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DEGENERATE ELLIPTIC SYSTEMS OF PSEUDO-DIFFERENTIAL EQUATIONS AND NON-COERCIVE BOUNDARY VALUE PROBLEMS

HIDEO SOGA

(Received June 18, 1976)

0. Introduction

In the present paper we shall study a class of degenerate elliptic systems of pseudo-differential equations, and apply the results obtained there to non-coercive boundary value problems of fourth order.

One of typical examples of non-coercive problems is the oblique derivative problem: Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) with a smooth boundary \( \Gamma \) and consider the problem

\[
\begin{aligned}
A(x, D_x)u &= f \text{ in } \Omega, \\
\frac{\partial u}{\partial v} &= g \text{ on } \Gamma,
\end{aligned}
\]

(0.1)

where \( A(x, D_x) \) is an elliptic differential operator of second order on \( \Omega \) and \( v \) is a non-vanishing real vector field tangent to \( \Gamma \) on its submanifold \( \Gamma_0 \). The behavior of \( v \) near \( \Gamma_0 \) has a crucial effect on this problem (for details, see [5], [14], etc.). We shall consider in §4 a similar problem for an elliptic operator \( L(x, D_x) \) of fourth order on \( \Omega \):

\[
\begin{aligned}
L(x, D_x)u &= f \text{ in } \Omega, \\
\frac{\partial^2 u}{\partial v_1 \partial n} &= g_1 \text{ on } \Gamma, \\
\frac{\partial u}{\partial v_2} &= g_2 \text{ on } \Gamma,
\end{aligned}
\]

(0.2)

where \( v_1, v_2 \) are vector fields of the same type as in (0.1). We study this problem by a usual method. Namely, let \( \mathcal{P} \) be the Poisson operator of the Dirichlet problem

\[
\begin{aligned}
L(x, D_x)u &= f \text{ in } \Omega, \\
D_x u &= h_1 \text{ on } \Gamma, \\
u &= h_2 \text{ on } \Gamma,
\end{aligned}
\]
where $D_n$ denotes the normal derivative $-i\frac{\partial}{\partial n}$ on $\Gamma$. Then the mapping $T: h=^t(h_1, h_2) \mapsto ^t \left( \frac{\partial^2}{\partial \nu_1 \partial n} \mathcal{P} h \big|_{\Gamma}, \frac{\partial}{\partial \nu_2} \mathcal{P} h \big|_{\Gamma} \right)$ is a pseudo-differential operator on $\Gamma$, whose principal symbol is the Lopatinski matrix (which is described in Chapter VI of [10]); the problem (0.2) can be reduced to investigation of the system of equations $T h = g$. The problem of such a system is not characterized even in the subelliptic case (which means that the estimate of the type (0.3) below holds), while scalar subelliptic operators are done completely by Egorov [3], [4].

We shall give in §2 a sufficient condition for the subellipticity. Let $A(x, D_x)$ be an $m \times m$-matrix of pseudo-differential operators on an open ball $U(\subset \mathbb{R}^n)$ and $\mathcal{A}(x, \xi)$ be homogeneous of order one in $\xi (|\xi| \leq 1)$. On some assumptions (cf. [A-I]∼[A-IV]) we derive the subelliptic estimate
\begin{equation}
|u|_{s+\epsilon_0, \nu'} \leq C \left( |Au|_{s, \nu'} + |u|_{s, \nu'} \right), \quad u \in C_0^\infty(U') (\epsilon_0 > 0),
\end{equation}
where $U'$ is an open ball $(U' \subset U)$, $|\cdot|_{s, \nu'}$ is the norm of the Sobolev space $H_s(U')$ and $C_0^\infty(U')$ is the set of $C^\infty$-functions in $U'$ with compact support. In §3 we consider the system of equations $Au = f$ on a compact manifold such that its symbol represented by local coordinates satisfies locally the same assumptions as in §2. Constructing the almost right inverse (i.e. right regularizer) in the same way as in [1], [6], [16], etc., we show that the equation $Au = f$ is of Fredholm type (i.e., the kernel and cokernel are finite-dimensional in the Sobolev space). Finally, in §4 we study the solvability of (0.2) by using the reduction stated earlier. If the vector fields $\nu_1, \nu_2$ satisfy several assumptions (cf. (4.2)∼(4.5)), (0.2) is of Fredholm type and the estimate
\begin{equation}
|u|_{s+3+\epsilon_0, \Omega} \leq C \left( ||Lu||_{s, \Omega} + \left| \frac{\partial u}{\partial \nu_1 \partial n} \right|_{s+3, l'} + \left| \frac{\partial u}{\partial \nu_2} \right|_{s+3, l'} \right) + |u|_{s+3, \Omega}, \quad u \in H_{s+3}(\Omega) (\epsilon_0 > 0)
\end{equation}
is obtained for $s \geq 0$.

Eschin in [6] investigated the degenerate elliptic system $Au = f$ when $\det A$ is of principal type. We note that in our class $\det A$ may have multi-characteristics.

The main results of this paper are stated in our previous note [15] without proofs.

1. Notations and properties of pseudo-differential operators

We denote by $S^m_{a,b}(W) (W \subset \mathbb{R}^n, m \in \mathbb{R}, 0 \leq \delta \leq \rho \leq 1, \delta < 1)$ the set of functions $\rho(x, \xi) \in C^\infty(W \times \mathbb{R}^n)$ satisfying for all multi-indices $\alpha, \beta$
\begin{equation}
|D^\alpha \partial^\beta \rho(x, \xi)| \leq C_{a,b} \langle \xi \rangle^{-\rho(|\alpha|+|\beta|)}, \quad x \in W, \xi \in \mathbb{R}^n,
\end{equation}
where $D_x^\xi = \left(-i \frac{\partial}{\partial x}\right)^\xi$, $\partial_x^\xi = \left(\frac{\partial}{\partial x}\right)^\xi$ and $\langle \xi \rangle = (|\xi|^2 + 1)^{1/2}$.

For $p(x, \xi) \in S^m_{p,\delta}(W)$ we define a pseudo-differential operator $p(x, D_x)$ by

$$p(x, D_x)u(x) = \int \exp{i\xi \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi, \quad u \in S,$$

where $d\xi = (2\pi)^{-n} d\xi$, $S$ is the space of rapidly decreasing functions and $\hat{u}(\xi)$ is the Fourier transform $\int e^{-i\xi \cdot x} u(x) dx$. We denote by $S^m_{p,\delta}(W)$ the set of these operators $p(x, D_x)$, and call $p(x, \xi)$ the symbol of $p(x, D_x)$. It is well known that the estimate

$$||p(x, D_x)u||_s \leq C ||p||_m ||u||_{s+m}, \quad u \in S (s \in \mathbb{R})$$

holds for any $p(x, \xi) \in S^m_{p,\delta}(-S^m_{p,\delta}(\mathbb{R}^n))$, where

$$||p||_m = \max_{|\alpha| \leq m} \inf_{\xi \neq 0} |D_\xi^\alpha \partial_x^\xi p(x, \xi) \cdot \langle \xi \rangle^{-|\alpha|} \cdot |\xi|^{1-m/2}|$$

and the constants $C, l$ do not depend on $p(x, \xi)$. This is proved in Calderón-Vaillancourt [2], Kumano-go [9], etc.

For $p(x, \xi) \in S^m_{p,\delta}$ and $q(x, \xi) \in S_{p,\delta}'$, we set

$$\sigma(p \circ q)(x, \xi) = \lim_{\varepsilon \to 0} \int \int e^{-i\eta \cdot y} \chi(\varepsilon \eta, \varepsilon y) p(x, \xi + \eta) q(x + y, \xi) dy d\eta,$$

where $\chi(\eta, y) \in \mathcal{S}(\mathbb{R}^n)$ and $\chi(0, 0) = 1$. Then we have $\sigma(p \circ q)(x, \xi) \in S^{m+\varepsilon}_{p,\delta}'$ and

$$\sigma(p \circ q)(x, D_x) u = p(x, D_x) \circ q(x, D_x) u = (p(x, D_x)(q(x, D_x) u)).$$

Furthermore the asymptotic expansion formula

$$(1.1) \quad \sigma(p \circ q)(x, \xi) = \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_\xi^\alpha p(x, \xi) D_\xi^\alpha q(x, \xi) \in S^{m+\varepsilon}_{p,\delta}'(\mathbb{R}^n)$$

is obtained for any integer $N (\geq 0)$. These are explained in Kumano-go [8], [10]. As is considered in [12], we have

**Proposition 1.1.** Let $p(x, \xi) \in S^m_{0,0}$ and $q(x, \xi) \in S^m_{0,\delta}$. If $\partial_\xi^\alpha q(x, \xi) \in S^{m+1}_{0,0}$ for any $j$, then it follows that

$$\sigma(p \circ q)(x, \xi) - p(x, \xi) q(x, \xi) \in S^m_{0,0}.$$

We can prove this proposition in the same way as in Chapter II of [10]. Replacing $\langle \xi \rangle$ in the above discussion with another basic weight function $\lambda(\xi)$ (i.e., $\lambda(\xi) \in C^\infty$, $1 \leq \lambda(\xi) \leq A_0 \langle \xi \rangle$ and $|\partial_\xi^\alpha \lambda(\xi)| \leq A_\lambda \lambda(\xi)^{-|\alpha|}$), we obtain the same results (cf. Chapter VII of [10]). We denote by $S^m_{\lambda,p,\delta}$ the set of symbols defined by $\lambda(\xi)$ ($\pm \langle \xi \rangle$).

In this paper we use pseudo-differential operators on a $C^\infty$ compact manifold.
Let $Q$ be a mapping: $C^\infty(M)\to C^\infty(M)$. Then for local coordinates $(\Phi_j, U_j)$, $(\Phi_k, U_k)$ ($\Phi_j$ is defined on an open set $U_j$) we have in a natural way a mapping $Q_{\Phi_k\Phi_j}: C^\infty(U_{\Phi_j})\to C^\infty(U_{\Phi_k}) (U_{\Phi_k}=\Phi_k(U_j))$. We say that a mapping $P: C^\infty(M)\to C^\infty(M)$ is a pseudo-differential operator on $M$ of order $m$ when there is a set of local coordinates $\{(\Phi_j, U_j)\}_{j=1,\ldots,N}$ covering $M$ such that (i) for any $\phi \in C^\infty(U_i), \psi \in C^\infty(U_j) (i \neq j)$ satisfying $\text{supp} (\phi) \cap \text{supp} (\psi) = \phi (\phi P \lambda |_{U_i})$ belongs to $S^0_{1,0}$, (ii) for any $\phi, \psi \in C^\infty(U_i) (\phi P \psi)_{\Phi_k \Phi_j}$ belongs to $S^0_{r,0}$ and (iii) the symbol $p(x, \xi)$ of $(\phi P \psi)_{\Phi_k \Phi_j}$ has a homogeneous asymptotic expansion, that is, there exist symbols $p_{m-j}(x, \xi)$ ($j=0, 1, \cdots \in S^m_{1,0}$) homogeneous of order $m-j$ in $\xi (|\xi| \geq 1)$ such that for any integer $N(\geq 0)$

$$p(x, \xi) - \sum_{j=0}^N p_{m-j}(x, \xi) \in S^m_{1,0} - N^{-1}.$$ 

We call $p(x, \xi)$ the local symbol of $P$ on $V$ when $\phi(x) = \psi(x) = 1$ on $V$. Using the principal part $p_m(x, \xi)$ of the local symbol, we can define a function $P_m$ on the cotangent space $T^*(M) - \{0\}$, which is called the principal symbol (part) of $P$. Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}$ be a matrix of pseudo-differential operators on $M$ such that the order of $a_{ij}$ is $-s_i + t_j$ ($s_i, t_j \in \mathbb{R}$ and $i, j=1, \cdots, m$). We say that $A$ is elliptic if its principal symbol is non-singular. In other words, the principal part $a_\phi(x, \xi)$ of the local symbol on $V$ always satisfies $\det a_\phi(x, \xi) \neq 0$ for $x \in \Phi(V)$ and $|\xi| \geq 1$.

2. The system of first order operators

We set for $\varepsilon, \rho > 0$

$$U_{t, \rho} = \{x = (t, y) \in \mathbb{R}^n; -\varepsilon < t < \varepsilon, |y| < \rho\}.$$ 

Let $A(x, \xi) = \begin{bmatrix} a_{11}(x, \xi) & \cdots & a_{1m}(x, \xi) \\ \vdots & \ddots & \vdots \\ a_{m1}(x, \xi) & \cdots & a_{mm}(x, \xi) \end{bmatrix}$ be a matrix of symbols belonging to $S^1_{1,0}(U_{t, \rho})$ and homogeneous of order one in $\xi$, that is,

$$a_{ij}(x, \mu \xi) = \mu a_{ij}(x, \xi), \quad \mu \geq 1, \quad |\xi| \geq 1.$$ 

We assume that when $t \neq 0$ $A(t, y; \tau, \eta)$ is elliptic (i.e., $\det A(t, y; \tau, \eta) \neq 0$ for $|\tau, \eta| = (\tau^2 + |\eta|^2)^{1/2} \geq 1$) and that when $t=0$ the ellipticity is degenerate in the following way: For $(t, y) \in U_{t, \rho}$ and $|\tau, \eta| \geq 1$

[A-I] $\det A(t, y; \tau, \eta) \neq 0$ when $t \neq 0$ or $t=0$ & $\tau \neq 0$;

[A-II] $A(0, y; 0, \eta) = [0]$ (zero-matrix), $|\eta| \geq 1$;

[A-III] $\frac{\partial A}{\partial \tau} (0, y; 0, \eta) \neq 0, \quad |\eta| \geq 1$;
there exist positive integers \(k_1, \ldots, k_t\) independent of \((t, y; \eta') \in \overline{U}_{t, t_1} \times S\) (\(\varepsilon > 0\) is small enough and \(S = \{\eta': |\eta'| = 1\}\) such that the following decomposition of the matrix

\[
A(t, y; \eta') = \frac{\partial A}{\partial \tau}(t, y; 0, \eta')^{-1} \cdot A(t, y; 0, \eta'), \quad (t, y; \eta') \in \overline{U}_{t, t_1} \times S
\]

is possible:

\[\begin{array}{c}
[A - IV] t^{- k_1} A(t, y; \eta')
\end{array}\]

is smooth on \(t = 0\) and has eigen-values \(\lambda_1(t, y; \eta'), \ldots, \lambda_{m_1}(t, y; \eta')\) whose imaginary parts do not vanish on \(\overline{U}_{t, t_1} \times S\). Other eigen-values vanish as \(t \to 0\). Define a projection \(P^i\) by

\[
P^i(t, y; \eta') = \frac{1}{2\pi i} \oint_{\Gamma_1} (\lambda - t^{- k_1} A(t, y; \eta'))^{-1} d\lambda,
\]

where \(\Gamma_1\) is a Jordan curve surrounding \(\lambda_1, \ldots, \lambda_{m_1}\) and having other eigen-values outside. Next for the matrix \(t^{- k_1} A(I - P^i)\) the same statements hold.

We can continue these decompositions one after another, and finally we have

\[
\sum_{i=1}^j \text{rank } P^i = m,
\]

where \(P^i\) is the projection for the eigen-values \(\lambda_1, \ldots, \lambda_{m_i}\) of \(t^{- k_1} \cdots t^{- k_i} A(I - P^i) \cdots (I - P^{i-1})\) with non-vanishing imaginary parts.

On the above assumptions we obtain

**Theorem 2.1.** If \(\text{Im } \lambda_1(0, y; \eta'), \ldots, \text{Im } \lambda_{m_i}(0, y; \eta')\) are all positive for every \(i\) such that \(k_1 + \cdots + k_t\) is odd, then we have the subelliptic estimate

\[
||u||_{s+\varepsilon_0} \leq C(||Au||_{s, \overline{U}_{t, t_1}} + ||u||_s), \quad u \in C^0_0(U_{t_2, t_2})(0 < \varepsilon_2 < \varepsilon_1),
\]

where \(\varepsilon_0 = \frac{1}{k_1 + \cdots + k_t + 1}\) and \(s \in \mathbb{R}\).

It is obvious that the estimate (2.1) for any \(s \in \mathbb{R}\) is derived from the one for \(s = 0\). In order to prove this theorem we state several lemmas. We write

\[
A(x; \tau, \eta) = A(x; 0, \eta) + \int_0^t \frac{\partial A}{\partial \tau}(x; \theta \tau, \eta)d\theta \tau,
\]

and set

\[
A^{(\tau)}(x; \tau, \eta) = \int_0^t \frac{\partial A}{\partial \tau}(x; \theta \tau, \eta)d\theta,
\]

which belongs to \(S^0_{\delta, 0}(U_{t_1, t_1})\).

**Lemma 2.1.** If \(\varepsilon > 0\) is sufficiently small, we have

(i) for all multi-indices \(\alpha, \beta\) (\(|\alpha| \geq 1\))
$$|D_{l,n}^\theta \partial_\eta^m A^{(l)}(t, y; \xi)| \leq C_{\alpha\beta} \langle \eta \rangle^{-1}, \quad (t, y) \in \bar{U}_{t, t}, \quad \xi \in \mathbb{R}^n,$$

(ii) \[ \det A^{(l)}(t, y; \xi) \geq \delta(\epsilon), \quad (t, y) \in \bar{U}_{t, t}, \quad |\xi| \geq R, \]

where \( R \) is a sufficiently large constant. (For a matrix \( A(x) = (a_{ij}(x)) \) \( |A(x)| \) denotes \( \max_{i,j} |a_{ij}(x)| \)).

Proof. (i) From the definition of \( A^{(l)} \), it follows that

$$|D_{l,n}^\theta \partial_\eta^m A^{(l)}(t, y; \xi)| \leq \int_0^1 |D_{l,n}^\theta \partial_\eta^m \left\{ \frac{\partial A}{\partial \tau}(t, y; \theta \tau, \eta) \right\} | \, d\theta$$

$$\leq C_1 \langle \eta \rangle^{-|\alpha|}.$$

Hence, if \( \langle \eta \rangle \geq \delta_1(\epsilon) > 0 \) we have the estimate (i). Let \( \langle \tau, \eta \rangle \leq \delta_2 \), then \( \langle \tau, \eta \rangle \leq C \langle \tau \rangle \quad |\tau| \leq 1 \) when \( \delta_1 \) is small. Therefore, we get

$$|D_{l,n}^\theta \partial_\eta^m A^{(l)}(t, y; \xi)| = \left| D_{l,n}^\theta \partial_\eta^m \left[ \frac{1}{\tau} \{ A(t, y; \tau, \eta) - A(t, y; 0, \eta) \} \right] \right|$$

$$\leq C_2 \frac{1}{|\tau|} + C_3 \langle \tau, \eta \rangle \langle \tau, \eta \rangle^{1/|\alpha|} \leq C \langle \tau, \eta \rangle^{-1} \quad (|\tau| \geq 1).$$

So we obtain the inequality (i) for any \( (\tau, \eta) \in \mathbb{R}^n \).

(ii) By Taylor's expansion

$$A^{(l)}(x; \tau, \eta) = A^{(l)}(x; 0, \eta) + \int_0^1 \frac{\partial A^{(l)}}{\partial \tau}(x; \theta \tau, \eta) \, d\theta \, \tau$$

and the estimate (i) we have

$$|\det A^{(l)}(t, y; \tau, \eta)| \geq |\det \frac{\partial A}{\partial \tau}(t, y; 0, \eta)| - C_3 \sum_{\tau=1}^n \left( \frac{|\tau|}{|\eta|} \right)^j, \quad (|\eta| \geq 1),$$

where \( C_3 \) does not depend on \( \tau, \eta \). Hence, from [A-III] the estimate (ii) holds when \( |\tau| \leq \rho \) and \( |t| < \epsilon \) (\( \epsilon, \rho \) are small enough). When \( |\tau| \geq \rho \), writing

$$A^{(l)}(t, y; \tau, \eta) = \frac{1}{\tau} \left\{ A(t, y; \tau, \eta) - A(t, y; 0, \eta) \right\}$$

$$= \pm A \left( t, y; \pm 1, \frac{\eta}{|\tau|} \right) - A(t, y; 0, \eta) \frac{1}{\tau}, \quad (|\tau| \geq 1)$$

(where \( \pm \) means the sign of \( \tau \)), we have

$$|\det A^{(l)}(t, y; \tau, \eta)| \geq |\det A(t, y; \pm 1, \frac{\eta}{|\tau|})|$$

$$- C_6 \left( \sum_{\tau=1}^n \left[ |A(t, y; 0, \eta)| \frac{1}{|\eta|} \rho^{-1} \right]^j \right),$$
where \( C_6 \) is independent of \( \tau, \eta \). From [A-II] it follows that

\[ |A(t, y; 0, \eta)| \leq C_7 |t| |\eta| \leq C_7 |\eta| \varepsilon \quad (|\eta| \geq 1). \]

Therefore, by [A-I] we obtain the estimate (ii) if \( \varepsilon \) is sufficiently small for \( \rho \). The proof is complete.

Now we set

\[ B^{(i)}(t, y; \xi) = A^{(i)}(t, y; \xi)^{-1} \theta(\xi), \]

where \((t, y) \in U_{\tau, \eta}, \theta(\xi) (\in C^\infty(\mathbb{R}^n))\) is equal to 0 for \(|\xi| \leq R\) and to 1 for \(|\xi| \geq R + 1\) (\(R\) is the constant in (ii) of Lemma 2.1). In view of Lemma 2.1 and Proposition 1.1 it is seen that \( B^{(i)}(x; \xi) \) satisfies the same inequality as in (i) of Lemma 2.1 and that \( B^{(i)}(x; D_x) \) is a local inverse of \( A^{(i)}(x; D_x) \) modulo \( S_{-1}^{0,0}(U_{\tau, \eta}) \), namely, for all \( \varphi(x), \psi(x) \in C^\infty(U_{\tau, \eta}) \) such that \( \psi(x) = 1 \) on a neighborhood of \( \text{supp}(\varphi) \), we have

\[ \varphi B^{(i)}(x; D_x)^{\circ} \psi A^{(i)}(x; D_x) \equiv \varphi \mod S_{-1}^{0,0}, \]

\[ \varphi A^{(i)}(x; D_x)^{\circ} \psi B^{(i)}(x; D_x) \equiv \varphi \mod S_{-1}^{0,0}. \]

This implies

\[ \varphi A(x; D_x) \equiv \varphi A^{(i)}(x; D_x)^{\circ} \psi [D_t + B^{(i)}(t, y; D_t, D_y) A(t, y; 0, D_y)] \mod S_{0,0}^{1}. \]

Therefore, noting that \( A(x; D_x) \) is elliptic when \( t \neq 0 \), we have only to examine the operator

\[ D_t + B^{(i)}(t, y; D_t, D_y) A(t, y; 0, D_y) \]

near \( t = 0 \). Furthermore, this is approximated by \( D_t + L(t, y; D_y) \), where \( L(t, y; \eta) \in S_{0,0}^{1}(U_{\tau, \eta}) \) has the form

\[ L(t, y; \eta) = \frac{\partial A}{\partial \tau}(t, y; 0, \eta)^{-1} A(t, y; 0, \eta), \quad (t, y) \in U_{\tau, \eta}, \quad |\eta| \geq 1. \]

More precisely, we have

**Lemma 2.2.** Let \( \varphi(t, y) \in C^\infty(U_{\tau, \eta}) \). Then we have

\[ \varphi(t, y) B^{(i)}(t, y; \tau, \eta) A(t, y; 0, \eta) - \varphi(t, y) L(t, y; \eta) = t \varphi(t, y) Q(t, y; \tau, \eta) + R(t, y; \tau, \eta), \]

where \( Q(x; \xi), R(x; \xi) \in S_{0,0}^{1} \) and \( Q(x; \xi) \) does not depend on \( \varphi \).

Proof. By Taylor's expansion of \( B^{(i)}(x; \tau, \eta) \) in \( \tau \) and \( B^{(i)}(x; 0, \eta) = \frac{\partial A}{\partial \tau}(t, y; 0, \eta)^{-1} (|\eta| \leq R + 1) \), we have
\[
\varphi(t, y)B(t, y; \tau, \eta)A(t, y; 0, \eta) - \varphi(t, y)L(t, y; \eta)
\equiv \varphi \int_0^1 \frac{\partial B(t, y; \theta \tau, \eta)}{\partial \tau} d\theta \tau A(t, y; 0, \eta) \mod S_{\varphi, 0}.
\]

[A-II] yields that
\[
A(t, y; 0, \eta) = t^\frac{1}{0} (\theta t, y; 0, \eta) d\theta, \quad |\eta| \geq 1.
\]

Therefore we obtain the lemma.

Proof of Theorem 2.1. Let \( \|u(t, y)\|_1, s'=\|D_1 \varphi u\|_0 + \|D_2 \varphi u\|_0 (s \geq 0, s' \geq 0) \).

It suffices to prove the following lemma:

Lemma 2.3. Let the assumptions in Theorem 2.1 be satisfied. Then we have
\[
\|\varphi u\|_{1, t_0} \leq C_1 ||(D_1 + \varphi L(t, y; D_j)) (\varphi u)||_0 + C_2 ||\varphi u||_0, \quad u \in C_0^\infty (R^n),
\]
where \( \varepsilon_0 = \frac{1}{k_1 + \cdots + k_j + 1} \), \( \varphi, \psi \in C_0^\infty (U_{s, t_1}), \psi(x)=1 \) on a neighborhood of \( \text{supp} (\varphi) \) and the constant \( C_1 \) is independent of \( \varphi, \psi \) and \( \varepsilon \).

In fact, combining Lemma 2.2 and this lemma, we obtain
\[
\|\varphi u\|_{1, t_0} \leq C_1 ||(D_1 + \varphi B(t, y; D_j))A(t, y; 0, D_j) (\varphi u)||_0 + C_2 \varepsilon ||D_1 (\varphi u)||_0 + C_3 ||\varphi u||_0, \quad u \in C_0^\infty (R^n).
\]
Here \( \varphi, \psi \) are the functions stated in Lemma 2.2 and Lemma 2.3. Since the above constant \( C_3 \) does not depend on \( \varepsilon \), if \( \varepsilon > 0 \) is small enough it follows (from (2.2)) that
\[
\|\varphi u\|_{1, t_0} \leq C_4 ||\psi A(x, D_j) \varphi u||_0 + ||\varphi u||_0,
\]
which proves Theorem 2.1.

Now let us derive Lemma 2.3. [A-IV] yields

Lemma 2.4. There exist a finite open covering \( \{V_a\} \) on \( S(= \{y': |y'| = 1\}) \) and a set of functions \( N_a(x; y) \in C^\infty (U_{s, t_1} \times V_a) \) such that for any \( (x, y) \in U_{s, t_1} \times V_a \)
\[(i) \quad \det N_a(x; y) \neq 0 \]
\[(ii) \quad N_a(x; y) A(x; y') = \begin{pmatrix} t^{k_1} \bar{A}_1(x; y') & t^{k_1+k_2} \bar{A}_2(x; y') & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \end{pmatrix} N_a(x; y'),
\]
where \( \bar{A}_i(x; y') \) is an \( m_i \times m_i \)-matrix with the eigen-values \( \lambda_1(x; y'), \ldots, \lambda_{m_i}(x; y') \).
stated in [A-IV].

Proof. Let us recall that

\[ P^i(t, y; \eta') = \frac{1}{2\pi i} \oint_{\Gamma_i} (\lambda - A_i(t, y; \eta'))^{-1} d\lambda. \]

Here \( A_i = t^{-k_1} \cdots t^{-k_l} \tilde{A}(I - P^0) \cdots (I - P^{m-1}) \) and \( \Gamma_i \) is a Jordan curve surrounding \( \lambda, \lambda', \ldots, \lambda_{m_i} \) and having other eigen-values outside. Obviously \( P^i(t, y; \eta') \) is infinitely differentiable on \( U_{t, \eta_i} \times S \). From the definition (2.3) it is easily seen that for all \( i, j = 1, \ldots, l \)

\[ P^i A = AP^i, \quad P^i A_j = A_j P^i, \quad P^i P^j = P^j P^i. \]

Set

\[ Q^i(t, y; \eta') = (I - P^0) \cdots (I - P^{m-1}) P^i \quad (i = 1, \ldots, l). \]

Then we have

\[ \text{rank } Q^i = \text{rank } P^i = m, \quad Q^i Q^j = [0] \quad \text{if } i \neq j, \]

\[ A Q^i = t^{k_1} \cdots t^{k_l} A_i Q^i. \]

Choose generalized eigen (row) vectors \( \psi_i(t, y; \eta') \), \( \psi_i(t, y; \eta') \) linearly independent such that \( \psi_i Q^i = \psi^i \). These vectors can be taken smoothly on \( U_{t, \eta_i} \times V \) \((V \text{ is an open set in } S)\). Put

\[ N(t, y; \eta') = \begin{pmatrix} \psi_i(t, y; \eta') \\ \vdots \\ \psi_{i_n}(t, y; \eta') \end{pmatrix}. \]

Then we see easily that \( N(t, y; \eta') \) satisfies (i) and (ii) of the lemma. The proof is complete.

**Proposition 2.1.** Let \( A'(\eta') \in C^m(S) \) \((S = \{ \eta' : |\eta'| = 1 \})\) be an \( m' \times m' \)-matrix whose eigen-values all have non-vanishing imaginary parts on \( S \), and let \( k \) be a constant positive integer. Set

\[ L'(t; \eta) = t^k A' \left( \frac{\eta}{|\eta|} \right) |\eta| \theta(\eta), \]

where \( \theta(\eta) \in C^\infty) = 1 \text{ for } |\eta| \geq 1 \text{ and } \theta(\eta) = 0 \text{ for } |\eta| \leq \frac{1}{2}. \) Then, we have

(i) If either 'k is even' or 'k is odd and every imaginary part of the eigen-value is positive', the following estimate holds:
(ii) If $k$ is even, there exists a continuous operator $R'$ from $L^2(\mathbb{R}^n)$ to $H^{(k)}(\mathbb{R}^n)$ such that $S' = (D_t + L')R' - I$ is continuous from $L^2(\mathbb{R}^n)$ to $H^0(\mathbb{R}^n) = \{u \in L^2; \partial_t u \in L^2\}$ for any $N > 0$.

We can prove this proposition in the same way as in [14]. A sketch of the proof is given in Appendix. In this section we do not need (ii) of the proposition, which is used in the next section.

Proof of Lemma 2.3. Let $\theta(\eta) \in C^\omega(\mathbb{R}^{n-1})$ be equal to 1 for $|\eta| \geq 1$ and to 0 for $|\eta| \leq \frac{1}{2}$. For any $\eta$ we define $N_\alpha(x; \eta)$ by $N_\alpha(x; \eta)\theta(\eta)$. Then $N_\alpha(x; \eta)$ belongs to $S^0_{(\eta), 1, 0}(U_{t, \xi})$. Let $\{\varphi_\alpha'(\eta')\}$ be a partition of unity on $S$ subject to the covering $\{V_{\alpha}\}$ stated in Lemma 2.4, and define $\varphi_\alpha(\eta) = \varphi_\alpha'(\frac{\eta}{|\eta|})\theta(\eta)$ for any $\eta$. By (i) of Lemma 2.4 and the asymptotic expansion formula (1.1) we have for $p(t, y; \eta) \in S^1_{(\eta), 1, 0}$ and $\varphi \in C^\omega(\mathbb{R}^n)$

\[
\|\varphi p(t, y; D_y)u\|_0 \leq C \sum_\alpha \|\varphi N_\alpha(t, y; D_y)\varphi_\alpha(D_y) \circ p(t, y; D_y)u\|_0 + C_2\|u\|_0.
\]

Similarly, from (ii) of Lemma 2.4 it follows that

\[
\varphi N_\alpha(x; D_y)\varphi_\alpha(D_y) \circ \varphi L(x; D_y) \equiv \varphi D(x; D_y) \circ \varphi N_\alpha(x; D_y) \varphi_\alpha(D_y)
\mod S^0_{(\eta), 1, 0}(\mathbb{R}^n).
\]

Here $D(x; \eta) \in S^1_{(\eta), 1, 0}(U_{t, \xi})$ is of the form

\[
D(x; \eta) = \begin{pmatrix}
t^k A_1(x; \eta & 0 & \\
& t^k A_2(x; \eta & 0 & \\
& & \ddots & 0 & \\
& & & t^{k_1 + \cdots + k_l} A_l(x; \eta & 0 & \\
\end{pmatrix}, |\eta| \geq 1.
\]

We get easily

\[
\|\partial_t^{(k+1)} v(t, y)\|_0 \leq C_0\|D_t v\|_0 + \|t^k \partial_x D_y v\|_0
\]

for $v(t, y) \in C^\omega(\mathbb{R}^n)$ whose support lies in $|t| \leq 1$ ($k$ is a positive integer). Combining this inequality and (2.4), we have
DEGENERATE ELLIPTIC SYSTEMS 547

\[\|D_x \psi \|^0 \leq C \sum_i \sum_k (\|D_i \psi_i \|^0 + \|D_k \psi_k \|^0),\]

where \((x, \psi) = \psi N_\alpha (x; D_\psi \phi_\alpha (D_\psi) (\psi) (\psi(x) = 1) in a neighborhood of supp (\phi)). \] (2.4) and (2.5) yield that

\[\|D_x \psi \|^0 \leq \|D_x \psi \|^0 + \|\phi L (\psi u)\|^0 \]

Noting that \(\|\psi D_x \phi_\alpha (\psi u)\|^0 \leq C \sum_i \|\psi D_x \phi_\alpha (\psi u)\|^0\), we have only to show

\[\|D_x \psi \|^0 + \|\psi D_x \phi_\alpha (\psi u)\|^0 \leq C \|D_x \psi \|^0 + \|\phi L (\psi u)\|^0\]

This is guaranteed by (i) of Proposition 2.1. The proof is complete.

3. The system on a compact manifold

Let \(M\) be a compact \(C^\infty\) manifold and let 

\[A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} \]

be a matrix of pseudo-differential operators on \(M\) such that the order of \(a_{ij}\) is \(-s_i + t_j\) \((s_i, t_j \in \mathbb{R})\). We define for \(s \in \mathbb{R}, s' = (s_1', \cdots, s_m') \) \((s_i' \in \mathbb{R})\)

\[\mathcal{H}^{s'}(M) = \prod_{i=1}^m \mathcal{H}_{s_i'}(M),\]

\[\|u\|_{s'} = \left( \sum_{i=1}^m \|u_i\|_{s_i'}^2 \right)^{1/2} \] \((u = (u_1, \cdots, u_m))\).

Then \(A\) is a continuous operator from \(\mathcal{H}^{s_0}(M)\) to \(\mathcal{H}^{s_0}(M)\).

In this section we consider the system of equations

\[Au = f\]

on the following assumptions. Let \(M (n = \dim M \geq 2)\) be separated into two connected components by a \(C^\infty\) submanifold \(M_0\). We assume that \(A\) is elliptic outside \(M_0\) and degenerate on \(M_0\) in the following way: Let \(\{x' = (x_0', x_1', \cdots, x_{n-1}') \}_{i=1}^N\) be a set of local coordinates such that each \(x'\) transforms an open set \(V_i\) to \(U_{t_1, t_2}(=\{(x_0', x_1', \cdots, x_{n-1}') : |x_0'| < \varepsilon_i, \{(x_1')^2 + \cdots + (x_{n-1}')^2\}^{1/2} < \varepsilon_i\} \) \((\varepsilon_i > 0)\) and that \(\bigcup_{i=1}^N V_i\) covers \(M_0\). Furthermore, let \(M_0\) be expressed by the equation \(x_0' = 0\) and the transition from \(x'\) to \(x_i\) in the domain where both \(x'\) and \(x_i\) are defined be given by the form

\[x_0' = x_0, \quad x_i = \varphi_i (x_1', \cdots, x_{n-1}') \] \((k = 1, \cdots, n-1)\).

We suppose that when \(A\) is represented near \(V\), by \(x' = (t, y_1, \cdots, y_{n-1}) (i = 1, \cdots,\)
the principal part $A(t, y; \tau, \eta)$ of the local symbol on $V_i$ always satisfies the assumptions $[A-I] \sim [A-IV]$ stated in §2, and that the constants $k_1, \ldots, k_i$ and $m_1, \ldots, m_i$ in $[A-IV]$ are all independent of a choice of $x'$.

Our purpose is to show the following theorem:

**Theorem 3.1.** (i) If $\text{Im} \lambda(t, y; \tau, \eta), \ldots, \text{Im} \lambda(t, y; \tau, \eta)$ are all positive for every $i$ such that $k_1 + \cdots + k_i$ is odd, then we have the estimate

$$|u|^s_{s+t_0-1} \leq C(|Au|^s_{s+t_0-1} + |u|^s_{s+t_0-1}), \quad u \in H^s(M)$$

for $s \in R \left( \varepsilon_0 = \frac{1}{k_1 + \cdots + k_i + 1} \right)$.

(ii) If every $k_i (i=1, \ldots, l)$ is even, $A$ is of Fredholm type as a mapping from $H^s(M)$ to $H^s(M)$ (i.e., the kernel and cokernel of (3.1) are finite-dimensional).

**Remark 3.1.** Let $A = A_\mu$ have a parameter $\mu > 0$ as a covariable and be elliptic including $\mu$ outside $M_0$. Furthermore, assume that the local symbol $A(t, y; \tau, \eta, \mu)$ inclusive of $y \in M_0$ satisfies the same hypotheses as in the theorem. Then the equation $A_\mu u = f$ is uniquely solvable in the same spaces if $\mu$ is large enough.

**Proof of Theorem 3.1.** Since $A$ is elliptic outside $M_0$, we have only to investigate the equation $Au = f$ locally near $M_0$. Let $A(t, y; \tau, \eta)$ be the local symbol of $A$ (on $V_i$) in the local coordinates $x' = (t, y) = x$. Set for $s' = (s_1', \ldots, s_m')$ ($s_i' \in R$)

$$\Lambda^{*'} = \begin{bmatrix} \langle D_x \rangle^{s_1'}, & 0 \\ 0 & \langle D_x \rangle^{s_m'} \end{bmatrix}.$$ 

Then $\Lambda^{*'}$ is a topological isomorphism from $H^s(R^n) = \prod_{j=1}^m H_{s+j}(R^n)$ to $H_s(R^n)$.

We examine

$$A'(x; D_x) = \Lambda^{1+s} \circ A(x; D_x) \circ \Lambda^{-t} \quad (1+s = (1+s_1, \ldots, 1+s_m))$$

instead of $A(x; D_x)$. Its principal part $A'_0(x; \xi)$ is of the form

$$A'_0(x; \xi) = A_0\left(x; \frac{\xi}{|\xi|} \right) \frac{1}{|\xi|}, \quad |\xi| \geq 1,$$

which is homogeneous of order one in $\xi$. $A'_0(x; \xi)$ satisfies all the assumptions for $A(x; \xi)$ stated in §2. Therefore, by Theorem 2.1 we obtain for $\psi(t, y) \in C_0(U_{r_1, r_2})$

$$|\psi u|^s_{s+t_0} \leq C_s(||A'_0(x; D_x) (\psi u)|^s_{s+t_0} + ||\psi u||_{s}), \quad u \in H_{s+1}(R^n),$$

which proves (i) of Theorem 3.1.
Now let us show (ii) of Theorem 3.1. From the estimate in (i) of the theorem the kernel of (3.1) is finite-dimensional. We shall show that the cokernel is also finite-dimensional by constructing the (right) regularizer $R$, that is, $R$ is continuous from $H^s_\ast(\mathcal{M})$ to $H^s_\ast(\mathcal{M})$ and $S=AR-I$ is a compact operator in $H^s_\ast(\mathcal{M})$. Obviously it suffices to do so for $s=0$. Furthermore, as is easily seen, we can make such an operator $R$ by the local analysis of $A$ in the same way as in Agranovich [1], Visik-Grušin [17], etc. Therefore we have only to construct a local regularizer of $D_t+B_0^{(i)}(t, y; D_t, D_y)A_0'(t, y; 0, D_y)$, where $B_0^{(i)}(x; \tau, \eta)=A_0^{(i)}(x; \tau, \eta)^{\sim} \left(= \left[ \int_0^{\tau} A_0'(x; \theta \tau, \eta) d\theta \right]^{-1} \right)$ for large $|\tau, \eta|$ (cf. Lemma 2.1 and (2.2)). We obtain the required local regularizer:

**Lemma 3.1.** Let $\varphi(t, y), \psi(t, y) \in C_0^\infty(U_{r, \tau})$ ($\varepsilon>0$ is small enough) and $\psi(t, y)=1$ in a neighborhood of $\text{supp} (\varphi)$. Then, there exists an operator $R_0$ continuous from $L^2(\mathbb{R}^n)$ to $H_{r,0}(\mathbb{R}^n) (= \{u \in L^2; D_t u \in L^2, \langle D_y \rangle^\alpha u \in L^2 \})$ such that

(i) the estimate

$$
\|D_t R_0 f\|_0 + \|\psi(D_0+L)^{\gamma}(R_0 f)\|_0 \leq C\|f\|_0
$$

holds for a constant $C$ independent of $\varphi, \varepsilon$, and

(ii) $\psi(D_t+L'(t, y; D_t, D_y)A_0'(t, y; 0, D_y))\varphi f = \varphi f + Q_0' \varphi f + \psi K_0' \varphi f$,

where $K_\psi$ is a continuous operator from $L^2(\mathbb{R}^n)$ to $H_{r,0}(\mathbb{R}^n)$ and $Q$ is a continuous one from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with a norm $\leq C'$ ($C'$ is independent of $\varepsilon, \varphi, \psi$).

Proof. We denote by $R_k'$ the operator $R'$ in Proposition 2.1 for $k=k_i$ and $A'(\eta')=\tilde{A}_i(0; \eta')$, and define the functions $N_\alpha(t, y; \eta), \varphi_\alpha(\eta), \varphi_\alpha(\eta)^{\sim}$ in the same way as in the proof of Lemma 2.3. Set

$$
R_0' = \sum_\alpha \psi_\alpha N_\alpha^{\sim}(t, y; D_y) \varphi_\alpha(D_y) \begin{bmatrix}
\psi R_0' \\
0
\end{bmatrix}_{\psi_\alpha(N_\alpha(t, y; D_y) \varphi_\alpha(D_y)).}
$$

Then Proposition 2.1 yields the estimate for $R_0'$ of the type (3.2). Moreover, using the asymptotic expansion formula (1.1), we have

$$
\psi(D_t+L'(t, y; D_y))R_0' \varphi f = \varphi f + Q_0' \varphi f + \psi K_0' \varphi f,
$$

where $L'(t, y; \eta)=\left[ \begin{array}{c}
\partial A_0'(t, y; 0, \eta) \\
\partial \tau
\end{array} \right]^{-1} \cdot A_0'(t, y; 0, \eta) \theta(\eta)$, $K_0'$ is continuous from $L^2(\mathbb{R}^n)$ to $H_{r,0}(\mathbb{R}^n)$ and $Q_0'$ is a continuous operator on $L^2$ with a norm $\leq \varepsilon C_1$ ($C_1$ does not depend on $\varepsilon$). Let $\chi_q(\tau, \eta) \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 if $|\tau| \geq (N+1)$

1) $\varphi_\alpha(\eta)$ is of the same type as $\varphi_\alpha(\eta)$ and equal to 1 on a neighborhood of $\text{supp} (\varphi_\alpha)$. 

• $|\eta| \& |\tau, \eta| \geq 1$ and to 0 if $|\tau| \leq N |\eta|$ or $|\tau, \eta| \leq \frac{1}{2}$. When $N$ is large enough, the symbol $R_0''(t, y; \tau, \eta) = \psi'(t, y)(\tau + L'(t, y; \eta))^{-\epsilon} \chi_N(\tau, \eta)$ belongs to $S_{\delta, 0}^{-1}$ ($\psi'(t, y) \in C_\sigma(U_{\tau, \eta})$ and $\psi'(t, y) = 1$ in a neighborhood of $\text{supp } (\psi)$ and it follows from Proposition 1.1 that

$$\psi(D_t + L'(t, y; D_j)) R_0''(t, y; D_t, D_j) \equiv \psi \chi_N(D_t, D_j) \mod S_{\delta, 0}^{-1}.$$  

On the other hand, $(1-\chi_N(D_t, D_j))$ is a continuous operator from $H_{\tilde{\nu}_{0}, \tilde{\nu}_0}(R^n)$ to $H_{\tilde{\nu}_{0}, \tilde{\nu}_0}(R^n)$. Therefore, setting $\tilde{R}_0 = R_0' - R_0'' K_0'$, we have

$$\psi(D_t + L'(t, y; D_j)) \tilde{R}_0 f = \varphi f + Q_0 \varphi f + \tilde{K} f,$$

where $\tilde{K}$ is continuous from $L^2(R^n)$ to $H_{\tilde{\nu}_{0}, \tilde{\nu}_0}(R^n)$. Hence, by Lemma 2.2 we easily obtain (ii) of Lemma 3.1. The proof is complete.

### 4. An application to boundary value problems

Let $\Omega$ be a bounded open set in $R^n (n \geq 3)$ with a $C^\infty$ boundary $\Gamma$, and consider the boundary value problem (mentioned in Introduction):

$$\begin{cases}
L(x, D_x) u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu_1, \partial n} = g_1 & \text{on } \Gamma, \\
\frac{\partial u}{\partial \nu_2} = g_2 & \text{on } \Gamma,
\end{cases}$$  

(4.1)

where $L(x, D_x)$ is an elliptic differential operator of fourth order on $\Omega$ with $C^\infty$ coefficients, $n$ is an inner (unit) normal vector of $\Gamma$ and $\nu_1, \nu_2$ are non-vanishing real vector fields on $\Gamma$. Let the following assumptions (4.2)~(4.5) be satisfied:

(4.2) $\Gamma$ is separated into two connected components by a $C^\infty$ submanifold $\Gamma_0$, and $\nu_1, \nu_2$ are tangent to $\Gamma$ on $\Gamma_0$ and transversal to $\Gamma_0$. We write

$$\nu_1 = \nu_1 + \nu_{1a} \quad (i = 1, 2)$$

where $\nu_{1a}$ is the normal component $(\langle \nu_1, n \rangle n)$ and $\nu_{1t}$ is the tangential component to $\Gamma$;

(4.3) the directions of $\nu_{1t}$ and $\nu_{2t}$ coincide in a neighborhood of $\Gamma_0$;

(4.4) the sigh of $\langle \nu_1, n \rangle (i = 1, 2)$ does not change near $\Gamma_0$ and $\langle \nu_i(x), n(x) \rangle$ has a zero of finite (even) order $\kappa_i$ along the curves on $\Gamma$ defined by the vector field $\nu_{1t}$ ($\kappa_i$ is constant on $\Gamma_0$).

Furthermore, the following inequality holds near $\Gamma_0$:

$$|\frac{\nu_2}{\nu_{2t}}| \cdot |\nu_{1a}| < \left| |\frac{\nu_1}{\nu_{1t}}| , n \right|.$$  

(4.4)
(The assumption (4.3) implies that \( \nu_i (i = 1, 2) \) belongs to the third class stated in Egorov-Kondrat’ev [5] or the author [14].) Let \( \omega^1_i (x; \zeta), \omega^2_i (x; \zeta) \) denote the roots of the equation \( L_0(x, \zeta + \omega n) = 0 \) in \( \omega \) with the positive imaginary part where \( L_0 \) is the principal part of \( L \) and \( \zeta \) is any vector \((\pm 0)\) parallel to \( \Gamma; \)

\[ (4.5) \quad \omega^1_i (x; \zeta) \text{ and } \omega^2_i (x; \zeta) \text{ are always purely imaginary in a neighborhood of } \Gamma_0. \]

On these assumptions we obtain

**Theorem 4.1.** Let the problem (4.1) be coercive outside \( \Gamma_0 \) (i.e., the Shapiro-Lopatinski condition (see [10], [11], etc.) is satisfied) and (4.2) \( \sim \) (4.5) hold. Then we have for any \( s \geq 0 \)

\[
(i) \quad \|u\|_{s+3+\tau_0, \Omega} \leq C \{\|L u\|_{s, \Omega} + \|B_1 u\|_{s+3/2, \Gamma} + \|B_2 u\|_{s+5/2, \Gamma} + \|u\|_{s+3, \Omega}\},
\]

\[
u(t) = \frac{1}{\kappa_1 + 1}, \quad \frac{1}{\kappa_2 + 1}, \quad B_1 u = \frac{\partial^2 u}{\partial \nu_1 \partial n} \bigg|_\Gamma \quad \text{and} \quad B_2 u = \frac{\partial u}{\partial \nu_2} \bigg|_\Gamma.
\]

(ii) The operator \( u \mapsto (L u, B_1 u, B_2 u) \) is of Fredholm type from \( H_{s+3+\tau_0} (\Omega) \) to \( H_0(\Omega) \times H_{s+3/2}(\Gamma) \times H_{s+5/2}(\Gamma) \).

**Remark.** Let \( \tilde{L}(x, D_x) \) be a strongly elliptic operator and replace \( L \) in (4.1) with \( \tilde{L} + \mu^4 \) (\( \mu \) is a parameter \((\geq 0))\). Then (4.1) is uniquely solvable for large \( \mu \), that is, for any \((f, g_1, g_2) \in H_0(\Omega) \times H_{s+3/2}(\Gamma) \times H_{s+5/2}(\Gamma) \) the solution \( u \in H_{s+3+\tau_0}(\Omega) \) is found uniquely in \( H_0(\Omega) \) if \( \mu \) is sufficiently large. This is obtained by means of Remark 3.1.

**Proof of Theorem 4.1.** Let \( \mathcal{P} \) be the Poisson operator, that is, \( \mathcal{P} \) is a continuous mapping from \( H^{(3+\tau_0)/2}(\Gamma) \) (\( \tau = (0, 1) \)) to \( H^{s+3+\tau_0}(\Omega) \) satisfying \( L(x, \partial_x^2) \mathcal{P} = 0 \) and \( D \mathcal{P} = I + K_\Gamma \) (where \( D u = \partial(D_x^2 u) \bigg|_\Gamma, u \bigg|_\Gamma \) and \( K_\Gamma \) is continuous from \( H^{(3+\tau_0)/2}(\Gamma) \) to \( H^{(3+\tau_0)/2}(\Gamma) \)). The construction of \( \mathcal{P} \) is described in [10]. Set

\[
T = B \mathcal{P} \quad (Bu = (B_1 u, B_2 u)).
\]

Then \( T \) is continuous from \( H^{(3+\tau_0)/2}(\Gamma) \) to \( H^{(3+\tau_0)/2}(\Gamma) \) (\( \tau = (-1, 0) \)). If we have for \( T \) the estimate and the regularizer of the same type as for \( A \) in (3.1), we can obtain the theorem. In fact, combining the inequalities

\[
\|u\|_{s+3+\tau_0, \Omega} \leq C_1 (\|L u\|_{s-\tau_0, \Omega} + \|D u\|_{s+3/2+\tau_0, \Gamma} + \|u\|_{s+3, \Omega})
\]

and

\[
\|h\|_{s+3/2+\tau_0, \Gamma} \leq C_2 (\|T h\|_{s+3/2, \Gamma} + \|h\|_{s+3/2, \Gamma}),
\]

we have

\[
\|u\|_{s+3+\tau_0, \Omega} \leq C_3 (\|L u\|_{s, \Omega} + \|B u\|_{s+3/2, \Gamma} + \|B u - D \mathcal{P} u\|_{s+3/2, \Gamma} + \|u\|_{s+3, \Omega}),
\]

which yields the estimate (i) of the theorem. There exists an operator \( R_D \) continuous from \( H_0(\Omega) \) to \( H_{s+4}(\Omega) \) such that \( LR_D = I + S_D \) and \( DR_D = 0 \) where \( S_D \) is
a continuous operator from $H_s(\Omega)$ to $H_{s+1}(\Omega)$. Using this $R_\rho$ and the regularizer $R$ of $T$, we set $R(f, g) = \partial R(g - BR_\rho f) + R_\rho f$ for $(f, g) \in H_s(\Omega) \times H_{s+1}(\Omega)$. Then $R$ is the (right) regularizer for the problem (4.1) (i.e., $R$ is continuous from $H_s(\Omega) \times H_{s+1}(\Omega)$ to $H_{s+1}(\Omega)$, and $1 - (L \rho, B \rho)$ is a compact operator on $H_s(\Omega) \times H_{s+1}(\Omega)$), which proves (ii) of the theorem. Therefore it suffices to examine $T$.

$T$ is a matrix of pseudo-differential operators on $\Gamma$ modulo a continuous operator from $H_{s+\frac{1}{2}}(\Gamma)$ to $H_{s+\frac{1}{2}}(\Gamma)$ ($N'$ is an arbitrary positive constant), whose $(i, j)$-element is of order $-s_i + t_j$ ($t=(t_1, t_2)=(0, 1), s=(s_1, s_2)=(-1, 0)$). The principal symbol $T_0$ of $T$ is expressed by the Lopatinski matrix of (4.1) on $\Gamma$. These are explained in Kumano-go [10].

Now, setting $M = \Gamma$ and $M_0 = \Gamma_0$, we shall show that $T$ satisfies all the assumptions for $A$ in (3.1) by choosing appropriate local coordinates near $\Gamma_0$. Let $\varphi(x) = \text{dis} (x, \Gamma)$ ($x \in \Omega$), which is a $C^\infty$-function when $\varphi$ is small enough).

From (4.2) and (4.3) we can take a set of local coordinates $\{(x^i, \varphi)\}_{i=1,...,N}$ covering $\Gamma_0$ such that (i) $\{x^i\}_{i=1,...,N}$ is of the same type as $\{x^i\}$ stated in §3 and (ii) $\partial a_i / \partial y_j$ $(j=1, 2)$ is transformed by $x^i=(t, y)$ $(i=1, \cdots, N)$ to

$$a_j(t, y) \frac{\partial}{\partial t} + t^i b_j(t, y) \frac{\partial}{\partial z},$$

where $a_j$ and $b_j$ are not equal to 0 near the origin. Representing $T$ locally by $x^i=(t, y)$ $(i=1, \cdots, N)$, its principal symbol $A_0(t, y; \tau, \eta)$ is of the following form (near $(t, y)=0$):

$$A_0(t, y; \tau, \eta) = \begin{bmatrix} -t^i b_i (\omega_1^i + \omega_2^i) - a_1 \tau & t^i b_i (\omega_1^i + \omega_2^i) \\ i t^i b_2 & i a_2 \tau \end{bmatrix}, \quad (|\tau, \eta| \geq 1).$$

From this form and (4.5) it is seen that $[A-I]$ and $[A-II]$ (see §2) are satisfied. Since

$$\frac{\partial A_0}{\partial \tau}(t, y; 0, \eta) = \begin{bmatrix} -t^i b_i (\omega_1^i + \omega_2^i) |_{\tau=0} - a_1 \tau & t^i b_i (\omega_1^i + \omega_2^i) |_{\tau=0} \\ 0 & i a_2 \end{bmatrix}, \quad (|\eta| \geq 1),$$

$[A-III]$ also holds. Finally let us check $[A-IV]$. We have for $\eta' \in S(= \{|\eta'| = 1\})$

$$A_0(t, y; \eta') = \frac{\partial A_0}{\partial \tau}(t, y; 0, \eta')^{-1} \cdot A_0(t, y; 0, \eta')$$

$$= \left\{ a_1 a_2 + t^i a_i b_i \partial, (\omega_1^i + \omega_2^i) |_{\tau=0} \right\}^{-1} \begin{bmatrix} t^2 \{ a_i b_i (\omega_1^i + \omega_2^i) + t^i b_i b_i \partial, (\omega_1^i + \omega_2^i) \} |_{\tau=0} - t^i a_i b_i (\omega_1^i + \omega_2^i) |_{\tau=0} \\ t^2 \{ a_i b_i + t^i b_i b_i \partial, (\omega_1^i + \omega_2^i) \} |_{\tau=0} \end{bmatrix}.$$ (4.4) yields

$$\frac{t^i b_i}{a_i} < \frac{t^i b_i}{a_i}$$

for small $t$, which implies that $\kappa_1 < \kappa_2$ or $\kappa_1 = \kappa_2$ &
In the latter case, by (4.5) we see that [A-IV] is satisfied ($l=1$ and $k_1=k_1=k_2$). Let us consider the former case. Set $k_1=k_1, k_2=k_2-k_1$ ($l=2$). Then $k_1$ and $k_2$ are positive even integers. Since

$$\det (\lambda-t^{-k_1}A_0(t, y; \eta))$$

$$= \lambda^2-(\alpha_2+t^{k_1}+\kappa_2)(a_{1}a_{2}+t^{k_1}+\alpha_3)(a_{1}a_{2}+t^{k_1}+\alpha_3)^{-2}$$

(where $\alpha_1=a_{1}b_{2}\partial,(\omega_{1}^{1}+\omega_{2}^{1})|_{\tau=0}, \alpha_2=a_{1}b_{1}(\omega_{1}^{1}+\omega_{2}^{1})|_{\tau=0}, \alpha_2=b_{2}d_{2}\partial,(\omega_{1}^{1}+\omega_{2}^{1})|_{\tau=0}, b_1=b_{2}d_{2}(\omega_{1}^{1}+\omega_{2}^{1})|_{\tau=0}$ and $\beta_2=a_{1}b_{2}d_{2}(\omega_{1}^{1}+\omega_{2}^{1})|_{\tau=0}$, the eigen-values $\lambda_1, \lambda_2$ of $t^{-k_1}A_0(t, y; \eta)$ are of the forms

$$\lambda_1(t, y; \eta') = (a_{1}a_{2}^{-1})^{-1}t^{k_1}+O(t^{k_1}),$$

$$\lambda_2(t, y; \eta') = -t^{k_2}a_{2}^{2}b_{2}\alpha_2^{-1}+O(t^{k_2+1})$$

where $O(t^{k})$ means that $t^{-k}O(t^{k})$ is smooth (in $t$, $y$ and $\eta'$). On the other hand, we have

$$(I-P^0) = \frac{1}{2\pi i} \oint \frac{(\lambda-t^{-k_1}A_0)d\lambda}{\lambda-\lambda_2-\lambda_2}$$

Therefore,

$$t^{-k_1}A_0(I-P^0) = \frac{\lambda_2}{\lambda_2-\lambda_1} \text{ cof} [\lambda_2-t^{-k_1}A_0]$$

$$= t^{k_2} [0 - a_{2}^{-1}b_{2}\alpha_2^{-2}+O(t^{k_2+1})]$$

Hence, by (4.5) we see that [A-IV] is satisfied. The proof is complete.

**Appendix. Proof of Proposition 2.1**

Proposition 2.1 is derived from the following lemma:

**Lemma A.** Let $A'$ be a constant $m' \times m'$-matrix whose eigen-values all have non-vanishing imaginary parts, and let $k$ be a constant positive integer.

(i) If either 'k is even' or 'k is odd and every imaginary part of the eigen-value is positive', we have the estimate

$$C^{-1}(\|D_{\tau}w(t)\|_{0, R^1}+\|t^{k}w\|_{0, R^1}) \leq \|\langle D_{\tau}+t^{k}A'\rangle w\|_{0, R^1}$$

$$\leq C(\|D_{\tau}w\|_{0, R^1}+\|t^{k}w\|_{0, R^1}), w(t) \in S,$$

where the constant $C$ can be taken uniformly in $\eta'$ when $A'=A'(\eta')$ ($A'(\eta')$ is stated in Proposition 2.1).

(ii) If $k$ is even, the operator

$$(D_{\tau}+t^{k}A'): W_{0}^{1}(R^1) \to L^{2}(R^1)$$
is a topological isomorphism \( W^1_t(R^n) = \{ w(t) \in H_1(R^n); t^k w(t) \in L^2(R^n) \} \).

Transforming \( A' \) to Jordan's normal form, we can prove this lemma in the same way as in the proof of Theorem 2.1 in the author [14].

Proof of Proposition 2.1. The idea of the proof is referred to Višik-Grušin [16], Grušin [7]. By the change of the variable: \( t = |\eta|^{-1/(k+1)} t' \), we have

\[
\| D_r w(t) \|_{\delta_1, R^1} = \| r^{1/(k+1)} D_r w'(t') \|_{\delta_1, R^1} \quad (w'(t') = w(|\eta|^{-1/(k+1)} t')) , \\
\| t^k |\eta| w(t) \|_{\delta_1, R^1} = \| r^{1/(k+1)} t^k w'(t') \|_{\delta_1, R^1} , \\
\| \{ D_r + t^k A' \left( \frac{\eta}{|\eta|} \right) |\eta| \theta(\eta) \} w(t) \|_{\delta_1, R^1} = \| r^{1/(k+1)} \{ D_r + t^k A' \left( \frac{\eta}{|\eta|} \right) \theta(\eta) \} w'(t') \|_{\delta_1, R^1} .
\]

Therefore, from (i) of Lemma A it follows that

\[
C^{-1} \left( \int_{|\eta| \geq 1} \| D_r \psi(t, \eta) \|_{\delta_1, R^1} d\eta + \int_{|\eta| \geq 1} \| t^k \psi(t, \eta) \|_{\delta_1, R^1} d\eta \right) \\
\leq \int_{|\eta| \geq 1} \| (D_r + L'(t; \eta)) \psi \|_{\delta_1, R^1} d\eta , \\
\leq C \left( \int_{|\eta| \geq 1} \| D_r \psi \|_{\delta_1, R^1} d\eta + \int_{|\eta| \geq 1} \| t^k \psi \|_{\delta_1, R^1} d\eta \right) , \psi(t, \eta) \in S ,
\]

which proves (i) of Proposition 2.1.

Let \( \psi(\eta) (\in C^\infty) = 1 \) for \( |\eta| \geq 2 \) and \( \psi(\eta) = 0 \) for \( |\eta| \leq 1 \). By (ii) of Lemma A the operator \( D_r + t^k A' \left( \frac{\eta}{|\eta|} \right) |\eta| \) has an inverse \( Q_n \) for any \( \eta (\neq 0) \). We define

\[
R' f(t, \eta) = \mathcal{E}_{\eta}^{-1} [Q_n \psi(\eta) \tilde{f}(t, \eta)] ,
\]

where \( \tilde{f}(t, \eta) = \int e^{-i\eta y} f(t, y) dy \). Then \( R' \) satisfies the requirement of (ii) in Proposition 2.1.

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References


