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Title	Nonlinear stability of viscous shock profile for a non-convex system of viscoelasticity
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Citation	Osaka Journal of Mathematics. 1997, 34(3), p. 589-603
Version Type	VoR
URL	https://doi.org/10.18910/4061
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Osaka J. Math.
34 (1997), 589–603

NONLINEAR STABILITY OF VISCOUS SHOCK PROFILE FOR A NON-CONVEX SYSTEM OF VISCOELASTICITY

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(Received September 9, 1996)

1. Introduction

This paper is concerned with an open problem proposed by Kawashima and Matsumura [4]. Namely, we study the nonlinear asymptotic stability of viscous shock profile for a one-dimensional non-convex system of viscoelasticity in the form

$$v_t - u_x = 0, \tag{1.1}$$

$$u_t - \sigma(v)_x = \mu u_{xx}, \quad x \in R, \quad t \geq 0, \tag{1.2}$$

with the initial data

$$(v, u)|_{t=0} = (v_0, u_0)(x) \tag{1.3}$$

which tend toward the given constant states $(v_\pm, u_\pm)$ as $x \to \pm\infty$. Here, v is the strain, u the velocity, $\mu > 0$ the viscous constant, and $\sigma(v)$ is the smooth stress function satisfying the condition

$$\sigma'(v) > 0 \quad \text{for all} \quad v \quad \text{under consideration}, \tag{1.4}$$

and the condition of non-convex (non-genuine) nonlinearity

$$\sigma''(v) \gtrless 0 \quad \text{for} \quad v \lesseqgtr 0 \quad \text{under consideration}, \tag{1.5}$$

which Kawashima and Matsumura [4] proposed as an unsolved case. Note that the condition (1.4) assures the strict hyperbolicity of the corresponding inviscid system, and the condition (1.5) yields the inviscid system neither genuinely nonlinear nor linearly degenerate around $v = 0$.

The stability of viscous shock profile for various one-dimesional viscous conservation laws has been studied by a number of authors (see [1–16] and the references therein). In 1985, an efficient energy method to prove the stability was first introduced independently by Matsumura and Nishihara [9] and Goodman [1] in the case of convex (genuine) nonlinearity, and later then, this case with effect of N-waves

was completed, via Liu [5,6], by Szepessy and Xin [16]. In the case of non-convex nonlinearity, Kawashima and Matsumura [4] proved the stability of viscous shock profile for the scalar non-convex conservation law $u_t + f(u)_x = \mu u_{xx}$, where the non-convexity of $f(u)$ means $f''(u) \lesseqgtr 0$ for $u \lesseqgtr 0$. When $f(u)$ satisfies the opposite sign relation like (1.5), they pointed out that a simple change of independent variable, $y = -x$, easily solves the problem in the scalar case, however does not this technique work to the system. Recently, Mei [11] and Matsumura and Nishihara [10] studied the stability as well as the time decay rate even for the degenerate shock (Oleinik's shock), and more genearal flux function. See also Jones and Gardner and Kapitula [2] by using the spectral analysis method, but their time decay rate is less sufficient than that of Kawashima and Matsumura [3,4], Mei [11] and Matsumura and Nishihara [10]. The time decay rates in [3,4,11,10] are optimal in a sense (cf. [15]). On the other hand, Kawashima and Matsumura [4] also treated the system case and proved the stability of viscous shock profile for system (1.1), (1.2) with the non-convexity condition

$$\sigma''(v) \lesseqgtr 0 \quad \text{for} \quad v \lesseqgtr 0. \tag{1.6}$$

Under the non-convexity condition (1.6), Nishihara [14] studied the stability of the degenerate viscous shock profile for the system (1.1), (1.2) at the first time. Furthermore, Mei and Nishihara [13] succeeded in improving the stability results in [4,14] with weaker conditions on nonlinear stress function, initial disturbance, and weight function. When the nonlinear stress function $\sigma(v)$ satisfies the opposite non-convex condition (1.5), remarkably different from the scalar case, the procedures in [4,10–14] can not be applied to our problem (1.1)–(1.5), so the stability remains still open as is stated in Kawashima and Matsumura [4] (also cf. [8]). In the present paper, to overcome this difficulty, we shall introduce a suitable transform function depending on the viscous shock profile of (1.1), (1.2) to transfer the original system into a new one, and then, following the technique in [4], choose a desired weight function to establish a basic energy estimate. The approach is due to an elementary but technical L^2-weighted energy method. Our plan of this paper is as follows. In Section 2, we shall give the main theorem and some basic properties of the viscous shock profile. In Section 3, we shall reformulate the system (1.1), (1.2) into an "integrated" system, and prove our stability theorem based on a basic energy estimate. Finally, Section 4 is the proof of the basic energy estimate by introducing a suitable transform function and a weight function, which plays a key role in the present paper.

NOTATIONS. L^2 denotes the space of measurable functions on R which are square integrable, with the norm

$$\|f\| = \left(\int |f(x)|^2 dx \right)^{1/2}.$$

$H^l(l \geq 0)$ denotes the Sobolev space of L^2-functions f on R whose derivatives $\partial_x^j f, j = 1, \cdots, l$, are also L^2-functions, with the norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2 \right)^{1/2}.$$

We note that $L^2 = H^0$ and $\|\cdot\| = \|\cdot\|_0$, and denote generic positive constants by C in what follows.

Let T and B be a positive constant and a Banach space, respectively. $C^k(0, T; B)$ $(k \geq 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0, T]$, and $L^2(0, T; B)$ denotes the space of B-valued L^2-functions on $[0, T]$. The corresponding spaces of B-valued function on $[0, \infty)$ are defined similarly.

2. Preliminaries and Main Theorem

In this section, before stating our main theorem, we now recall the properties of traveling wave solution with shock profile.

We call a traveling wave solution with shock profile, or say, a viscous shock profile, for (1.1) and (1.2) if and only if it is a smooth solution of (1.1) and (1.2) in the form

$$(v, u)(t, x) = (V, U)(\xi), \quad \xi = x - st, \tag{2.1}$$

$$(V, U)(\xi) \to (v_\pm, u_\pm), \quad \xi \to \pm\infty, \tag{2.2}$$

where s is the shock speed and $(v_\pm, u_\pm)$ are constant states at $\pm\infty$ satisfying the Rankine-Hugoniot condition

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) - (\sigma(v_+) - \sigma(v_-)) = 0, \end{cases} \tag{2.3}$$

and the generalized shock condition

$$\frac{1}{s}h(v) \equiv \frac{1}{s}[-s^2(v - v_\pm) + \sigma(v) - \sigma(v_\pm)] \begin{cases} < 0, & \text{if } v_+ < v < v_- \\ > 0, & \text{if } v_- < v < v_+. \end{cases} \tag{2.4}$$

We note that the condition (2.4) with (1.4) and (1.5) implies

$$\lambda(v_+) < s \leq \lambda(v_-) \quad \text{or} \quad -\lambda(v_+) \leq s < -\lambda(v_-), \tag{2.5}$$

where $\lambda(v) = \sqrt{\sigma'(v)}$ is the positive characteristic root, and that, especially when $\sigma''(v) > 0$, the condition (2.4) is equivalent to

$$\lambda(v_+) < s < \lambda(v_-) \quad \text{or} \quad -\lambda(v_+) < s < -\lambda(v_-), \tag{2.6}$$

which is well-known as the Lax's shock condition. We call the condition (2.5) with $s = \lambda(v_-)$ or $= -\lambda(v_+)$ (resp. the condition (2.6)) the degenerate (resp. non-degenerate) shock condition. If $(v,u)(t,x) = (V,U)(\xi)$ $(\xi = x - st)$ is the viscous shock profile, then $(V,U)(\xi)$ must satisfy

$$\begin{cases} -sV' - U' = 0, \\ -sU' - \sigma(V)' = \mu U''. \end{cases} \tag{2.7}$$

Integrating (2.7) and eliminating U, we obtain a single ordinary differential equation for $V(\xi)$:

$$\mu s V' = -s^2 V + \sigma(V) - a \equiv h(V), \tag{2.8}$$

where

$$a = -s^2 v_\pm + \sigma(v_\pm). \tag{2.9}$$

Let $(v_+,u_+) \neq (v_-,u_-)$ and $s > 0$ (the case $s < 0$ can be treated similarly). We are now ready to summarize a characterization of the generalized shock condition (2.4) and the results on the existence of viscous shock profile that can be easily proved by the same procedure as in [4]:

Proposition 2.1. *Suppose that* (1.4) *and* (1.5) *hold. Then the following statements are equivalent to each other.*

(i) *The generalized shock condition* (2.4) *holds.*

(ii) $\sigma'(v_-) \geq s^2$, *i.e.,* $\lambda(v_-) \geq s$.

(iii) $\sigma'(v_+) < s^2 \leq \sigma'(v_-)$, *i.e.,* $\lambda(v_+) < s \leq \lambda(v_-)$.

(iv) *There exists uniquely a* $v_* \in (v_+,v_-)$ *such that* $\sigma'(v_*) = s^2$ *and it holds*

$$\sigma'(v) < s^2 \quad \textit{for} \quad v \in (v_+,v_*), \quad s^2 < \sigma'(v) \quad \textit{for} \quad v \in (v_*,v_-). \tag{2.10}$$

i.e.,

$$h'(v_*) = 0, \quad h'(v) < 0 \quad \textit{for} \quad v \in (v_+,v_*), \quad h'(v) > 0 \quad \textit{for} \quad v \in (v_*,v_-). \tag{2.11}$$

Moreover, if one of the above four conditions holds, then we must have $v_+ \neq 0$. *In addition,* $v_+ \lesseqgtr v_-$ *and* $v_* \gtreqless 0$ *hold when* $v_+ \gtreqless 0$, *i.e.,* $v_* v_+ > 0$.

Proposition 2.2. *Suppose that* (1.4) *and* (1.5) *hold.*

(i) *If* (1.1), (1.2) *admits a viscous shock profile* $(V(x-st), U(x-st))$ *connecting* $(v_\pm,u_\pm)$, *then* $(v_\pm,u_\pm)$ *and* s *must satisfy the Rankine-Hugoniot condition* (2.3) *and the generalized shock condition* (2.4).

(ii) *Conversely, suppose that* (2.3) *and* (2.4) *hold, then there exists a viscous shock profile* $(V,U)(x-st)$ *of* (1.1), (1.2) *which connects* $(v_\pm, u_\pm)$. *The* $(V,U)(\xi)(\xi = x-st)$ *is unique up to a shift in* ξ *and is a monotone function of* ξ. *In particular, when* $v_+ \lesseqgtr v_-$ *(and hence* $u_+ \gtreqless u_-$*) we have*

$$u_+ \gtreqless U(\xi) \gtreqless u_-, \quad U_\xi(\xi) \gtreqless 0, \tag{2.12}$$

$$v_+ \lesseqgtr V(\xi) \lesseqgtr v_-, \quad V_\xi(\xi) \lesseqgtr 0, \tag{2.13}$$

for all $\xi \in R$. *Moreover,* $(V,U)(\xi) \to (v_\pm, u_\pm)$ *exponentially as* $\xi \to \pm\infty$, *with the following exceptional case: when* $\lambda(v_-) = s$, $(V,U)(\xi) \to (v_-, u_-)$ *at the rate* $|\xi|^{-1}$ *as* $\xi \to -\infty$, *and* $|h(V)| = |\mu s V_\xi| = O(|\xi|^{-2})$ *as* $\xi \to -\infty$.

In this paper, our aim is to show the stability of viscous shock profile in the non-degenerate case (2.5). Now, without loss of generality, we restrict our attention to the case

$$s > 0 \quad \text{and} \quad v_+ < 0 < v_-, \quad i.e., \quad \mu s V_\xi = h(V) < 0. \tag{2.14}$$

Let $(V,U)(x-st)$ be a viscous shock profile connecting $(v_\pm, u_\pm)$, we assume the integrability of $(v_0 - V, u_0 - U)(x)$ over R and

$$\int_{-\infty}^{\infty} (v_0 - V, u_0 - U)(x)dx = x_0(v_+ - v_-, u_+ - u_-), \tag{2.15}$$

for some $x_0 \in R$. Then it is easily seen that the shifted function $(V,U)(x-st+x_0)$ is also a viscous shock profile conneting $(v_\pm, u_\pm)$ such that

$$\int_{-\infty}^{\infty} (v_0(x) - V(x+x_0), u_0(x) - U(x+x_0))dx = 0. \tag{2.16}$$

In what follows, set $x_0 = 0$ for simplicity. Let us define (ϕ_0, ψ_0) by

$$(\phi_0, \psi_0)(x) = \int_{-\infty}^{x} (v_0 - V, u_0 - U)(y)dy. \tag{2.17}$$

Our main theorem is the following.

Theorem 2.3 (Stability). *Suppose* (1.4), (1.5), (2.3), (2.5), (2.16), *and* $(\phi_0, \psi_0) \in H^2$. *Furthermore, assume that*

$$s^2 < \sigma'(v_-) + \tfrac{1}{2}\sigma''(v_-)[2(v_* - v_+) + v_- - v_+], \tag{2.18}$$

$$\sigma'''(v) < 0, \quad v \in (v_+, 0) \cup (0, v_-). \tag{2.19}$$

Then there exists a positive constant δ_1 *such that if* $\|(\phi_0, \psi_0)\|_2 < \delta_1$, *then* (1.1)–(1.3) *has a unique global solution* $(v, u)(t, x)$ *satisfying*

$$v - V \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^1),$$
$$u - U \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^2),$$

and the asymptotic behavior

$$\sup_{x \in R} |(v, u)(t, x) - (V, U)(x - st)| \to 0 \quad as \quad t \to \infty. \tag{2.20}$$

REMARKS 1. First note that our condition (2.18) is, as easily seen, much stronger than Lax's shock condition. We get the stability of *any viscous shock* (weak shock or not) as long as the condition (2.18) is satisfied. This means that we don't necessarily assume that the viscous shock profile is weak, i.e., $|v_+ - v_-| \ll 1$, which is a sufficient condition in the previous works.

2. An important example is $\sigma(v) = \alpha v - \beta v^3$ for $v \in [v_+, v_-]$, where α, β are any given positive constants. It is easy to see that $\sigma(v)$ satisfies (1.4), (1.5) for some v_+ and v_-. In this case, the Lax's entropy condition is equivalent to $v_+ < -2v_-$, and our condition (2.18) to $v_+ < -a_* v_-$, where $a_* = 7.418190\cdots$ is a unique positive root of

$$-x^2 + 10x + 5 = 2\sqrt{3}\sqrt{x^2 - x + 1}.$$

3. For the general stress $\sigma(v)$, if viscous shock is weak, i.e., $|v_+ - v_-| \ll 1$, and suppose $\sigma'''(0) \neq 0$, then the condition (2.18) is equivalent to the condition $v_+ < -a_* v_-$. A significant example is $\sigma(v) = v/\sqrt{1 + v^2}$.

3. Reformulation of Problem and Proof of Theorem

In this section, we shall prove the Stability Theorem 2.3 by means of a key estimate which will be proved in the next section. In order to show the stability, we first make a reformulation for the problem (1.1)–(1.3) by changing unknown variables

$$(v, u)(t, x) = (V, U)(\xi) + (\phi_\xi, \psi_\xi)(t, \xi), \quad \xi = x - st \tag{3.1}$$

Then the problem (1.1)–(1.3) is reduced to the following "integrated" system

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0 \\ \psi_t - s\psi_\xi - \sigma'(V)\phi_\xi - \mu\psi_{\xi\xi} = F \\ (\phi, \psi)(0, \xi) = (\phi_0, \psi_0)(\xi) \end{cases} \tag{3.2}$$

where

$$F = \sigma(V + \phi_\xi) - \sigma(V) - \sigma'(V)\phi_\xi.$$

For any interal $I \subset [0, \infty)$, we define the solution space of (3.2) as

$$X(I) = \{(\phi, \psi) \in C^0(I; H^2), \phi_\xi \in L^2(I; H^1), \psi_\xi \in L^2(I; H^2)\},$$

and set

$$N(t) = \sup_{0 \le \tau \le t} \|(\phi, \psi)(\tau)\|_2.$$

It is well-known in the previous papers that Theorem 2.3 can be proved by the following theorem to the problem (3.2).

Theorem 3.1. *Suppose that the assumptions in Theorem* 2.3 *hold. Then there exist positive constants* δ_2 *and* C *such that if* $\|(\phi_0, \psi_0)\|_2 < \delta_2$, *then* (3.2) *has a unique global solution* $(\phi, \psi) \in X([0, \infty))$ *satisfying*

$$\|(\phi, \psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\} d\tau \le C\|(\phi_0, \psi_0)\|_2^2 \tag{3.3}$$

for all $t \ge 0$. *Moreover, the stability holds in the following sense*:

$$\sup_{\xi \in R} |(\phi_\xi, \psi_\xi)(t, \xi)| \to 0 \quad as \quad t \to \infty. \tag{3.4}$$

By the same continuation procedure as in [4], we can prove Theorem 3.1 combining the following local existence and *a priori* estimate.

Proposition 3.2 (Local existence). *For any* $\delta_0 > 0$, *there exists a positive constant* T_0 *depending on* δ_0 *such that, if* $(\phi_0, \psi_0) \in H^2$ *and* $\|(\phi_0, \psi_0)\|_2 \le \delta_0$, *then the problem* (3.2) *has a unique solution* $(\phi, \psi) \in X([0, T_0])$ *satisfying* $\|(\phi, \psi)(t)\|_2 \le c_0\delta_0$ *for* $0 \le t \le T_0$, *where* c_0 *is a positive constant independent of* δ_0.

Proposition 3.3 (A priori estimate). *Under the assumptions in Theorem* 2.3, *let* $(\phi, \psi) \in X([0, T])$ *be a solution of* (3.2) *for a positive* T. *Then there exist positive constants* δ_3 *and* C *which are independent of* T *such that if* $N(T) < \delta_3$, *then* (ϕ, ψ) *satisfies the a priori estimate* (3.3) *for* $0 \le t \le T$.

The proof of Proposition 3.2 is standard, so we here omit it. In the rest of this paragraph, our purpose is to prove Proposition 3.3 by using the following key

estimate. In what follows, we assume that $(\phi, \psi) \in X([0,T])$ is a solution of (3.2) for a positive T and $N(T) \leq 1$.

Lemma 3.4 (Basic Estimate). *Suppose the assumptions in Theorem* 2.3. *Then it holds*

$$\|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau \leq C\left(\|(\phi_0,\psi_0)\|^2 + N(t)\int_0^t \|\phi_\xi(\tau)\|^2 d\tau\right), \tag{3.5}$$

for $t \in [0,T]$.

Proof of Proposition 3.3. Since the proof is given exactly in the same way as in [4], we only show its rough sketch. From the equations (3.2), we have

$$\mu\phi_{\xi t} - s\mu\phi_{\xi\xi} + \sigma'(V)\phi_\xi + s\psi_\xi - \psi_t = -F. \tag{3.6}$$

Multiplying (3.6) by ϕ_ξ and integrating the resultant equality over $[0,t] \times R$, using the basic estimate of Key Lemma 3.4, we obtain

$$\|\phi_\xi(t)\|^2 + (1 - CN(t))\int_0^t \|\phi_\xi(\tau)\|^2 d\tau \leq C(\|(\phi_0,\psi_0)\|^2 + \|\phi_{0,\xi}\|^2). \tag{3.7}$$

For the estimates of ψ_ξ, we may differentiate the equations (3.2) in ξ, and mutiply the frist equation by $\sigma'(V)\phi_\xi$ and the second one by ψ_ξ respectively, then may add them up and integrate the resultant equality over $[0,t] \times R$. Then, combined with (3.5) and (3.7), it consequently gives us

$$\|(\phi_\xi,\psi_\xi)(t)\|^2 + \int_0^t \|\psi_{\xi\xi}(\tau)\|^2 d\tau \leq C\|(\phi_0,\psi_0)\|_1^2 \tag{3.8}$$

provided $N(T)$ is suitably small. Similarly, for the estimates of $\phi_{\xi\xi}$, differentiating equation (3.6) in ξ, multiplying it by $\phi_{\xi\xi}$, and integrating the resultant equality over $[0,t] \times R$, we then obtain, combined with (3.5), (3.7) and (3.8),

$$\|\phi_{\xi\xi}(t)\|^2 + \int_0^t \|\phi_{\xi\xi}(\tau)\|^2 d\tau \leq C(\|(\phi_0,\psi_0)\|_1^2 + \|\phi_{0,\xi\xi}\|^2) \tag{3.9}$$

provided $N(T)$ is suitably small. Furthermore, we differentiate the second equation of (3.2) in ξ twice, and mutiply it by $\psi_{\xi\xi\xi}$. Then, for suitably small $N(T)$, we can similarly show

$$\|(\phi_{\xi\xi},\psi_{\xi\xi})(t)\|^2 + \int_0^t \|\psi_{\xi\xi\xi}(\tau)\|^2 d\tau \leq C\|(\phi_0,\psi_0)\|_2^2. \tag{3.10}$$

Combining (3.5)–(3.10) yields

$$\|(\phi,\psi)(t)\|_2^2 + \int_0^t \{\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2\}d\tau \le C\|(\phi_0,\psi_0)\|_2^2$$

for suitably small $N(T)$, say $N(T) < \delta_3$. Thus, we have completed the proof of Proposition 3.3. □

4. Proof of Basic Estimate

To prove the stability by energy method, the key step is to establish the basic estimate (3.5) in Key Lemma 3.4. Since the previous procedures in [4, 12, 13, 14] are invalid for the non-convexity condition (1.5), so we have to find another way to arrive at our goal. Here, our idea is that after transforming the system (3.2) into a new one by selecting a suitable transform function, we prove the basic estimate (3.5) by the weighted energy method with a suitable weight.

Let us introduce a transform function $T(v)$ and a weight function $w(v)$ as follows

$$T(v) = \begin{cases} C_0(v+b), & v \in [0, v_-], \\ \sqrt{\sigma'(v)}(v+b), & v \in [v_+, 0], \end{cases} \tag{4.1}$$

$$w(v) = (v+b)^2, \quad v \in [v_+, v_-], \tag{4.2}$$

where $C_0 = \sqrt{\sigma'(0)}$ and b is a positive constant choosen as

$$0 < 2v_* - 3v_+ < b < 2(s^2 - \sigma'(v_-))\sigma''(v_-)^{-1} - v_- \tag{4.3}$$

corresponding to the assumption (2.18) and Proposition 2.1. It is noted that $T(v)$ and $w(v)$ are bounded and positive on $[v_+, v_-]$, and are in $C^1[v_+, v_-]$, but without the continunity of $T''(v)$ at the point $v = 0$, for example, in the case of $\sigma'''(0) \neq 0$. We shall show how to choose $T(v)$ and $w(v)$ in the following procedure.

Let $(\phi,\psi)(t,\xi)$ be the solution of equations (3.2). We define a transformation in the form

$$(\phi,\psi)(t,\xi) = T(V(\xi))(\Phi(t,\xi), \Psi(t,\xi)), \tag{4.4}$$

where $V(\xi)$ is the viscous shock profile. We denote ξ_0 as a number in R such that $V(\xi_0) = 0$. It is easily seen that ξ_0 is unique because of the monotonicity of $V(\xi)$,

i.e., $V_\xi(\xi) < 0$. Then the equations (3.2) can be transformed into

$$
\begin{cases}
\Phi_t - s\Phi_\xi - \Psi_\xi - s\dfrac{T_\xi}{T}\Phi - \dfrac{T_\xi}{T}\Psi = 0, \\
\Psi_t - \left(s + 2\mu\dfrac{T_\xi}{T}\right)\Psi_\xi - \sigma'(V)\Phi_\xi - \mu\Psi_{\xi\xi} \\
\quad - \left(s\dfrac{T_\xi}{T} + \mu\dfrac{T_{\xi\xi}}{T}\right)\Psi - \sigma'(V)\dfrac{T_\xi}{T}\Phi = F/T(V), \\
(\Phi, \Psi)(0, \xi) = (\phi_0, \psi_0)(\xi)/T(V),
\end{cases} \tag{4.5}
$$

in respect of two spatial parts $\xi \in (-\infty, \xi_0]$ and $\xi \in [\xi_0, \infty)$ due to the discontinuity of $T_{\xi\xi}$ at the point ξ_0, where T denotes the transform function $T(V)$, $T_\xi = \partial T(V)/\partial\xi$ and $T_{\xi\xi} = \partial^2 T(V)/\partial\xi^2$.

Multiplying the first equation of (4.5) by $\sigma'(V)w(V)\Phi$ and the second one by $w(V)\Psi$ respectively, noting $\mu V_\xi = h(V)$, we have

$$
\begin{aligned}
&\frac{1}{2}\{(w\sigma')(V)\Phi^2 + w(V)\Psi^2\}_t - \{\cdots\}_\xi + \mu w(V)\Psi_\xi^2 \\
&\quad + \frac{|V_\xi|}{2s}w(V)Y(V)(s\Phi + \Psi)^2 + \frac{|V_\xi|}{s}w(V)Z(V)\Psi^2 \\
&\quad = Fw(V)\Psi/T(V),
\end{aligned} \tag{4.6}
$$

in respect of the two spatial parts $(-\infty, \xi_0]$ and $[\xi_0, \infty)$, where

$$
\begin{aligned}
\{\cdots\} = &\frac{s}{2}\sigma'(V)w(V)\Phi^2 + \sigma'(V)w(V)\Phi\Psi \\
&+ \left[\left(\frac{s}{2} + \mu\frac{T_\xi}{T}\right)w(V) + \frac{\mu}{2}w(V)_\xi\right]\Psi^2,
\end{aligned} \tag{4.7}
$$

$$
Y(V) = -\sigma''(V) - \sigma'(V)\left(\frac{w'(V)}{w(V)} - 2\frac{T'(V)}{T(V)}\right), \tag{4.8}
$$

$$
\begin{aligned}
Z(V) = &\frac{\sigma''(V)}{2} + h'(V)\frac{w'(V)}{w(V)} - h(V)\frac{w'(V)}{w(V)}\frac{T'(V)}{T(V)} - h'(V)\frac{T'(V)}{T(V)} \\
&+ h(V)\left(\frac{T'(V)}{T(V)}\right)^2 + \frac{h(V)}{2}\left(\frac{w'}{w}\right)'(V) + \frac{h(V)}{2}\left(\frac{w'(V)}{w(V)}\right)^2.
\end{aligned} \tag{4.9}
$$

We see that the coefficient functions in (4.7) are continuous in R since $w(V(\xi))$ and $T(V(\xi))$ are in $C^1(-\infty, \infty)$, so $\{\cdots\}_\xi$ will disappear after integration over $(-\infty, \infty)$. The most essential point of this paper is to choose $w(V)$ and $T(V)$ properly so that both $Y(V)$ and $Z(V)$ are non-negative in (4.6).

Lemma 4.1. *Under the sufficient conditions* (2.18) *and* (2.19), *let* $T(v)$ *and*

$w(v)$ be choosen as in (4.1) *and* (4.2). *Then it holds*

$$Y(v) \geq 0, \qquad Z(v) \geq C_1|\sigma''(v)| \tag{4.10}$$

for all $v \in [v_+, v_-]$, *where* $C_1 > 0$ *is a constant.*

Proof. Since $\sigma''(v)$ changes its sign depending on the sign of v, we have to divide the region of v into two parts as follows.

Part 1. When $v \in [0, v_-]$, i.e., $\sigma''(v) < 0$ and $h'(v) = \sigma'(v) - s^2 > 0$, we find $T(v)$ and $w(v)$ satisfy

$$\frac{w'(v)}{w(v)} = 2\frac{T'(v)}{T(v)},$$

which yields $Y(v) = -\sigma''(v) > 0$, and

$$\begin{aligned} Z(v) &= \frac{\sigma''(v)}{2} + \frac{h'(v)}{2}\frac{w'(v)}{w(v)} + \frac{h(v)}{2}\left(\frac{w'}{w}\right)'(v) + \frac{h(v)}{4}\left(\frac{w'(v)}{w(v)}\right)^2 \\ &= \frac{q_1(v)}{v+b}, \end{aligned} \tag{4.11}$$

where

$$q_1(v) \equiv h'(v) + \frac{1}{2}\sigma''(v)(v+b). \tag{4.12}$$

Therefore, in order to see (4.10), we should show $q_1(v)$ is positive on $[0, v_-]$. We first note that $q_1(v)$ is monotonicaly decreasing since $q_1'(v) = (3/2)\sigma''(v) + (1/2)\sigma'''(v)(v+b) < 0$, $\sigma''(v) < 0$, $\sigma'''(v) < 0$ and $v + b > 0$ (see (2.18) and (4.3)). Then we have $q_1(v) \geq q_1(v_-) = h'(v_-) + (1/2)\sigma''(v_-)(v_- + b) > 0$ by (4.3). Thus, we observe that

$$Z(v) \geq \frac{q_1(v_-)}{v_- + b} \geq \frac{q_1(v_-)|\sigma''(v)|}{(v_- + b)|\sigma''(v_-)|}, \quad v \in [0, v_-].$$

Part 2. When $v \in [v_+, 0]$, i.e., $\sigma''(v) > 0$ and $h'(v) = \sigma'(v) - s^2 \lesseqgtr 0$ for $v \lesseqgtr v_*$, see (2.11) in Sect. 2, we find $T(v)$ and $w(v)$ satisfy

$$\frac{w'(v)}{w(v)} = 2\frac{T'(v)}{T(v)} - \frac{\sigma''(v)}{\sigma'(v)},$$

which yields $Y(v) \equiv 0$, and

$$Z(v) = \frac{s^2}{2}\frac{\sigma''(v)}{\sigma'(v)} + \frac{h(v)}{4}\left(\frac{\sigma''(v)}{\sigma'(v)}\right)^2 + \frac{\sigma'(v) - s^2}{v+b}. \tag{4.13}$$

To show $Z(v) > C\sigma''(v)$, we further devide the region $[v_+, 0]$ into $[v_*, 0]$ and $[v_+, v_*]$.

When $v \in [v_*, 0]$, since $\sigma'(v) - s^2 > 0$ and $b + v > 0$ for $v \in [v_*, 0]$, we have

$$Z(v) > \frac{s^2}{4}\frac{\sigma''(v)}{\sigma'(v)}\left(2 + \frac{h(v)\sigma''(v)}{s^2\sigma'(v)}\right) > C\sigma''(v). \tag{4.14}$$

Here we used the fact

$$0 < q_2(v) \equiv -\frac{h(v)\sigma''(v)}{s^2\sigma'(v)} < 1, \quad \text{for} \quad v \in [v_+, 0]. \tag{4.15}$$

To see (4.15), making use of $h(v_+) = \sigma''(0) = 0$, and $\sigma'''(v) < 0$, we observe that $q_2(v_+) = q_2(0) = 0$, and $q_2(v) > 0$ for $v \in (v_+, 0)$. Consequently, $q_2(v)$ attains its maximum over $[v_+, 0]$ at a point $v = \bar{v}$ in $(v_+, 0)$, and hence $\bar{q}_2 = \max_{v\in[v_+,0]} q_2(v) = q_2(\bar{v}) > 0$ and $q_2'(\bar{v}) = 0$. Rewriting $q_2(v)$ as $-h(v)\sigma''(v) = s^2\sigma'(v)q_2(v)$ and differentiating it with respect to v at $v = \bar{v}$, we have

$$\bar{q}_2 = 1 - \frac{\sigma'(\bar{v})}{s^2} - \frac{h(\bar{v})\sigma'''(\bar{v})}{s^2\sigma''(\bar{v})} < 1.$$

When $v \in [v_+, v_*]$, i.e., $\sigma'(v) - s^2 < 0$, there exists a point $\tilde{v} \in (v, v_*)$ such that

$$\sigma'(v) - s^2 = \sigma''(\tilde{v})(v - v_*) > \sigma''(v)(v - v_*), \tag{4.16}$$

because of $\sigma'''(v) < 0$. Substituting (4.16) back into (4.13), we have

$$\begin{aligned} Z(v) &\geq \frac{\sigma''(v)}{4\sigma'(v)^2}\left\{2s^2\sigma'(v) + h(v)\sigma''(v) + 4\sigma'(v)^2\frac{v - v_*}{v + b}\right\} \\ &\equiv \frac{\sigma''(v)}{4\sigma'(v)^2}q_3(v). \end{aligned} \tag{4.17}$$

Differentiating $q_3(v)$ with repect to v, and making use of $h(v) < 0$ and $\sigma'''(v) < 0$, we have

$$\begin{aligned} q_3'(v) &= h(v)\sigma'''(v) + \sigma''(v)q_4(v) + 4\frac{\sigma'(v)}{(v + b)^2}q_5(v) \\ &\geq \sigma''(v)q_4(v) + 4\frac{\sigma'(v)}{(v + b)^2}q_5(v), \end{aligned} \tag{4.18}$$

where

$$q_4(v) = s^2 + \sigma'(v)\frac{5v - 4v_* + b}{v + b}, \quad q_5(v) = \sigma'(v)(v_* + b) + (v + b)(v - v_*)\sigma''(v).$$

Making use of $s^2 \geq \sigma'(v)$ and $v + b > 0$ on $[v_+, v_*]$, and (2.18) and (4.3), we know $q_4(v) > 0$ for $v \in [v_+, v_*]$. On the other hand, we can see that

$$q_5(v) \geq q_5(-b) = \sigma'(-b)(v_* + b) > 0. \tag{4.19}$$

In fact, by $\sigma''(v) > 0$, $v + b > 0$, (2.18) (see also (4.3)) and (2.19), we have

$$q_5'(v) = 2\sigma''(v)(v+b) + (v+b)(v-v_*)\sigma'''(v) \geq 0$$

for $v_* \geq v \geq -b$, which implies (4.19). Consequently, we have proved $q_3'(v) > 0$ for $v \in [v_+, v_*]$ in (4.18). Thus using $s^2 > \sigma'(v_+)$, (2.18) and (4.3), we obtain

$$q_3(v) \geq q_3(v_+) \geq 2\sigma'(v_+)^2 \left\{ 1 + 2\frac{v_+ - v_*}{v_+ + b} \right\} > 0, \quad v \in [v_+, v_*].$$

Therefore, by (4.17), we can see that $Z(v) > \text{Const.}\sigma''(v) > 0$ for $v \in [v_+, v_*]$. Combining Parts 1 and 2, we have completed the proof of (4.10). □

Integrating (4.6) over $[0,t] \times (-\infty, \xi_0]$ and $[0,t] \times [\xi_0, \infty)$ respectively, and adding them, we obtain by Lemma 4.1

Lemma 4.2. *Suppose the assumptions in Theorem* 2.3. *Then it holds*

$$\begin{aligned} (4.20) \qquad & \|(\Phi,\Psi)(t)\|^2 + \int_0^t \|\Psi_\xi(\tau)\|^2 d\tau + \int_0^t \int_{-\infty}^{\infty} |V_\xi| w(V) Z(V) \Psi^2 d\xi d\tau \\ & \leq C \left(\|(\Phi_0,\Psi_0)\|^2 + \int_0^t \int_{-\infty}^{\infty} \frac{w(V)}{T(V)} |F| |\Psi| d\xi d\tau \right) \end{aligned}$$

for $t \in [0,T]$.

Proof of Key Lemma 3.4. Since $\|(\Phi,\Psi)\| \sim \|(\phi,\psi)\|$ by the boundedness of $T(V)$, and $|F| = O(|\phi_\xi|^2)$, we have due to Lemma 4.2

$$\begin{aligned} (4.21) \qquad & \|(\phi,\psi)(t)\|^2 + \int_0^t \|\Psi_\xi(\tau)\|^2 d\tau + \int_0^t \int_{-\infty}^{\infty} |V_\xi| w(V) Z(V) \Psi^2 d\xi d\tau \\ & \leq C \left(\|(\phi_0,\psi_0)\|^2 + N(t) \int_0^t \|\phi_\xi(\tau)\|^2 d\tau \right). \end{aligned}$$

Furthermore, multiplying the first equation of (3.2) by ϕ and the second one by $\psi\sigma'(V)^{-1}$ respectively, and adding them, we have

$$\begin{aligned} (4.22) \qquad & \left\{ \frac{\phi^2}{2} + \frac{\psi^2}{2\sigma'(V)} \right\}_t - \left\{ \frac{s\phi^2}{2} + \frac{s\psi^2}{2\sigma'(V)} + \phi\psi + \frac{\mu}{\sigma'(V)} \psi\psi_\xi \right\}_\xi \\ & + \frac{\mu}{\sigma'(V)} \psi_\xi^2 + \frac{s\sigma''(V)V_\xi}{\sigma'(V)} \psi^2 + \frac{\mu\sigma''(V)V_\xi}{\sigma'(V)^2} \psi\psi_\xi = \frac{F\psi}{\sigma'(V)}. \end{aligned}$$

We note that

$$(4.23) \qquad \left| \frac{\mu\sigma''(V)V_\xi}{\sigma'(V)^2} \psi\psi_\xi \right| \leq \varepsilon \frac{\mu\psi_\xi^2}{\sigma'(V)} + \frac{\mu\sigma''(V)^2 V_\xi^2 \psi^2}{4\varepsilon\sigma'(V)^3}$$

for $0 < \varepsilon < 1$. Substituting (4.23) into (4.22), and integrating the resultant inequality over $[0,t] \times R$, we have

$$(4.24)\quad \|(\phi,\psi)(t)\|^2 + \int_0^t \|\psi_\xi(\tau)\|^2 d\tau \leq C\left(\|(\phi_0,\psi_0)\|^2 + \int_0^t \int_{-\infty}^{\infty} |\sigma''(V)V_\xi|\psi^2 d\xi d\tau + N(t)\int_0^t \|\phi_\xi(\tau)\|^2 d\tau\right).$$

Making use of Lemma 4.1 and $w(V) \sim T(V) \sim$ Const., we obtain

$$(4.25)\quad \int_0^t \int_{-\infty}^{\infty} |\sigma''(V)V_\xi|\psi^2 d\xi d\tau \leq C\int_0^t \int_{-\infty}^{\infty} |V_\xi| Z(V) w(V) \Psi^2 d\xi d\tau.$$

Applying (4.25) and (4.21) to (4.24), we finally have (3.5). This completes the proof of Key Lemma 3.4. □

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