2)(P_j) &= -((d-j)(j+1))^{1/2}P_{j+1} + (j(d-j+1))^{1/2}P_{j-1}, & 0 \leq j \leq d, \\ (1.2)_3 \quad \rho(X_3)(P_j) &= -(-(d-j)(j+1))^{1/2}P_{j+1} - (-j(d-j+1))^{1/2}P_{j-1}, & 0 \leq j \leq d, \end{aligned}$$

where we put $P_{-1} = P_{d+1} = 0$.

2. Real irreducible representations of $SU(2)$

In this section we give a brief review on real irreducible representations of $SU(2)$.

Let G be a compact Lie group and (V, ρ) be a complex irreducible representation of G . Then (V, ρ) is said to be self-conjugate if V has a structure map j , i.e., a conjugate linear map on V such that

$$\begin{aligned} j(\rho(g)v) &= \rho(g)j(v), \quad g \in G, \quad v \in V, \\ j(\alpha v + \beta w) &= \bar{\alpha}j(v) + \bar{\beta}j(w), \quad \alpha, \beta \in \mathbf{C}, \quad v, w \in V, \\ j^2 &= \pm 1. \end{aligned}$$

A self-conjugate representation (V, ρ) is said to be of index 1 (resp. -1) if $j^2=1$ (resp. $j^2=-1$). For simple Lie groups self-conjugate representations and their indices are known [13]. We denote by $(V_{\mathbf{R}}, \rho)$ the representation of G over \mathbf{R} obtained by the restriction of the coefficient field from \mathbf{C} to \mathbf{R} .

Let (V, ρ) be a self-conjugate representation of G of index -1 . Then $(V_{\mathbf{R}}, \rho)$ is also irreducible. But $(V_{\mathbf{R}}, \rho)$ is reducible if (V, ρ) is a self-conjugate representation of G of index 1. Namely $(1+j)V_{\mathbf{R}}$ and $(1-j)V_{\mathbf{R}}$ are mutually equivalent real irreducible representation of G and

$$V_{\mathbf{R}} = (1+j)V_{\mathbf{R}} + (1-j)V_{\mathbf{R}}, \text{ (direct sum).}$$

For these facts we refer, for instance, to [1].

Now we confine our attention to the case $G=SU(2)$.

Let j be a conjugate-linear automorphism on \mathbf{C}^2 defined by

$$j(z, w) = (-\bar{w}, \bar{z}), \quad z, w \in \mathbf{C}.$$

Extend j to an automorphism on $V(d)$ by

$$(jP)(z, w) = \overline{P(j(z, w))}, \quad z, w \in \mathbf{C}.$$

Then j is a structure map on $V(d)$ with $j^2=(-1)^d$. So $(V(d)_{\mathbf{R}}, \rho)$ is a self-conjugate representation of index $(-1)^d$. Let d be an even integer $d=2d'$ and put $Q_i=(-1)^{i/2}P_i$, $0 \leq i \leq d$. Then

$$jP_i = (-1)^i P_{d-i}, \quad jQ_i = -(-1)^i Q_{d-i}, \quad 0 \leq i \leq d.$$

Since $P_0, P_1, \dots, P_d, Q_0, Q_1, \dots, Q_d$ are basis of $V(d)_{\mathbf{R}}$, $(1+j)P_i, (1+j)Q_i$, $0 \leq i \leq d$, are generators of $(1+j)V(d)_{\mathbf{R}}$. It is easily seen that $(1+j)P_i, (1-j)Q_i$, $0 \leq i \leq d-1$ and $(1+j)P_{d'}$ [resp. $(1+j)Q_{d'}$] are basis of $(1+j)V(d)_{\mathbf{R}}$ if d' is an even [resp. odd] integer. We denote $(1+j)V(d)_{\mathbf{R}}$ by $V_0(d)$.

Lemma 2.1. *Let d be an even integer, $d=2d'$. Then $\sum_{i=0}^d z_i P_i$ is contained in $V_0(d)$ if and only if*

$$z_i = (-1)^i \bar{z}_{d-i}, \quad 0 \leq i \leq d'.$$

Proof.

$$\begin{aligned} \sum_{i=0}^d z_i P_i &= (\operatorname{Re} z_0 P_0 + \operatorname{Re} z_d P_d) + (\operatorname{Im} z_0 Q_0 + \operatorname{Im} z_d Q_d) \\ &\quad + (\operatorname{Re} z_1 P_1 + \operatorname{Re} z_{d-1} P_{d-1}) + (\operatorname{Im} z_1 Q_1 + \operatorname{Im} z_{d-1} Q_{d-1}) \\ &\quad + \dots \\ &\quad + z_{d'} P_{d'}. \end{aligned}$$

Remember that $P_j + (-1)^j P_{d-j}, Q_j - (-1)^j Q_{d-j}$, $0 \leq j \leq d'-1$ and $P_{d'}$ [resp. $Q_{d'}$] are basis of $V_0(d)$ if d' is an even [resp. odd] integer. So $\sum_{i=0}^d z_i P_i$ is contained in $V_0(d)$ if and only if

$$\begin{aligned} \operatorname{Re} z_i &= (-1)^i \operatorname{Re} z_{d-i}, & \operatorname{Im} z_i &= -(-1)^i \operatorname{Im} z_{d-i}, & 0 \leq i \leq d'-1. \\ \operatorname{Im} z_{d'} &= 0 \text{ [resp. } \operatorname{Re} z_{d'} = 0 \text{] if } d' \text{ is even [resp. odd].} \end{aligned}$$

So we get the Lemma.

Q.E.D.

3. Orbits in a sphere

Let G be a Lie subgroup in $SO(N+1)$. Then G acts on the unit sphere S_1^N in \mathbf{R}^{N+1} centered at the origin in a natural manner. Take a point p_0 in S_1^N and let M be the orbit of the action of G through p_0 .

Let \mathfrak{g} be the Lie algebra of G . We denote by A^* the vector field on S_1^N defined by

$$(3.1) \quad A^*|_p = d/dt|_{t=0} \exp(tA)(p), \quad p \in S_1^N.$$

We consider elements of \mathfrak{g} as skew symmetric $(N+1) \times (N+1)$ -matrices in a natural manner. Then we get from (3.1) the following

$$A^*|_p = A(p), \quad A \in \mathfrak{g}, \quad p \in S_1^N.$$

So the tangent space of M at p is

$$T_p(M) = \{A(p) \mid A \in \mathfrak{g}\}.$$

Let $N_p(M)$ be the normal space at p in S_1^N . Consider the tangent space $T_p(M)$ and the normal space $N_p(M)$ as a subspace in \mathbf{R}^{N+1} . Then \mathbf{R}^{N+1} is decomposed into the direct sum

$$(3.2) \quad \mathbf{R}^{N+1} = \mathbf{R}p + T_p(M) + N_p(M).$$

For a vector A in \mathbf{R}^{N+1} , we denote A^T and A^N the $T_p(M)$ -component and $N_p(M)$ -component of A in the decomposition (3.2) respectively.

Lemma 3.1. *Let G be a Lie subgroup in $SO(N+1)$. Let α be the second fundamental form of the orbit $G \cdot p$ in S_1^N . Then*

$$(3.3) \quad \alpha(A^*, B^*)|_p = (A(B(p)))^N,$$

$$(3.4) \quad \nabla_{B^*} A^*|_p = (A(B(p)))^T, \quad A, B \in \mathfrak{g}.$$

where ∇ is the Riemannian connection on M .

Proof. Let D be the Riemannian connection in \mathbf{R}^{N+1} . Then

$$\begin{aligned} D_{B^*} A^*|_p &= d/dt|_{t=0} A^*|_{\exp(tB)(p)} \\ &= d/dt|_{t=0} A(\exp(tB)(p)) \\ &= A(B(p)). \end{aligned}$$

Since $\alpha(A^*, B^*)|_p = (D_{B^*} A^*|_p)^N$ and $\nabla_{B^*} A^*|_p = (D_{B^*} A^*|_p)^T$, we get the Lemma. Q.E.D.

4. Left invariant metrics on $SU(2)$ and $SO(3)$

In this section we denote by G the Lie group $SU(2)$ or $SO(3)$. The Lie algebras of $SU(2)$ and $SO(3)$ are mutually isomorphic. We denote them by $\mathfrak{su}(2)$.

Let B be the Killing form of $\mathfrak{su}(2)$. Then X_1, X_2, X_3 defined in §1 are orthonormal with respect to $-B/8$. Let g_0 be the Riemannian metric on G which is the bi-invariant extension of $-B/8$.

Lemma 4.1. [11]. *Let g be an inner product on $\mathfrak{su}(2)$. Then there exists an element σ in G such that*

- (i) $X'_i = Ad(\sigma)(X_i)$, $i=1, 2, 3$, are mutually orthogonal with respect to g .
- (ii) $g = \lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2$, where λ_i are positive constants and $\omega_i(\cdot) = g_0(X'_i, \cdot)$, $i=1, 2, 3$.

Let g be the Riemannian metric on G which is the left invariant extension of the inner product g on $\mathfrak{su}(2)$. Extend $X'_i/(\lambda_i)^{1/2}$, $1 \leq i \leq 3$, to the left invariant vector fields Y_i , $1 \leq i \leq 3$. Let θ_i , $1 \leq i \leq 3$, be the dual coframe fields on G to Y_i , $1 \leq i \leq 3$. Let θ_{ij} (resp. Ω_{ij}) be the connection (resp. curvature) form of (G, g) with respect to the orthonormal frame fields Y_1, Y_2, Y_3 . Then we get easily

$$\begin{aligned} \theta_{12} &= -(\lambda_1 + \lambda_2 - \lambda_3)/(\lambda_1 \lambda_2 \lambda_3)^{1/2} \theta_3, \\ \theta_{23} &= -(\lambda_2 + \lambda_3 - \lambda_1)/(\lambda_1 \lambda_2 \lambda_3)^{1/2} \theta_1, \\ \theta_{31} &= -(\lambda_3 + \lambda_1 - \lambda_2)/(\lambda_1 \lambda_2 \lambda_3)^{1/2} \theta_2, \\ \Omega_{12} &= (((\lambda_1 - \lambda_2)^2 - 3\lambda_3^2 + 2\lambda_3(\lambda_1 + \lambda_2))/\lambda_1 \lambda_2 \lambda_3) \theta_1 \wedge \theta_2, \\ \Omega_{23} &= (((\lambda_2 - \lambda_3)^2 - 3\lambda_1^2 + 2\lambda_1(\lambda_2 + \lambda_3))/\lambda_1 \lambda_2 \lambda_3) \theta_2 \wedge \theta_3, \\ \Omega_{31} &= (((\lambda_3 - \lambda_1)^2 - 3\lambda_2^2 + 2\lambda_2(\lambda_3 + \lambda_1))/\lambda_1 \lambda_2 \lambda_3) \theta_3 \wedge \theta_1. \end{aligned}$$

So (G, g) is a space of constant curvature k if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 1/k$, i.e., $g = (1/k)g_0$.

Let (V, ρ) be a real representation of G and \langle, \rangle be a G -invariant inner product on V . Then an orbit M of G through a unit vector $p \in V$ is contained in the unit sphere S_1 (in V centered at the origin).

Lemma 4.2. (i) *The orbit M is a 3-dimensional space of constant curvature k if and only if*

$$\langle \rho(X_i)(p), \rho(X_j)(p) \rangle = \delta_{ij}/k, \quad 1 \leq i, j \leq 3.$$

(ii) *Assume that the orbit M is a 3-dimensional space of constant curvature k .*

Then M is a minimal submanifold in S_1 if and only if

$$\sum_{j=1}^3 \rho(X_j)^2(p) = -3kp.$$

Proof. Define a map $f: G \rightarrow S_1$ by

$$f(\sigma) = \rho(\sigma)(p), \quad \sigma \in S_1.$$

Then

$$f_*(X_i) = \rho(X_i)(p).$$

Let g be the induced metric on G of f_* . Then g is a left invariant metric. So (G, g) is a 3-dimensional space of constant curvature k if and only if $g=(1/k)g_0$. By definition of g

$$\begin{aligned} g(X_i, X_j) &= \langle \rho(X_i)(p), \rho(X_j)(p) \rangle \\ &= g_0(X_i, X_j)/k \\ &= \delta_{ij}/k, \quad 1 \leq i, j \leq 3, \end{aligned}$$

if and only if $g=(1/k)g_0$.

(ii) Since (G, g) is a space of constant curvature, $\exp tX_i$ are geodesics in (G, g) . By Lemma 3.1, $(\rho(X_i))^2(p)$ is normal to M . Consider the vector $\sum_{i=1}^3 (\rho(X_i))^2(p)$ in V , which is normal to M . Then its $N_p(M)$ -components in the decomposition (3.2) is the mean curvature vector of M in S_1 at p . Since M is an orbit of a representation of G , M is a minimal submanifold in S_1 if and only if the mean curvature vector of M in S_1 at one point is 0. So M is a minimal submanifold if and only if

$$(4.1) \quad \sum_{i=1}^3 (\rho(X_i))^2(p) = cp,$$

for some constant c . Assume that (4.1) holds, then

$$\begin{aligned} c &= \langle \sum_{i=1}^3 (\rho(X_i))^2(p), p \rangle \\ &= -\sum_{i=1}^3 \langle \rho(X_i)(p), \rho(X_i)(p) \rangle \\ &= -3k. \end{aligned}$$

Q.E.D.

5. Proof of Theorems

For each integer d , there exists a (complex) irreducible linear representation of $SU(2)$. We denote by $(V(d)_{\mathbf{R}}, \rho)$ the real representation of $SU(2)$ obtained by the restriction of the coefficient field. Then $(V(d)_{\mathbf{R}}, \rho)$ is irreducible if d is odd. $(V(d)_{\mathbf{R}}, \rho)$ is reducible if d is even and we denote by $V_0(d)$ one of the irreducible component of $V(d)_{\mathbf{R}}$. In this section we study whether there exists an orbit of constant curvature which is a minimal submanifold in the unit sphere in $V(d)_{\mathbf{R}}$ or $V_0(d)$.

Let \langle , \rangle be the real part of the $SU(2)$ -invariant Hermitian inner product $(,)$ on $V(d)$ defined in (1.1). Then \langle , \rangle is an $SU(2)$ -invariant inner product on $V(d)_{\mathbf{R}}$.

Let $p = \sum_{i=0}^d z_i P_i \in S_1^{2d+1}$, i.e.,

$$(5.1) \quad \sum_{i=0}^d z_i \bar{z}_i = 1.$$

By a formula of Freudenthal [14], we have

$$(5.2) \quad \rho(X_1)^2 + \rho(X_2)^2 + \rho(X_3)^2 = -d(d+2)1.$$

Then the following is an immediate consequence of Lemma 4.2.

Lemma 5.1. *If an orbit $M = \rho(SU(2)) (p)$ is a space of constant curvature k , then*

- (i) $k = 3/d(d+2)$,
- (ii) M is a minimal submanifold in S_1^{2d+1} .

By virtue of the above Lemma, we have only to verify the existence of an orbit of constant curvature in S_1^{2d+1} to prove Theorem A.

Extend $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{gl}(d+1, \mathbf{C})$ to $\mathfrak{sl}(2, \mathbf{C}) = (\mathfrak{su}(2))^{\mathbf{C}}$ and put

$$H = -(-1)^{1/2} X_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = X_2 - (-1)^{1/2} X_3 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},$$

$$Y = -X_2 - (-1)^{1/2} X_3 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

Then from (1.2), we get

$$(5.3)_1 \quad \rho(H)(P_j) = (2j-d)P_j, \quad 0 \leq j \leq d,$$

$$(5.3)_2 \quad \rho(X)(P_j) = -2((d-j)(j+1))^{1/2} P_{j+1}, \quad 0 \leq j \leq d,$$

$$(5.3)_3 \quad \rho(Y)(P_j) = -2(j(d-j+1))^{1/2} P_{j-1}, \quad 0 \leq j \leq d.$$

where we put $P_{-1} = P_{d+1} = 0$.

Lemma 5.2. *An orbit $M = \rho(SU(2)) (p)$ is a space of constant curvature $3/d(d+2)$ if and only if*

$$(5.4)_1 \quad (\rho(H)(p), \rho(X)(p)) + \overline{(\rho(H)(p), \rho(Y)(p))} = 0,$$

$$(5.4)_2 \quad (\rho(X)(p), \rho(Y)(p)) = 0,$$

$$(5.4)_3 \quad (\rho(H)(p), \rho(H)(p)) = d(d+2)/3.$$

Proof. By definition of H , X and Y

$$X_1 = (-1)^{1/2} H, \quad X_2 = X - Y, \quad X_3 = (-1)^{1/2} (X + Y).$$

A simple computation shows

$$\begin{aligned} & \langle \rho(X_1)(p), \rho(X_2)(p) \rangle \\ &= \langle (-1)^{1/2} \rho(H)(p), \rho(X)(p) - \rho(Y)(p) \rangle \\ &= -\text{Im}(\rho(H)(p), \rho(X)(p)) + \text{Im}(\rho(H)(p), \rho(Y)(p)). \end{aligned}$$

Similarly

$$\begin{aligned} & \langle \rho(X_1)(p), \rho(X_3)(p) \rangle \\ &= \text{Re}(\rho(H)(p), \rho(X)(p)) + \text{Re}(\rho(H)(p), \rho(Y)(p)), \\ & \quad \langle \rho(X_2)(p), \rho(X_3)(p) \rangle \\ &= 2 \text{Im}(\rho(X)(p), \rho(Y)(p)), \\ & \quad \langle \rho(X_1)(p), \rho(X_1)(p) \rangle \\ &= (\rho(H)(p), \rho(H)(p)), \\ & \quad \langle \rho(X_2)(p), \rho(X_2)(p) \rangle \\ &= (\rho(X)(p), \rho(X)(p)) + (\rho(Y)(p), \rho(Y)(p)) - 2 \text{Re}(\rho(X)(p), \rho(Y)(p)), \\ & \quad \langle \rho(X_3)(p), \rho(X_3)(p) \rangle \\ &= (\rho(X)(p), \rho(X)(p)) + (\rho(Y)(p), \rho(Y)(p)) + 2 \text{Re}(\rho(X)(p), \rho(Y)(p)). \end{aligned}$$

An orbit $M = \rho(SU(2))(p)$ is a space of constant curvature $3/d(d+2)$ if and only if

$$\langle \rho(X_i)(p), \rho(X_j)(p) \rangle = d(d+2)/3 \delta_{ij}, \quad 1 \leq i, j \leq 3,$$

by Lemma 4.2. Taking (5.2) into account, the Lemma is an immediate consequence. Q.E.D.

Proof of Theorems. Let $p = \sum_{j=0}^d z_j P_j$ be a point in S^{2d+1}_1 , i.e.,

$$(5.1) \quad \sum_{j=0}^d z_j \bar{z}_j = 1.$$

From (5.3)₁, (5.3)₂ and (5.3)₃, we get

$$\begin{aligned} \rho(H)(p) &= \sum_{j=0}^d (2j-d) z_j P_j, \\ \rho(X)(p) &= -2 \sum_{j=0}^{d-1} ((d-j)(j+1))^{1/2} z_j P_{j+1}, \\ \rho(Y)(p) &= -2 \sum_{j=1}^d (j(d-j+1))^{1/2} z_j P_{j-1}. \end{aligned}$$

Then

$$\begin{aligned} & \rho(H)(p), \rho(X)(p) + \overline{(\rho(H)(p), \rho(Y)(p))} \\ &= -2 \sum_{j=1}^d (2j-d)(j(d-j+1))^{1/2} z_j \bar{z}_{j-1} - 2 \sum_{j=0}^{d-1} (2j-d)((j+1)(d-j))^{1/2} z_j \bar{z}_{j+1}, \\ & \quad (\rho(X)(p), \rho(Y)(p)) \\ &= 4 \sum_{j=0}^{d-1} (j(j+1)(d-j+1)(d-j))^{1/2} z_{j-1} \bar{z}_{j+1}, \\ & \quad (\rho(H)(p), \rho(H)(p)) \\ &= \sum_{j=0}^d (d^2 - 4dj + 4j^2) z_j \bar{z}_j. \end{aligned}$$

So (5.4)₁ and (5.4)₂ is equivalent to the following

$$(5.5)_1 \quad \sum_{j=1}^d (2j-d)(j(d-j+1))^{1/2} z_j \bar{z}_{j-1} + \sum_{j=0}^{d-1} (2j-d)((j+1)(d-j))^{1/2} z_j \bar{z}_{j+1} = 0,$$

$$(5.5)_2 \quad \sum_{j=1}^{d-1} (j(j+1)(d-j+1)(d-j))^{1/2} z_{j-1} \bar{z}_{j+1} = 0.$$

Taking (5.1) into account, (5.4)₃ is equivalent to

$$(5.5)_3 \quad \sum_{j=0}^d (6j^2 - 6dj + d^2 - d) z_j \bar{z}_j = 0.$$

Now we prove the system of equations (5.5)₁, (5.5)₂ and (5.5)₃ has a solution under the condition (5.1)

When $d=4$ we put

$$z_i = \begin{cases} 1/2 & , & \text{if } i = 0, 4, \\ (-2)^{1/2}/2, & \text{if } i = 2, \\ 0 & , & \text{if } i = 1, 3. \end{cases}$$

When d is an even integer $d=2d'$ and $d \geq 6$, we put

$$z_i = \begin{cases} ((d'+1)/6d')^{1/2} & , & \text{if } i = 0, d, \\ (-1)^{d'/2}((2d'-1)/3d')^{1/2}, & \text{if } j = d', \\ 0 & , & \text{if otherwise.} \end{cases}$$

When d is an odd integer $d=2d'+1$, $d' \geq 2$, we put

$$z_i = \begin{cases} ((d'+2)/(3d'+3))^{1/2} & , & \text{if } i = 0, \\ ((2d'+1)/(3d'+3))^{1/2}, & \text{if } i = d'+1, \\ 0 & , & \text{if otherwise.} \end{cases}$$

Then it is easily verified that (z_0, z_1, \dots, z_d) is a solution of the equation. So Theorem A is proved.

When d is an even integer, $d \geq 6$, $\sum_{i=0}^d z_i P_i$ is contained in $V_0(d)$ by Lemma 2.1. So the orbit passing this point must be contained in the unit sphere in $V_0(d)$. So we get Theorem B. Q E.D.

In Theorem B the case $d=4$ is excluded. But this is a natural consequence of the following

Theorem 5.7 (J.D. Moore, [10]). *Let M be a connected n -dimensional Riemannian manifold of constant curvature k isometrically and minimally immersed in a simply connected $(2n-1)$ -dimensional Riemannian manifold N of constant curvature K . Then either M is totally geodesic or it is flat.*

Recently Li [9] proved the following

Theorem. *If $\Phi: S^m \rightarrow S_1^n$ is an isometric minimal immersion, then $\Phi(S^m)$ is either an embedded sphere or an embedded projective space.*

But this is not true if the codimension is not maximal. Let M be the orbit passing $(2^{1/2}P_0 - (-5)^{1/2}P_3 + 2^{1/2}P_6)/3$ in $V_0(6)$. As we proved, M is a space of constant curvature $1/16$ and is a minimal submanifold in S_1^6 . But the orbit is neither an embedded sphere nor an embedded projective space in S_1^6 . Namely we have the following

Proposition 5.8. *Let π be the covering map*

$$\pi: SU(2) \rightarrow M; g \rightarrow \rho(g)((2^{1/2}P_0 - (-5)^{1/2}P_3 + 2^{1/2}P_6)/3).$$

Then π is at least 6-fold.

Proof. Put $g = \begin{bmatrix} \alpha & \\ & \alpha^{-1} \end{bmatrix}$, $\alpha = e^{(-1)^{1/2}k\pi/3}$ ($0 \leq k \leq 5$). Then

$$\begin{aligned} & \rho(g)((2^{1/2}P_0 - (-5)^{1/2}P_3 + 2^{1/2}P_6)/3) \\ &= (2^{1/2}\alpha^{-6}P_0 - (-5)^{1/2}\alpha^{-3}\alpha^3P_3 + 2^{1/2}\alpha^6P_6)/3 \\ &= (2^{1/2}P_0 - (-5)^{1/2}P_3 + 2^{1/2}P_6)/3 \end{aligned}$$

So the covering π is at least 6-fold.

Q.E.D.

References

- [1] J.F. Adams: Lectures on Lie groups, The University of Chicago Press, Chicago & London, 1969.
- [2] M. Berger, P. Gauduchon et E. Mazet: Le spectre d'une variété Riemannienne, Lecture Notes in Math., Springer Verlag, Berlin-Heiderberg-New York, 1971.
- [3] E. Calabi: *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geom. **1** (1967), 111-126.
- [4] S.S. Chern: *Brief survey of minimal submanifolds*, Berichte aus dem Math, Forschungsinstitut Oberwolfach, **4** (1971), 43-59.
- [5] M.P. do Carmo: *Brief survey of minimal submanifolds II*, *ibid*, 9-23.
- [6] M.P. do Carmo and N. Wallach: *Representations of compact Lie groups and minimal immersions into spheres*, J. Differential Geom. **4** (1970), 91-104.
- [7] ———: *Minimal immersions of spheres into spheres*, Ann. of Math. **93** (1971), 43-62.
- [8] N. Ejiri: *Totally real submanifold in a 6-sphere*, Proc. Amer. Math. Soc. **83** (1981), 759-763.
- [9] P. Li: *Minimal immersions of compact irreducible homogeneous Riemannian manifolds*, J. Differential Geom. **16** (1981), 105-115.
- [10] J.D. Moore: *Isometric immersions of space forms in space forms*, Pacific J. Math. **40** (1972), 157-166.

- [11] K. Sugahara: *The sectional curvature and the diameter estimate*, Math. Japon. **26** (1981), 153–159.
- [12] M. Sugiura: *Unitary representations and harmonic analysis*, John Wiley and Sons, New York-London-Sidney-Toronto, 1975.
- [13] J. Tits: *Tabellen zu den einfachen Lieschen Gruppen und ihre Darstellungen*, Springer Verlag, Berlin, 1967.
- [14] N. Wallach: *Harmonic analysis on homogeneous spaces*, Marcel Dekker, New York, 1973.

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