



Title	Fixed point sets of orientation reversing involutions on 3-manifolds
Author(s)	Kobayashi, Masako
Citation	Osaka Journal of Mathematics. 1988, 25(4), p. 877-879
Version Type	VoR
URL	<a href="https://doi.org/10.18910/4051">https://doi.org/10.18910/4051</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## FIXED POINT SETS OF ORIENTATION REVERSING INVOLUTIONS ON 3-MANIFOLDS

Dedicated to Professor Junzo Tao on his sixtieth birthday

MASAKO KOBAYASHI

(Received August 18, 1987)

### 1. Introduction

Let  $M$  be a closed orientable 3-manifold admitting an orientation reversing involution  $\tau$  (i.e.  $\tau^2 = \text{identity}$  and  $\tau_*[M] = -[M]$  for the fundamental class  $[M]$  of  $M$ ). Let  $\text{Fix } \tau$  be the fixed point set of  $\tau$  on  $M$ .

According to Smith theory (cf. [1]), each component of  $\text{Fix } \tau$  is a point or a closed surface, and the Euler characteristic number  $\chi(\text{Fix } \tau) \equiv 0 \pmod{2}$ . A. Kawauchi has shown in [4] that  $\text{Tor } H_1(M; Z) \cong A \oplus A$  or  $Z_2 \oplus A \oplus A$  for some finite abelian group  $A$  and  $\dim_{Z_2} H_1(\text{Fix } \tau; Z_2) \equiv 0 \pmod{2}$  iff  $\text{Tor } H_1(M; Z) \cong A \oplus A$ . J. Hempel has proved in [2] that if  $\text{Fix } \tau = \emptyset$  or contains a closed orientable surface  $S \neq S^2$ , then  $\beta_1(M) > 0$ . And in [3], he has shown that if  $\pi_1(M)$  is not isomorphic to  $\{1\}$  or  $Z_2$ , and  $\pi_1(M)$  is not virtually representable to  $Z$ , then  $\text{Fix } \tau$  is a 2-sphere or two points or  $\text{Fix } \tau$  contains a projective plane. The author proved in [6] that for rational homology 3-spheres  $M$  with an orientation reversing involution  $\tau$ ,  $\dim_{Z_2} H_1(\text{Fix } \tau; Z_2) \leq \dim_{Z_2} H_1(M; Z_2)$ . In this paper, we shall give a generalization of this inequality in the case of general closed 3-manifolds.

**Theorem.** *For any closed orientable 3-manifold  $M$  admitting an orientation reversing involution  $\tau$ , we have*

$$\dim_{Z_2} H_1(\text{Fix } \tau; Z_2) \leq \dim_{Z_2} H_1(M; Z_2) + \beta_1(M)$$

where  $\beta_1(M)$  is the first Betti number of  $M$ .

This inequality is best possible. For example, consider a double of handlebody with involution interchanging handlebodies. Another example is in the proof of theorem 1 of [5].

Throughout this paper, we will work in the piecewise-linear category.

The author would like to express her hearty thanks to Professor A. Kawauchi for helpful advice.

## 2. Proof

We may assume that  $\text{Fix } \tau$  contains surfaces. Let  $F$  be the union of surfaces contained in  $\text{Fix } \tau$  and  $i_*: H_1(F; Z) \rightarrow H_1(M; Z)$  the homomorphism induced by the inclusion map. Note that  $H_1(F; Z) \cong H_1(\text{Fix } \tau; Z)$ . For each 1-cycle  $C$  on  $F$  such that  $[C] \in \text{Ker } i_*$ , there exists a 2-chain  $D$  in  $M$  such that  $\partial D = C$ . Note that  $D - \tau D$  is a 2-cycle in  $M$ . Now define a subgroup  $G$  of  $\text{Ker } i_*$  as follows;

$$G = \{x \in \text{Ker } i_* \mid \text{there exists a 2-chain } D \text{ in } M \text{ such that } [\partial D] = x \text{ and } [D - \tau D] = 0 \in H_2(M; Z)\}.$$

Let  $\phi: \text{Ker } i_* \rightarrow \text{Ker } i_*/G$  be a canonical homomorphism. For each element  $y$  of  $\text{Ker } i_*/G$ , let  $\bar{y}$  be an element of  $\text{Ker } i_*$  such that  $\phi(\bar{y}) = y$ . Then there exists a 2-chain  $D$  such that  $[\partial D] = \bar{y}$ . Note that  $[\partial(D + \tau D)] = 2\bar{y}$  and  $[(D + \tau D) - \tau(D + \tau D)] = 0$ . Hence  $2\bar{y} \in G$ . It follows that each element of  $\text{Ker } i_*/G$  has at most order 2. Thus we have  $\text{Ker } i_*/G \cong \bigoplus^m Z_2$  for some integer  $m$ .

Let  $y_1, y_2, \dots, y_m$  be a basis of  $\text{Ker } i_*/G \cong \bigoplus^m Z_2$  and  $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_m$  elements of  $\text{Ker } i_*$  such that  $\phi(\bar{y}_i) = y_i$  ( $i = 1, 2, \dots, m$ ). Then there exist 2-chains  $D_1, D_2, \dots, D_m$  in  $M$  such that  $[\partial D_i] = \bar{y}_i$  ( $i = 1, 2, \dots, m$ ). We will prove that  $[D_1 - \tau D_1], [D_2 - \tau D_2], \dots, [D_m - \tau D_m]$  are linearly independent in  $H_2(M; Z)$ . If  $\sum_{i=1}^m a_i [D_i - \tau D_i] = 0$  for some integers  $a_1, a_2, \dots, a_m$ , we may assume the greatest common divisor of  $a_1, a_2, \dots, a_m$  is one, since  $H_2(M; Z)$  is torsion free. Then,  $[\partial(\sum_{i=1}^m a_i D_i)] = \sum_{i=1}^m a_i \bar{y}_i$  and  $[\sum_{i=1}^m a_i D_i - \tau(\sum_{i=1}^m a_i D_i)] = 0$ . Hence  $\sum_{i=1}^m a_i \bar{y}_i \in G$  and  $\phi(\sum_{i=1}^m a_i \bar{y}_i) = \sum_{i=1}^m a_i y_i = 0$ . It shows that each  $a_i$  is even ( $i = 1, 2, \dots, m$ ) and this is a contradiction.

Therefore,  $[D_1 - \tau D_1], [D_2 - \tau D_2], \dots, [D_m - \tau D_m]$  are linearly independent in  $H_2(M; Z)$ . Hence  $m \leq \beta_2(M) = \beta_1(M)$ , where  $\beta_i(M)$  is the  $i$ -th Betti number of  $M$ .

On the other hand, we will have  $G < 2H_1(F; Z)$ . For each element  $x$  of  $G$ , there exists a 2-chain  $D$  in  $M$  such that  $[\partial D] = x$  and  $[D - \tau D] = 0 \in H_2(M; Z)$ . Let  $E$  be a 3-chain in  $M$  such that  $\partial E = D - \tau D$ . Then we can see that  $0 = [\partial(E \cap F)]_2 = [x]_2 \in H_1(F; Z_2) \cong H_1(F; Z)/2H_1(F; Z)$ . Hence,  $x \in 2H_1(F; Z)$ . For detail, see the proof of theorem 4 of [5].

Hence we have,

$$\dim_{Z_2} \text{Im } i_* \otimes Z_2 \geq \dim_{Z_2} H_1(F; Z) \otimes Z_2 - m \geq \dim_{Z_2} H_1(\text{Fix } \tau; Z_2) - \beta_1(M).$$

On the other hand,

$$\dim_{Z_2} \text{Im } i_* \otimes Z_2 \leq \dim_{Z_2} H_1(M; Z) \otimes Z_2 = \dim_{Z_2} H_1(M; Z_2).$$

Thus, we have,

$$\dim_{Z_2} H_1(\text{Fix } \tau; Z_2) \leq \dim_{Z_2} H_1(M; Z_2) + \beta_1(M).$$

This completes the proof.

---

### References

- [1] G.E. Bredon: Introduction to compact transformation groups, Pure and applied mathematics, 46, Academic press, New York, London, 1972.
- [2] J. Hempel: *Orientation reversing involutions and the first Betti number for finite coverings of 3-manifolds*, Invent. Math., **67** (1982), 133–142.
- [3] ———: *Virtually Haken manifolds*, in Combinatorial methods in topology and algebraic geometry, J.R. Harper and R. Mandelbaum, Ed. Comtemp. Math. **44**, Amer. Math. Soc. Providence R.I., 1985, 149–155.
- [4] A. Kawauchi: *On 3-manifolds admitting orientation-reversing involutions*, J. Math. Soc. Japan, **33** (1981), 571–589.
- [5] M. Kobayashi: *A construction of certain 3-manifolds with orientation reversing involution*, J. Math. Soc. Japan, **40** (1988), 151–159.
- [6] ———: *Rational homology 3-spheres with orientation reversing involution*, Kobe J. Math., **5** (1988), 109–116.

Department of Mathematics  
Osaka City University  
Sugimoto, Sumiyoshi-ku  
Osaka, 558, Japan

