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FIXED POINT SETS OF ORIENTATION REVERSING INVOLUTIONS ON 3-MANIFOLDS

Dedicated to Professor Junzo Tao on his sixtieth birthday

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1. Introduction

Let M be a closed orientable 3-manifold admitting an orientation reversing involution τ (i.e. τ^2=identity and $\tau_*[M]=-[M]$ for the fundamental class $[M]$ of M). Let $\mathrm{Fix}\,\tau$ be the fixed point set of τ on M.

According to Smith theory (cf. [1]), each component of $\mathrm{Fix}\,\tau$ is a point or a closed surface, and the Euler characteristic number $\chi(\mathrm{Fix}\,\tau)\equiv 0 \pmod 2$. A. Kawauchi has shown in [4] that $\mathrm{Tor}\,H_1(M;Z)\cong A\oplus A$ or $Z_2\oplus A\oplus A$ for some finite abelian group A and $\dim_{Z_2} H_1(\mathrm{Fix}\,\tau; Z_2)\equiv 0 \pmod 2$ iff $\mathrm{Tor}\,H_1(M;Z)\cong A\oplus A$. J. Hempel has proved in [2] that if $\mathrm{Fix}\,\tau=\phi$ or contains a closed orientable surface $S\neq S^2$, then $\beta_1(M)>0$. And in [3], he has shown that if $\pi_1(M)$ is not isomorphic to $\{1\}$ or Z_2, and $\pi_1(M)$ is not virtually representable to Z, then $\mathrm{Fix}\,\tau$ is a 2-sphere or two points or $\mathrm{Fix}\,\tau$ contains a projective plane. The author proved in [6] that for rational homology 3-spheres M with an orientation reversing involution τ, $\dim_{Z_2} H_1(\mathrm{Fix}\,\tau; Z_2)\leqq \dim_{Z_2} H_1(M; Z_2)$. In this paper, we shall give a generalization of this inequality in the case of general closed 3-manifolds.

Theorem. *For any closed orientable 3-manifold M admitting an orientation reversing involution τ, we have*

$$\dim_{Z_2} H_1(\mathrm{Fix}\,\tau; Z_2)\leqq \dim_{Z_2} H_1(M; Z_2)+\beta_1(M)$$

where $\beta_1(M)$ is the first Betti number of M.

This inequality is best possible. For example, consider a double of handlebody with involution interchanging handlebodies. Another example is in the proof of theorem 1 of [5].

Throughout this paper, we will work in the piecewise-linear category.

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2. Proof

We may assume that $\mathrm{Fix}\,\tau$ contains surfaces. Let F be the union of surfaces contained in $\mathrm{Fix}\,\tau$ and $i_*\colon H_1(F;Z)\to H_1(M;Z)$ the homomorphism induced by the inclusion map. Note that $H_1(F;Z)\cong H_1(\mathrm{Fix}\,\tau;Z)$. For each 1-cycle C on F such that $[C]\in \mathrm{Ker}\,i_*$, there exists a 2-chain D in M such that $\partial D=C$. Note that $D-\tau D$ is a 2-cycle in M. Now define a subgroup G of $\mathrm{Ker}\,i_*$ as follows;

$$G=\{x\in \mathrm{Ker}\,i_* \mid \text{there exists a 2-chain } D \text{ in } M \text{ such that } [\partial D]=x \text{ and } [D-\tau D]=0\in H_2(M;Z)\}.$$

Let $\phi\colon \mathrm{Ker}\,i_*\to \mathrm{Ker}\,i_*/G$ be a canonical homomorphism. For each element y of $\mathrm{Ker}\,i_*/G$, let $\bar{y}$ be an element of $\mathrm{Ker}\,i_*$ such that $\phi(\bar{y})=y$. Then there exists a 2-chain D such that $[\partial D]=\bar{y}$. Note that $[\partial(D+\tau D)]=2\bar{y}$ and $[(D+\tau D)-\tau(D+\tau D)]=0$. Hence $2\bar{y}\in G$. It follows that each element of $\mathrm{Ker}\,i_*/G$ has at most order 2. Thus we have $\mathrm{Ker}\,i^*/G\cong \overset{m}{\oplus} Z_2$ for some integer m.

Let $y_1, y_2, \cdots, y_m$ be a basis of $\mathrm{Ker}\,i_*/G\cong \overset{m}{\oplus} Z_2$ and $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m$ elements of $\mathrm{Ker}\,i_*$ such that $\phi(\bar{y}_i)=y_i$ $(i=1, 2, \cdots, m)$. Then there exist 2-chains $D_1, D_2, \cdots, D_m$ in M such that $[\partial D_i]=\bar{y}_i$ $(i=1, 2, \cdots, m)$. We will prove that $[D_1-\tau D_1]$, $[D_2-\tau D_2]$, $\cdots$, $[D_m-\tau D_m]$ are linearly independent in $H_2(M;Z)$. If $\sum_{i=1}^m a_i[D_i-\tau D_i]=0$ for some integers $a_1, a_2, \cdots, a_m$, we may assume the greatest common divisor of $a_1, a_2, \cdots, a_m$ is one, since $H_2(M;Z)$ is torsion free. Then, $[\partial(\sum_{i=1}^m a_iD_i)]=\sum_{i=1}^m a_i\bar{y}_i$ and $[\sum_{i=1}^m a_iD_i-\tau(\sum_{i=1}^m a_iD_i)]=0$. Hence $\sum_{i=1}^m a_i\bar{y}_i\in G$ and $\phi(\sum_{i=1}^m a_i\bar{y}_i)=\sum_{i=1}^m a_iy_i=0$. It shows that each a_i is even $(i=1, 2, \cdots, m)$ and this is a contradiction.

Therefore, $[D_1-\tau D_1]$, $[D_2-\tau D_2]$, $\cdots$, $[D_m-\tau D_m]$ are linearly independent in $H_2(M;Z)$. Hence $m\leqq \beta_2(M)=\beta_1(M)$, where $\beta_i(M)$ is the i-th Betti number of M.

On the other hand, we will have $G<2H_1(F;Z)$. For each element x of G, there exists a 2-chain D in M such that $[\partial D]=x$ and $[D-\tau D]=0\in H_2(M;Z)$. Let E be a 3-chain in M such that $\partial E=D-\tau D$. Then we can see that $0=[\partial(E\cap F)]_2=[x]_2\in H_1(F;Z_2)\cong H_1(F;Z)/2H_1(F;Z)$. Hence, $x\in 2H_1(F;Z)$. For detail, see the proof of theorem 4 of [5].

Hence we have,

$$\dim_{Z_2} \mathrm{Im}\,i_*\otimes Z_2 \geqq \dim_{Z_2} H_1(F;Z)\otimes Z_2-m \geqq \dim_{Z_2} H_1(\mathrm{Fix}\,\tau;Z_2)-\beta_1(M).$$

On the other hand,

$$\dim_{Z_2} \mathrm{Im}\,i_*\otimes Z_2 \leqq \dim_{Z_2} H_1(M;Z)\otimes Z_2=\dim_{Z_2} H_1(M;Z_2).$$

Thus, we have,

$$\dim_{Z_2} H_1(\mathrm{Fix}\,\tau;\, Z_2) \leqq \dim_{Z_2} H_1(M;\, Z_2) + \beta_1(M)\,.$$

This completes the proof.

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