

Title	On the stable James numbers of complex projective spaces
Author(s)	Ōshima, Hideaki
Citation	Osaka Journal of Mathematics. 11(2) P.361-P.366
Issue Date	1974
Text Version	publisher
URL	https://doi.org/10.18910/4059
DOI	10.18910/4059
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ON THE STABLE JAMES NUMBERS OF COMPLEX PROJECTIVE SPACES

HIDEAKI ŌSHIMA

(Received November 14, 1973)

1. Introduction

For a pointed finite CW-pair $i: A \subset X$ where A is a connected oriented topological manifold, a (stable) map $f: X \rightarrow A$ is of type r if the composite $A \xrightarrow{i} X \xrightarrow{f} A$ has degree r . $j(X, A)$ and $j_s(X, A)$ denote the sets of integers r for which there exists a map $f: X \rightarrow A$ of type r and a stable map of type r respectively. When $j(X, A)$ forms an ideal $(k(X, A))$ in the ring of integers Z —here $k(X, A)$ denotes the non-negative generator, we call $k(X, A)$ the James number of the pair (X, A) . In the stable case $j_s(X, A)$ is always an ideal of Z . So we may define the stable James number $k_s(X, A)$.

James [3] has posed the problem of determining $j(SP^m(S^n), S^n)$, where $SP^m(S^n)$ is the m -fold symmetric product of an n -sphere S^n with a base point x_0 and $i: S^n \rightarrow SP^m(S^n)$ is the axial embedding $x \rightarrow [x, x_0, \dots, x_0]$. James showed for example $j(SP^m(S^n), S^n)$ forms an ideal of Z and, for an even dimensional sphere S^{2n} , $k(SP^m(S^{2n}), S^{2n}) = 0$. On the contrary $k_s(SP^m(S^n), S^n) \neq 0$ for any positive integers m and n . From now on we introduce the notation $k_s^{m,n}$ instead of $k_s(SP^m(S^n), S^n)$.

In this note we give lower bounds and an upper bound of $k_s^{m,2}$. That is, we prove

Theorem. For positive integers m and n

- (1) $k_s^{m,n} \neq 0$;
- (2) $k_s^{m+1,n}$ is a multiple of $k_s^{m,n}$;
- (3) $k_s^{m,2}$ is divisible by all the integers $m, m-1, \dots, 2$;
- (4) $k_s^{2^m-1,2}$ is divisible by 2^m for $m \geq 2$;
- (5) $k_s^{m,2}$ is a divisor of $m!(m-1)! \cdots 2!$, in particular none of the prime factors of $k_s^{m,2}$ is greater than m .

Corollary. The above lower estimates (3) and (4) are best possible for $m \leq 4$. That is

$$k_s^{1,2} = 1, k_s^{2,2} = 2, k_s^{3,2} = k_s^{4,2} = 12.$$

There is a homeomorphism $SP^m(S^2) \simeq CP^m$, the m -dimensional complex projective space. Under this identification the natural inclusions $S^2 \subset SP^m(S^2) \subset SP^{m+1}(S^2)$ become the standard ones $CP^1 \subset CP^m \subset CP^{m+1}$. $k_s^{m,2}$ is just the same as Conner-Smith's $d(m)$ [1], Example 4.

The author wishes to express his thanks to Professor S. Araki for his kind advices.

2. Proofs of (1) and (2)

Using the group multiplication of S^1 , we know $k_s^{m,1} = 1$. So we assume $n \geq 2$. First we prove (1). Consider the stable Puppe exact sequence

$$\cdots \rightarrow \{SP^m(S^n), S^n\} \xrightarrow{i^*} \{S^n, S^n\} \rightarrow \{SP^m(S^n)/S^n, S^{n+1}\} \rightarrow \cdots,$$

here $\{X, Y\}$ denotes the set of stable homotopy classes of stable maps $X \rightarrow Y$. Since $\{S^n, S^n\} \cong Z$, if i^* is non-trivial, then $k_s^{m,n} = \text{index of image } i^* \neq 0$. So, for our purpose, it suffices to show that $\{SP^m(S^n)/S^n, S^{n+1}\}$ is finite. Notice that $\{SP^m(S^n)/S^n, S^{n+1}\} = \pi_s^{n+1}(SP^m(S^n)/S^n)$ is the reduced framed cobordism group. Let $E_2^{u,v} = \hat{H}^u(SP^m(S^n)/S^n; G_{-v}) \Rightarrow \pi_s^*(SP^m(S^n)/S^n)$ be the Atiyah-Hirzebruch spectral sequence for $SP^m(S^n)/S^n$, where G_k is the stable k -stem of spheres. Since $\hat{H}^u(SP^m(S^n)/S^n; Z) = 0$ for $u \leq n+1$, $\sum_{u+v=n+1} E_2^{u,v}$ is finite. Then $\sum_{u+v=n+1} E_\infty^{u,v}$ and hence $\pi_s^{n+1}(SP^m(S^n)/S^n)$ are finite. This implies (1).

From the equality $k_s^{m,n} = \text{index of image } i^*$, (2) is obvious. Thus (1) and (2) follow.

3. Proof of (3)

We use the complex K -theory. Let η_m be the canonical complex line bundle over CP^m and $g_C = \eta_1 - 1 \in \tilde{K}(S^2)$ be the Bott generator. Then $K(CP^m)$ is the truncated polynomial ring with generator $\eta_m - 1$ and the relation $(\eta_m - 1)^{m+1} = 0$. Choose $f \in \{CP^m, S^2\}$ such that $i^*(f) = k_s^{m,2} \iota$, where ι denotes the identity map of S^2 . Let $f^*: K^*(S^2) \rightarrow K^*(CP^m)$ and $f^*: H^*(S^2; Q) \rightarrow H^*(CP^m; Q)$ be the induced homomorphisms. Put

$$f^*(g_C) = \sum_{j=1}^m a_j (\eta_m - 1)^j$$

where $a_j \in Z$. Since $i^*(f) = k_s^{m,2} \iota$, we have $a_1 = k_s^{m,2}$. Let $t \in H^2(CP^m; Z)$ be the first Chern class of η_m . We apply the Chern character, ch , for $\tilde{K}^*(CP^m)$ and $\tilde{K}^*(S^2)$. Then

$$a_1 t = k_s^{m,2} t = (f^* \circ ch)(g_C) = (ch \circ f^*)(g_C) = \sum_{j=1}^m a^j (\exp(t) - 1)^j$$

that is

$$t = \sum_{j=1}^m (a_j/a_1)(\exp(t)-1)^j$$

in $H^*(CP^m; Q)$, where we used the fact that ch is “stable”. On the other hand

$$t = \log(1+(\exp(t)-1)) = \sum_{j=1}^{\infty} ((-1)^{j+1}/j)(\exp(t)-1)^j.$$

Hence

$$k_s^{m,2} = a_1 = (-1)^{j+1}j a_j; j = 1, 2, \dots, m.$$

This implies (3).

4. Proof of (4)

In this section we use KO -theory. We introduce the following notations: η_H = the canonical symplectic line bundle over S^4 ; $g_H = \eta_H - 1 \in \widetilde{KSp}(S^4)$; $g_R = g_H \wedge g_H \in \widetilde{KO}(S^8)$; $\rho: K^*() \rightarrow KO^*()$, the real restriction; $\varepsilon: KO^*() \rightarrow K^*()$, the complexification; $\mu_3 = \rho(g_C^3 \wedge (\eta_m - 1)) \in \widetilde{KO}^{-6}(CP^m)$; $\mu_0 = \rho(\eta_m - 1) \in \widetilde{KO}(CP^m)$. We require the following theorem of Fujii [2]:

$\widetilde{KO}^{-6}(CP^m)$ is the free module with basis $\mu_3, \mu_3\mu_0, \dots, \mu_3\mu_0^{u-1}$, and also, in case m is odd, $\mu_3\mu_0^u$ (if $m \equiv 3 \pmod{4}$) or τ (if $m \equiv 1 \pmod{4}$), where $2\tau = \mu_3\mu_0^u$ and $u = [m/2]$ ($[]$ is the Gauss notation).

Choose $f \in \{CP^m, S^2\}$ such that $i^*(f) = k_s^{m,2}i$. Let $f^*: KO^*(S^2) \rightarrow KO^*(CP^m)$ be the induced homomorphism. By Fujii's theorem we may write

$$f^*(g_R) = \begin{cases} \sum_{j=0}^{\binom{m/2}{2}-1} a_j \mu_3 \mu_0^j & \text{if } m \equiv 0 \pmod{2} \\ \sum_{j=0}^{[m/2]} a_j \mu_3 \mu_0^j & \text{if } m \equiv 3 \pmod{4} \\ \sum_{j=0}^{[m/2]-1} a_j \mu_3 \mu_0^j + a_{[m/2]} \tau & \text{if } m \equiv 1 \pmod{4} \end{cases}$$

where $a_j \in Z$. In case $m=1$, we have $\mu_3 = 2g_R \in \widetilde{KO}^{-6}(S^2)$. This and $i^*(f) = k_s^{m,2}i$ imply $2a_0 = k_s^{m,2}$. We write ch for $ch \circ \varepsilon$. Then we have

$$ch(\mu_3) = \exp(t) - \exp(-t) = 2 \sinh(t)$$

and

$$ch(\mu_0) = \exp(t) + \exp(-t) - 2 = 2(\cosh(t) - 1).$$

Since

$$2a_0 t = k_s^{m,2} t = (f^* \circ ch)(g_R) = (ch \circ f^*)(g_R),$$

we obtain

$$(\#) a_0 t = \begin{cases} \sinh(t) \sum_{j=0}^{(m/2)-1} 2^j a_j (\cosh(t)-1)^j & \text{if } m \equiv 0 \pmod 2 \\ \sinh(t) \sum_{j=0}^{\lfloor m/2 \rfloor} 2^j a_j (\cosh(t)-1)^j & \text{if } m \equiv 3 \pmod 4 \\ \sinh(t) \left\{ \sum_{j=0}^{\lfloor m/2 \rfloor - 1} 2^j a_j (\cosh(t)-1)^j \right. \\ \quad \left. + 2^{\lfloor m/2 \rfloor - 1} a_{\lfloor m/2 \rfloor} (\cosh(t)-1)^{\lfloor m/2 \rfloor} \right\}, & \text{if } m \equiv 1 \pmod 4 \end{cases}$$

in $H^*(CP^m; Q)$. In case $m \equiv 0 \pmod 2$, if we differentiate the two sides of (#) by t , then we have

$$j! a_0 = (-1)^j 2^j \cdot 3 \cdot 5 \cdots (2j+1) a_j \quad \text{for } 1 \leq j \leq m/2 - 1,$$

and elementary calculation shows that we obtain the same information on $k_s^{m,2} = 2a_0$ as (3). This and (#) imply that we obtain the same information about $k_s^{4j+1,2}$ and $k_s^{4j+1,2}$ for $j \geq 1$. Hence, in case $m \equiv 1 \pmod 4$, we obtain nothing more than (3). In case $m \equiv 3 \pmod 4$, that is $m = 4j - 1$ for some j , we have the same information about $k_s^{4j-1,2}$ and $k_s^{4j,2}$. If j is a power of two, 2^q , from (3) we see that 2^{q+1} divides $k_s^{2^{q+2}-1,2}$, but the aboves imply that 2^{q+2} divides $k_s^{2^{q+2}-1,2}$. Thus (4) follows. Remark, in case j is not a power of two, we obtain nothing more than (3).

5. Proofs of (5) and Corollary

Choose $f_{m-1} \in \{CP^{m-1}, S^2\}$ such that $i^*(f_{m-1}) = k_s^{m-1,2} \iota$. Let $p_{m-1}: S^{2m-1} \rightarrow CP^{m-1}$ be the canonical fibration and $\text{ord}(p_{m-1})$ be its order as a stable map. The composite $(\text{ord}(p_{m-1}))\iota \circ f_{m-1} \circ p_{m-1}$ is null homotopic. Hence there exists $f \in \{CP^m, S^2\}$ such that $f \circ j = (\text{ord}(p_{m-1}))\iota \circ f_{m-1} \in \{CP^{m-1}, S^2\}$, where $j: CP^{m-1} \subset CP^m$. This implies that $k_s^{m,2}$ is a divisor of $\text{ord}(p_{m-1}) \cdot k_s^{m-1,2}$. Inductively we know that $k_s^{m,2}$ is a divisor of $\text{ord}(p_{m-1}) \cdot \text{ord}(p_{m-2}) \cdots \text{ord}(p_1) k_s^{1,2}$. Obviously $k_s^{1,2} = 1$. By Toda [4], page 1103, $\text{ord}(p_{m-1})$ is a divisor of $m!$. Thus (5) follows. And we complete the proof of Theorem.

We prove Corollary. For $m \leq 3$, the estimates (3), (4) and (5) imply that $k_s^{1,2} = 1$, $k_s^{2,2} = 2$ and $k_s^{3,2} = 12$. We show $k_s^{4,2} = 12$. Choose $f_3 \in \{CP^3, S^2\}$ such that $i^*(f_3) = 12\iota$. The composite $f_3 \circ p_3: S^7 \rightarrow S^2$ represents an element of G_5 , five-stem of spheres. It is well known that $G_5 = 0$. Hence there exists $f \in \{CP^4, S^2\}$ such that the composite $CP^3 \subset CP^4 \xrightarrow{f} S^2$ coincides with f_3 . This implies that $k_s^{4,2}$ is a divisor of 12. By (2) $k_s^{4,2}$ is a multiple of $k_s^{3,2} = 12$. Therefore $k_s^{4,2} = 12$. This completes the proof of Corollary.

6. Addendum

The same technique is applicable to the stable James number $d_H(m) = k_s$

(HP^m, S^4) of the pair of symplectic projective spaces. Using the complex K -theory, a lower bound of $d_H(m)$ can be obtained from

$$a_1 t = \sum_{j=1}^m 2^j a_j (\sum_{k=1}^{\infty} t^k / (2k)!)^j, \quad a_1 = d_H(m),$$

where $t \in H^4(HP^m; Z)$ is a generator and $a_j \in Z$. For example, we have $12 | d_H(2)$. Since the order of the canonical fibration $S^7 \rightarrow S^4$ as a stable map is 24, we have $d_H(2) = 24$. So that this estimate is not best possible.

The unstable James numbers of the pairs (RP^m, S^1) , (CP^m, S^2) , (HP^m, S^4) and the stable James number of (RP^m, S^1) are all zero for $m \geq 2$, where RP^m denotes the m -dimensional real projective space.

OSAKA CITY UNIVERSITY

References

[1] P.E. Conner and L. Smith: *On the complex bordism of finite complexes*, II, *J. Differential Geometry* **6** (1971), 135–174.
 [2] M. Fujii: *K_0 -groups of projective spaces*, *Osaka J. Math.* **4** (1967), 141–149.
 [3] I.M. James: *Symmetric functions of several variables, whose range and domain is a sphere*, *Bol. Soc. Mat. Mexicana.* **1** (1956), 85–88.
 [4] H. Toda: *On unstable homotopy of spheres and classical groups*, *Proc. Nat. Acad. Sci. USA* **46** (1960), 1102–1105.

Added in proof. After completed this manuscript, the author has found a paper of J. Ucci “Symmetric maps of spheres of least positive James number, *Indiana Univ. Math. J.* (1972), 709–714” which gives an upper bound of unstable James numbers $k^{m,n} = k(SP^m(S^n), S^n)$. Combining his estimate with ours, we obtain

Theorem A.

- (i) $\beta_2(m) \leq \nu_2(k_s^{m,2}) \leq 2\beta_2(m)$,
- (ii) $m \leq \nu_2(k_s^{2^m-1,2}) \leq 2m-2$ for $m \geq 2$,
- (iii) $\nu_p(k_s^{m,2}) = \beta_p(m)$ for an odd prime p ,

where $\nu_p(n)$ denotes the exponent of p in the prime factorization of n and $\beta_p(m)$ is defined by $p^{\beta_p(m)} \leq m < p^{\beta_p(m)+1}$.

Proof. Identifying $S(S^n)$ with S^{n+1} , $S(SP^m(S^n))$ can be embedded in $SP^m(S^{n+1})$ so that the inclusion $S^{n+1} \xrightarrow{i} SP^m(S^{n+1})$ factorizes as the composition $S(S^n) \xrightarrow{S(i)} S(SP^m(S^n)) \subset SP^m(S^{n+1})$, where $S(X)$ denotes the reduced suspension of a pointed space X . This implies that $k_s^{m,n}$ is a factor of $k_s^{m,n+1}$. By definition, $k_s^{m,n}$ is a factor of $k^{m,n}$ for odd n . So, in particular, $k_s^{m,2}$ is a factor of $k^{m,3}$. Ucci's

estimates of $k^{m,3}$ are $\nu_2(k^{m,3}) \leq 2\beta_2(m)$ and $\nu_p(k^{m,3}) = \beta_p(m)$ for an odd prime p . Therefore we have $\nu_2(k_s^{m,2}) \leq 2\beta_2(m)$ and $\nu_p(k_s^{m,2}) \leq \beta_p(m)$ for an odd prime p . On the other hand the estimates (3) and (4) imply that $\beta_p(m) \leq \nu_p(k_s^{m,2})$ for a prime p and $n \leq \nu_2(k_s^{2^n-1,2})$ for $n \geq 2$. Thus Theorem A follows.