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INTEGRODIFFERENTIAL EQUATION WHICH INTERPOLATES THE HEAT EQUATION AND THE WAVE EQUATION

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Introduction

Recently many authors have studied the following integrodifferential equation:

$$(0.1)\qquad u(t, x) = \phi(x)+\int_0^t h(t-s)\Delta u(s, x)ds \qquad t>0,\ x\in \boldsymbol{R}$$

where $\Delta=(\partial/\partial x)^2$. (cf. [3], [5], [7], [16], [17]). The equation (0.1) describes the heat conduction with memory ([5], [7]). In the present paper, we shall consider the case $h(t)=\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ $(\equiv h_\alpha(t))$ for $1\leq\alpha\leq 2$. Here $\Gamma(x)$ is the gamma function. Thus, the equation (0.1) becomes

$$(\mathrm{IDE})_\alpha\qquad u(t, x) = \phi(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Delta u(s, x)ds\,.$$

For the selection of $\{h_\alpha(t)\}_{1\leq\alpha\leq 2}$, we have two reasons. The first reason is that the operator

$$(0.2)\qquad I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$$

defines the Riemann-Liouville integral of order α ([11]). As a result, $(\mathrm{IDE})_\alpha$ $(1<\alpha<2)$ interpolates the heat equatoin $(\mathrm{IDE})_1$ and the wave equation $(\mathrm{IDE})_2$. Formally, $(\mathrm{IDE})_\alpha$ corresponds to "partial differential equation"

$$(\partial/\partial t)^\alpha u(t, x) = \Delta u(t, x)\,.$$

The second reason is that $\{h_\alpha(t)\}_{1<\alpha<2}$ represents memory of a long-time tail of the power order ([14]).

The aim of the present paper is to show the following for $1<\alpha<2$:

1) The fundamental solution $\dfrac{1}{\alpha}P_\alpha(t, |x|)$ of $(\mathrm{IDE})_\alpha$ takes its maximum at $x=$

$\pm c_\alpha t^{\alpha/2}$ for each $t>0$. Here $c_\alpha>0$ is the constant determined by α (for the definition of $P_\alpha(t, x)$, see § 1).

2) If $\phi(x)\geq 0$ for $x\in \boldsymbol{R}$, the solution $u_\alpha(t, x)$ of $(\mathrm{IDE})_\alpha$ is also nonnegative for $(t, x)\in(0, \infty)\times \boldsymbol{R}$; the support of $u_\alpha(t, \cdot)$ is not compact in $\boldsymbol{R}$ for each $t>0$, even if $\phi(x)$ $(\not\equiv 0)$ has compact support. Moreover $t^{\alpha/2}u_\alpha(t, x)$ tends to a constant for every $x\in \boldsymbol{R}$, as $t\to\infty$.

By 2), the solution of $(\mathrm{IDE})_\alpha$ $(1<\alpha<2)$ has the same properties as that of the heat equation. On contrast to this fact, by 1), the fundamental solution of $(\mathrm{IDE})_\alpha$ has the similar property to that of the wave equation: the points, where the fundamental solution takes its maximum, propagate with finite speed. Therefore, $(\mathrm{IDE})_\alpha$ $(1<\alpha<2)$ is resonable interpolation of the heat equation and the wave equation.

The present paper consists of two sections. In § 1, we give the representation of the solution by the fundamental solution. In § 2, we show the above statement 1) and 2).

Finally we mention the work of Schneider and Wyss [12]. After most of this work was completed, we learned the existence of it. Although they also gave the representation of the solution, our methods for it are independent of theirs. We emphasize the following point: our aim of the present paper is not only to give the representation of the solution but also to show the statment 1) and 2).

1. Representation of solution

Let $C([0, \infty): \mathcal{S}(\boldsymbol{R}))$ be the space consisting of $\mathcal{S}(\boldsymbol{R})$-valued continuous functions on $[0, \infty)$. Here $\mathcal{S}(\boldsymbol{R})$ is the space of the rapidly decreasing functions of Schwartz. Throughout this paper we assume that ϕ of $(\mathrm{IDE})_\alpha$ belongs to $\mathcal{S}(\boldsymbol{R})$.

DEFINITION. For $1\leq\alpha\leq 2$, the function u_α in $C([0, \infty): \mathcal{S}(\boldsymbol{R}))$ is called the solution of $(\mathrm{IDE})_\alpha$, if it satisfies $(\mathrm{IDE})_\alpha$ for every $(t, x)\in(0, \infty)\times \boldsymbol{R}$.

The reason why we consider the solution in $C([0, \infty): \mathcal{S}(\boldsymbol{R}))$ is that this space is convenient for treating Fourier transform.

Before stating the theorem, we need some notations. For $1\leq\alpha<2$ and $t>0$, let

$$q_\alpha(t, \xi) = \exp\left[-t|\xi|^\delta e^{-\gamma\pi i\cdot\mathrm{sgn}(\xi)/2}\right] \tag{1.1}$$

where $\delta=\dfrac{2}{\alpha}$, $\gamma=2-\dfrac{2}{\alpha}$ and

$$\mathrm{sgn}(\xi) = \begin{cases} \dfrac{\xi}{|\xi|} & (\xi\neq 0) \\ 0 & (\xi=0). \end{cases} \tag{1.2}$$

Define $P_\alpha(t, x)$ by

$$P_\alpha(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q_\alpha(t, \xi)e^{-ix\xi}d\xi\,. \tag{1.3}$$

Then, $P_\alpha(t, x)$ is a probability density for each $t>0$, i.e.,

$$\begin{cases} P_\alpha(t, x)\geq 0 & (t, x)\in(0, \infty)\times \boldsymbol{R} \\ \displaystyle\int_{-\infty}^{\infty} P_\alpha(t, x)dx = 1 & t\in(0, \infty) \end{cases} \tag{1.4}$$

(cf. Theorem 5.7.3 of [6]). Moreover, it holds that

$$P_\alpha(t, x) = P_\alpha(xt^{-\alpha/2})t^{-\alpha/2} \qquad (t, x)\in(0, \infty)\times \boldsymbol{R} \tag{1.5}$$

where $P_\alpha(x)\equiv P_\alpha(1, x)$.

Now we state the theorem of this section.

Theorem A. *For $1\leq\alpha\leq 2$, $(\mathrm{IDE})_\alpha$ has a unique solution $u_\alpha(t, x)$ given by*

$$u_\alpha(t, x)=\begin{cases} \displaystyle\frac{1}{\alpha}\int_{-\infty}^{\infty} \phi(x-y)P_\alpha(t, |y|)dy & (1\leq\alpha<2) \\ \displaystyle\frac{1}{2}[\phi(x+t)+\phi(x-t)] & (\alpha=2)\,. \end{cases} \tag{1.6}$$

The representation (1.6) is proved in [4] and [12]. Although the representation in § 3 of [12] is slightly different from (1.6), they coincide mutually by Theorems 5.8.3 and 5.8.4 of [6]. Since the proofs of [4] and [12] are too complicated, we shall give a simple proof for the self-containedness.

In the proof of Theorem A below, we shall use the following symbols:

Re z = the real part of $z\in \boldsymbol{C}$,
Im z = the imaginary part of $z\in \boldsymbol{C}$,
Res (f, a) = the residue of the function f at a,
$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$ (Fourier transform),
$\overline{\mathcal{F}}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\xi}dx$ (Fourier inverse transform).

Proof of Theorem A (uniqueness). It is sufficient to show that if $u\in C([0, \infty): \mathcal{S}(\boldsymbol{R}))$ satisfies the equality

$$u(t, x) = I^\alpha[(\partial/\partial x)^2u](t, x) \qquad (t, x)\in(0, \infty)\times \boldsymbol{R}\,, \tag{1.7}$$

then $u\equiv 0$. Here I^α denotes the Riemann-Liouville integral defined by (0.2). Applying the Fourier transform to (1.7), we have

$$\mathcal{F}u(t, \xi) = -\xi^2 I^{\alpha}[\mathcal{F}u](t, \xi)\,.$$

Fix $T>0$ arbitrarily. Since $\mathcal{F}u(t, \xi)$ is continuous in $(t, \xi)\in[0, T]\times\boldsymbol{R}$, we have for $(t, \xi)\in[0, T]\times\boldsymbol{R}$

$$|\mathcal{F}u(t, \xi)| \leq \frac{|\xi|^2 T^{\alpha-1}}{\Gamma(\alpha)}\int_0^t |\mathcal{F}u(s, \xi)|\,ds\,.$$

Gronwall's inequality yields $\mathcal{F}u(t, \xi)=0$ for $(t, \xi)\in[0, T]\times\boldsymbol{R}$. Hence we get $u(t, x)\equiv 0$ on $[0, T]\times\boldsymbol{R}$. Since $T>0$ is arbitrary, we have proved the uniqueness.

To show the existence, we need four lemmas. Put

$$E_\alpha(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(k\alpha+1)} \qquad (y\in\boldsymbol{R})\,. \tag{1.8}$$

This is called the Mittag-Leffler function and investigated by many authors (cf. [2], [9]). Define the function $F_\alpha(\xi)$ by

$$F_\alpha(\xi) = E_\alpha(-\xi^2) = \sum_{k=0}^{\infty}\frac{(-\xi^2)^k}{\Gamma(k\alpha+1)}\,. \tag{1.9}$$

The function F_α plays an important role in the construction of the solution. To investigate the properties of F_α, we shall give an integral representation of it. We set for $1<\alpha<2$

$$\begin{cases} a_\alpha(\xi) = |\xi|^{2/\alpha}\exp\left[\dfrac{\pi i}{\alpha}\,\mathrm{sgn}(\xi)\right] \\ b_\alpha(\xi) = |\xi|^{2/\alpha}\exp\left[-\dfrac{\pi i}{\alpha}\,\mathrm{sgn}(\xi)\right] \end{cases} \qquad (\xi\in\boldsymbol{R}) \tag{1.10}$$

and

$$f_\alpha(\xi) = \begin{cases} \dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^\infty \frac{\xi^2 t^{\alpha-1}e^{-t}}{t^{2\alpha}+2\xi^2 t^\alpha\cos(\alpha\pi)+\xi^4}\,dt & (\xi\neq 0) \\ 1-\dfrac{2}{\alpha} \quad (\xi=0)\,. \end{cases} \tag{1.11}$$

Lemma 1.1. *For* $1<\alpha<2$:

(I) *All of the functions* $\exp[a_\alpha(\xi)]$, $\exp[b_\alpha(\xi)]$ *and* $f_\alpha(\xi)$ *are continuous on* $\boldsymbol{R}$ *and belong to* $L^1(\boldsymbol{R})$.

(II) $F_\alpha(\xi)=\dfrac{1}{\alpha}\{\exp[a_\alpha(\xi)]+\exp[b_\alpha(\xi)]\}+f_\alpha(\xi)$ *for* $\xi\in\boldsymbol{R}$. *Therefore* $F_\alpha(\xi)$ *also belongs to* $L^1(\boldsymbol{R})$.

Proof. (I) First we shall show that $\exp[a_\alpha(\xi)]$, $\exp[b_\alpha(\xi)]$ and $f_\alpha(\xi)$ are continuous on $\boldsymbol{R}$. It is easy to see that $\exp[a_\alpha(\xi)]$ and $\exp[b_\alpha(\xi)]$ are continuous on $\boldsymbol{R}$. Since $f_\alpha(\xi)$ is continuous at $\xi \neq 0$ clearly, it remains to check that $f_\alpha(\xi)$ is continuous at $\xi=0$. Using the change of variable $t=|\xi|^{2/\alpha}s^{1/\alpha}$ $(\xi \neq 0)$ in (1.11), we have

$$f_\alpha(\xi)=\frac{\sin(\alpha\pi)}{\alpha\pi}\int_0^\infty \frac{\exp[-|\xi|^{2/\alpha}s^{1/\alpha}]}{s^2+2s\cos(\alpha\pi)+1}ds \qquad (\xi \neq 0)\,.$$

The dominated convergence theorem yields

$$\begin{aligned}\lim_{\xi\to 0} f_\alpha(\xi) &= \frac{\sin(\alpha\pi)}{\alpha\pi}\int_0^\infty \frac{1}{s^2+2s\cos(\alpha\pi)+1}ds \\ &= -\frac{1}{\alpha\pi}\int_{-\cot(\alpha\pi)}^\infty \frac{1}{\eta^2+1}d\eta \\ &= -\frac{1}{\alpha\pi}[\tan^{-1}\eta]_{-\cot(\alpha\pi)}^\infty = 1-\frac{2}{\alpha}\,,\end{aligned}$$

since $\tan^{-1}[-\cot(\alpha\pi)]=\left(\alpha-\frac{3}{2}\right)\pi$ for $1<\alpha<2$. Hence $f_\alpha(\xi)$ is continuous at $\xi=0$.

Next we shall show that $\exp[a_\alpha(\xi)]$, $\exp[b_\alpha(\xi)]$ and $f_\alpha(\xi)$ belong to $L^1(\boldsymbol{R})$. We have

$$|\exp[a_\alpha(\xi)]| = |\exp[b_\alpha(\xi)]| = \exp\left[|\xi|^{2/\alpha}\cos\frac{\pi}{\alpha}\right] \in L^1(\boldsymbol{R})\,,$$

since $\cos\frac{\pi}{\alpha}<0$ for $1<\alpha<2$. On the other hand, by the dominated convergence theorem, we get

$$\begin{aligned}\lim_{|\xi|\to\infty}|\xi|^2 f_\alpha(\xi) &= \frac{\sin(\alpha\pi)}{\pi}\lim_{|\xi|\to\infty}\int_0^\infty \frac{\xi^4 e^{-t}t^{\alpha-1}}{t^{2\alpha}+2\xi^2 t^\alpha\cos(\alpha\pi)+\xi^4}dt \\ &= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-t}t^{\alpha-1}dt = \frac{\Gamma(\alpha)}{\pi}\sin(\alpha\pi)\,.\end{aligned}$$

Since $f_\alpha(\xi)$ is continuous on $\boldsymbol{R}$, $f_\alpha(\xi)$ belongs to $L^1(\boldsymbol{R})$. This completes the proof of (I).

(II) We shall prove (II) following the idea of [9]. The case $\xi=0$ is trivial. Therefore, assume that $\xi \neq 0$. Let $R>0$ be sufficiently large and $\varepsilon>0$ sufficiently small such that $R^\alpha>|\xi|^2>\varepsilon^\alpha$. By (3) of chapter 18 of [2] we have

$$F_\alpha(\xi)=\frac{1}{2\pi i}\int_{C_1}\frac{e^t t^{\alpha-1}}{t^\alpha+\xi^2}dt\,. \tag{1.12}$$

Here C_1 is the path defined as follows; we draw, first, a straight line from $\infty e^{-\pi i}$ to $\mathrm{R}e^{-\pi i}$ along the real axis, then the cricle of center 0 and radius R in the positive sense, and, finally, a straight line from $\mathrm{R}e^{\pi i}$ to $\infty e^{\pi i}$ along the real axis. (Figure 1).

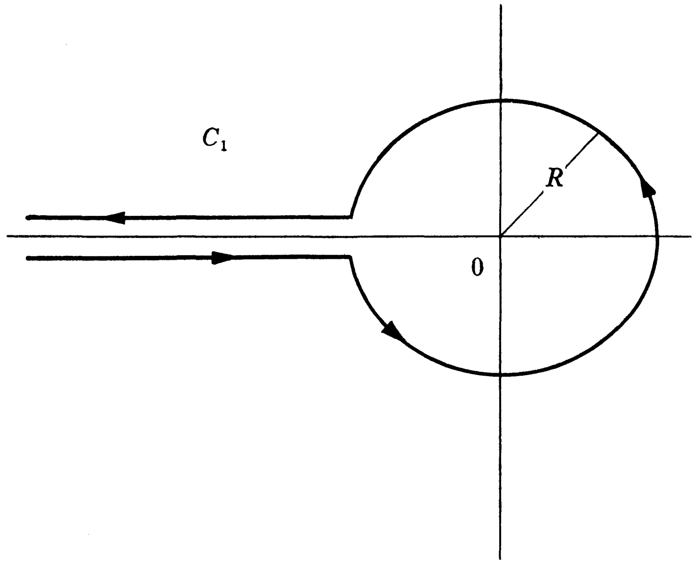

Fig. 1

Next, let C_2 be the closed path defined as follows; we draw, first, a straight line from $\varepsilon e^{-\pi i}$ to $Re^{-\pi i}$ along the real axis, then the circle of center 0 and radius R in the positive sense, then a straight line from $Re^{\pi i}$ to $\varepsilon e^{\pi i}$ along the real axis, and, finally, the circle of center 0 and radius ε in the negaitve sense (Figure 2).

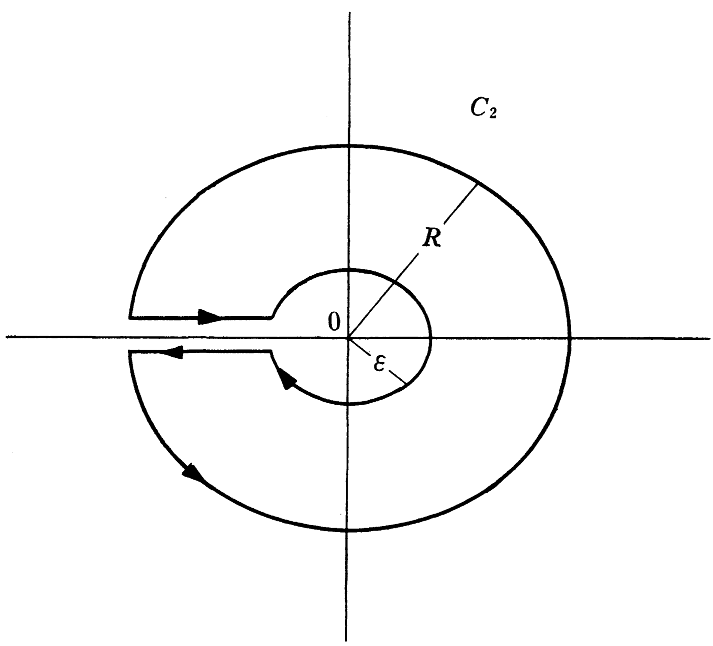

Fig. 2

In the domain surrounded by C_2, the function $\Phi(t)=\dfrac{e^t t^{\alpha-1}}{t^\alpha+\xi^2}$ takes its singularities at $a_\alpha(\xi)$ and $b_\alpha(\xi)$ of (1.10). By residue theorem we have

$$
\begin{aligned}
(1.13)\qquad & 2\pi i\{\mathrm{Res}\,(\Phi,\ a_\alpha(\xi))+\mathrm{Res}\,(\Phi,\ b_\alpha(\xi))\} = \int_{C_2}\Phi(t)\,dt \\
&= \int_\varepsilon^R \Phi(te^{-\pi i})d(te^{-\pi i})+\int_{-\pi}^{\pi}\Phi(\mathrm{Re}^{i\theta})d(\mathrm{Re}^{i\theta})+\int_R^\varepsilon \Phi(te^{\pi i})d(te^{\pi i}) \\
&\quad +\int_\pi^{-\pi}\Phi(\varepsilon e^{i\theta})d(\varepsilon e^{i\theta})\,.
\end{aligned}
$$

Thus we get by (1.12) and (1.13)

$$
\begin{aligned}
2\pi iF_\alpha(\xi) &= \int_\infty^R \Phi(te^{-\pi i})d(te^{-\pi i})+2\pi i\{\mathrm{Res}\,(\Phi,\ a_\alpha(\xi))+\mathrm{Res}\,(\Phi,\ b_\alpha(\xi))\} \\
&\quad -\int_\varepsilon^R \Phi(te^{-\pi i})d(te^{-\pi i})-\int_R^\varepsilon \Phi(te^{\pi i})d(te^{\pi i})-\int_\pi^{-\pi}\Phi(\varepsilon e^{i\theta})d(\varepsilon e^{i\theta})+\int_R^\infty \Phi(te^{\pi i})d(te^{\pi i}) \\
&\equiv I_1(R)+2\pi i\times\{A\}-I_2(\varepsilon,\ R)-I_3(\varepsilon,\ R)-I_4(\varepsilon)+I_5(R)\,.
\end{aligned}
$$

It is easy to see that $\lim_{R\to\infty} I_1(R)=\lim_{R\to\infty} I_5(R)=\lim_{\varepsilon\downarrow 0} I_4(\varepsilon)=0$ and $\lim_{R\to\infty}\lim_{\varepsilon\downarrow 0}\,[-I_2(\varepsilon,\ R)-I_3(\varepsilon,\ R)]=2\pi i f_\alpha(\xi)$. Since $\mathrm{Res}(\Phi,\ a_\alpha(\xi))=\dfrac{1}{\alpha}\exp[a_\alpha(\xi)]$ and $\mathrm{Res}(\Phi,\ b_\alpha(\xi))=\dfrac{1}{\alpha}\exp[b_\alpha(\xi)]$, we get the desired result. This completes the proof of Lemma 1.1.

Lemma 1.2. *The function* $\overline{\mathcal{F}}[F_\alpha(\xi t^{\alpha/2})\mathcal{F}\phi(\xi)](x)$ *is a solution of* $(\mathrm{IDE})_\alpha$ *for* $1\leq\alpha\leq 2$.

Proof. Put $v(t,\ x)=\overline{\mathcal{F}}[F_\alpha(\xi t^{\alpha/2})\mathcal{F}\phi(\xi)](x)$. Since F_α and its derivatives of any order are bounded and continuous on $\boldsymbol{R}$, it is easy to see that v belongs to $C([0,\ \infty):\ \mathcal{S}(\boldsymbol{R}))$. We have

$$
\begin{aligned}
I^\alpha[(\partial/\partial x)^2 v](t,\ x) &= I^\alpha\overline{\mathcal{F}}[(-\xi^2)\mathcal{F}v](t,\ x) \\
&= \overline{\mathcal{F}}\left[(-\xi^2)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_\alpha(\xi s^{\alpha/2})\,ds\,\mathcal{F}\phi(\xi)\right](x)\,.
\end{aligned}
$$

By (1.9), it holds that

$$
(-\xi^2)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_\alpha(\xi s^{\alpha/2})\,ds = F_\alpha(\xi t^{\alpha/2})-1
$$

for every $\xi\in\boldsymbol{R}$. Hence

$$
\begin{aligned}
& I^\alpha[(\partial/\partial x)^2 v](t,\ x) \\
&= \overline{\mathcal{F}}[F_\alpha(\xi t^{\alpha/2})\mathcal{F}\phi(\xi)-\mathcal{F}\phi(\xi)](x) = (t,\ x)-\phi(x)\,,
\end{aligned}
$$

so that v is a solution of $(\mathrm{IDE})_\alpha$. This complets the proof of Lemma 1.2.

By Lemmas 1.1 and 1.2, $(\mathrm{IDE})_\alpha$ has a unique solution $u_\alpha(t, x)=\overline{\mathcal{F}}[F_\alpha(\xi t^{\alpha/2})\mathcal{F}\phi(\xi)](x)$. It remains to show the expression (1.6). In Lemmas 1.3 and 1.4 below, we shall calculate the inverse Fourier transform of $F_\alpha(\xi)$.

Lemma 1.3. *For* $1<\alpha<2$,

$$\int_{-\infty}^{\infty}\frac{\xi^2}{\xi^4+2\xi^2\cos\alpha\pi+1}e^{ix\xi}d\xi=-\frac{\pi}{\sin\alpha\pi}\operatorname{Im}\left[\exp\left\{|x|e^{\alpha\pi i/2}+\frac{\alpha\pi i}{2}\right\}\right].$$

Proof. We shall show only the case $x>0$, since the case $x<0$ can be proved similarly. Let C_3 be the closed path defined as follows; we describe, first, the segment $[-R, R]$, then the circular arc $z=\mathrm{Re}^{i\theta}$ with $0\leq\theta\leq\pi$ (Figure 3).

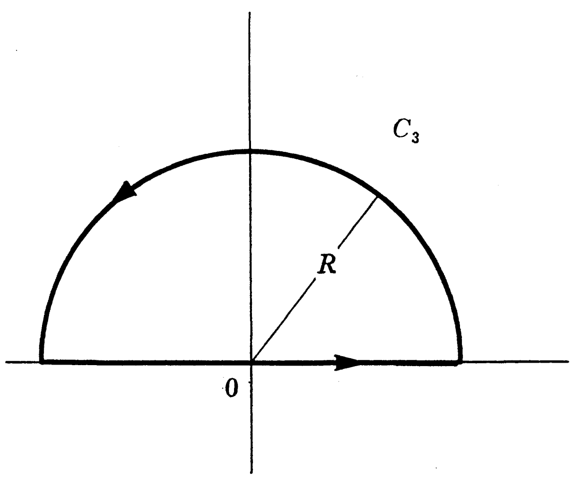

Fig. 3

In the domain surrounded by C_3, the function

$$\Psi(z)=\frac{z^2}{z^4+2z^2\cos\alpha\pi+1}e^{ixz}$$

takes its sungularities at $z_1=\exp\left[\frac{\alpha-1}{2}\pi i\right]$ and $z_2=\exp\left[\frac{3-\alpha}{2}\pi i\right]$. By residue theorem, we have

$$\int_{C_3}\Psi(z)dz=2\pi i\sum_{j=1}^{2}\operatorname{Res}(\Psi, z_j)=\frac{-\pi}{\sin(\alpha\pi)}\operatorname{Im}\left[\exp\left\{xe^{\alpha\pi i/2}+\frac{\alpha\pi i}{2}\right\}\right].$$

Since the integral over the circular arc tends to 0 as $R\to\infty$, we have the desired result. This completes the proof.

Lemma 1.4. $\overline{\mathscr{F}}[F_\alpha](x)=\dfrac{1}{\alpha}\mathrm{P}_\alpha(|x|)\ (1\leq\alpha<2)$.

Proof. When $\alpha=1$, this lemma follows easily from (1.3) and the equalities $F_1(\xi)=q_1(1, \xi)=\exp[-\xi^2]$, where $q_1(t, \xi)$ is the function of (1.1). Therefore, assume that $\alpha>1$. It follows from (1.3) that

$$\begin{cases} \overline{\mathscr{F}}[\exp(a_\alpha)](x)=P_\alpha(-x) \\ \overline{\mathscr{F}}[\exp(b_\alpha)](x)=P_\alpha(x)\,. \end{cases}$$

Now we shall consider $\overline{\mathscr{F}}[f_\alpha](x)$. By Fubini theorem and Lemma 1.3, we have

$$\begin{aligned} (1.14)\qquad & \overline{\mathscr{F}}[f_\alpha](x) \\ &=\frac{\sin(\alpha\pi)}{2\pi^2}\int_0^\infty t^{\alpha-1}e^{-t}dt\int_{-\infty}^\infty \frac{\xi^2}{\xi^4+2\xi^2t^\alpha\cos(\alpha\pi)+t^{2\alpha}}e^{ix\xi}d\xi \\ &=\frac{\sin(\alpha\pi)}{2\pi^2}\int_0^\infty t^{(\alpha/2)-1}e^{-t}\,dt\int_{-\infty}^\infty \frac{\eta^2}{\eta^4+2\eta^2\cos(\alpha\pi)+1}\exp[ix\eta t^{\alpha/2}]\,d\eta \\ &=-\frac{1}{2\pi}\,\mathrm{Im}\int_0^\infty t^{(\alpha/2)-1}\exp\left[-t+|x|t^{\alpha/2}e^{\alpha\pi i/2}+\frac{\alpha\pi i}{2}\right]dt \\ &=-\frac{1}{\alpha\pi}\,\mathrm{Im}\int_0^\infty \exp\left[-r^{2/\alpha}+|x|re^{\alpha\pi i/2}+\frac{\alpha\pi i}{2}\right]dr\,. \end{aligned}$$

Cauchy's integral theorem enables us to change the variable r into $w\exp\left[\dfrac{1-\alpha}{2}\pi i\right]$ in the last expression of (1.14). We have, thereofre, by (1.3)

$$\begin{aligned} & \overline{\mathscr{F}}[f_\alpha](x) \\ &=-\frac{1}{\alpha\pi}\,\mathrm{Re}\{\int_0^\infty \exp[-w^{2/\alpha}e^{-\gamma\pi i/2}+|x|wi]dw\} \qquad \left(\gamma=2-\frac{2}{\alpha}\right) \\ &=-\frac{1}{\alpha}P_\alpha(-|x|)\,. \end{aligned}$$

Hence we obtain by (II) of Lemma 1.1

$$\overline{\mathscr{F}}[F_\alpha](x)=\frac{1}{\alpha}[\boldsymbol{P}_\alpha(-x)+\boldsymbol{P}_\alpha(x)-\boldsymbol{P}_\alpha(-|x|)]=\frac{1}{\alpha}\boldsymbol{P}_\alpha(|x|)\,.$$

This completes the proof.

Proof of Theorem A. (expression (1.6))

Since the existence of the solution has been proved in Lemma 1.2, we shall prove the expression (1.6). The case $\alpha=2$ is widely known. Therefore we shall treat the case $1\leq\alpha<2$. Put $\boldsymbol{Q}_\alpha(t, x)=\dfrac{1}{\alpha}\boldsymbol{P}_\alpha(t, |x|)$. By Lemma 1.4 and

(1.5), $\overline{\mathcal{F}}_\alpha[F_\alpha(\cdot\ t^{\alpha/2})](x)=\boldsymbol{Q}_\alpha(t, x)$. Fourier inverse theorem yields $F_\alpha(\xi t^{\alpha/2})=\mathcal{F}[\boldsymbol{Q}_\alpha(t, \cdot)](\xi)$. Hence we get by Lemma 1.2

$$u_\alpha(t, x)=\overline{\mathcal{F}}[\mathcal{F}[\boldsymbol{Q}_\alpha(t, \cdot)](\xi)\mathcal{F}\phi(\xi)](x)=\int_{-\infty}^{\infty}\boldsymbol{Q}_\alpha(t, y)\phi(x-y)dy\,.$$

This completes the proof of (1.6). We have, therefore, proved Theorem A.

2. Properties of the solution

We recall that $\frac{1}{\alpha}\boldsymbol{P}_\alpha(t, |x|)$ is the fundamental solution of $(\mathrm{IDE})_\alpha$: the integral kernel of the operator from ϕ (the initial data) to $u_\alpha(t, x)$ (the solution) of $(\mathrm{IDE})_\alpha$ (cf. (1.6)).

Theorem B. *For* $1<\alpha<2$,

(B1) $\boldsymbol{P}_\alpha(t, |x|)$ *is continuous in* $(t, x)\in(0, \infty)\times\boldsymbol{R}$.

(B2) $\boldsymbol{P}_\alpha(t, |x|)$ *takes its extreme values at* $x=\pm c_\alpha t^{\alpha/2}$ (*maximum*) *and* $x=0$ (*minimum*); *it is monotone elsewhere. Here* $c_\alpha>0$ *is the constant determined by* α,

(B3) $\boldsymbol{P}_\alpha(t, |x|)$ *never vanishes for* $(t, x)\in(0, \infty)\times\boldsymbol{R}$ (Figure 4).

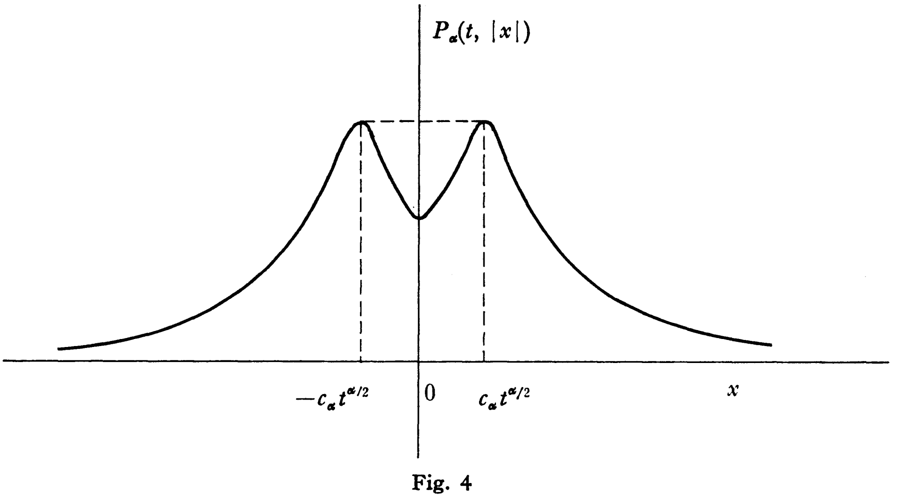

Fig. 4

By (B2), the points, where the fundamental solution takes its maximum, propagate with finite speed. Thus the fundamental solution of $(\mathrm{IDE})_\alpha$ ($1<\alpha<2$) has the similar property to that of the wave equation.

Theorem C. *For* $1\leq\alpha<2$,

(C1) *if* $\phi(x)\geq 0$ *for* $x\in\boldsymbol{R}$, *then* $u_\alpha(t, x)\geq 0$ *for* $(t, x)\in(0, \infty)\times\boldsymbol{R}$, (C2) $\lim_{t\to\infty} t^{\alpha/2}u_\alpha(t, x) = \frac{1}{\alpha}\boldsymbol{P}_\alpha(0)\int_{-\infty}^{\infty}\phi(z)dz \qquad x\in\boldsymbol{R}$,

(C3) *if* $\phi(x)\geq 0$ *and* $\phi(x)\not\equiv 0$, *the support of* $u_\alpha(t, \cdot)$ *is not compact in* $\boldsymbol{R}$ *for each* $t>0$.

The statement (C1)~(C3) shows that the solution of $(\text{IDE})_\alpha$ $(1<\alpha<2)$ inherits the properties of the solution of the heat equation $(\text{IDE})_1$.

Proof of Theorem B. (B1) Since $\boldsymbol{P}_\alpha(t, x)$ is continuous by (1.3), this is trivial.
(B2) By Lemma 5.10.2 of [6] and its proof, $\boldsymbol{P}_\alpha(x)$ $(=\boldsymbol{P}_\alpha(1, x))$ takes its maximum at $x=c_\alpha$ and is monotone elsewhere (refer to (5.8.2c) of [6]). Here $c_\alpha>0$ is the constant determined by α. By (1.5), we have the desired result.
(B3) By (1.5) and (B2), it is sufficient to show that $\boldsymbol{P}_\alpha(0)\neq 0$ and $\boldsymbol{P}_\alpha(|x|)\neq 0$ $(|x|>M)$ for some sufficiently large $M>0$. First, by Theorem 5.8.2 of [6], we have

$$\boldsymbol{P}_\alpha(0) = \frac{1}{\pi}\Gamma\left(\frac{\alpha}{2}+1\right)\sin\left(\frac{\alpha\pi}{2}\right) = -\frac{1}{\Gamma(-\alpha/2)}\neq 0\,.$$

Next, by [13] (see also Theorem 2.1.6 of [8]), we can show

$$\lim_{|x|\to\infty}\boldsymbol{P}_\alpha(|x|)\{|x|^{-(\alpha-1)/(2-\alpha)}\exp\,[A_\alpha|x|^{2/(2-\alpha)}]\} = B_\alpha\,.$$

Here A_α and B_α are positive constants determined by α. Hence $\boldsymbol{P}_\alpha(|x|)$ never vanishes for $x\in\boldsymbol{R}$. We have, therefore, proved Theorem B.

Proof of Theorem C. The statement (C1) and (C2) follows easily from (1.7), (1.8), (B1) and dominated convergence theorem.

To prove (C3), we need a lemma. Choose $M>0$ sufficiently large so that $\phi(x)\not\equiv 0$ on $[-M, M]$. Put

$$\phi_M(x) = \begin{cases} \phi(x) & |x|\leq M \\ 0 & |x|>M\,. \end{cases}$$

We recall the following notation:

$$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx \qquad \text{(Fourier transform)}.$$

Lemma 2.1. (i) $\phi(x)\geq\phi_M(x)\geq 0$ $(x\in\boldsymbol{R})$.
(ii) $\mathcal{F}\phi_M$ *can be extended to an entire function and it satisfies*

$$\lim_{\xi\to\infty}[|\mathcal{F}\phi_M(i\xi)|+|\mathcal{F}\phi_M(-i\xi)|] = \infty\,.$$

Proof. (i) This is trivial.
(ii) By the Paley-Wiener theorem (see [1], pp. 158), $\mathcal{F}\phi_M$ can be extended to

an analytic function and satisfies $|\mathcal{F}\phi_M(z)|\leq Ae^{C|z|}$ ($z\in\boldsymbol{C}$) for some A, $C>0$. Suppose that

$$\varlimsup_{\xi\to\infty}[|\mathcal{F}\phi_M(i\xi)|+|\mathcal{F}\phi_M(-i\xi)|]<\infty\,.$$

Then, by the Phragmèn-Lindelöf theorem ([1] pp. 155), $\mathcal{F}\phi_M(z)$ must be bounded on $\boldsymbol{C}$. Since $\mathcal{F}\phi_M(z)$ is an entire function and $\lim_{\xi\to\infty}\mathcal{F}\phi_M(\xi)=0$, we find that $\mathcal{F}\phi_M(z)\equiv 0$ on $\boldsymbol{C}$. This contradicts the fact such that $\phi_M\not\equiv 0$. This completes the proof.

Proof of (C3). We get by (1.4) and i) of Lemma 2.1

$$u_\alpha(t,x)=\frac{1}{\alpha}\int_{-\infty}^{\infty}\phi(x-y)\boldsymbol{P}_\alpha(t,|y|)dy$$
$$\geq\frac{1}{\alpha}\int_{-\infty}^{\infty}\phi_M(x-y)\boldsymbol{P}_\alpha(t,|y|)dy\ (\equiv u_{\alpha,M}(t,x))\geq 0\,.$$

Hence it is sufficient to show (C3) for $u_{\alpha,M}(t,x)$. By (1.10), $F_\alpha(\xi)$ can be extended to an entire function. Therefore, Lemma 1.4 and the Fourier transform lead to

$$\mathcal{F}u_{\alpha,M}(t,z)=\mathcal{F}\phi_M(z)F_\alpha(zt^{\alpha/2})\qquad(z\in\boldsymbol{C})\,.$$

Now, fix $t>0$ arbitrarily. By (10) of Chapter 18 of [2] we find that

$$\lim_{\xi\to\pm\infty}\exp[-t|\xi|^{2/\alpha}]E_\alpha(\xi^2t^\alpha)=\frac{1}{\alpha}\,.$$

Since $F_\alpha(i\xi t^{\alpha/2})=F_\alpha(-i\xi t^{\alpha/2})=E_\alpha(\xi^2t^\alpha)$ for $\xi\in\boldsymbol{R}$, we have by ii) of Lemma 2.1

$$\lim_{\xi\to\infty}\exp[-t|\xi|^{2/\alpha}]\{|\mathcal{F}u_{\alpha,M}(t,i\xi)|+|\mathcal{F}u_{\alpha,M}(t,-i\xi)|\}=\infty\,.$$

By Paley-Wiener theorem ([1], pp 158), the support of $u_{\alpha,M}(t,\cdot)$ is not compact in $\boldsymbol{R}$ for each $t>0$. This completes the proof of (C3). Thus Theorem C has been proved.

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