

Title	On irreducible unitary representations of some special linear groups of the second order. I
Author(s)	Tanaka, Shun'ichi
Citation	Osaka Journal of Mathematics. 1966, 3(2), p. 217-227
Version Type	VoR
URL	<a href="https://doi.org/10.18910/4030">https://doi.org/10.18910/4030</a>
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## ON IRREDUCIBLE UNITARY REPRESENTATIONS OF SOME SPECIAL LINEAR GROUPS OF THE SECOND ORDER, I

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(Received July 4, 1966)

### 1. Introduction

Let  $\mathbf{K}$  be a non-discrete locally compact field. In this paper we shall discuss the construction of unitary representations of discrete series of  $SL(2, \mathbf{K})$ . For each quadratic extension  $\mathbf{K}(\sqrt{\tau})$ , we define the imbedding of  $SL(2, \mathbf{K})$  into the symplectic group associated with  $\mathbf{K}(\sqrt{\tau})$ . A natural projective unitary representation of the symplectic group associated with a locally compact abelian group  $G$  on  $L^2(G)$  was constructed by A. Weil [5]. If we restrict it to  $SL(2, \mathbf{K})$ , we can show that it constitute a unitary representation (the unitary operator corresponding to  $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  coincide with the Fourier transform on  $\mathbf{K}(\sqrt{\tau})$ ).

If we denote with  $C$  the multiplicative group of elements of norm 1 in  $\mathbf{K}(\sqrt{\tau})$ , the operator induced by the transformation  $u \rightarrow tu$ ,  $t \in C$  on  $\mathbf{K}(\sqrt{\tau})$  commute with operators of the representation. So by Fourier transformation with respect to the compact abelian group  $C$ , the representation is decomposed into invariant subspaces (this procedure for the case  $\mathbf{K} = \mathbf{R}$  and for the operator corresponding to  $s$  is the classical construction of Fourier-Bessel transform). They are shown to be equivalent to discrete series constructed by I.M. Gel'fand and M.I. Graev.

Our method works also well when  $\mathbf{K}$  is a finite field  $\mathbf{F}$  and the representations thus obtained (decomposed further if necessary) together with the representations obtained by E. Hecke [1] are shown to constitute all irreducible representations of  $SL(2, \mathbf{F})$  (The problem of construction of all irreducible representations of  $SL(2, \mathbf{F})$  was solved by another method by I.M. Gel'fand and M.I. Graev in [2]). We do not discuss this problem in this paper.

We can also reconstruct the representation of the modular congruence group obtained by H.D. Kloosterman [4] who used the transformation formula of theta functions. This representation does not give all irreducible representations of the modular congruence group when decomposed. Modifying a little the construction, we obtain a new representation which may give some of irreducible representations absent in H.D. Kloosterman's work. This problem will be

discussed in the forthcoming part II of the present work.

The discrete series of  $SL(2, \mathbf{R})$  was constructed by V. Bargman. The problem of constructing discrete series of  $SL(2, \mathbf{K})$ , where  $\mathbf{K}$  is a non-discrete totally disconnected locally compact field, was solved by I.M. Gel'fand and M.I. Graev. They defined Bessel functions of the second kind over a non-discrete locally compact field by integral representation, gave the family of operators expressed by them and showed that they constitute an irreducible representation (when  $\mathbf{K}=\mathbf{R}$ , Bessel functions of the second kind are Bessel functions of integral index in the usual sense and the representations constructed by them are another realization of the discrete series constructed by V. Bargman). Their construction is described in [3].

In Section 2, we collect the results of Chapter I in [5] which we need later. Section 3 contains some known facts about non-discrete totally disconnected locally compact fields. We consider these fields only and omit the case when  $\mathbf{K}=\mathbf{R}$ . An auxiliary computation needed in 5 will be made in Section 4. In Section 5 we construct a unitary representation of  $SL(2, \mathbf{K})$ . The space of the constructed representation is decomposed into invariant subspaces and the representations obtained by the decomposition are shown to be equivalent to the discrete series constructed by I.M. Gel'fand and M.I. Graev (Section 6).

Professor H. Yoshizawa informed the author that the problem of constructing representations of the modular congruence group had been undertaken with considerable progress by J.A. Shalika in his unpublished work.

## 2. Summary of results of A. Weil

Let  $G$  be a commutative locally compact group and let  $G^*$  be its dual. For  $u \in G$  and  $u^* \in G^*$ , put  $\langle u, u^* \rangle = u^*(u)$ . Let  $du$  and  $du^*$  be Haar measures on  $G$  and  $G^*$  respectively. Define Fourier transform of  $\Phi(u)$  by

$$\Phi^*(u^*) = \int \Phi(u) \langle u, u^* \rangle du .$$

Inversion formula and Plancherel formula are written as follows:

$$\Phi(u) = c \int \Phi^*(u^*) \langle -u, u^* \rangle du^*$$

and

$$\int |\Phi(u)|^2 du = c \int |\Phi^*(u)|^2 du^* ,$$

where  $c$  is a positive constant.

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Added in Proof.

Professor M. Kuga informed the author that J.A. Shalika, in 1965, had stated the connection of these problems with the work of A. Weil.

Let  $G$  and  $H$  be commutative locally compact groups with Haar measures  $du$  and  $dv$ . If  $u \rightarrow u\alpha$  is a homomorphism of  $G$  into  $H$ , its dual  $\alpha^*$  is a homomorphism of  $H^*$  into  $G^*$  defined by the formula  $\langle u\alpha, v^* \rangle = \langle u, v^*\alpha^* \rangle$  for any  $u \in G$  and  $v^* \in H^*$ . The module of an isomorphism  $\alpha$  of  $G$  onto  $H$  is the number  $|\alpha| = d(u\alpha)/du$  defined by the formula

$$\int F(v)dv = |\alpha| \int F(u\alpha)du,$$

where  $F \in L^1(H)$ . If  $H^* = G$  and  $\alpha = \alpha^*$ ,  $\alpha$  is called a symmetric homomorphism.

Let  $T$  be the multiplicative group of the complex numbers of module 1. A continuous complex-valued function  $f$  on  $G$  is called a character of the second degree, if its value lie in  $T$  and  $f(u_1 + u_2)f(u_1)^{-1}f(u_2)^{-1}$  is an ordinary character in  $u_1$  for each fixed  $u_2$ . If  $f$  is any such, then there exists a unique symmetric homomorphism  $\rho_f$  of  $G$  into  $G^*$  such that

$$\langle u_1, u_2\rho_f \rangle = f(u_1 + u_2)/f(u_1)f(u_2).$$

$f$  is called non-degenerate if  $\rho_f$  is an isomorphism.

Let  $w \rightarrow w\sigma$  be an automorphism of  $G \times G^*$ . Putting  $w = (u, u^*)$ ,  $\sigma$  can be represented by matrix:

$$(u, u^*) \rightarrow (u, u^*) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (u\alpha + u^*\gamma, u\beta + u^*\delta),$$

where  $\alpha, \beta, \gamma$  and  $\delta$  are homomorphisms of  $G$  into  $G$ , of  $G$  into  $G^*$ , of  $G^*$  into  $G$  and of  $G^*$  into  $G^*$  respectively. An automorphism of  $G \times G^*$  is called symplectic if  $\sigma\sigma^I = I$  where

$$\sigma^I = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}.$$

Put  $F(w_1, w_2) = \langle u_1, u_2^* \rangle$  for  $w_i = (u_i, u_i^*) \in G \times G^*$  ( $i=1, 2$ ).  $\sigma$  is symplectic if and only if

$$F(w_1\sigma, w_2\sigma)F(w_2\sigma, w_1\sigma)^{-1} = F(w_1, w_2)F(w_2, w_1)^{-1}$$

for any  $w_1, w_2 \in G \times G^*$ . The group of all symplectic automorphism is denoted with  $Sp(G)$ .

Let  $A(G)$  be the group of all pairs  $w, t$ , where  $w = (u, u^*) \in G \times G^*$ ,  $t \in T$  and  $(w_1, t_1)(w_2, t_2) = (w_1 + w_2, F(w_1, w_2)t_1t_2)$ . For  $w = (v, v^*)$  and  $t \in T$ , we define the unitary operator  $U(w, t)$  in  $\mathfrak{H} = L^2(G)$  by the formula  $[U(w, t)\Phi](u) = t\Phi(u + v)\langle u, v^* \rangle$ ,  $\Phi \in \mathfrak{H}$ .  $U(w, t)$  is a unitary representation of  $A(G)$ . Let  $\mathbf{A}(G)$  denote the group of all unitary operators  $U(w, t)$ , which is clearly isomorphic to  $A(G)$ , and  $\mathbf{B}_0(G)$  denote the group of all unitary operators  $V$  in  $\mathfrak{H}$  such that  $V\mathbf{A}(G)V^{-1} = \mathbf{A}(G)$ . Each member of  $\mathbf{B}_0(G)$  induces an automorphism of  $A(G)$  which have the

form  $(w, t) \rightarrow (w\sigma, f(w)t)$ , where  $\sigma$  is an automorphism of  $G \times G^*$  and  $f$  is a continuous function from  $G \times G^*$  to  $T$  related with  $\sigma$  by the following identity:

$$F(w_1\sigma, w_2\sigma) = F(w_1, w_2)f(w_1+w_2)/f(w_1)f(w_2).$$

Conversely, any pair  $\sigma, f$  with this relation defines an automorphism of  $A(G)$ , and the group of all such is denoted by  $B_0(G)$  ( $(\sigma, f) \in B_0(G)$  implies that  $\sigma$  is symplectic and  $f$  is a character of the second degree). Let  $\pi_0$  denote the homomorphism of  $B_0(G)$  into  $B_0(G)$  defined by the foregoing considerations.  $\pi_0$  is surjective and its kernel is the group of constant multiples of the identity.

Now define the following elements in  $B_0(G)$ :

(i)  $d_0(\alpha) = \left( \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}, 1 \right),$

where  $\alpha$  is an automorphism of  $G$ .

(ii)  $t_0(f) = \left( \begin{pmatrix} 1 & \rho_f \\ 0 & 1 \end{pmatrix}, f \right),$

where  $f$  is a character of the second degree on  $G$ .

(iii)  $d'_0(\gamma) = \left( \begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix}, \langle u, -u^* \rangle \right),$

where  $\gamma$  is an isomorphism of  $G^*$  onto  $G$ .

Let  $\alpha, \gamma, f$  be as above. Define the following operators on  $\mathfrak{H}$ :

(i)  $d_0(\alpha)\Phi(u) = |\alpha|^{1/2}\Phi(u\alpha)$

(ii)  $t_0(f)\Phi(u) = \Phi(u)f(u) \tag{(\Phi \in \mathfrak{H})}$

(iii)  $d'_0(\gamma)\Phi(u) = c^{1/2}|\gamma|^{-1/2}\Phi^*(-u\gamma^{*-1})$

where  $c$  is the positive constant appearing in the inversion formula.

Then they are in  $B_0(G)$  and  $d_0(\alpha) = \pi_0 \circ d_0(\alpha)$ ,  $t_0(f) = \pi_0 \circ t_0(f)$  and  $d'_0(\gamma) = \pi_0 \circ d'_0(\gamma)$ .

For a non-degenerate character of the second degree  $f$  on  $G$ , there exists the constant  $\gamma(f) \in T$  such that

$$(1) \quad \int \left( \int \Phi(u-v)f(v)dv \right) du = \gamma(f)c^{-1/2}|\rho_f|^{-1/2} \int \Phi(u)du,$$

where  $\Phi \in S(G)$  (space of the functions of type Schwartz and Bruhat).

For  $s = (\sigma, f)$  and  $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ , put  $\gamma = \gamma(s)$ . Let  $\Omega_0(G)$  be the set of elements  $s \in B_0(G)$  such that  $\gamma(s)$  is an isomorphism of  $G^*$  onto  $G$ . Each element  $s$  in  $\Omega_0(G)$  can be written uniquely in the form

$$s = t_0(f_1)d'_0(\gamma)t_0(f_2).$$

Put

$$r_0(s) = t_0(f_1)d'_0(\gamma)t_0(f_2).$$

Let  $s=(\sigma, f)$ ,  $s'=(\sigma', f')$  and  $s''=(\sigma'', f'')$  be elements in  $\Omega_0(G)$  such that  $s''=ss'$ , where  $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ ,  $\sigma' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$  and  $\sigma'' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$ . Then

$$r_0(s)r_0(s') = \gamma(f_0)r_0(s''),$$

where  $f_0$  is the character of the second degree on  $G$  defined by the formula

$$f_0(u) = f(0, u\gamma^{-1})f'(u, -u\alpha'\gamma'^{-1}).$$

For  $s \in \Omega_0(G)$ ,  $s^{-1}$  is contained in  $\Omega_0(G)$  and  $r_0(s)r_0(s^{-1})=I$ .

If  $u \rightarrow 2u$  is an automorphism of  $G$ , a natural injective homomorphism  $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rightarrow (\sigma, f_\sigma)$  of  $S_p(G)$  into  $B_0(G)$  exists, where

$$f_\sigma(u, u^*) = \langle u, 2^{-1}u\alpha\beta^* \rangle \langle 2^{-1}u^*\gamma\delta^*, u^* \rangle \langle u^*\gamma, u\beta \rangle.$$

$(\sigma, f_\sigma)$  is simply denoted with  $\sigma$ . Let  $\sigma, \sigma'$  and  $\sigma''$  be elements of  $\Omega_0(G)$  such that  $\sigma''=\sigma\sigma'$ , where  $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  etc. Then

$$r_0(\sigma)r_0(\sigma') = \gamma(f_0)r_0(\sigma''),$$

where

$$(2) \quad f_0(u) = \langle 2^{-1}u, u\gamma^{-1}\gamma''\gamma'^{-1} \rangle$$

### 3. Properties of a non-discrete totally disconnected locally compact field

Let  $K$  be a non-discrete totally disconnected locally compact field. Let  $K^+$  be the additive group of  $K$ ,  $K^*$  its multiplicative group and  $dx$  be a Haar measure on  $K$ . Define a function  $|a|$  on  $K$  by the formula  $d(ax) = |a|dx$  and call  $|a|$  norm of the element  $a$ . Normalize  $dx$  such that  $\int_{|x| \leq 1} dx = 1$ .

$K^*$  is isomorphic to the direct product  $Z \times Z_q \times A$  of an infinite cyclic group  $Z$ , a cyclic group  $Z_{q-1}$  of order  $q-1$ ,  $q=p^n$  ( $p$  a prime number) and group  $A$  of elements  $x$  in  $K$  such that  $|x-1| < 1$ . Assume that  $p$  is odd.  $K$  has three quadratic extension  $K(\sqrt{p})$ ,  $K(\sqrt{\varepsilon})$  and  $K(\sqrt{\varepsilon p})$ , where  $p$  and  $\varepsilon$  are generating elements of  $Z$  and  $Z_q$  respectively.

Let us fix a quadratic extension  $L=K(\sqrt{\tau})$ . For  $z=x+\sqrt{\tau}y$  define  $\bar{z}=x-\sqrt{\tau}y$ ,  $S(z)=z+\bar{z}$  and  $N(z)=z\bar{z}$ . We take  $dz=dx dy$  for Haar measure on  $L^+$ . Define for  $x \in K$   $\text{sign}_\tau x=1$  if  $x$  can be expressed as  $x=N(z)$  for some

$z \in \mathbf{L}$  and  $\text{sign}_\tau x = -1$  otherwise.  $\text{sign}_\tau x$  is a character of  $\mathbf{K}^*$ . Put  $\mathbf{K}_\pm = \{x \in \mathbf{K}; \text{sign}_\tau x = \pm 1\}$ .

A set of elements  $t$  in  $\mathbf{L}$  which satisfy  $N(t) = c$  is called a circle in  $\mathbf{L}$ . We denote with  $C$  the circle with  $c = 1$ . On a circle there exists a measure  $d^*t$  which is invariant under the multiplication of elements of  $C$ , normalized by the condition that total measure on the circle is 1.

If a function  $\Phi(u)$  on  $\mathbf{L}$  satisfies  $\Phi(tu) = \Phi(u)$  for  $t \in C$ , then  $\Phi(u) = \varphi(N(u))$ , where  $\varphi$  is a function defined on  $\mathbf{K}_+$ . We have

$$\int \Phi(u) du = a_\tau \int_{\mathbf{K}_+} \varphi(x) dx,$$

where

$$a_\tau = 2(1+q^{-1})(1+|\tau|)^{-1}.$$

Let  $\chi(x)$  be a non-trivial character of  $\mathbf{K}^+$  such that  $\chi(x)$  is trivial on additive group  $\{x; |x| \leq 1\}$  and non trivial on  $\{x; |x| \leq q^n\}$  for  $n \geq 1$ . For  $a \in \mathbf{K}^*$ , put  $\chi_a(x) = \chi(ax)$ .

Detailed discussion of the facts listed above can be found in [3, Chapter II, §1 and §2].

#### 4. Computation of a constant

Put  $\langle u, v \rangle = \chi_a(S(u\bar{v}))$  for  $u, v \in \mathbf{L}$ .  $\langle u, v \rangle$  defines a self-duality of  $\mathbf{L}^+$ . Taking  $G = \mathbf{L}^+$ , we shall apply the general theory described in 2. In this section we compute  $\gamma(f)$  for  $f(u) = \chi(bN(u))$ , where  $b \in \mathbf{K}^*$ .

By (1), for  $\Phi \in S(G)$ , we have

$$(3) \quad \int \left( \int \Phi(u-v) \chi(bN(v)) dv \right) du = \gamma(f) |b|^{-1} |\tau|^{-1/2} \int \Phi(u) du.$$

Put

$$L_n = \left\{ u = u_1 + \sqrt{\tau} u_2 \in \mathbf{L}; |u_1| \leq q^{-n} \text{ and } |u_2| \leq q^{-n} \right\},$$

where  $n$  is an integer satisfying  $|b|q^{-2n} \leq 1$ . Let  $\Phi$  be the indicator function of  $L_n$ , then (3) reduces to

$$\int \chi(bN(u)) \left( \int_{L_n} \chi(-bS(u\bar{v})) dv \right) du = \gamma(f) |b|^{-1} |\tau|^{-1/2} \int_{L_n} du.$$

So we have

$$\int_{L_n^*} \chi(bN(u)) du = \gamma(f) |b|^{-1} |\tau|^{-1/2},$$

where

$$L_n^* = \left\{ u = u_1 + \sqrt{\tau} u_2 \in \mathbf{L}; |bu_1| \leq q^n, |b\tau u_2| \leq q^n \right\}.$$

If we notice that  $|N(x + \sqrt{\tau}y)| \leq 1$  if and only if  $|x| \leq 1$  and  $|y| \leq 1$ , we have

$$L_n^* = \left\{ u; |b^2\tau N(u)| \leq q^{2n} \right\}.$$

So by putting  $b_n = |b^2\tau|^{-1}q^{2n}$ , we have

$$\begin{aligned} \int_{L_n^*} \chi(bN(u)) du &= a_\tau \int_{K_+, |x| \leq b_n} \chi(bx) dx \\ &= a_\tau |b|^{-1} \int_{K_\pm, |x| \leq b_n|b|^{-1}} \chi(x) dx \quad (\text{sing}_\tau b = \pm 1) \\ &= \frac{a_\tau |b|^{-1}}{2} \left\{ \int_{|x| \leq b_n|b|^{-1}} \chi(x) dx + \text{sign}_\tau b \int_{|x| \leq b_n|b|^{-1}} \chi(x) \text{sign}_\tau x dx \right\}. \end{aligned}$$

For sufficiently large  $n$ , the first term in bracket vanishes and the second term does not depend on  $n$  (for this and for what follows in this section, see [3, pp. 198–206]).

So we have for such  $n$

$$\int_{L_n^*} \chi(bN(u)) du = \text{sign}_\tau b |b|^{-1} c_\tau^{-1},$$

where

$$c_\tau^{-1} = \frac{a_\tau}{2} \int \chi(x) \text{sign}_\tau x dx.$$

Therefore

$$(4) \quad \gamma(f) = \text{sign}_\tau b \cdot c_\tau^{-1} |\tau|^{1/2}.$$

$c_\tau$  satisfies the following identity:

$$c_\tau^2 = |\tau| \text{sign}_\tau(-1).$$

### 5. Construction of a unitary representation of $SL(2, K)$

Let  $\alpha \in K$ , then multiplication by  $\alpha: u \rightarrow u\alpha$  defines a homomorphism of  $G = L^+$  which we will denote with  $\alpha$ . Each element of  $SL(2, K)$  can be considered as an element of  $Sp(G)$ . So  $SL(2, K)$  can be imbedded homomorphically into  $\mathbf{B}_0(G)$ .

If  $\alpha, \gamma \in K^*$  and  $\beta \in K$ , then for  $\Phi \in \mathfrak{F} = L^2(G)$

$$r_0 \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \Phi(u) = |\alpha| \Phi(u\alpha),$$

$$r_0 \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \Phi(u) = \Phi(u) \chi_a(\beta N(u)),$$

$$r_0 \begin{pmatrix} 0 & -\gamma^{-1} \\ \gamma & 0 \end{pmatrix} \Phi(u) = |a| |\tau|^{1/2} |\gamma|^{-1} \Phi^*(-u\gamma^{-1}).$$

Let  $\Omega$  be the subset of  $SL(2, \mathbf{K})$  consisting of elements  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  with  $\gamma \neq 0$ .  
By the formula

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \alpha\gamma^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\gamma^{-1} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} 1 & \delta\gamma^{-1} \\ 0 & 1 \end{pmatrix},$$

we have for  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Omega$  and  $\Phi \in \mathfrak{F}$ ,

$$r_0(g)\Phi(u) = |a| |\tau|^{1/2} |\gamma|^{-1} \int \chi_a \left( \frac{\alpha N(u) + \delta N(v)}{\gamma} \right) \chi_a \left( -\frac{S(u\bar{v})}{\gamma} \right) \Phi(v) dV.$$

Let  $g_1 g_2 = g_3$ , where  $g_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$  ( $i=1, 2, 3$ ). We have

$$(5) \quad r_0(g_1) r_0(g_2) = \gamma(f_0) r_0(g_3),$$

where

$$f_0(u) = \chi_a(\gamma_3 \gamma_1^{-1} \gamma_2^{-1} N(u)).$$

So by (4) we have

$$\gamma(f_0) = \text{sign}_\tau(\gamma_3 \gamma_1 \gamma_2) \cdot c_\tau^{-1} |\tau|^{1/2}.$$

Now define a unitary operator  $T(g)$  in  $\mathfrak{F}$ , for  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Omega$  and  $\Phi \in \mathfrak{F}$ , by the formula

$$(6) \quad T(g)\Phi(u) = |a| \text{sign}_\tau a \cdot c_\tau \frac{\text{sign}_\tau \gamma}{|\gamma|} \int \chi_a \left( \frac{\alpha N(u) + \delta N(v)}{\gamma} \right) \chi_a \left( \frac{-S(u\bar{v})}{\gamma} \right) \Phi(v) dv.$$

Then (5) reduces to

$$T(g_1) T(g_2) = T(g_3).$$

For  $g \in \Omega$

$$T(g) T(g^{-1}) = I$$

is a consequence of  $r_0(g) r_0(g^{-1}) = I$  and  $c_\tau^2 = |\tau| \text{sign}_\tau(-1)$ . Let  $G_0$  be the subgroup of  $SL(2, \mathbf{K})$  consisting of elements  $g_0 = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$ , then the operators defined for  $\Phi \in \mathfrak{F}$  by the formula

$$(7) \quad T(g_0)\Phi(u) = \text{sign}_\tau \alpha |\alpha| \chi_a(\alpha \beta N(u)) \Phi(\alpha u)$$

give a unitary representation of  $G_0$ . It can easily be verified that

$$T(g_0') T(g) = T(g_0' g)$$

if  $g' \in G_0$  and  $g \in \Omega$ . So the family of unitary operators defined by (6) and (7) is a unitary representation of  $SL(2, \mathbf{K})$ . This representation is obviously continuous, because a homomorphism between groups which is continuous on an open subset is continuous everywhere.

**6. Decomposition into invariant subspaces**

For  $\Phi \in \mathfrak{H}$ , let the representation constructed in 5 be written as follows:

$$T(g)\Phi(u) = \int K(g|u, v)\Phi(v)dv,$$

For  $t \in C$ , define the operators  $R_t$  in  $\mathfrak{H}$  as follows:

$$R_t\Phi(u) = \Phi(tu).$$

Then  $R_t$  commute with  $T(g)$ . Let  $\pi$  be an element of  $\tilde{C}$ , character group of  $C$ , and  $\mathfrak{H}_\pi$  be the subspace of  $\mathfrak{H}$  consisting of elements  $\Phi$  which satisfy, for all  $t \in C$ ,

$$R_t\Phi = \pi(t)\Phi.$$

By above mentioned commutativity,  $\mathfrak{H}_\pi$  is an invariant subspace of the representation space  $\{T(g), \mathfrak{H}\}$ . Put  $T_\pi(g) = T(g)|_{\mathfrak{H}_\pi}$ . If we define

$$\Phi_\pi(u) = \int_C \Phi(tu)\overline{\pi(t)}d^*t,$$

then we have the inversion formula,

$$\Phi(u) = \sum_{\pi \in \tilde{C}} \Phi_\pi(u),$$

and the Plancherel formula

$$\int |\Phi(u)|^2 du = \sum_{\pi \in \tilde{C}} \int |\Phi_\pi(u)|^2 du.$$

So we have constructed the decomposition of  $\{T(g), \mathfrak{H}\}$  into  $\{T_\pi(g), \mathfrak{H}_\pi\}$ . Extend  $\pi$  to a character of  $L^*$  and put

$$\Phi'_\pi(u) = \Phi_\pi(u)\overline{\pi(u)}.$$

Then  $\Phi'_\pi(tu) = \Phi'_\pi(u)$  for all  $t \in C$ , so  $\Phi'_\pi(u) = \varphi(N(u))$ , where  $\varphi$  is a function defined on  $\mathbf{K}_+$ . The mapping  $\Phi_\pi \rightarrow \varphi$  is an isometric transformation of  $\mathfrak{H}_\pi$  onto  $L^2(\mathbf{K}_+)$ , because

$$\int |\Phi_\pi(u)|^2 du = a_\tau \int_{\mathbf{K}_+} |\varphi(x)|^2 dx.$$

We have

$$\begin{aligned} (T(g)\Phi)_\pi(u) &= \int K(g|u, v)\Phi_\pi(v) dv \\ &= \int K(g|u, tv)\Phi_\pi(tv) dv \quad (t \in C) \\ &= \int \left( \int_C K(g|u, tv)\pi(t) d^*t \right) \Phi_\pi(v) dv . \end{aligned}$$

Because

$$\begin{aligned} &\int_C \chi_a\left(-\frac{S(uv\bar{t})}{\gamma}\right)\pi(t) d^*t \\ &= \pi^{-1}\left(\frac{v}{u}\right) \int_{t\bar{t}=N(v)N(u)^{-1}} \chi_a\left(-\frac{N(u)t+N(v)t^{-1}}{\gamma}\right)\pi(t) d^*t , \end{aligned}$$

we have , for  $g \in \Omega$ ,

$$\begin{aligned} (T(g)\Phi)'_\pi(u) &= |a| \operatorname{sign}_\tau a \cdot c_\tau \frac{\operatorname{sign}_\tau \gamma}{|\gamma|} \int \chi_a\left(\frac{\alpha N(u) + \delta N(v)}{\gamma}\right) \\ &\quad \times \left( \int_{t\bar{t}=N(v)N(u)^{-1}} \chi_a\left(-\frac{N(u)t+N(v)t^{-1}}{\gamma}\right)\pi(t) d^*t \right) \Phi'_\pi(u) du . \end{aligned}$$

For  $g_0 \in G_0$ ,

$$(T(g_0)\Phi)'_\pi(u) = \pi(\alpha) \operatorname{sign}_\tau \alpha \cdot |\alpha| \chi_a(\alpha\beta N(u)) \Phi'_\pi(\alpha u) .$$

So the induced action of  $T_\pi(g)$  on  $\varphi(x)$  can be written as follows:

$$T_\pi(g)\varphi(x) = \int_{K_+} K_\pi^a(g|x, y)\varphi(y) dy ,$$

where for  $g \in \Omega$

$$\begin{aligned} K_\pi^a(g|x, y) &= a_\tau |a| \operatorname{sign}_\tau a \cdot c_\tau \frac{\operatorname{sign}_\tau \gamma}{|\gamma|} \chi_a\left(\frac{\alpha x + \delta y}{\gamma}\right) \\ &\quad \times \int_{t\bar{t}=yx^{-1}} \chi_a\left(-\frac{xt+yt^{-1}}{\gamma}\right)\pi(t) d^*t \end{aligned}$$

and for  $g_0 \in G_0$

$$K_\pi^a(g_0|x, y) = \pi(\alpha) \operatorname{sign}_\tau \alpha \cdot |\alpha| \chi_a(\alpha\beta x) \delta(y - \alpha^2 x) .$$

Put  $\psi(x) = \varphi(a^{-1}x)$ , then  $\psi$  is defined on  $K_\pm$  according to  $\operatorname{sign}_\tau a = \pm 1$  and

$$\int_{K_\pm} |\psi(x)|^2 dx = |a| \int_{K_+} |\varphi(x)|^2 dx .$$

The induced action of  $T_\pi(g)$  on  $\psi(x)$  can be written as follows:

$$T_\pi^\pm(g)\psi(x) = \int_{K_\pm} K_\pi^\pm(g|x, y)\psi(y) dy ,$$

where

$$K_{\pi}^{\pm}(g|x, y) = \pm a_{\tau, c_{\tau}} \frac{\text{sign}_{\tau} \gamma}{|\gamma|} \chi\left(\frac{\alpha x + \delta \gamma}{\gamma}\right) \int_{t\bar{t}=yx^{-1}} \chi\left(-\frac{xt + yt^{-1}}{\gamma}\right) \pi(t) d^*t$$

for  $g_0 \in \Omega$ , and

$$K_{\pi}^{\pm}(g_0|x, y) = \pi(\alpha) \text{sign}_{\tau} \alpha |\alpha| \chi(\alpha \beta x) \delta(y - \alpha^2 x)$$

for  $g_0 \in G_0$ .

So we have constructed the discrete series obtained by I.M. Gel'fand and M.I. Graev.

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**References**

- [1] E. Hecke: *Über ein Fundamentalproblem aus der Theorie der elliptischen Modul-funktionen*, Abh. Math. Sem. Hamburgischen Univ. **6** (1928), 235–257.
- [2] I.M. Gel'fand and M.I. Graev: *Categories of group representations and the classification problem of irreducible representations* (in Russian), Dokl. Akad. Nauk SSSR **146** (1962), 757–760.
- [3] I.M. Gel'fand, M.I. Graev and I. Pyatetzki-Shapiro: *Theory of representations and automorphic functions* (in the Series 'Generalized functions', Vol. 6, in Russian), Moscow, 1966.
- [4] H.D. Kloosterman: *The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups*, Ann. of Math. **47** (1946), 317–447.
- [5] A. Weil: *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.

