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ON THE STABLE HOMOTOPY OF A Z2-MOORE SPACE

Dedicated to Professor A. Komatu for his 60th birthday

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Introduction

This paper is a continuation of [5].

Denote by $M \ a Z_2$ -Moore space. We take $M = S^1 \cup {}_2e^2$, which is obtained from a 1-sphere S^1 by attaching a 2-cell e^2 , using a map $S^1 \rightarrow S^1$ of degree 2. Let π_k be the k-th group of the stable homotopy of M, i.e., $\pi_k = \underset{n \rightarrow \infty}{\text{Dir Lim }} [S^{n+k}M, S^nM]$, where the direct limit is taken with respect to suspensions. Put $\pi_* = \sum \pi_k$, then it admits a ring structure with respect to the composition. In fact, it forms an algebra over Z_4 .

In [5] we determined the additive structure of π_* in dim ≤ 21 . In this paper we shall investigate compositions of elements in π_* and the ring structure of π_* in dim ≤ 21 . Our main theorems are Theo. 4.1 and 4.2. Our methods deeply depend on the results and the methods of Toda [6].

In §1 we shall state the general formulas obtained from composing elements of $\pi_j(2) = \text{Dir Lim} [S^{n+j-2}M, S^n], \pi_k^*(2) = \text{Dir Lim} [S^{n+k+1}, S^nM] \text{ and } \pi_l$.

In §2 we fix the generators of the above groups by use of the formulas of §1 and we examine compositions of the generators.

In 3 we prove the theorem in which the relations in the secondary or tertiary compositions are mentioned. They hold the key to the discussions in 2.

Our main theorems are stated in §4.

§5 is devoted to the improvement in Theo. 5.1 of [5].

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Notations and conventions

The notations of [5] are carried over the present work with making a few changes and adding new one.

In [5] we did not distinguish between a representative of a set of the

(co-) extensions and the set itself. In this paper this distinction is essential.

Suppose given $\alpha = \{f\} \in [Y, Z], \beta = \{g\} \in [X, Y] \text{ and } \gamma = \{h\} \in [W, X] \text{ such that } \alpha\beta = 0 \text{ in } [X, Z] \text{ and } \beta\gamma = 0 \text{ in } [W, Y].$ Then we have two maps $F: Y \cup_{\beta} CX \rightarrow Z$ and $H: SW \rightarrow Y \cup_{\beta} CX$ which are characterized by the following homotopy commutative diagram,



where i_{β} is an inclusion and p_{β} a map shrinking Y to the base point of SX. F and H are called the extension of f, the coextension of h respectively.

We shall fix the following notations.

- i) $\overline{\alpha}_{\beta} = \{F\}$: the extension of α with respect to β .
- ii) $\tilde{\gamma}_{\beta} = \{H\}$: the coextension of γ with respect to β .
- iii) Ext_{β} α : the set of $\overline{\alpha}_{\beta}$.
- iv) Coext_{β} γ : the set of $\tilde{\gamma}_{\beta}$.

We note that the Toda bracket $\{\alpha, \beta, \gamma\} = (\text{Ext}_{\beta} \alpha)(\text{Coext}_{\beta} \gamma)$ as a double coest of two subgroups $\alpha[SW, Y]$ and $[SX, Z](S\gamma)$ in [SW, Z] (see Prop. 1.7 in p. 13 of [6]).

If $\beta = 2 \in G_0$, then we put $i_{\beta} = i$, $p_{\beta} = p$, $\overline{\alpha}_{\beta} = \overline{\alpha}$, $\tilde{\gamma}_{\beta} = \tilde{\gamma}$, $\text{Ext}_{\beta} \alpha = \text{Ext} \alpha$ and $\text{Coext}_{\beta} \gamma = \text{Coext} \gamma$ respectively.

Assume that $\alpha \in G_k$ is of order 2 and that $\eta \alpha \in G_{k+1}$ is divisible by 2, where η is the generator of G_1 . Then we can define Coext ($\overline{\alpha}$) and Ext ($\widetilde{\alpha}$) by use of Prop. 2.5 of [5]. We take

v) $(\widetilde{\alpha}) \in \operatorname{Coext}(\overline{\alpha}) \text{ and } (\widetilde{\alpha}) \in \operatorname{Ext}(\overline{\alpha}).$

We put the following new notation.

vi) $K_s = \ker i^* \cap \ker p_*$, where $i^*: \pi_s \to \pi_s^*(2)$ and $p_*: \pi_s \to \pi_s(2)$ are natural homomorphisms induced by $i: S^1 \to S^1 \cup {}_2e^2$ and $p: S^1 \cup {}_2e^2 \to S^2$.

Our conventions are the following.

In discussions of this paper we often use those properties of Toda brackets which are stated in Prop. 1.2, Prop. 1.4 and i) of (3.9) of [6] and those results about $(G_*; 2)$ which are stated in Theo. B of [5] and Theo. 14.1 of [6] without any reference.

1. Compositions of elements of $\pi_j(2)$, $\pi_k^*(2)$ and π_l

In this section we shall state some general formulas obtained from composing elements of $\pi_i(2)$, $\pi_i^*(2)$ and π_i .

Throughout this section we take α and β in $(G_i; 2)$ and $(G_k; 2)$ respectively.

Proposition 1.1. If α is neither of order 2 nor divisible by 2, then $\widetilde{\alpha p} \equiv \overline{i\alpha} \mod i\pi_{i+1}(2) + \pi_{j+1}^*(2)p$

and

 $\operatorname{Coext}(\alpha p)i \equiv i\alpha \bmod i\pi_{j+1}(2)i.$

Proof. By ii) and iii) of Prop. 1.2 of [6], $\alpha p \in \{i, 2, \alpha p\} \subseteq \{i, 2\alpha, p\} \supseteq \{i\alpha, 2, p\} \supseteq \overline{i\alpha}$. Since the bracket $\{i, 2\alpha, p\}$ is a coset of $i\pi_{j+1}(2) + \pi_{j+1}(2)p$, we have the first assertion.

The second assertion is a direct consequence of the first one.

Proposition 1.2. Assume that α is of order 2 and that $\eta \alpha$ is divisible by 2. Then we have the following.

i)
$$(\widetilde{\alpha}) \equiv (\widetilde{\alpha}) \mod \sum_{1 \leq s \leq m} \{\widetilde{\gamma_s p}\} + i\pi_{j+2}(2) + \pi_{j+2}^*(2)p$$
,

where $\gamma_1, \dots, \gamma_m$ are the elements of $(G_{j+1}; 2)$ which are neither of order 2 nor divisible by 2.

ii) Supose given $\overline{\alpha}$ and $\widetilde{\alpha}$ such that $i\eta \overline{\alpha} = \widetilde{\alpha} \eta p$. Then

Coext $(\overline{\alpha})i \equiv \widetilde{\alpha} \mod \sum \{i\gamma_t\} + i\pi_{j+2}(2)i$,

where t runs over the subset of $\{1, 2, \dots, m\}$ which consists of s satisfying the equation $i\eta\gamma_s p=0$.

Proof. Obviously, $(\widetilde{\alpha}) \equiv (\widetilde{\alpha}) \mod p_*^{-1} i^{*-1}(0)$. By use of Prop. 1.1, Prop. 1.2 and Prop. 1.3 of [5], it is easy to see that $p_*^{-1} i^{*-1}(0) = p_*^{-1}(G_{j+1}p)$ and that this equals the given subgroup of π_{j+1} in i). So, i) is proved.

By Theo. A of [5], $2(\tilde{\alpha}) = i\eta\bar{\alpha}$, $2(\tilde{\alpha}) = \tilde{\alpha}\eta p$ and $2\gamma_s p = i\eta\gamma_s p$. So, i) and the assumption of ii) lead us to the assertion of ii).

Proposition 1.3. Assume that α is of order 2 and that β is neither of order 2 nor divisible by 2. Then we have the following.

- i) In case $\alpha\beta \neq 0$:
 - a) $\overline{\alpha} \operatorname{Coext} (\beta p) \equiv \beta \overline{\alpha} \mod \alpha \pi_{k+1}(2) + G_{j+k+1}p$,
 - b) Coext $(\beta p)(i\overline{\alpha}) \equiv i\beta\overline{\alpha} \mod i\pi_{k+1}(2)i\overline{\alpha}$.
- ii) In case $\alpha\beta=0$:
 - a) $\overline{\alpha} \operatorname{Coext} (\beta p) \equiv 0 \mod \alpha \pi_{k+1}(2) + G_{j+k+1}p$,
 - b) Coext $(\beta p)(i\overline{\alpha}) \equiv i\{\beta, \alpha, 2\} p \mod i\pi_{k+1}(2)i\overline{\alpha} + i\beta G_{j+1}p$.

Proof. Clearly, $\overline{\alpha} \operatorname{Coext} (\beta p) \subseteq \{\alpha, 2, \beta p\} \supseteq \{\alpha, 2\beta, p\}$. This bracket contains $\beta \overline{\alpha}$ or 0 according as $\alpha \beta \neq 0$ or $\alpha \beta = 0$ and it is a coset of $\alpha \pi_{k+1}(2) + G_{j+k+1}p$. So, a) of i) and ii) are proved.

By Prop. 1.1, Coext $(\beta p)(i\overline{\alpha})$ contains $i\beta\overline{\alpha} \mod i\pi_{k+1}(2)i\overline{\alpha}$. So, b) of i) is proved.

If $\alpha\beta=0$, $i\beta\bar{\alpha}\equiv i\beta\{\alpha, 2, p\}=i\{\beta, \alpha, 2\}p \mod i\beta G_{j+1}p$. This leads us to b) of ii).

Proposition 1.4. Let α and β be same as the above proposition. Then we have the following.

- i) In case $\alpha\beta \neq 0$:
 - a) Coext $(\beta p) \tilde{\alpha} \equiv \tilde{\alpha} \beta \mod i G_{j+k+1}$,
 - b) $(\tilde{\alpha}p)$ Coext $(\beta p) = \tilde{\alpha}\beta p$.
- ii) In case $\alpha\beta=0$:
 - a) Coext $(\beta p) \tilde{\alpha} \equiv 0 \mod i G_{j+k+1}$,
 - b) $(\tilde{\alpha}p)$ Coext $(\beta p) \subseteq i\{2, \alpha, \beta\}p \mod iG_{j+1}\beta p$.

Proof. If $\alpha\beta \neq 0$, Coext $(\beta p)\tilde{\alpha} = \{i, 2, \beta p\}\tilde{\alpha} \subseteq \{i, 2, \beta\alpha\} \supseteq \{i, 2, \alpha\}\beta \supseteq \tilde{\alpha}\beta$. Since the bracket $\{i, 2, \beta\alpha\}$ is a coset of iG_{j+k+1} , we have a) of i).

The others are obvious.

Proposition 1.5. Assume that α and β are of order 2 respectively and that $\eta \alpha$ is divisible by 2. Let $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\alpha}$ be fixed. Then we have the following.

- i) In case $\beta \tilde{\alpha} \neq 0$:
 - a) If $i\eta \overline{\alpha} = \widetilde{\alpha} \eta p$,

 $\overline{\beta} \operatorname{Coext}(\overline{\alpha}) \equiv (\overline{\beta} \widetilde{\alpha}) \mod \sum_{i} \{\gamma_t \overline{\beta}\} + \beta \pi_{j+2}(2) + G_{j+k+2} p$,

where t runs over the subset of $\{1, 2, \dots, m\}$ which consists of s satisfying the equation $i\eta\gamma_s p=0$.

b) If $\alpha\beta=0$ and $\alpha\bar{\beta}=0$ and if there exists $\gamma \in (G_{j+k+1}; 2)$ which satisfies $\bar{\alpha}\beta=i\gamma$,

 $\tilde{\alpha}\bar{\beta}\equiv i\bar{\gamma} \mod K_{j+k+1}$.

- ii) In case $\bar{\beta}\tilde{\alpha}=0$:
 - a) $\overline{\beta}$ Ext $(\overline{\alpha}) = \{\overline{\beta}, \overline{\alpha}, 2\} p$.
 - b) If $\alpha\beta=0$ and $\{\alpha, \beta, 2\}=0$,

 $\tilde{\alpha}\bar{\beta} \equiv i\{2, \alpha, \beta, 2\} p \mod iG_{i+1} \operatorname{Ext} \beta + (\operatorname{Coext} \alpha)G_{k+1}p.$

Proof. a) of i) is a direct consequence of ii) of Prop. 1.2. b) of i) and a) of ii) are obvious.

By use of ii) of (3.9) of [6], $\{\beta, 2, \alpha\} + \{2, \alpha, \beta\} + \{\alpha, \beta, 2\} \ni 0$. So, we can take $\gamma \equiv \overline{\beta} \overline{\alpha} \mod \beta G_{j+1} + \alpha G_{k+1} + 2G_{j+k+1}$ in b) of i) and we have $\{2, \alpha, \beta\} \ni 0$ under the assumption of b) of ii).

Now we can construct the tertiary composition $\{2, \alpha, \beta, 2\}$ by use of the Mimura's methods (see [2]). Namely, from the fact $\{2, \alpha, \beta\} \ni 0$, we can choose $\overline{2}_{\alpha}$ and β_{α} such that $\overline{2}_{\alpha}\beta_{\alpha}=0$. It is clear that $\beta_{\alpha}2 \in \{i_{\alpha}, \alpha, \beta\}2 = i_{\alpha}\{\alpha, \beta, 2\}=0$. So, we can define the Toda bracket $\{\overline{2}_{\alpha}, \beta_{\alpha}, 2\}$. We put

 $\{2, \alpha, \beta, 2\} \equiv \{\overline{2}_{\alpha}, \widetilde{\beta}_{\alpha}, 2\} \mod \overline{2}_{\alpha}[S^{n+j+k+2}, S^n \cup_{\alpha} e^{n+j+1}] + [S^{n+j+1} \cup_{\beta} e^{n+j+k+2}, S^n]\widetilde{2}_{\beta},$ where *n* is sufficiently large.

It follows that $i\bar{2}_{\alpha} \in i\{2, \alpha, p_{\alpha}\} = \{i, 2, \alpha\} p_{\alpha} = (\operatorname{Coext} \alpha)p_{\alpha}$. Similarly, we obtain $\tilde{2}_{\beta}p \in i_{\beta} \operatorname{Ext} \beta$. Therefore, we have $i\{2, \alpha, \beta, 2\}p \equiv i\{\bar{2}_{\alpha}, \tilde{\beta}_{\alpha}, 2\}p = i\bar{2}_{\alpha}\{\tilde{\beta}_{\alpha}, 2, p\} \subseteq (\operatorname{Coext} \alpha)(\operatorname{Ext} \beta) \mod iG_{j+1} \operatorname{Ext} \beta + (\operatorname{Coext} \alpha)G_{k+1}p$. This leads us to the assertion of b) of ii).

Proposition 1.6. α and β are same as the above proposition. Let $\tilde{\beta}$ and $\bar{\alpha}$ be fixed. Then we have the following.

- i) In case $\overline{\alpha}\beta \neq 0$:
 - a) Coext $(\bar{\alpha})\tilde{\beta} \equiv (\bar{\alpha}\tilde{\beta}) \mod iG_{j+k+2}$.
 - b) If $\alpha\beta = 0$ and $\beta\alpha = 0$ and if there exists $\gamma \in (G_{j+k+1}; 2)$ which satisfies $\beta\overline{\alpha} = \gamma p$,

 $\tilde{\beta} \bar{\alpha} \equiv \tilde{\gamma} p \mod K_{i+k+1}$.

- ii) In case $\bar{\alpha}\tilde{\beta}=0$:
 - a) Coext $(\bar{\alpha})\tilde{\beta} = i\{2, \bar{\alpha}, \tilde{\beta}\}.$
 - b) If $\alpha\beta=0$ and $\{2, \beta, \alpha\}=0$,

 $\tilde{\beta}\bar{\alpha} \equiv i\{2, \beta, \alpha, 2\} p \mod iG_{k+1} \operatorname{Ext} \alpha + (\operatorname{Coext} \beta)G_{j+1}p.$

The proof is quite similar to the one of the above proposition and we omit it.

Proposition 1.7. Assume that α and β are neither of order 2 nor divisible by 2 respectively.

i) If $\alpha\beta$ is neither of order 2 nor divisible by 2,

 $\widetilde{\alpha p \beta p} \in \operatorname{Coext} (\alpha \beta p).$

- ii) Suppose given αp and βp such that $\alpha pi=i\alpha$ and $\beta pi=\beta i$, then we have the following.
 - a) If $\alpha\beta$ is divisible by 2, $\widetilde{\alpha \rho \beta \rho} \equiv 0 \mod K_{i+k}$.
 - b) If $\alpha\beta$ is not divisible by 2 but of order 2, $\widetilde{\alpha p \beta p} \equiv i \overline{\alpha \beta} + \widetilde{\alpha \beta p} \mod K_{i+k}$.

The proof is left to the reader.

Proposition 1.8. Assume that α is neither of order 2 nor divisible by 2 and that β is of order 2 and $\eta\beta$ is divisible by 2. Let $\overline{\beta}$ be fixed.

i) If $\alpha\beta \neq 0$, $\widetilde{\alpha p}(\widetilde{\beta}) \in \text{Coext}(\alpha \overline{\beta})$.

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- ii) If $\alpha\beta=0$ and if there exists $\gamma \in (G_{j+k+1}; 2)$ which satisfies $\alpha\overline{\beta}=\gamma p$, we have the following.
 - a) $\widetilde{\alpha p}(\widetilde{\beta}) \in \operatorname{Coext}(\gamma p).$
 - b) If $\tilde{\beta}$, $\widetilde{\alpha p}$ and $(\widetilde{\beta})$ are fixed such that $\widetilde{\alpha p i} = i\alpha$ and $(\widetilde{\beta})i = \tilde{\beta}$ and if $\widetilde{\alpha p \beta} = \tilde{\beta}\alpha = i\gamma$ and $\overline{\beta \alpha p} = \gamma p$, we have $(\widetilde{\beta})\widetilde{\alpha p} \equiv \widetilde{\alpha p}(\widetilde{\beta}) \mod K_{j+k+1}$.

The proof is left to the reader.

In ii) of the above proposition we can take $\gamma \in \{\alpha, \beta, 2\}$.

Proposition 1.9. Assume that α and β are of order 2 respectively and that $\eta\alpha$ and $\eta\beta$ are divisible by 2 respectively. Let $\overline{\alpha}$, $\overline{\beta}$, $\overline{\beta}$ and $(\overline{\beta})$ be fixed such that $(\overline{\beta})i=\overline{\beta}$. Then we have the following.

i) If $\bar{\alpha}\tilde{\beta} \neq 0$,

 $(\widetilde{\overline{\alpha}})(\widetilde{\overline{\beta}}) \in \operatorname{Coext}((\overline{\overline{\alpha}}\widetilde{\beta}))$.

ii) If $\bar{\alpha}\tilde{\beta}=0$ and if $\{\bar{\alpha}, \tilde{\beta}, 2\}$ and $\{2, \bar{\alpha}, \tilde{\beta}\}$ consist of the elements which are not divisible by 2 but of order 2 respectively, we have

$$(\widetilde{\overline{\alpha}})(\widetilde{\overline{\beta}}) \equiv i\{\overline{2, \overline{\alpha}, \overline{\beta}}\} + \{\widetilde{\overline{\alpha}, \overline{\beta}, 2}\} p \mod K_{j+k+2}.$$

Proof. i) is obvious.

ii) follows from a) of ii) of Prop. 1.5 and Prop. 1.6.

2. Generators and relations in $\pi_i(2)$, $\pi_k^*(2)$ and π_l

In this section we shall use the general formulas of §1 and choose the generators of $\pi_j(2)$, $\pi_k^*(2)$ and π_l . We shall compute compositions of elements of π_* .

The Toda brackets which appear in this section are the following.

Theorem 2.1. (Toda).

i)	$\{\eta, 2, \eta\} = \pm 2\nu, \{\nu, \eta, 2\} = 0,$
	$\{\eta, 2, \eta^*\} \equiv \pm 2\nu^* \mod \eta \overline{\mu}, \{\nu^*, \eta, 2\} \equiv 0 \mod 2G_{\scriptscriptstyle 20}.$
ii)	$\{\eta, 2, \nu^2\} = \{\eta, \nu^2, 2\} \equiv \varepsilon \mod \eta \sigma, \{\nu^2, \eta, 2\} = 0,$
	$\{\eta, 2, 8\sigma\} = \{\eta, 8\sigma, 2\} \equiv \mu \mod \{\eta^2 \sigma, \eta \mathcal{E}\}, \{8\sigma, \eta, 2\} = 0,$
	$\{\eta, 2, \sigma^2\} = \{\eta, \sigma^2, 2\} \equiv \eta^* \mod \eta \rho, \{\sigma^2, \eta, 2\} = 0,$
	$\{\eta, 2, \overline{\sigma}\} = \{\eta, \overline{\sigma}, 2\} \equiv 0 \mod \eta \overline{\kappa}, \{\overline{\sigma}, \eta, 2\} = 0,$
	$\{\mu, 2, 8\sigma\} = \{\mu, 8\sigma, 2\} \equiv \overline{\mu} \mod \eta^2 \rho, \{8\sigma, \mu, 2\} = 0.$
•••	

iii)
$$\{\eta, 2, \varepsilon\} \equiv 0 \mod \eta \mu, \{\eta, 2, \kappa\} \equiv 0 \mod \eta \rho,$$

 $\{\mu, 2, \varepsilon\} \equiv 0 \mod \eta \overline{\mu}.$

$$\begin{split} \text{iv}) & \{\sigma, \nu^2, 2\} = \{\sigma, 2\nu, \nu\} \equiv 0 \mod \sigma^2, \\ & \{\nu, \sigma^2, 2\} \equiv \{\nu, 2\sigma, \sigma\} = \nu^* \mod 2\nu^*, \\ & \{\sigma^2, \eta, \nu\} = \{\sigma, \eta\sigma, \nu\} = \bar{\sigma}, \{\sigma, \varepsilon, \nu\} = 0. \\ \text{v}) & \{\eta\varepsilon, \eta, 2\} = \{\gamma^2\sigma, \eta, 2\} \equiv \zeta \mod 2G_{11}, \\ & \{\eta^2\rho, \eta, 2\} \equiv \overline{\zeta} \mod 2G_{19}, \\ & \{\nu, 8\sigma, 2\} \supset \{\nu, 2\sigma, 8\} \supseteq \zeta \mod 2G_{19}, \\ & \{\zeta, 8\sigma, 2\} \supset \{\zeta, 2\sigma, 8\} \supseteq \zeta \mod 2G_{19}, \\ & \{\sigma, 8\sigma, 2\} \supset \{\sigma, 2\sigma, 8\} \supseteq \rho \mod 2G_{15}, \\ & \{\varepsilon, 8\sigma, 2\} \supset \{\sigma, 2\sigma, 8\} \supseteq \rho \mod 2G_{15}, \\ & \{\varepsilon, 8\sigma, 2\} \supseteq \{\eta\sigma, 8\sigma, 2\} = \eta\rho, \{8\sigma, 2, 8\sigma\} = 16\rho. \\ \text{vi}) & \{\eta\kappa, \eta, 2\} = \nu\kappa, \{\kappa, 2, \nu^2\} = \eta\overline{\kappa}. \\ & \text{vii}) & \{2, \nu^2, \rho\} = 0, \{\nu, \eta, \eta^2\sigma\} = 0, \{\sigma, \nu, \zeta\} = 0. \\ & \text{viii}) & \{\kappa, 8\sigma, 2\} \equiv 0 \mod \eta^2 \overline{\kappa}, \{\sigma, \kappa, 2\} = \nu \overline{\sigma}, \\ & \{\overline{\nu}^2, \widetilde{\nu}^2, 2\} = \{2, \overline{\nu^2}, \widetilde{\nu^2}\} = \kappa, \\ & \{\overline{\eta}, \overline{\kappa}\nu, 2\} = \{2, \nu(\overline{\kappa}), \overline{\eta}\} \equiv 0 \mod 2G_{20}. \\ & \text{ix}) & \{2, 4\nu, \eta, 2\} = 0, \\ & \{2, \overline{\sigma}, \eta, 2\} = \{2, \eta, \overline{\sigma}, 2\} \equiv 0 \mod \eta^2 \overline{\kappa}. \\ \end{split}$$

This theorem will be proved in the next section.

Throughout this section we denote by Roman letters x, y, z, etc. integers 0 or 1.

2.1. First we define $\delta \in \pi_{-1}$ by

 $(2.1) \qquad \qquad \delta = ip \,.$

We choose

(2.2)
$$\overline{\eta} \in \operatorname{Ext} \eta \quad and \quad \widetilde{\eta} \in \operatorname{Coext} \eta$$

arbitrarily. Then remark that Ext $\eta = \{\overline{\eta}, -\overline{\eta}\}$ and Coext $\eta = \{\widetilde{\eta}, -\widetilde{\eta}\}$. We define η_1 and η_2 in π_1 and $\eta_3 \in \pi_3$ by

(2.3)
$$\eta_1 = i\overline{\eta}, \quad \eta_2 = \widetilde{\eta}p \quad and \quad \eta_3 = \widetilde{\eta}\overline{\eta}.$$

Take

(2.4)
$$\nu_1 \in \operatorname{Coext}(\nu p) \subset \pi_3$$

arbitrarily.

Proposition 2.1.

i) $\delta^2 = 0$, $\delta \eta_1 = \eta_2 \delta = 0$ and $\eta_1 \delta = \delta \eta_2 = i\eta p = 2.1$, where 1 is a generator of π_0 and of order 4.

ii)
$$\delta\eta_{3} = \eta_{1}^{2} = i\eta\overline{\eta}, \eta_{3}\delta = \eta_{2}^{2} = \widetilde{\eta}\eta p \text{ and } \delta\eta_{3}\delta = 0.$$

iii) $\nu_{1}\delta = \delta\nu_{1} = i\nu p.$
iv) $\eta_{1}\eta_{2} = \eta_{2}\eta_{1} = 0 \text{ and } \eta_{1}\eta_{3} = \eta_{3}\eta_{2} = 0.$
v) $\eta_{3}\eta_{1} = \eta_{2}\eta_{3} = \widetilde{\eta}\eta\overline{\eta}.$
vi) $\eta_{2}^{2}\eta_{3} = 0.$
vii) $\eta_{1}\nu_{1} = \nu_{1}\eta_{1} = \eta_{2}\nu_{1} = \nu_{1}\eta_{2} = 0 \text{ and } \eta_{3}\nu_{1} = \nu_{1}\eta_{3} = \eta_{3}^{2} = 0.$

Proof. i), ii) and v) are obvious (see Theo. A of [5]).

By Theo. 3.1 of [5], $\pi_4(2) = \{\eta^2 \overline{\eta}\}$. Since $i\eta^2 \overline{\eta} i = i\eta^3 = i(4\nu) = 0$, we have $i\pi_4(2)i=0$. So, we have, by use of Prop. 1.1,

(2.5)
$$\operatorname{Coext}(\nu p)i = i\nu$$

From this we have the assertion of iii).

iv) follows from the fact $\bar{\eta}\tilde{\eta} = \pm 2\nu$ of i) of Theo. 2.1.

By Theo. 3.1 and Theo. 3.2 of [5], $\pi_5(2)=0$ and $\pi_5^*(2)=0$. So, we have

(2.6)
$$\overline{\eta}\nu_1 = \nu\overline{\eta} = 0 \quad and \quad \nu_1\overline{\eta} = \overline{\eta}\nu = 0$$

From (2.5) and (2.6) we have vii).

Finally, we shall prove vi). By use of b) of ii) of Prop. 1.5, $\eta_2^2 \eta_3 = \tilde{\eta} \eta^2 \bar{\eta} = \tilde{\eta}^3 \bar{\eta} = \tilde{4}\nu \bar{\eta} \equiv i\{2, 4\nu, \eta, 2\}p \mod iG_5 \operatorname{Ext} \eta + \operatorname{Coext} (4\nu)G_2p = 0$. By ix) of Theo. 2.1, $\{2, 4\nu, \eta, 2\}$ consists of 0. This leads us to vi).

2.2. By ii) of Theo. 2.1, $\{\nu^2, 2, \eta\} = \{\eta, 2, \nu^2\} \equiv \varepsilon \mod \eta \sigma$. So, we can choose $\overline{\nu^2} \in \text{Ext } \nu^2$ and $\overline{\nu^2} \in \text{Coext } \nu^2$ such that

(2.7)
$$\overline{\nu^2}\widetilde{\eta} = \overline{\eta}\widetilde{\nu^2} = \mathcal{E}.$$

It follows from (2.7) that

(2.8)
$$\overline{\eta \nu^2} = \varepsilon p \quad and \quad \widetilde{\nu^2} \eta = i \varepsilon$$
.

We shall prove the first. Since $\{\eta, \nu^2, 2\} \equiv \varepsilon \mod \eta \sigma$ by ii) of Theo. 2.1, we have $\eta \overline{\nu^2} \in \eta \{\nu^2, 2, p\} = \{\eta, \nu^2, 2\} p \equiv \varepsilon p \mod \eta \sigma p$. So, we can put $\eta \overline{\nu^2} = \varepsilon p + x\eta \sigma p$. Multiply this equality by $\tilde{\eta}$ on the right, then we have x=0 by (2.7) and by the result $\eta^2 \sigma \neq 0$.

Since $\{8\sigma, 2, \eta\} = \{\eta, 2, 8\sigma\} \equiv \mu \mod \{\eta^2 \sigma, \eta \mathcal{E}\}$ by ii) of Theo. 2.1, we can take $\overline{8\sigma} \in \text{Ext}(8\sigma)$ and $\widetilde{8\sigma} \in \text{Coext}(8\sigma)$ such that

(2.9)
$$\overline{8\sigma}\tilde{\eta} = \bar{\eta}\widetilde{8\sigma} = \mu .$$

From the results that $\{\varepsilon, 2, \eta\} = \{\eta, 2, \varepsilon\} \equiv 0 \mod \eta \mu$ of iii) of Theo. 2.1, we can choose $(\overline{\varepsilon}) \in \text{Ext } \varepsilon$ and $\tilde{\varepsilon} \in \text{Coext } \varepsilon$ such that

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(2.10)
$$\overline{(\varepsilon)}\tilde{\eta} = \bar{\eta}\tilde{\varepsilon} = 0$$

If follows that

(2.11)
$$\eta(\overline{\varepsilon}) = \varepsilon \overline{\eta} \quad and \quad \widetilde{\varepsilon} \eta = \widetilde{\eta} \varepsilon.$$

We shall prove the second. Clearly, $\hat{\epsilon}\eta - \tilde{\eta}\epsilon \in iG_{10} = \{i\eta\mu\}$. Namely, we can put $\hat{\epsilon}\eta = \tilde{\eta}\epsilon + xi\eta\mu$. Multiply this equality by $\bar{\eta}$ on the left, then we have x=0. For $\bar{\eta}\epsilon\eta=0$ by (2.10), $\bar{\eta}\eta\epsilon=2\nu\epsilon=0$ and $\eta^2\mu\pm0$.

By Theo. 3.1 of [5], $i\pi_{s}(2)i=i\{\overline{8\sigma}, \eta\sigma p, \varepsilon p\}i=0$. So, we have, by use of Prop. 1.1,

(2.12)
$$\operatorname{Coext} (\sigma p) i = i\sigma .$$

Since $\eta \overline{8\sigma} \in \{\eta, 8\sigma, 2\} p \equiv \mu p \mod \{\eta^2 \sigma p, \eta \in p\}$ by ii) of Theo. 2.1, we have $\eta \pi_s(2) = G_{\mathfrak{g}} p$. Therefore, we can choose $\sigma_1 \in \text{Coext}(\sigma p) \subset \pi_{\eta}$, by use of a) of i) of Prop. 1.3, such that

$$(2.13) \qquad \bar{\eta}\sigma_1 \equiv \pm \sigma \bar{\eta} \mod \eta \mathcal{E}p .$$

By use of a) of i) of Prop. 1.4, we can put $\sigma_1 \tilde{\eta} = \pm \tilde{\eta} \sigma + x i \eta \mathcal{E} + y i \mu$. Multiply this equality by $\bar{\eta}$ on the left, then we have y=0. For $\bar{\eta}\sigma_1 \tilde{\eta} = \sigma \bar{\eta} \tilde{\eta} = \bar{\eta} \tilde{\eta} \sigma = 2\nu\sigma = 0$, $\eta^2 \mathcal{E} = 0$ and $\eta \mu \neq 0$. So, we have

(2.14)
$$\sigma_1 \tilde{\eta} \equiv \pm \tilde{\eta} \sigma \mod i \eta \mathcal{E} .$$

Since $i\pi_{s}(2)i=0$, $i\eta\sigma p=2\sigma_{1}\pm 0$ by Theo. 3.3 of [5] and $i\eta\overline{\nu^{2}}=\widetilde{\nu^{2}}\eta p=i\varepsilon p$ by (2.8), we have, by use of ii) of Prop. 1.2,

(2.15)
$$\operatorname{Coext}(\overline{\nu^2})i = \overline{\nu^2}.$$

Since $\eta \pi_{s}(2) = G_{9}p$ we can choose $\nu_{2} \in \text{Coext}(\overline{\nu^{2}}) \subset \pi_{7}$, by use of a) of i) of Prop. 1.5, such that

(2.16)
$$\overline{\eta}\nu_2 \equiv \pm(\overline{\varepsilon}) \mod \eta^2 \sigma p$$
.

By the similar arguments to (2.14), we have

(2.17)
$$\nu_2 \tilde{\eta} \equiv \pm \tilde{\varepsilon} \mod i \eta^2 \sigma$$

Proposition 2.2.

i)
$$\sigma_1 \delta = \delta \sigma_1 = i \sigma p$$

- ii) $\delta v_2 = i \overline{v^2} \text{ and } v_2 \delta = \widetilde{v^2} p.$
- iii) $\nu_1^2 = \delta \nu_2 + \nu_2 \delta$.
- iv) $\sigma_1\eta_1 = \eta_1\sigma_1 = i\sigma\bar{\eta} \text{ and } \sigma_1\eta_2 = \eta_2\sigma_1 = \tilde{\eta}\sigma p.$
- v) $\nu_2\eta_1 = \eta_1\nu_2 = i(\overline{\varepsilon}) \text{ and } \nu_2\eta_2 = \eta_2\nu_2 = \tilde{\varepsilon}p.$

vi) $\eta_3 \sigma_1 = \pm \tilde{\eta} \sigma \bar{\eta}$ and $\sigma_1 \eta_3 = \pm \eta_3 \sigma_1$.

vii) $\eta_3 \nu_2 = \pm \tilde{\eta}(\overline{\epsilon})$ and $\nu_2 \eta_3 = \pm \eta_3 \nu_2$.

Proof. i) and ii) are direct consequences of (2.12) and (2.15) respectively. By use of b) of ii) of Prop. 1.7, $\nu_1^2 \equiv i\overline{\nu^2} + \widetilde{\nu^2}p \mod K_6$. Since $K_6 = \{i\sigma p\}$ by Theo. 3.3 of [5], we can put $\nu_1^2 = i\overline{\nu^2} + \widetilde{\nu^2}p + xi\sigma p$. Multiply this equality by $\overline{\eta}$ on the left, then we have x=0. For $\overline{\eta}\nu_1^2=0$ by (2.6), $\eta\overline{\nu^2}=\overline{\eta}\overline{\nu^2}p=\varepsilon p$ by (2.7) and (2.8) and $\eta\sigma p \neq 0$ by Theo. 3.1 of [5]. This proves iii).

We shall prove the first assertion of iv). The equality $\sigma_1 \eta_1 = i\sigma \bar{\eta}$ is a direct consequence of (2.12). We have $\eta_1 \sigma_1 = i\sigma \bar{\eta}$ by (2.13) since $i\eta \varepsilon p = i(2(\overline{\varepsilon})) = 0$.

We shall prove the second assertion of v). By (2.17) we have $\nu_2 \eta_2 \equiv \tilde{\epsilon} p \mod i \eta^2 \sigma p = 0$.

By use of b) of i) of Prop. 1.6, we have $\eta_2 v_2 = \tilde{\eta} v^2 \equiv \tilde{\epsilon} p \mod K_s$ since $\tilde{\eta} v^2 = 0$ by (2.6) and $\eta v^2 = \epsilon p$ by (2.8). It follows from Theo. 3.3 of [5] that $K_s = \{i \mu p\}$. So, we can put $\tilde{\eta} v^2 = \tilde{\epsilon} p + xi\mu p$. Multiply this equality by $\tilde{\eta}$ on the right, then we have x = 0. For $\tilde{\eta} v^2 \tilde{\eta} = \tilde{\eta} \epsilon$ by (2.7), $\tilde{\eta} \epsilon = \tilde{\epsilon} \eta$ by (2.11) and $i\eta\mu \pm 0$ by Theo. 3.2 of [5].

The first assertions of vi) and vii) are obtained from (2.13) and (2.16) respectively since $2\eta_3\nu_2 = \tilde{\eta}\eta \varepsilon p = i\{2, \eta, \eta\varepsilon\}p = i\zeta p = i\{2, \eta, \eta^2\sigma\}p = \tilde{\eta}\eta^2\sigma p = 2\eta_3\sigma_1$ by v) of Theo. 2.1.

Similarly, we have $i\eta\varepsilon\bar{\eta}=i\gamma^2\sigma\bar{\eta}=i\zeta p$. By (2.11) and by Theo. 3.3 of [5], we have $\tilde{\varepsilon}\bar{\eta}\equiv\bar{\eta}(\bar{\varepsilon}) \mod K_{11}=\{i\zeta p\}$. Therefore, we obtain the second assertions of vi) and vii) by (2.14) and (2.17) respectively.

2.3. By use of ii) of Prop. 1.2, Coext $(\overline{8\sigma})i \equiv \widetilde{8\sigma} \mod i\pi_{\mathfrak{s}}(2)i$. So, we can choose $A \in \operatorname{Coext}(\overline{8\sigma}) \subset \pi_{\mathfrak{s}}$ such that

From this and (2.9), we can choose

(2.19)
$$(\overline{\mu}) = \overline{\eta} A \in \operatorname{Ext} \mu \quad and \quad \widetilde{\mu} = A \widetilde{\eta} \in \operatorname{Coext} \mu.$$

Since $\{8\sigma, 2, 8\sigma\} = 16\rho$ by v) of Theo. 2.1, we can take

(2.20)
$$\overline{16\rho} = \overline{8\sigma}A \in \text{Ext}(16\rho) \text{ and } \overline{16\rho} = A\widetilde{8\sigma} \in \text{Coext}(16\rho).$$

By use of i) of Prop. 1.9, we can take

As $\overline{\eta}A^2 i = (\overline{\mu}) \widetilde{8\sigma} \equiv \overline{\mu} \mod \eta^2 \rho$ and $p(A^2 \tilde{\eta}) = \overline{8\sigma} \tilde{\mu} \equiv \overline{\mu} \mod \eta^2 \rho$ by ii) of Theo. 2.1, we can choose

(2.22) $(\overline{\mu}) \equiv \overline{\eta} A^2 \mod \eta \rho \overline{\eta} \quad and \quad (\widetilde{\mu}) \equiv A^2 \overline{\eta} \mod \overline{\eta} \eta \rho.$

Proposition 2.3.

- i) $\delta A = i \overline{\delta \sigma}$ and $A \delta = \widetilde{\delta \sigma} p$.
- ii) $\delta A = A \delta + x i \eta \sigma p + y i \varepsilon p$ and $\delta A^2 = A^2 \delta$.
- iii) $\eta_1 A = i(\overline{\mu}), A\eta_1 = \eta_1 A + xi\eta\sigma\overline{\eta} + yi\varepsilon\overline{\eta} \text{ and } \eta_1^2 A = A\eta_1^2.$
- iv) $A\eta_2 = \tilde{\mu}p, \eta_2 A = A\eta_2 + x \tilde{\eta}\eta\sigma p + y \tilde{\eta}\varepsilon p \text{ and } \eta_2^2 A = A\eta_2^2.$
- v) $\eta_1 A^2 \equiv i(\overline{\mu}) \mod i\eta\rho\bar{\eta} \text{ and } A^2\eta_1 = \eta_1 A^2.$
- vi) $A^2\eta_2 \equiv (\widetilde{\mu})p \mod \widetilde{\eta}\eta\rho p \text{ and } \eta_2 A^2 = A^2\eta_2.$

Proof. i) is obvious.

Since $i \operatorname{Ext} (8\sigma) = i \{8\sigma, 2, p\} = i \{2, 8\sigma, p\} = \{i, 2, 8\sigma\}p = \operatorname{Coext} (8\sigma)p$ and $i\{2, 8\sigma, p\}$ is a coset of $iG_{s}p = \{i\eta\sigma p, i\varepsilon p\}$, we can put $\delta A = A\delta + xi\eta\sigma p + yi\varepsilon p$. We have $A(i\eta\sigma p) = \widetilde{8\sigma}\eta\sigma p = i\{2, 8\sigma, \eta\sigma\}p = i\eta\rho p = i\{2, 8\sigma, \varepsilon\}p = A(i\varepsilon p)$ by v) of Theo. 2.1. Similarly, we have $(i\eta\sigma p)A = (i\varepsilon p)A = i\eta\rho p$. So, we obtain $\delta A^{2} = A\delta A + (x+y)i\eta\rho p = A^{2}\delta$.

By the above proof of ii) and (2.9), $\delta \sigma \eta = i\mu + xi\eta^2 \sigma + yi\eta \varepsilon$. As $K_9 = 0$ by Theo. 3.3 of [5], we have, by use of b) of i) of Prop. 1.5, $A\eta_1 = \delta \sigma \bar{\eta} = i(\overline{\mu}) + xi\eta\sigma\bar{\eta} + yi\varepsilon\bar{\eta}$. Therefore, the first assertions of iv) and v) of Prop. 2.2, (2.11) and (2.19) imply iii).

The first of (2.22) implies the first of v).

By the similar arguments to the above proof of ii), $A(i\eta\sigma\bar{\eta})=A(i\epsilon\bar{\eta})=i\eta\rho\bar{\eta}$. On the other hand, $(i\eta\sigma\bar{\eta})A=i\eta\sigma(\mu)=i\sigma\mu\bar{\eta}=i\eta\rho\bar{\eta}$ since $\sigma\zeta=0$ and $\eta(\mu)\equiv\mu\bar{\eta}$ mod ζp . We have $(i\epsilon\bar{\eta})A=i\epsilon(\mu)=i\eta\rho\bar{\eta}+zi\nu^*p$ since $\epsilon\mu=\eta^2\rho$ and $i\eta\bar{\mu}p=0$. Multiply this equality by ν_1 on the left, then we obtain z=0. For $\nu_1(i\epsilon)=i\nu\epsilon=0, \nu_1(i\eta)=i\nu\eta=0$ and $\nu_1(i\nu^*p)=i\nu\nu^*p=i\sigma^3p\pm 0$ by Theo. 3.3 of [5] (see (7.16) and Prop. 7.2 of [3]). Consequently, the second of v) is proved.

The proofs of iv) and vi) are quite similar to the ones of iii) and v) respectively and we omit them.

2.4. From the results that $\{\nu, 8\sigma, 2\} \equiv \zeta \mod 2G_{11}, \{\sigma, 8\sigma, 2\} \equiv \rho \mod 2G_{15}$ and $\{\zeta, 8\sigma, 2\} \equiv \overline{\zeta} \mod 2G_{19}$ of v) of Theo. 2.1, we can take, by use of a) of ii) of Prop. 1.8,

- (2.23) $\nu_1 A \in \operatorname{Coext} (\zeta p) \subset \pi_{11},$
- (2.24) $\sigma_1 A \in \operatorname{Coext}(\rho p) \subset \pi_{15}$

and

(2.25) $\nu_1 A^2 \in \operatorname{Coext} (\overline{\zeta} p) \subset \pi_{19}.$

Proposition 2.4.

- i) $A\sigma_1 = \pm \sigma_1 A$.
- ii) $A\nu_1 = \nu_1 A$.
- iii) $\eta_3 A = \tilde{\eta}(\mu)$ and $A\eta_3 \equiv \pm \eta_3 A \mod \nu_1 A$.
- iv) $\eta_3 A^2 \equiv \pm \tilde{\eta}(\overline{\mu}) \mod \nu_1 A^2 \text{ and } A^2 \eta_3 = \eta_3 A^2.$

Proof. Sinc $\rho \in \{\sigma, 2\sigma, 8\} \subset \{\sigma, 8\sigma, 2\} \mod 2G_{15}$ by v) of Theo. 2.1, $p(A\sigma_1) = \overline{8\sigma\sigma_1} \in \{8\sigma, 2, \sigma p\} = \{8, 2\sigma, \sigma p\} = \{8, 2\sigma, \sigma\} p = \rho p$ and similarly $\sigma_1 A i = A\sigma_1 i = i\rho$. Therefore, we have, by use of b) of ii) of Prop. 1.8, $A\sigma_1 \equiv \sigma_1 A \mod K_{15} = \{i\eta\rho p, i\eta^*p, i\overline{16\rho}\}$. Namely, we can put $A\sigma_1 = \pm \sigma_1 A + xi\eta^*p + yi\overline{16\rho}$. Multiply this equality by $\overline{\eta}$ on the left and by $\overline{\eta}$ on the right at the same time, then we have x = y = 0. For it is clear that $\overline{\eta} A\sigma_1 \overline{\eta} = \overline{\eta} \sigma_1 A \overline{\eta} = 0$ and that $\eta^2 \eta^*$ and $\eta \overline{\mu}$ are linearly independent in $(G_{18}; 2)$.

By the similar arguments to the above, we obtain $A\nu_1 = \nu_1 A + zi\eta\mu\bar{\eta}$. By i) $\nu_1 A\sigma_1 = \nu_1\sigma_1 A$ and this equals $A\nu_1\sigma_1$ since $\nu_1\sigma_1 \in K_{10} = \{i\zeta p\}$ and $(i\zeta p)A = A(i\zeta p) = i\bar{\zeta}p$. On the other hand, $i\eta\mu\bar{\eta}\sigma_1 = i\eta\mu\sigma\bar{\eta} = i\bar{\gamma}^2\rho\bar{\eta} = i\bar{\zeta}p$ since $\{\eta^2\rho, \eta, 2\} \equiv \bar{\zeta} \mod 2G_{19}$ by v) of Theo. 2.1. This leads us to the assertion z=0.

It is clear that $2\eta_3 A = 2A\eta_3 = i\eta\mu\bar{\eta}$. So, iii) follows from Theo. 3.3 of [5]. The proof of iv) is left to the reader.

2.5. From the results that $\{\kappa, 2, \eta\} = \{\eta, 2, \kappa\} \equiv 0 \mod \eta \rho$ of iii) of Theo. 2.1, we can choose $(\bar{\kappa}) \in \text{Ext } \kappa$ and $\tilde{\kappa} \in \text{Coext } \kappa$ such that

(2.26)
$$(\overline{\kappa})\tilde{\eta} = \overline{\eta}\tilde{\kappa} = 0.$$

By the similar arguments to (2.11), we obtain

(2.27)
$$\eta(\overline{\kappa}) = \kappa \overline{\eta} \quad and \quad \widetilde{\kappa} \eta = \widetilde{\eta} \kappa .$$

We define $\kappa_1 \in \pi_{14}$ and $\kappa_2 \in \pi_{16}$ by

(2.28)
$$\kappa_1 = i(\overline{\kappa}) \quad and \quad \kappa_2 = \tilde{\eta}(\overline{\kappa}).$$

Proposition 2.5.

i)
$$\delta \kappa_1 = 0$$
 and $\kappa_1 \delta = i \kappa p$.

- ii) $\kappa_1 \eta_2 = \eta_2 \kappa_1 = 0$ and $\delta \kappa_2 = \kappa_1 \eta_1 = \eta_1 \kappa_1 = i \eta(\overline{\kappa})$.
- iii) $\eta_1 \kappa_2 = \kappa_2 \eta_2 = \kappa_1 \eta_3 = 0$ and $\eta_3 \kappa_2 = \kappa_2 \eta_3 = 0$.
- iv) $\nu_2^2 + \kappa_1 = \tilde{\kappa} p$.
- v) $\kappa_2 \eta_1 = \eta_2 \kappa_2 = \eta_3 \kappa_1 = \tilde{\eta} \kappa \bar{\eta} \equiv i \nu (\kappa) + \tilde{\kappa} \nu p \mod i \nu^* p.$
- vi) $\nu_1 \kappa_1 = i\nu(\overline{\kappa})$ and $\kappa_1 \nu_1 \equiv \nu_1 \kappa_1 \mod i\nu^* p$.
- vii) $\kappa_2 \nu_1 = \nu_1 \kappa_2 = 0.$

- viii) $\nu_2 \kappa_1 = \widetilde{\nu^2(\kappa)} \text{ and } \kappa_1 \nu_2 \equiv i \overline{\kappa} \overline{\eta} \mod i \nu_{\overline{\sigma}} p.$ ix) $\delta \nu_2^2 = \nu_2^2 \delta = \nu_2 \delta \nu_2 = \kappa_1 \delta \text{ and } \kappa_2 \delta = \eta_2 \nu_2^2.$

Proof. i), ii) and iii) are obvious.

By viii) of Theo. 2.1, $\{\overline{\nu^2}, \widetilde{\nu^2}, 2\} = \{2, \overline{\nu^2}, \widetilde{\nu^2}\} = \kappa$. So, we obtain, by use of ii) of Prop. 1.9 and Theo. 3.3 of [5], $\nu_2^2 = \kappa_1 + \tilde{\kappa} p + \kappa i \rho p$. Multiply this equality by $\bar{\eta}$ on the left and by η_2 on the right at the same time, then we have $\kappa = 0$. For $\bar{\eta}\nu_2^2\eta_2 = \bar{\eta}\eta_2\nu_2^2 = 0$, $\bar{\eta}\kappa_1\eta_2 = 0$, $\bar{\eta}\tilde{\kappa}p\eta_2 = 0$ and $\eta^2\rho p \neq 0$ in $\pi_{17}(2)$. Thus, iv) is proved.

By iv) of Theo. 2.1, $p(\tilde{\eta}\kappa\bar{\eta})=\eta\kappa\bar{\eta}=\{\eta\kappa,\eta,2\}p=\nu\kappa p$ and similarly $(\tilde{\eta}\kappa\bar{\eta})i$ = $i\nu\kappa$. So, we have $\tilde{\eta}\kappa\bar{\eta}\equiv i\nu(\overline{\kappa})+\tilde{\kappa}\nu p \mod K_{17}=\{i\nu^*p\}$. From this and (2.27), we obtain v).

The first of vi) is obvious.

By use of a) of i) of Prop. 1.3, $(\overline{\kappa})\nu_1 \equiv \nu(\overline{\kappa}) \mod \kappa \pi_4(2) + G_{18}p = \{\nu^* p, \eta \overline{\mu} p\}$. By the similar arguments to (2.14), we have

(2.29)
$$(\overline{\kappa})\nu_1 \equiv \nu(\overline{\kappa}) \mod \nu^* p$$
.

From this we have the second of vi).

By (2.6), $\nu_1 \kappa_2 = \nu_1 \tilde{\eta}(\overline{\kappa}) = 0$. By (2.29), (2.6) and i) of Theo. 2.1, $\kappa_2 \nu_1 = \tilde{\eta}(\overline{\kappa})\nu_1 \equiv \tilde{\eta}\nu(\overline{\kappa}) = 0 \mod \tilde{\eta}\nu^* p = i\{2, \eta, \nu^*\}p = 0$. So, vii) is proved.

The first of viii) is obvious.

By use of a) of i) of Prop. 1.5, $\kappa_1 \nu_2 \equiv i \bar{\kappa} \bar{\eta} \mod i \kappa \pi_8(2) + i G_{22} p = \{i \nu \bar{\sigma} p\}$ since $(\bar{\kappa}) \tilde{\nu}^2 = \bar{\kappa} \eta$ and $i \kappa \bar{8} \sigma \equiv 0 \mod i \kappa \varepsilon p = 0$ by vi) and viii) of Theo. 2.1.

i), ii) and iv) imply ix) except for the relation $\nu_2 \delta \nu_2 = \delta \nu_2^2$. This will be proved in Prop. 2.9.

2.6. We have the relations

(2.30)
$$\overline{(\varepsilon)}\tilde{\mu} = (\overline{\mu})\tilde{\varepsilon} = 0$$

We shall prove the first. By iii) of Theo. 2.1, $(\overline{\varepsilon})\tilde{\mu} = x\eta\overline{\mu}$. Multiply this by η on the left, then we have x=0 since $\eta(\overline{\varepsilon})\tilde{\mu} = \varepsilon\eta\overline{\mu} \in \varepsilon G_{11} = 0$ and $\eta^2\overline{\mu} \pm 0$.

Proposition 2.6.

- i) $A\nu_2 \equiv \nu_2 A \equiv \pm \sigma_1 A \mod \{i\kappa\bar{\eta}, \,\tilde{\eta}\kappa\,p\}.$
- ii) $\delta A \nu_2 = A \delta \nu_2 = A \nu_2 \delta = \delta \nu_2 A = \nu_2 \delta A = \nu_2 A \delta = \delta \sigma_1 A.$

Proof. By v) of Theo. 2.1, $2A\nu_2=2\nu_2A=i\{\varepsilon, 8\sigma, 2\}p=i\eta\rho p=i\{\eta\sigma, 8\sigma, 2\}p=2\sigma_1A$. So, we have, by Theo. 3.3 of [5], $A\nu_2\equiv\nu_2A\equiv\sigma_1A \mod \{i\eta\rho p, i\eta^*p, i\overline{16\rho}, i\kappa\overline{\eta}, \eta\kappa p\}$. Multiply these by $\overline{\eta}$ on the left and by η on the right at the same time, then we have i) by (2.30).

By use of iii) of Prop. 2.2, ii) of Prop. 2.3 and Prop. 2.4 and i), we obtain ii).

2.7. Lemma 2.7
$$\sigma \overline{\nu^2} = \overline{\nu^2} \sigma_1 = 0$$
 and $\widetilde{\nu^2} \sigma = \sigma_1 \widetilde{\nu^2} = 0$.

Proof. By iv) of Theo. 2.1, $\sigma \overline{\nu^2} \in \{\sigma, \nu^2, 2\} p \equiv 0 \mod \sigma^2 p$. Assume that $\sigma \overline{\nu^2} = \sigma^2 p$, then we have, by the definition of $\sigma \in (G_{19}; 2)$ (see iv) of Theo. 2.1) and by the relation $\tilde{\eta}\nu = 0$ of (2.6),

This contradicts to the result that $\bar{\sigma} \neq 0$ in G_{19} . Thus the first relation is proved.

By Theo. 3.1 of [5], $\nu \pi_{11}(2) = \nu \{\eta(\overline{\mu}), \zeta p\} = 0$. So, we have $\overline{\nu^2} \sigma_1 \in \{\nu^2, 2, \sigma p\} = \{\nu, 2\nu, \sigma p\} = \{\nu, 2\nu, \sigma\} p \equiv 0 \mod \sigma^2 p$ by iv) of Theo. 2.1. Namely, we can put $\overline{\nu^2} \sigma_1 = x \sigma^2 p$. Multiply this by σ on the left, then we have x = 0 since $\sigma \overline{\nu^2} \sigma_1 = 0$ by the first relation and $\sigma^3 p \neq 0$ by Theo. 3.1 of [5].

By the quite similar arguments to the above, we obtain the other relations. As an immediate consequence of this lemma we have

Corollary. $\sigma_1 \nu_2 = \nu_2 \sigma_1 = 0.$

2.8. Proposition 2.8. $\kappa_1 \sigma_1 = \sigma_1 \kappa_1 = i \nu \overline{\sigma} p$.

Proof. By viii) of Theo. 2.1, $\sigma_1 \kappa_1 = i\sigma(\overline{\kappa}) = i\{\sigma, \kappa, 2\} p = i\nu \overline{\sigma} p$.

On the other hand, we have $\kappa_1 \sigma_1 \in i\{\kappa, 2, \sigma p\}$. The bracket $\{\kappa, 2, \sigma p\}$ is a coset of $\kappa \pi_s(2) + G_{15} \sigma p = \{\eta^2 \bar{\kappa} p\}$ since $\sigma \rho = \eta \sigma \kappa = 0$, $\varepsilon \kappa = \eta^2 \bar{\kappa}$ and $\kappa 8 \sigma \equiv 0 \mod \varepsilon \kappa p$. By use of Prop. 1.5 of [6], $\{\{2, \overline{\nu^2}, \widetilde{\nu^2}\}, 2, \sigma p\} + \{2, \{\overline{\nu^2}, \widetilde{\nu^2}, 2\}, \sigma p\} + \{2, \overline{\nu^2}, \widetilde{\nu^2}, 2, \sigma p\} = \{2, \overline{\nu^2}, \{\overline{\nu^2}, \sigma p\}\} \equiv 0$. By (2.15) and Corollary of 2.7, we have $\{\widetilde{\nu^2}, 2, \sigma p\} \equiv \nu_2 \sigma_1 = 0 \mod \widetilde{\nu^2} \pi_s(2) + \pi_8^*(2) \sigma p = \{i \rho p\}$. By vii) of Theo. 2.1, $\{2, \overline{\nu^2}, i \rho p\} \subseteq \{2, \nu^2, \rho\} p = 0 \mod 2\pi_{22}(2) = \{\eta^2 \bar{\kappa} p\}$. Therefore, we have, by viii) of Theo. 2.1, $\{\kappa, 2, \sigma p\} = \{2, \kappa, \sigma p\} \supseteq \{2, \kappa, \sigma\} p = \nu \sigma p \mod \eta^2 \bar{\kappa} p$. This leads us to the first relation.

2.9. By ii) of (1.4) of [6], we have $(1 \# \nu)(1 \# \sigma) = 1 \# \nu \sigma = 0$ and $(1 \# \sigma)(1 \# \nu) = 1 \# \sigma \nu = 0$, where 1 is the generator of π_0 and $\alpha \# \beta$ is the reduced join (see p. 6 of [6]). Clearly, we have $1 \# \nu \in \text{Coext}(\nu p)$ and $1 \# \sigma \in \text{Coext}(\sigma p)$. Since $\text{Coext}(\nu p)$ is a coset of $i\pi_4(2) = \{2\eta_3\}$ and $\text{Coext}(\sigma p)$ is a coset of $i\pi_8(2) = \{\delta A, 2\sigma_1, 2\nu_2\}$, we have $1 \# \nu = \nu_1 + 2x\eta_3$ and $1 \# \sigma = \pm \sigma_1 + 2y\nu_2 + z\delta A$. So, by the above two relations and the ones that $\nu_1 \delta A =$

 $\delta A \nu_1 = 2\eta_3 \sigma_1$, we have $\nu_1 \sigma_1 = \sigma_1 \nu_1 = 2(x+y)\eta_3 \sigma_1$.

Now we change the definition of ν_1 . We replace ν_1 by $\nu_1+2(x+y)\eta_3$. Then we have

We note that $\nu_1 + 2(x+y)\eta_3$ is contained in Coext (νp) since this is a coset of $\{2\eta_3\}$.

Proposition 2.9.

- i) $\nu_1 \nu_2 = \nu_2 \nu_1 = \eta_3 (\nu_2 \pm \sigma_1)$
- ii) $\eta_2\eta_3\nu_2=\eta_2\eta_3\sigma_1=\nu_1A.$
- iii) $\nu_1 \nu_2^2 = \eta_3 \nu_2^2 = \eta_3 \kappa_1$.
- iv) $\nu_2 \delta \nu_2 = \delta \nu_2^2$.

Proof. Since $p\nu_1\nu_2i=p\nu_2\nu_1i=\nu^3=\eta(\varepsilon+\sigma\eta)=p\eta_3(\nu_2+\sigma_1)i$, we have, by use of Theo. 3.3 of [5], $\nu_1\nu_2\equiv\nu_2\nu_1\equiv\eta_3(\nu_2+\sigma_1) \mod \{2 \eta_3\sigma_1, \eta_1^2A, \eta_2^2A\}$. Multiply these by η_1 on the left and by η_2 on the right respectively, then we have $\nu_1\nu_2\equiv\nu_2\nu_1\equiv\eta_3(\nu_2+\sigma_1) \mod 2\eta_3\sigma_1$. Furthermore, multiply the equality $\nu_1\nu_2=\nu_2\nu_1+2x\eta_3\sigma_1$ by A, then we have x=0 by i) of Prop, 2.6. This leads us to i).

By vii) of Prop. 2.1 and iii) of Prop. 2.2 and by i), $0=\nu_1^2\nu_2=\delta\nu_2^2+\nu_2\delta\nu_2$. Namely, iv) is proved.

The proofs of ii) and iii) are left to the reader.

2.10. From the results that $\{\sigma^2, 2, \eta\} = \{\eta, 2, \sigma^2\} \equiv \eta^* \mod \eta \rho$ of ii) of Theo. 2.1, we can choose $\overline{\sigma^2} \in \text{Ext } \sigma^2$ and $\overline{\sigma^2} \in \text{Coext } \sigma^2$ such that

(2.32)
$$\overline{\sigma^2}\tilde{\eta} = \bar{\eta}\widetilde{\sigma^2} = \eta^*.$$

By the similar arguments to (2.8), we have

(2.33)
$$\eta \overline{\sigma^2} = \eta^* p \quad and \quad \widetilde{\sigma^2} \eta = i \eta^*.$$

From the results that $\{\eta^*, 2, \eta\} = -\{\eta, 2, \eta^*\} \equiv \pm 2\nu^* \mod \eta \overline{\mu}$ of i) of Theo. 2.1, we can choose $\overline{\eta^*} \in \text{Ext } \eta^*$ and $\overline{\eta^*} \in \text{Coext } \eta^*$ such that

(2.34)
$$\overline{\eta^*}\tilde{\eta} = \pm 2\nu^* \quad and \quad \bar{\eta}\tilde{\eta^*} = \pm 2\nu^*.$$

It is clear that

(2.35)
$$\eta \overline{\eta^*} = \eta^* \overline{\eta} \quad and \quad \overline{\eta^*} \eta = \overline{\eta} \eta^*.$$

By Theo. 3.1 of [5], it is clear that $i\pi_{16}(2)i=\{i\eta\kappa\}$ and $\eta\pi_{16}(2)=G_{17}p$. By use of ii) of Prop. 1.2 and a) of i) of Prop. 1.5, we can choose $\sigma_2 \in \text{Coext}(\overline{\sigma^2}) \subset \pi_{15}$ such that

(2.36)
$$\sigma_2 i = \widetilde{\sigma}^2$$

and

(2.37)
$$\bar{\eta}\sigma_2 \equiv \pm \overline{\eta^*} \mod \{\nu \kappa p, \eta^2 \rho p\}.$$

Obviously, we have

(2.38)
$$\sigma_2 \tilde{\eta} \equiv \pm \tilde{\eta^*} \mod \{i\nu\kappa, i\eta^2\rho\}.$$

Since $\{\nu, \sigma^2, 2\} \equiv \nu^* \mod 2\nu^*$ by iv) of Theo. 2.1, we can choose, by a) of ii) of Prop. 1.8,

(2.39)
$$\nu_1 \sigma_2 \in \operatorname{Coext} (\nu^* p) \subset \pi_{18}.$$

Proposition 2.10.

- i) $\delta \sigma_2 = i \overline{\sigma^2} \text{ and } \sigma_2 \delta = \widetilde{\sigma^2} p.$
- ii) $\sigma_1^2 = \delta \sigma_2 + \sigma_2 \delta$.
- iii) $\eta_1 \sigma_2 \equiv i \overline{\eta^*} \mod i \nu \kappa p \text{ and } \sigma_2 \eta_1 \equiv \eta_1 \sigma_2 \mod i \nu \kappa p.$
- iv) $\sigma_2 \eta_2 \equiv \widetilde{\eta^*} p \mod i\nu \kappa p \text{ and } \eta_2 \sigma_2 \equiv \sigma_2 \eta_2 \mod i\nu \kappa p.$
- v) $\eta_3 \sigma_2 \equiv \pm \tilde{\eta} \eta^* \mod i \bar{\zeta} p \text{ and } \sigma_2 \eta_3 \equiv \pm \eta_3 \sigma_2 \mod i \bar{\zeta} p.$
- vi) $\sigma_2 \nu_1 \equiv \nu_1 \sigma_2 \mod \{2\eta_3 \sigma_2, i \xi p, i \sigma p\}.$

The proof is similar to the one of Prop. 2.2 and we omit it. We note that the following relations hold.

(2.40)
$$\overline{\nu^2}\,\widetilde{\sigma^2} = \overline{\sigma^2}\,\widetilde{\nu^2} = \sigma^3$$

We shall prove the second relation. By Lemma 2.7 and ii) of Prop. 2.10, $0 = \sigma_1^2 \widetilde{\nu^2} = \delta \sigma_2 \widetilde{\nu^2} + \sigma_2 \delta \widetilde{\nu^2}$. So, we have $i \overline{\sigma^2} \widetilde{\nu^2} = \widetilde{\sigma^2} \nu^2 = i \{2, \sigma^2, \nu\} \nu = i \nu^* \nu = i \sigma^3$. Therefore we obtain $\overline{\sigma^2} \widetilde{\nu^2} - \sigma^3 \in 2G_{21} = 0$ (see [3]).

2.11 By use of ii) of Prop. 1.2, we can choose $\kappa_3 \in \text{Coext}(\nu(\kappa)) \subset \pi_{18}$ such that

(2.41)
$$\kappa_3 i = \tilde{\kappa} \nu + \mathrm{x} i \nu^* \,.$$

Since $\nu_1 \sigma_2 i = i\nu^*$, we have

(2.42)
$$\kappa_3 + x\nu_1\sigma_2 \in \operatorname{Ext}\left(\tilde{\kappa}\nu\right).$$

Since $\{\bar{\eta}, \tilde{\kappa}\nu, 2\} = \{2, \nu(\bar{\kappa}), \tilde{\eta}\} \equiv 0 \mod 2G_{20}$ by viii) of Theo. 2.1, we have, by sue of a) of ii) of Prop. 1.5 and 1.6 and by (2.6),

(2.43)
$$\bar{\eta}\kappa_3 = 0 \quad and \quad \kappa_3 \tilde{\eta} = 0.$$

Proposition 2.11.

i) $\delta \kappa_3 = i\nu(\overline{\kappa})$ and $\kappa_3 \delta \equiv \tilde{\kappa}\nu p \mod i\nu^*p$.

- ii) $\eta_1 \kappa_3 = \kappa_3 \eta_1 = \eta_2 \kappa_3 = \kappa_3 \eta_2 = 0$ and $\eta_3 \kappa_3 = \kappa_3 \eta_3 = 0$.
- iii) $\nu_1 \kappa_3 \equiv \nu_2 \kappa_1 + \kappa_1 \nu_2 \mod \{i\sigma \overline{\sigma^2}, i\nu \overline{\sigma}p\} \text{ and } \kappa_3 \nu_1 \equiv \nu_1 \kappa_3 \mod \{i\sigma \overline{\sigma^2}, \overline{\sigma^2} \sigma p, i\nu \overline{\sigma}p\}.$

The proof is easy and left to the reader.

2.12. From the results that $\{\overline{\sigma}, 2, \eta\} = \{\eta, 2, \overline{\sigma}\} \equiv 0 \mod \eta \overline{\kappa}$ of ii) of Theo. 2.1, we can choose $(\overline{\sigma}) \in \text{Ext}(\overline{\sigma})$ and $(\overline{\sigma}) \in \text{Coext}(\overline{\sigma})$ such that

(2.44)
$$\overline{(\overline{\sigma})}\tilde{\eta} = \bar{\eta}(\widetilde{\overline{\sigma}}) = 0$$
.

It is clear that

(2.45)
$$\eta(\overline{\sigma}) = 0 \quad and \quad (\widetilde{\sigma})\eta = 0.$$

Since $i\pi_{21}(2)i=0$ by Theo. 3.1 of [5], we have, by ii) of Prop. 1.2,

(2.46)
$$\operatorname{Coext}\left(\overline{(\overline{\sigma})}\right)i = \widetilde{(\overline{\sigma})}.$$

Choose $\bar{\sigma}_1 \in \text{Coext}\left(\overline{(\bar{\sigma})}\right) \subset \pi_{20}$ arbitrarily, then we have

(2.47) $\bar{\eta}\bar{\sigma}_1 \equiv 0 \mod \{\nu\bar{\sigma}p, \eta^2\bar{\kappa}p\} \text{ and } \bar{\sigma}_1\tilde{\eta} \equiv 0 \mod \{i\nu\bar{\sigma}, i\eta^2\bar{\kappa}\}.$

Proposition 2.12. $\eta_1 \bar{\sigma}_1 \equiv \bar{\sigma}_1 \eta_2 \equiv \eta_2 \bar{\sigma}_1 = \bar{\sigma}_1 \eta_1 \equiv 0 \mod i \nu \bar{\sigma} p.$

Proof. The first two relations are obivous.

For the proofs that $\eta_2 \bar{\sigma}_1 = \bar{\sigma}_1 \eta_1 = 0$, we use the facts that $\{2, \eta, \bar{\sigma}, 2\} = \{2, \bar{\sigma}, \eta, 2\} \equiv 0 \mod \eta^2 \bar{\kappa}$ of ix) of Theo. 2.1. The details are left to the reader.

2.13. Since $i\pi_{21}(2)i=0$ by Theo. 3.1 of [5], we have, by use of Prop. 1.1, (2.48) Coext $(\bar{\kappa}p)i=i\bar{\kappa}$.

Let $\bar{\kappa}_1$ be a representative of Coext $(\bar{\kappa}p) \subset \pi_{20}$, then we have, by use of a) of i) of Prop. 1.3 and 1.4,

(2.49) $\bar{\eta}\bar{\kappa}_1 \equiv \pm \bar{\kappa}\bar{\eta} \mod \nu \bar{\sigma} p \quad and \quad \bar{\kappa}_1 \bar{\eta} \equiv \pm \bar{\eta}\bar{\kappa} \mod i\nu \bar{\sigma} .$

Proposition 2.13.

i)
$$\bar{\kappa}_1 \delta = \delta \bar{\kappa}_1 = i \bar{\kappa} p$$
.

- ii) $\bar{\kappa}_1 \eta_1 = i \bar{\kappa} \bar{\eta} = and \ \eta_1 \bar{\kappa}_1 \equiv \bar{\kappa}_1 \eta_1 \mod i \nu \bar{\sigma} p.$
- iii) $\eta_2 \bar{\kappa}_1 = \tilde{\eta} \bar{\kappa} p \text{ and } \bar{\kappa}_1 \eta_2 \equiv \eta_2 \bar{\kappa}_1 \mod i \nu \bar{\sigma} p.$
- iv) $\nu_2^3 \equiv \nu_2 \kappa_1 + \eta_2 \bar{\kappa}_1 \mod i \nu \bar{\sigma} p$.

The proof is obvious.

3. Proof of Theorem 2.1

In this section we shall prove Theorem 2.1 which holds the key to our computations in the previous section.

We can find almost all of the results of Theo. 2.1 in [3], [4] and [6]. The ones which we can not find there will be proved by use of the methods and the results of [6].

3.1. **Proof of i**)

 $\{\eta, 2, \eta\} = \pm 2\nu$ by (5,4) in p. 41 of [6].

 $\{\nu, \eta, 2\} = 0$ since $G_5 = 0$.

 $\{\eta, 2, \eta^*\} \equiv \pm 2\nu^* \mod \eta \overline{\mu} \text{ since } 2\{\eta, 2, \eta^*\} = \{2, \eta, 2\} \eta^* = \eta^2 \eta^* = 4\nu^* \text{ by}$ Cor. 3.7 in p. 31 of [6].

 $\{\nu^*, \eta, 2\} \equiv 0 \mod 2G_{20} \text{ since } \eta \bar{\kappa} \neq 0 \text{ and } \{\nu^*, \eta, 2\} \eta = \nu^* \{\eta, 2, \eta\} = 2\nu\nu^* = 0.$

3.2. Proof of ii)

 $\{\eta, 2, \nu^2\} = \{\eta, \nu^2, 2\} \equiv \varepsilon \mod \eta \sigma$ by (6.1) in p. 51 of [6].

 $\{\nu^2, \eta, 2\} = \nu\{\nu, \eta, 2\} = 0$ by i).

 $\{\eta, 2, 8\sigma\} = \{\eta, 8\sigma, 2\} \equiv \mu \mod \{\eta^2 \sigma, \eta \mathcal{E}\}$. See p. 189 of [6].

 $\{8\sigma, \eta, 2\}=0$ since $\{8\sigma, \eta, 2\}\subseteq\{2, 0, 2\}=2G_{9}=0$.

 $\{\eta, 2, \sigma^2\} = \{\eta, \sigma^2, 2\} \equiv \eta^* \mod \eta \rho \text{ and } \{\sigma^2, \eta, 2\} = 0.$ See the proof of (2) of Lemma 4.2 in p. 279 of [5].

 $\{\eta, 2, \bar{\sigma}\} = \{\eta, \bar{\sigma}, 2\} \equiv 0 \mod \eta \bar{\kappa} \text{ and } \{\bar{\sigma}, \eta, 2\} = 0.$ See the proof of (4) of Lemma 4.2 in p. 280 of [5].

 $\{\mu, 2, 8\sigma\} = \{\mu, 8\sigma, 2\} \equiv \overline{\mu} \mod \eta^2 \rho.$ See p. 189 of [6]. $\{8\sigma, \mu, 2\} \subseteq \{2, 0, 2\} = 2G_{17} = 0$

3.3. Proof of iii)

 $\{\eta, 2, \varepsilon\} \equiv 0 \mod \eta \mu$. We know that $\{\eta, 2, \overline{\nu}\} \equiv 0 \mod \eta \mu$ by (10.1) in p. 95 of [6]. Since $\overline{\nu} = \eta \sigma + \varepsilon$ and $\{\eta, 2, \eta\sigma\} = \{\eta, 2, \eta\}\sigma = 2\nu\sigma = 0$, we have the assertion. $\{\eta, 2, \kappa\} \equiv 0 \mod \eta \rho$ by Lemma 15.2 in p. 39 of [4].

 $\{\mu, 2, \varepsilon\} \equiv 0 \mod \eta \overline{\mu} \text{ since } (G_{1s}; 2) = \{\nu^*, \eta \overline{\mu}\}, \nu\{\mu, 2, \varepsilon\} = \{\nu, \mu, 2\} \varepsilon \in G_{13} \varepsilon = 0$ and $\nu \nu^* = \sigma^3 \neq 0$.

3.4. **Proof of iv**)

 $\{\sigma, \nu^2, 2\} = \{\sigma, \nu, 2\nu\} \equiv 0 \mod \sigma^2$ by the fact $\{\nu, \sigma, \nu\} = \sigma^2$ (see Example 4 in p. 85 of [6]) and by (3.10) in p. 33 of [6].

 $\{\nu, \sigma^2, 2\} \supset \{\nu, \sigma, 2\sigma\} = \{\nu, 2\sigma, \sigma\} = \nu^* \mod 2\nu^*$. See p. 153 of [6].

 $\{\sigma^2, \eta, \nu\} = \{\sigma, \eta\sigma, \nu\} = \bar{\sigma}$ by the definition of $\bar{\sigma}$ (see p. 189 of [6]).

 $\{\sigma, \varepsilon, \nu\}=0. \text{ It is sufficient to prove } \{\sigma, \overline{\nu}, \nu\}=\overline{\sigma}. \text{ By use of (3.7) in p. 33} \text{ of [6], } \{\{\nu, \sigma, \nu\}, \eta, \nu\}-\{\nu, \{\sigma, \nu, \eta\}, \nu\}+\{\nu, \sigma, \{\nu, \eta, \nu\}\} \ge 0. \text{ Since } \{\nu, \sigma, \nu\}=\sigma^2, \\ \{\sigma, \nu, \eta\}\subseteq (G_{12}; 2)=0 \text{ and } \{\nu, \eta, \nu\}=\overline{\nu} \text{ (see p. 53 of [6]), we have } \{\nu, \sigma, \overline{\nu}\}=\\ \{\sigma^2, \eta, \nu\}=\overline{\sigma}. \text{ By use of ii) of (3.9) in p. 33 of [6], } \{\nu, \sigma, \overline{\nu}\}-\{\sigma, \overline{\nu}, \nu\}+\\ \{\overline{\nu}, \nu, \sigma\} \ge 0. \text{ Since } \{\nu^2, 2, \eta\}\equiv\overline{\nu} \mod \eta\sigma \text{ and } \{\eta\sigma, \nu, \sigma\}=\sigma\{\eta, \nu, \sigma\}=0, \text{ we can put } \{\overline{\nu}, \nu, \sigma\}=\{\{\nu^2, 2, \eta\}, \nu, \sigma\}. \text{ By use of (3.7) of [6], } \{\{\nu^2, 2, \eta\}, \nu, \sigma\}+\\ \{\nu^2, \{2, \eta, \nu\}, \sigma\}+\{\nu^2, 2, \{\eta, \nu, \sigma\}\} \ge 0. \text{ So, we have } \{\overline{\nu}, \nu, \sigma\}=0. \text{ From this } \{\nu, \nu, \sigma\}=0. \text{ From this } \{\nu, \nu, \sigma\}=0. \text{ From this } \{\nu, \nu, \sigma\}=0. \text{ and } \{\nu, \nu, \nu, \sigma\}=0. \text{ and } \{\nu, \nu, \sigma\}=0. \text{ and } \{\nu, \nu, \sigma\}=0. \text{ and } \{\nu, \nu,$

and the above, we have $\{\sigma, \bar{\nu}, \nu\} = \{\nu, \sigma, \bar{\nu}\} = \bar{\sigma}$.

3.5. Proof of v)

 $\{\eta \varepsilon, \eta, 2\} = \{\eta^2 \sigma, \eta, 2\} \equiv \zeta \mod 2G_{11}$ by Lemma 9.1 in p. 91 of [6].

 $\{\eta^2 \rho, \eta, 2\} \equiv \xi \mod 2G_{19}$. See (3) of Lemma 4.2 in p. 278 of [5].

 $\{\nu, 8\sigma, 2\} \supset \{\nu, 8, 2\sigma\} = \zeta \mod 2G_{11}$. See p. 189 of [6].

 $\{\zeta, 8\sigma, 2\} \supset \{\zeta, 2, 8\sigma\} = \tilde{\zeta} \mod 2G_{19}$. See p. 189 of [6].

 $\{\sigma, 8\sigma, 2\} \supset \{\sigma, 2\sigma, 8\} \ni \rho \mod 2G_{15}$ by Lemma 10.9 in p. 110 of [6].

 $\{\varepsilon, 8\sigma, 2\} = \{\eta\sigma, 8\sigma, 2\} = \eta\rho.$ We have $\{\overline{\nu}, 8\sigma, 2\}\eta = \overline{\nu}\{8\sigma, 2, \eta\} \equiv \overline{\nu}\mu = 0$ mod $\{\overline{\nu}\eta^2\sigma, \overline{\nu}\eta\varepsilon\}=0.$ Since $G_{16}=\{\eta\rho, \eta^*\}=Z_2+Z_2$ and $\eta^2\rho$ and $\eta\eta^*$ are linearly independent in G_{17} , we obtain $\{\overline{\nu}, 8\sigma, 2\}\equiv 0$ mod $\{\overline{\nu}\eta\sigma, \overline{\nu}\varepsilon\}+2G_{17}=0.$ On the other hand, $\{\overline{\nu}, 8\sigma, 2\}=\{\varepsilon, 8\sigma, 2\}+\{\eta\sigma, 8\sigma, 2\}$ and $\{\eta\sigma, 8\sigma, 2\}=\eta\{\sigma, 8\sigma, 2\}=\eta\rho$. This leads us to the assertion.

 $\{8\sigma, 2, 8\sigma\} = 16\rho$. See p. 103 of [6].

3.6. Proof of vi)

 $\{\eta\kappa, \eta, 2\} = \nu\kappa$. By Lemma 15.1 in p. 39 of [4], $\{\eta\kappa, \eta, 2\} \subset \{\eta, \eta\kappa, 2\} \equiv \nu\kappa$ mod $\{\eta\eta^*, \eta^2\rho\}$. Since $\{\eta\kappa, \eta, 2\}$ is a coset of 0, we can put $\{\eta\kappa, \eta, 2\} =$ $\nu\kappa + x\eta\eta^* + y\eta^2\rho$, where x and y are 0 or 1 respectively. Multiply this equality by η , then we have x=0 since $\{\eta\kappa, \eta, 2\}\eta = \eta\kappa\{\eta, 2, \eta\} = 2\nu\eta\kappa = 0, \eta\nu\kappa = \eta^3\rho = 0$ and $\eta^2\eta^* \neq 0$ in G_{18} . Multiply it by η on the right, then we have y=0. For $\kappa\nu\eta = \kappa\{\nu, \eta, 2\}p = 0$ by i), $\eta^2\rho\eta = \{\eta^2\rho, \eta, 2\}p = \xi p \neq 0$ by v) and Theo. 3.1 of [5] and $\{\eta\kappa, \eta, 2\}\eta \subseteq \{\eta\kappa, \eta, \eta^2p\} \subseteq \{\kappa, 4\nu, \eta p\} \supseteq \{\kappa, 4, 0\} \equiv 0 \mod \kappa\pi_5(2) + G_{18}\eta p = 0$.

 $\{\kappa, 2, \nu^2\} = \eta \bar{\kappa}$. By the definition of $\bar{\kappa}$ (see p. 44 of [4]) and by the fact $\nu^2 = \{\eta, \nu, \eta\}$ (see Example 4 in p. 85 of [6]), we have $\eta \bar{\kappa} = \eta \{\nu, \bar{\eta}, \bar{\kappa}\} = \{\eta, \nu, \bar{\eta}\} \bar{\kappa} = \{\nu^2, 2, \kappa\}$.

3.7. Proof of vii)

 $\{2, \nu^2, \rho\} = 0.$ By use of (3.7) in p. 33 of [6], $\{2, \nu^2, \{\sigma, 2\sigma, 8\}\} + \{2, \{\nu^2, \sigma, 2\sigma\}, 8\} + \{\{2, \nu^2, \sigma\}, 2\sigma, 8\} \equiv 0.$ Since $\{\nu^2, \sigma, 2\sigma\} = \nu \{\nu, \sigma, 2\sigma\} = \nu \nu^* = \sigma^3, \{2, \nu^2, \sigma\} \equiv 0 \mod \sigma^2$ and $\{\sigma^2, 2\sigma, 8\} = \sigma \{\sigma, 2\sigma, 8\} = \sigma \rho = 0$, we have $\{2, \{\nu^2, \sigma, 2\sigma\}, 8\} = \{2, \sigma^3, 8\} = 4\{2, \sigma^3, 2\} = 0$ and $\{\{2, \nu^2, \sigma\}, 2\sigma, 8\} = 0.$ This leads us to the assertion.

 $\{\nu, \eta, \eta^2 \sigma\} = \{\nu, 4\nu, \sigma\} = 2\{\nu, 2\nu, \sigma\} = 0.$

 $\{\sigma, \nu, \zeta\}=0.$ By use of (3.7) in p. 33 of [6], $\{\sigma, \nu, \{\eta, \eta^2\sigma, 2\}\}=$ $\{\sigma, \{\nu, \eta, \eta^2\sigma\}, 2\}+\{\{\sigma, \nu, \eta\}, \eta^2\sigma, 2\} \ge 0.$ We have $\{\sigma, \{\nu, \eta, \eta^2\sigma\}, 2\}=\{\sigma, 0, 2\}=$ $=\sigma G_{15}+2G_{22}=0$ and $\{\{\sigma, \nu, \eta\}, \eta^2\sigma, 2\}=\{0, \eta^2\sigma, 2\}=2G_{22}=0.$ This leads us to the assertion.

3.8. Proof. of viii)

 $\{\bar{\eta}, \bar{\kappa}\nu, 2\} \equiv 0 \mod 2G_{20}$. By the proof of Lemma 15.3 in p. 43 and Lemma 15.4 in p. 44 of [4], $\{\eta\kappa, \eta, \nu\} = \pm 2\bar{\kappa}$. On the other hand $\{\eta\kappa, \eta, \nu\} = \{\bar{\eta}, i\eta\kappa, \nu\} = \{\bar{\eta}, 2\bar{\kappa}, \nu\} \supseteq \{\bar{\eta}, \bar{\kappa}, 2\nu\} \subseteq \{\bar{\eta}, \bar{\kappa}\nu, 2\}$. Therefore we have the assertion.

 $\{2, \nu(\overline{\kappa}), \tilde{\eta}\} \equiv 0 \mod 2G_{20}$. The proof is quite similar to the above and we omit it.

 $\{\overline{\nu^2}, \overline{\nu^2}, 2\} = \{2, \overline{\nu^2}, \overline{\nu^2}\} = \kappa$. The proof that $\{\overline{\nu^2}, \overline{\nu^2}, 2\} \equiv \{2, \overline{\nu^2}, \overline{\nu^2}\} \equiv \kappa \mod \sigma^2$ is quite similar to the dicussions in p. 40 of [4] and we omit it. By Lemma of $\{2, \sigma\{\overline{\nu^2}, \overline{\nu^2}, 2\} = 2\{\sigma, \overline{\nu^2}, \overline{\nu^2}\} = 2G_{21} = 0$ and $\{2, \overline{\nu^2}, \overline{\nu^2}\} \sigma = 2\{\overline{\nu^2}, \overline{\nu^2}, \sigma\} = 0$. So, the fact $\sigma^3 \neq 0$ leads us to the assertion.

 $\{\kappa, 8\sigma, 2\} = \{\kappa, 2, 8\sigma\} \equiv 0 \mod \eta^2 \bar{\kappa}.$ By use of Prop. 1.5 in p. 12 of [6], $\{\{2, \bar{\nu^2}, \tilde{\nu^2}\}, 2, 8\sigma\} + \{2, \{\bar{\nu^2}, \tilde{\nu^2}, 2\}, 8\sigma\} + \{2, \bar{\nu^2}, \{\tilde{\nu^2}, 2, 8\sigma\}\} \equiv 0.$ We have $\{\bar{\nu^2}, 2, 8\sigma\} \subseteq iG_{15} = \{i\rho, i\eta\kappa\} \text{ since } \{\nu^2, 2, 8\sigma\} = \{\nu, 2\nu, 8\sigma\} = 8\{\nu, 2\nu, \sigma\} = 0.$ It follows that $\{2, \bar{\nu^2}, i\rho\} \subseteq \{2, \nu^2, \rho\} = 0$ and $\{2, \bar{\nu^2}, i\eta\kappa\} \subseteq \{2, \nu^2, \eta\kappa\} = \{2, \nu^2, \eta\}\kappa = \varepsilon\kappa$ $= \eta^2 \bar{\kappa}.$ Therefore, we have $\{\kappa, 2, 8\sigma\} \equiv \{2, \kappa, 8\sigma\} = 8\{2, \kappa, \sigma\} = 8G_{22} = 0 \mod \eta^2 \bar{\kappa}.$

 $\{\sigma, \kappa, 2\} = \nu \overline{\sigma}$. Since $\{\nu, \eta, \eta^2 \sigma\} = 0$ and $\{\nu, \eta, \nu^3\} = \{\nu, \eta, \nu\} \nu^2 = \overline{\nu} \nu^2 = 0$, the tertial composition $\{\nu, \eta, 2, \overline{\nu}\}$ is a coset of 0. Obviously, $\sigma\{\nu, \eta, 2, \overline{\nu}\} = \sigma\{(\overline{\nu})_{\eta}, \tilde{2}_{\eta}, \overline{\nu}\} = \{\sigma, (\overline{\nu})_{\eta}, \tilde{2}_{\eta}\} \overline{\nu} = G_{13} \overline{\nu} = 0$. So, by the definition of κ , we can take $\kappa = \{(\overline{\nu})_{\eta}, \tilde{2}_{\eta}, \overline{\nu}\}$ (see p. 96 of [6]).

By use of Prop. 1.5 in p. 12 of [6], $\{\sigma, \{(\overline{\nu})_{\eta}, \tilde{2}_{\eta}, \bar{\nu}\}, 2\} + \{\sigma, (\overline{\nu})_{\eta}, \{\tilde{2}_{\eta}, \bar{\nu}, 2\}\} + \{\{\sigma, (\overline{\nu})_{\eta}, \tilde{2}_{\eta}\}, \bar{\nu}, 2\} \ge 0$. Since $\{\sigma, (\overline{\nu})_{\eta}, \tilde{2}_{\eta}\} \subseteq G_{13} \approx Z_3$, we have $\{\sigma, \kappa, 2\} = \{\sigma, (\overline{\nu})_{\eta}, \{\tilde{2}_{\eta}, \bar{\nu}, 2\}\}$. Since $\{2, \bar{\nu}, 2\} = \eta \bar{\nu} = \nu^3$ by Cor. 3.7 in p. 31 of [6], we can take $(\tilde{\nu}_{\eta})\nu^2 \equiv \{\tilde{2}_{\eta}, \bar{\nu}, 2\} \mod i_{\eta}\zeta$. Since $\{\sigma, (\overline{\nu})_{\eta}, i_{\eta}\zeta\} \subseteq \{\sigma, \nu, \zeta\} = 0$, we have $\{\sigma, \kappa, 2\} = \{\sigma, (\overline{\nu})_{\eta}, \tilde{\nu}_{\eta}\nu^2\} = \{\sigma, (\overline{\nu})_{\eta}, \tilde{\nu}_{\eta}, \nu^2\} = \{\sigma, \bar{\nu}, \nu\} = \{\sigma, \bar{\nu}, \nu\} = \nu \bar{\sigma}.$

3.9. Proof of ix)

 $\{2, 4\nu, \eta, 2\}=0$. By the definition of $\bar{\kappa}$, we have $0=8\bar{\kappa}=8\{(\bar{\nu})_{\eta}, \tilde{2}_{\eta}, \kappa\}\subseteq 2\{4(\nu)_{\eta}, \tilde{2}_{\eta}, \kappa\}=\{2, 4(\bar{\nu})_{\eta}, \tilde{2}_{\eta}\}\kappa=\{2, 4\nu, \eta, 2\}\kappa$. It is clear that $\{2, 4\nu, \eta, 2\}$ is a coset of 0. So, we have $\{2, 4\nu, \eta, 2\}\kappa=0$. Since $G_6=\{\nu^2\}$ and $\nu^2\kappa=4\bar{\kappa}\pm0$ (see Lemma 15.4 in p. 44 of [4]), we have the assertion.

 $\{2, \bar{\sigma}, \eta, 2\} \equiv 0 \mod \eta^2 \bar{\kappa}$. The proof that $\{2, \bar{\sigma}, \eta, 2\}$ is a coset of $\eta^2 \bar{\kappa}$ is left to the reader.

Since $\bar{\sigma} = \{\eta\sigma, \sigma, \nu\}$ and $\{\sigma, \nu, \eta\} = 0$, we can choose $(\bar{\sigma})_{\eta} \in \{\eta\sigma, \sigma, (\bar{\nu})_{\eta}\}$. So, we can put $\{2, \bar{\sigma}, \eta, 2\} \equiv \{2, (\bar{\sigma})_{\eta}, \tilde{2}_{\eta}\} = \{2, \{\eta\sigma, \sigma, (\bar{\nu})_{\eta}\}, \tilde{2}_{\eta}\} \mod \eta^{2}\bar{\kappa}$. By use of Prop. 1.5 of [6], $\{2, \{\eta\sigma, \sigma, (\bar{\nu})_{\eta}\}, \tilde{2}_{\eta}\} + \{2, \eta\sigma, \{\sigma, (\bar{\nu})_{\eta}, \tilde{2}_{\eta}\}\} + \{\{2, \eta\sigma, \sigma\}, (\bar{\nu})_{\eta}, \tilde{2}_{\eta}\} \equiv 0$. Since $\{\sigma, (\bar{\nu})_{\eta}, \tilde{2}_{\eta}\} \subseteq G_{13} \approx Z_{3}$, $\{2, \eta\sigma, \sigma\} \equiv \{2, \eta, \sigma^{2}\} = 0$ mod $\mu\sigma$ and $\{\mu\sigma, (\bar{\nu})_{\eta}, \tilde{2}_{\eta}\} = \mu\{\sigma, (\bar{\nu})_{\eta}, \tilde{2}_{\eta}\} = 0$, we have the assertion.

{2, η , $\bar{\sigma}$, 2} $\equiv 0 \mod \eta^2 \bar{\kappa}$. The proof is quite similar to the above and we omit it.

4. The ring structure of π_*

In this section we shall state our main theorems.

By use of the discussions in §2 and Theo. 3.3 of [5], we obtain the following

Theorem 4.1. A set of additive generators for π_* is as follows in dim ≤ 21 :

$$\begin{split} \delta, \ 1, \ \eta_1, \ \eta_2, \ \eta_1^2, \ \eta_2^2, \ \delta\nu_1, \ \eta_3, \ \nu_1, \ \eta_2\eta_3, \ \delta\nu_2\delta, \ \delta\nu_2, \ \nu_2\delta, \ \delta\sigma_1, \ \nu_2, \ \sigma_1, \ \delta A, \ A, \ \eta_1\nu_2, \ \eta_2\nu_2, \\ \eta_1\sigma_1, \ \eta_2\sigma_1, \ \eta_1^2\nu_2, \ \eta_2^2\nu_2, \ \eta_1^2\sigma_1, \ \eta_2\sigma_1, \ \eta_1A, \ \eta_2A, \ \eta_1^2A, \ \eta_2^2A, \ \eta_3\sigma_1, \ \eta_3(\nu_2+\sigma_1), \ \eta_3A, \ \nu_1A, \\ \eta_2\eta_3A, \ \delta\sigma_2\delta, \ \kappa_1\delta, \ \delta\sigma_2, \ \sigma_2\delta, \ \kappa_1, \ \nu_2^2, \ \delta\sigma_1A, \ \sigma_2, \ \sigma_1A, \ \eta_1\kappa_1, \ \eta_2\nu_2^2, \ \delta A^2, \ A^2, \ \kappa_2, \ \eta_1\sigma_2, \ \eta_2\sigma_2, \\ \eta_1\sigma_1A, \ \eta_2\sigma_1A, \ \eta_1A^2, \ \eta_2A^2, \ \delta\kappa_3, \ \kappa_3\delta, \ \eta_1^2\sigma_2, \ \eta_2^2\sigma_2, \ \eta_1^2\sigma_1A, \ \eta_2^2\sigma_1A, \ \delta\nu_1\sigma_2, \ \kappa_3, \ \nu_1\sigma_2, \ \eta_1^2A^2, \\ \eta_2^2A^2, \ \eta_3\sigma_2, \ \eta_3\sigma_1A, \ \delta\bar{\sigma}_1\delta, \ \eta_2\eta_3\sigma_2, \ \eta_3A^2, \ \nu_1A^2, \ \delta\bar{\sigma}_1, \ \bar{\sigma}_1\delta, \ \delta\bar{\kappa}_1, \ \bar{\sigma}_1, \ \bar{\kappa}_1, \ \eta_2\eta_3A^2, \ \delta\nu_2\kappa_1, \ \delta\sigma_1\sigma_2\delta, \\ \nu_2\kappa_1, \ \eta_1\bar{\kappa}_1, \ \eta_2\bar{\kappa}_1, \ \delta\sigma_1\sigma_2\delta, \ \delta\nu_1\bar{\sigma}_1\delta. \end{split}$$

The ring structure of π_* , in dim ≤ 21 , is given by the following

Theorem 4.2. The ring π_* , in dim ≤ 21 , is generated by δ , η_1 , η_2 , η_3 , ν_1 , ν_2 , σ_1 , A, κ_1 , σ_2 , κ_2 , κ_3 , $\bar{\sigma}_1$, $\bar{\kappa}_1$, with the following relations:

$$\begin{split} \delta^{2} = 0, \\ \delta\eta_{1} = \eta_{2}\delta = 0, \ \delta\eta_{2} = \eta_{1}\delta = 2.1, \\ \eta_{1}\eta_{2} = \eta_{2}\eta_{1} = 0, \ \delta\eta_{3} = \eta_{1}^{2}, \eta_{3}\delta = \eta_{2}^{2}, \nu_{1}\delta = \delta\nu_{1}, \\ \eta_{1}\eta_{3} = \eta_{3}\eta_{2} = \eta_{1}\nu_{1} = \nu_{1}\eta_{1} = \eta_{2}\nu_{1} = \nu_{1}\eta_{2} = 0, \ \eta_{3}\eta_{1} = \eta_{2}\eta_{3}, \\ \eta_{2}^{2}\eta_{3} = 0, \\ \eta_{3}^{2} = \eta_{3}\nu_{1} = \nu_{1}\eta_{3} = 0, \ \nu_{1}^{2} = \delta\nu_{2} + \nu_{2}\delta, \ \sigma_{1}\delta = \delta\sigma_{1}, \\ A\delta = \delta A + 2x\sigma_{1} + 2y\nu_{2}, \\ \nu_{2}\eta_{1} = \eta_{1}\nu_{2}, \nu_{2}\eta_{2} = \eta_{2}\nu_{2}, \ \sigma_{1}\eta_{1} = \eta_{1}\sigma_{1}, \ \sigma_{1}\eta_{2} = \eta_{2}\sigma_{1}, \\ A\eta_{1} = \eta_{1}A + x\eta_{1}^{2}\sigma_{1} + y\eta_{1}^{2}\nu_{2}, \ A\eta_{2} = \eta_{2}A + x\eta_{2}^{2}\sigma_{1} + y\eta_{2}^{2}\nu_{2}, \\ \nu_{2}\eta_{3} = \pm \eta_{3}\nu_{2}, \ \sigma_{1}\eta_{3} = \pm \eta_{3}\sigma_{1}, \ \nu_{2}\nu_{1} = \nu_{1}\nu_{2} = \eta_{3}(\nu_{2}\pm\sigma_{1}), \ \nu_{1}\sigma_{1} = \sigma_{1}\nu_{1} = 0. \\ \eta_{2}\eta_{3}\sigma_{1} = A\nu_{1} = \nu_{1}A, \ A\eta_{3} = \pm \eta_{3}A + (x+y)\nu_{1}A, \\ \delta\kappa_{1} = 0, \ \delta\nu_{2}^{2} = \kappa_{1}\delta, \\ \sigma_{1}\nu_{2} = \nu_{2}\sigma_{1} = 0, \ \sigma_{1}^{2} = \delta\kappa_{2} + \sigma_{2}\delta, \\ \eta_{2}\kappa_{1} = \kappa_{1}\eta_{2} = 0, \ \eta_{1}\nu_{2}^{2} = \delta\kappa_{2} = \kappa_{1}\eta_{1} = \eta_{1}\kappa_{1}, \ \kappa_{2}\delta = \eta_{2}\nu_{2}^{2}, \ A\sigma_{1} = \pm\sigma_{1}A, \ A\nu_{2} \equiv \nu_{2}A \equiv \\ \sigma_{1}A \ \operatorname{mod} \left\{ 2\sigma_{1}A, \ \eta_{1}\kappa_{1}, \ \eta_{2}\nu_{2}^{2} \right\}, \\ \sigma_{2}\eta_{1} \equiv \eta_{1}\sigma_{2} \ \operatorname{mod} 2\kappa_{2}, \ \sigma_{2}\eta_{2} \equiv \eta_{2}\sigma_{2} \ \operatorname{mod} 2\kappa_{2}, \\ \eta_{1}\kappa_{2} = \kappa_{2}\eta_{2} = \kappa_{1}\eta_{3} = 0, \ \kappa_{1}\nu_{1} \equiv \nu_{1}\kappa_{1} = \delta\kappa_{3} \ \operatorname{mod} \delta\nu_{1}\sigma_{2}, \ \eta_{3}\nu_{2}^{2} = \eta_{3}\kappa_{1} = \eta_{2}\kappa_{2} = \kappa_{2}\eta_{1} \equiv \delta\kappa_{3} + \\ \kappa_{3}\delta \ \operatorname{mod} \delta\nu_{1}\sigma_{2}, \\ \sigma_{2}\eta_{3} \equiv \pm \eta_{3}\sigma_{2} \ \operatorname{mod} 2\eta_{3}\sigma_{1}A, \ \sigma_{2}\nu_{1} \equiv \nu_{1}\sigma_{2} \ \operatorname{mod} \left\{ 2\eta_{3}\sigma_{2}, \ 2\eta_{3}\sigma_{1}A, \ \delta\sigma_{1}\delta \right\}, \\ \eta_{1}\kappa_{3} = \kappa_{3}\eta_{3} = \eta_{2}\kappa_{3} = \kappa_{3}\eta_{2} = \eta_{3}\kappa_{2} = \kappa_{2}\eta_{3} = \nu_{1}\kappa_{2} = \kappa_{2}\nu_{1} = 0, \ \kappa_{1}\delta_{1}\delta_{1}\delta, \\ \eta_{1}\kappa_{3} = \kappa_{3}\eta_{3} = \eta_{2}\sigma_{3} = \sigma_{1}\eta_{1} = 0, \ \eta_{1}\sigma_{1} \equiv \sigma_{1}\eta_{2} \equiv 0 \ \operatorname{mod} \delta\nu_{1}\sigma_{1}\delta, \ \sigma_{1}\kappa_{1} = \kappa_{1}\sigma_{1} = \delta\nu_{1}\sigma_{1}\delta, \\ \kappa_{1}\nu_{2} \equiv \kappa_{1}\eta_{1} \equiv \eta_{1}\kappa_{1} \ \operatorname{mod} \delta\nu_{1}\sigma_{1}\delta, \ \kappa_{1}\eta_{2} \equiv \eta_{2}\kappa_{1} \ \operatorname{mod} \delta\nu_{1}\sigma_{1}\delta, \ \kappa_{1}\nu_{2} \equiv \nu_{2}\kappa_{1} + \eta_{2}\kappa_{1} \ \operatorname{mod} \delta\nu_{1}\sigma_{1}\delta, \\ \nu_{1}\kappa_{3} \equiv \nu_{2}\kappa_{1}$$

Proof. The relations hold by use of our propositions in §2.

To complete the proof of this theorem, we must construct the table obtained

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Table of relations, I.

	δ	η_1	7/2	η_3	ν ₁	ν2	σ_1	A	κ_1	σ_2	κ_2	κ3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
δ	0	0	2	η_1^2	$\delta \nu_1$	$\delta \nu_2$	$\delta \sigma_1$	δA	0	$\delta \sigma_2$	$\eta_1 \kappa_1$	$\delta \kappa_3$	$\delta \bar{\sigma}_1$	$\delta \bar{\kappa}_1$
η_1	2	η_1^2	0	0	0	$\eta_1 \nu_2$	$\eta_1 \sigma_1$	$\eta_1 A$	$\eta_1 \kappa_1$	$\eta_1 \sigma_2$	0	0	$\begin{array}{c} 0 \\ \text{mod} \\ \delta \nu_1 \overline{\sigma}_1 \delta \end{array}$	$\eta_1 \bar{\kappa}_1$
η_2	0	0	η_2^2	$\eta_2\eta_3$	0	$\eta_2 \nu_2$	$\eta_2 \sigma_1$	$\eta_2 A$	0	$\eta_2 \sigma_2$	$\delta \kappa_3 + \kappa_3 \delta \mod \delta \nu_1 \sigma_2$	0	0	$\eta_2 \bar{\kappa}_1$
η_1^2	0	$2\eta_3$	0	0	0	$\eta_1^2 \nu_2$	$\eta_1^2 \sigma_1$	$\eta_1^2 A$	$2\kappa_2$	$\eta_1^2 \sigma_2$	0	0	0	$\eta_1^2 ar{\kappa}_1$
η_2^2	0	0	$2\eta_3$	0	0	$\eta_2^2 \nu_2$	$\eta_2^2 \sigma_1$	$\eta_2^2 A$	0	$\eta_2^2 \sigma_2$	0	0	0	$\eta_2^2 \bar{\kappa}_1$
$\delta \nu_1$	0	0	0	0	$\delta \nu_2 \delta$	$\eta_1^2(\nu_2 + \sigma_1)$	0	$2\eta_3\sigma_1$	0	$\delta \nu_1 \sigma_2$	0	$\delta \nu_2 \kappa_1$	$\delta \nu_1 \bar{\sigma}_1$	
η_3	η_2^2	$\eta_2\eta_3$	0	0	0	$\eta_3 \nu_2$	$\eta_3 \sigma_1$	$\eta_3 A$	$\eta_2 \kappa_2$	$\eta_3 \sigma_2$	0	0		$\eta_3 \bar{\kappa}_1$
ν ₁	$\delta \nu_1$	0	0	0	$\delta \nu_2 + \nu_2 \delta$	$\eta_3(\nu_2\pm\sigma_1)$	0	$\nu_1 A$	δκ3	$\nu_1 \sigma_2$	0	$ \begin{array}{c} \overline{\nu_2 \kappa_1 + \eta_1 \overline{\kappa}_1} \\ \text{mod} \\ \{ \delta \sigma_1 \sigma_2, \\ \delta \nu_1 \overline{\sigma}_1 \delta \} \end{array} $	$ u_1 \overline{\sigma}_1$	
$\eta_2\eta_3$	$2\eta_3$	0	0	0	0	$\nu_1 A$	$\nu_1 A$	$\eta_2\eta_3 A$	0	$\eta_2\eta_3\sigma_2$	0	0	0	
$\delta \nu_z \delta$	0	0	0	0	0	0	0	0	0	$\delta\sigma_1\sigma_2\delta$	0			
$\delta \nu_2$	δν ₂ δ	0	$2\nu_2$	$\eta_1^2 \nu_2$	$\eta_1^2(\nu_2 + \sigma_1)$	κιδ	0	$\delta\sigma_1 A$	$\delta \nu_2 \kappa_1$	$\delta \sigma_1 \sigma_2 \ \mathrm{mod} \ \delta u_1 \overline{\sigma}_1 \delta$	$\eta_1^2 ar \kappa_1$			
ν ₂ δ	0	0	$2\nu_2$	$\eta_1^2 \nu_2$	$\eta_2^2(\nu_2+\sigma_1)$	$\kappa_1\delta$	0	$\delta\sigma_1 A$	0	$\sigma_1 \sigma_2 \delta$ mod $\delta \nu_1 \overline{\sigma}_1 \delta$	$\eta_1^2 ar \kappa_1$			
$\delta\sigma_1$	0	0	$2\sigma_1$	$\eta_1^2 \sigma_1$	0	0	$\delta \sigma_2 \delta$	$\delta\sigma_1 A$	0	$\delta\sigma_1\sigma_2$	0			
ν_2	$\nu_2 \delta$	$\eta_1 \nu_2$	$\eta_2 \nu_2$	$\pm \eta_3 \nu_2$	$\nu_1 \nu_2$	ν_2^2	0	$\sigma_1 A$	$\nu_2 \kappa_1$					

Table of rleatiotions, II.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
σ_1	$\delta\sigma_1$	$\eta_1 \sigma_1$	$\eta_2 \sigma_1$	$\pm \eta_3 \sigma_1$	0	0	$\delta \sigma_2 \ + \sigma_2 \delta$	$\sigma_1 A$	$\delta \nu_1 \bar{\sigma}_1 \delta$	$\sigma_1 \sigma_2$				
δA	0	0	2 <i>A</i>	$\eta_1^2 A + 2 \ (x+y)\eta_3\sigma_1$	$2\eta_3\sigma_1$	$\delta \sigma_1 A$	$\delta \sigma_1 A$	δA^2	0					
A	$\delta A + 2x\sigma_1 + 2y u_2$	$\eta_1A+x\eta_1^2\sigma_1\ +y\eta_1^2 u_2$	$\eta_2 A + x \eta_2^2 \sigma_1 \ + y \eta_2^2 u_2$	$\frac{\pm \eta_3 A +}{(x+y)\nu_1 A}$	$\nu_1 A$	$\begin{array}{c} \pm \sigma_1 A \\ \text{mod} \\ \{\eta_1 \kappa_1, \eta_2 \nu_2^2\} \end{array}$	$\pm \sigma_1 A$	A ²						
$\eta_1 \nu_2$	$2\nu_2$	$\eta_1^2 \nu_2$	0	0	0	$\eta_1 \kappa_1$	0	$\eta_1 \sigma_1 A \mod 2\kappa_2$	$\eta_1^2 \bar{\kappa}_1$					
$\eta_2 \nu_2$	0	0	$\eta_2^2 \nu_2$	$\nu_1 A$	0	$\eta_2 u_2^2$	0	$\eta_2 \sigma_1 A \mod 2\kappa_2$	0					
$\eta_1 \sigma_1$	$2\sigma_1$	$\eta_1^2 \sigma_1$	0	0	0	0	0	$\eta_1 \sigma_1 A$	0					
$\eta_2 \sigma_1$	0	0	$\eta_2^2 \sigma_1$	$\nu_1 A$	0	0	0	$\eta_2 \sigma_1 A$	0					
$\eta_1^2 u_2$	0	$2\eta_3\sigma_1$	0	0	0	$2\kappa_2$	0	$\eta_1^2 \sigma_1 A$						
$\eta_2^2 \nu_2$	0	0	$2\eta_3\sigma_1$	0	0	$2\kappa_2$	0	$\eta_2^2 \sigma_1 A$	0					
$\eta_1^2 \sigma_1$	0	$2\eta_3\sigma_1$	0	0	0	0	0	$\eta_1^2 \sigma_1 A$	0					
$\eta_2^2 \sigma_1$	0	0	$2\eta_3\sigma_1$	0	0	0	0	$\eta_2^2 \sigma_1 A$	0					
$\eta_1 A$	2 <i>A</i>	$\eta_1^2 A + 2\eta_3 \ (x\sigma_1 + y\nu_2)$	0	0	0	$\eta_1 \sigma_1 A \mod 2\kappa_2$	$\eta_1 \sigma_1 A$	$\eta_1 A^2$	-	-				
$\eta_2 A$	0	0	$\eta_2^2 A + 2\eta_3 \ (x\sigma_1 + y\nu_2)$	$\eta_2\eta_3 A$	0	$\eta_2 \sigma_1 A \mod 2\kappa_2$	$\eta_2 \sigma_1 A$	$\eta_2 A^2$						
$\eta_1^2 A$	0	$2\eta_3 A$	0	0	0	$\eta_1^2 \sigma_1 A$	$\eta_1^2 \sigma_1 A$	$\eta_1^2 A^2$						

	δ	η_1	η_2	η_3	ν ₁	ν2	σ1	A	κ_1	σ_2	κ2	κ3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
$\eta_2^2 A$	0	0	$2\eta_3 A$	0	0	$\eta_2^2 \sigma_1 A$	$\eta_2^2 \sigma_1 A$	$\eta_2^2 A$						
$\eta_3 \sigma_1$	$\eta_2^2 \sigma_1$	$\nu_1 A$	0	0	0	0	0	$\eta_3 \sigma_1 A$	0					
$\eta_3(\nu_2+\sigma_1)$	$\eta_2^2(\nu_2\!+\!\sigma_1)$	0	0	0	0	$\eta_2 \kappa_2$	0	$0 \mod 2\eta_3\sigma_1 A$						
$\eta_3 A$	$\frac{\eta_2^2 A + 2\eta_3}{(x\sigma_1 + y\nu_2)}$	$\eta_2\eta_3 A$	0	0	0	$\pm \eta_3 \sigma_1 A$	$\pm \eta_3 \sigma_1 A$	$\eta_3 A^2$						
$\nu_1 A$	$2\eta_3\sigma_1$	0	0	0	0	0	0	$ u_1 A^2 $						
$\eta_2\eta_3A$	$2\eta_3 A$	0	0	0	0	$ u_1 A^2 $	$\nu_1 A^2$	$\eta_2\eta_3A^2$						
$\delta\sigma_2\delta$	0	0	0	0	0	0	$\delta \sigma_1 \sigma_2 \delta$							
$\kappa_1\delta$	0	0	0	$2\kappa_2$	2 <i>k</i> ₂	$\delta \nu_2 \kappa_1 + 2 \bar{\kappa}_1$	0							
$\delta\sigma_2$	$\delta \sigma_2 \delta$	0	$2\sigma_2$	$\eta_1^2 \sigma_2$	$\delta \nu_1 \sigma_2$	$\delta \sigma_1 \sigma_2 \\ \text{mod } \delta \nu_1 \overline{\sigma}_1 \delta$	$\delta \sigma_1 \sigma_2$ mod $\delta \nu_1 \overline{\sigma}_1 \delta$							
$\sigma_2 \delta$	0	0	$2\sigma_2$	$\eta_1^2 \sigma_2$	$\delta \nu_1 \sigma_2$	$\delta \sigma_1 \sigma_2$ mod $\delta \nu_1 \overline{\sigma}_1 \delta$	$\sigma_1 \sigma_2 \delta$ mod $\delta \nu_1 \overline{\sigma}_1 \delta$							
κ1	κιδ	$\eta_1 \kappa_1$	0	0	$\delta \kappa_3 \mod \delta \nu_1 \sigma_2$	$\eta_1 \bar{\kappa}_1$ mod $\delta \nu_1 \bar{\sigma}_1 \delta$	$\delta \nu_1 \bar{\sigma}_1 \delta$							
ν_2^2	κιδ	$\eta_1 \kappa_1$	$\eta_2 \nu_2^2$	$\eta_2 \kappa_2$	$\eta_2 \kappa_2$	$\frac{\nu_2 \kappa_1 + \eta_2 \bar{\kappa}_1}{\mod \delta \nu_1 \bar{\sigma}_1 \delta}$	0							
$\delta\sigma_1 A$	0	0	$2\sigma_1 A$	$\eta_1^2 \sigma_1 A$	0	0								
σ2	$\sigma_2 \delta$	$\eta_1 \sigma_2 \mod 2\kappa_2$	$\eta_2 \sigma_2 \mod 2\kappa_2$	$\eta_3\kappa_2 \ \mathrm{mod} \ 2\eta_3\sigma_1 A$										

Table of relations, III.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ1	A	κ1	σ_2	κ2	κ3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
$\sigma_1 A$	$\delta\sigma_1 A$	$\eta_1 \sigma_1 A$	$\eta_2 \sigma_1 A$	$\pm\eta_3\sigma_1A$	0	0								
$\eta_1 \kappa_1$	0	$2\kappa_2$	0	0	0	$\eta_1^2 \bar{\kappa}_1$	0							
$\eta_2 \nu_2^2$	0	0	$2\kappa_2$	0	0	$\eta_2^2 \bar{\kappa}_1$	0							
δA^2	0	0	$2A^2$	$\eta_1^2 A^2$	$2\eta_3\sigma_1A$		-							
A^2	δA^2	$\eta_1 A^2$	$\eta_2 A^2$	$\eta_3 A^2$	$\nu_1 A^2$									
κ2	$\eta_2 \nu_2^2$	$\eta_2 \kappa_2$	0	0	0		-							
$\eta_1 \sigma_2$	$2\sigma_2$	$\eta_1^2 \sigma_2$	0	0	0									
$\eta_2 \sigma_2$	0	0	$\eta_2^2 \sigma_2$	$\eta_2\eta_3\sigma_2$	0									
$\eta_1 \sigma_1 A$	$2\sigma_1 A$	$\eta_1^2 \sigma_1 A$	0	0	0	0								
$\eta_2 \sigma_1 A$	0	0	$\eta_2^2 \sigma_1 A$	$\nu_1 A^2$	0	0	_							
$\eta_1 A^2$	$2A^2$	$\eta_1^2 A^2$	0	0	0		-							
$\eta_2 A^2$	0	0	$\eta_2^2 A^2$	$\eta_2\eta_3A^2$	0									
$\delta \kappa_3$	$2\kappa_2$	0	0	0	$\delta \nu_2 \kappa_1$ mod $\delta \sigma_1 \sigma_2 \delta$			P. 1993. 1994		1				
κ ₃ δ	0	0	0	0	$\delta \nu_2 \kappa_1 \\ \mod \delta \sigma_1 \sigma_2 \delta$									

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ_3	$\bar{\sigma}_1$	<i>k</i> ₁
$\eta_1^2 \sigma_2$	0	$2\eta_3\sigma_2$	0	0	0									
$\eta_2^2 \sigma_2$	0	0	$2\eta_3\sigma_2$	0	0									
$\eta_1^2 \sigma_1 A$	0	$2\eta_3\sigma_1A$	0	0	0									
$\eta_2^2 \sigma_1 A$	0	0	$2\eta_3\sigma_1A$	0	0									
$\delta \nu_1 \sigma_2$	0	0	0	0	$\delta\sigma_1\sigma_2\delta$									
κ3	κ ₃ δ	0	0	0	$\begin{array}{c}\nu_{2}\kappa_{1}+\eta_{1}\bar{\kappa}_{1}\\ \mod\\ \{\delta\sigma_{1}\sigma_{2},\\\sigma_{1}\sigma_{2}\delta,\delta\nu_{1}\bar{\sigma}_{1}\delta\}\end{array}$									
$\nu_1 \sigma_2$	$\delta \nu_1 \sigma_2$	0	0	0	$\begin{array}{c} 0 \\ \mathrm{mod} \ \delta \nu_1 \overline{\sigma} \delta \end{array}$									
$\eta_1^2 A^2$	0	$2\eta_3 A^2$	0	0	0									
$\eta_2^2 A^2$	0	0	$2\eta_3 A^2$	0	0									
$\eta_3 \sigma_2$	$\eta_2^2 \sigma_2$	$\eta_2\eta_3\sigma_2$	0	0	0									
$\eta_3 \sigma_1 A$	$\eta_2^2 \sigma_1 A$	$\nu_1 A^2$	0	0	0									
$\delta \sigma_1 \delta$	0	0	0	0	$\delta u_1 \bar{\sigma}_1 \delta$									
$\eta_2 \eta_3 \sigma_2$	$2\eta_3\sigma_2$	0	0	0	0									
$\eta_3 A^2$	$\eta_2^2 A^2$	$\eta_2\eta_3A^2$	0	0	0									

Table of relations, V.

Table of relations, VI.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	κ1	σ_2	κ2	κ_3	$\bar{\sigma}_1$	<i>κ</i> ₁ .
$\nu_1 A^2$	$2\eta_3\sigma_1A$	0	0	0	0	0	0						
$\delta \bar{\sigma}_1$	$\delta \bar{\sigma}_1$	0	0	0									
$\bar{\sigma}_1 \delta$	0	0	0	0									
$\delta \bar{\kappa}_1$	0	0	$2\bar{\kappa}_1$	$\eta_1^2 ec \kappa_1$	-								
$\bar{\sigma}_1$	$\bar{\sigma}_1 \delta$	0	$\begin{array}{c} 0\\ \mathrm{mod} \ \delta \nu_1 \overline{\sigma}_1 \delta \end{array}$										
$\bar{\kappa}_1$	$\delta \bar{\kappa}_1$	$\eta_1 \bar{\kappa}_1 \\ \mod \delta \nu_1 \bar{\sigma}_1 \delta$	$\eta_2 \bar{\kappa}_1 \\ \mod \delta \nu_1 \bar{\sigma}_1 \delta$					 					
$\eta_2\eta_3A^2$	$2\eta_3 A^2$	0	0	0	0								
$\delta \nu_2 \kappa_1$	0	0	0	0									
$\delta \sigma_1 \sigma_2 \delta$	0	0	0	0	0	0							
$\nu_2 \kappa_1$	$\delta \nu_2 \kappa_1 + 2 \bar{\kappa}_1$	$\eta_1^2 ar{\kappa}_1$	0	0	-								
$\eta_1 \bar{\kappa}_1$	$2\bar{\kappa}_1$	$\eta_1^2 ar \kappa_1$	0	0				 					
$\eta_2 ar\kappa_1$	0	0	$\eta_2^2 ar{\kappa}_1$	0				 					
$\delta\sigma_1\sigma_2$	$\delta\sigma_1\sigma_2\delta$	0	· · · · · · · · · · · · ·										
$\sigma_1 \sigma_2 \delta$	0	0											
$\delta u_1 ar{\sigma}_1 \delta$	0	0	0	0				 					

from multiplying all of the additive generators of π_* except for the unit by themselves. But this whole work is too long and tedious. So, we write down a part of the table which plays an essential role for this work. Really, it finishes the proof.

REMARK. In Theo. 4.2 we can take y=0 by use of the results that $4\nu\bar{\kappa}=\eta^3\bar{\kappa}\pm 0$ in G_{23} .

Finally we mention the following relations in secondary or tertiary compositions.

Proposition 4.3.

$\eta_1 = \{\delta, \delta, \eta_2^2\},$	$\eta_2 = \{\eta_1^2, \delta, \delta\},$
$\pm\eta_{3}=\{\eta_{2}, \eta_{1}, 2\},\$	$\nu_2 \in \{\eta_1, 2, \nu_1, \eta_1\},$
$\kappa_1 \in \{\delta, \delta, \eta_2 \nu_2^2\},$	$\kappa_2 \in \{\eta_2, \kappa_1, 2\},$
$\kappa_3 \in \{\nu_1, 2, \nu_2^2\},$	$\bar{\kappa}_1 \in \{\nu_1, \eta_1, 2, \nu_2^2 + \kappa_1\}.$

Proof. We shall prove the first and the fourth relation.

Clearly, $\{\delta, \delta, \eta_2^2\} = \{ip, ip, \eta_2^2\} \supseteq i\{p, i, \eta^2 p\} = i\{p, i, 2\overline{\eta}\} \supseteq i\{p, i, 2\}\overline{\eta} = \eta_1$ since $\{p, i, 2\} \equiv 1 \mod 2G_0$. As $\{\delta, \delta, \eta_2^2\}$ is a coset of $\delta \pi_2 + \pi_{-1}\eta_2^2 = 0$, we obtain the first.

Since $\{\eta_1, 2, \nu_1\}$ is a coset of $\eta_1 \pi_4 + \pi_2 \nu_1 = \{\delta \nu_1^2\} = \pi_5$, we have $\eta_1(\overline{2})_{\nu_1} \in \{\eta_1, 2, \nu_1\} p_{\nu_1} = \{\delta \nu_1^2 p_{\nu_1}\} = 0$ for any $\overline{(2)}_{\nu_1} \in \operatorname{Ext}_{\nu_1} 2$.

As $\{2, \nu_1, \eta_1\} = \{2, \nu_1, i\bar{\eta}\} \subseteq \{2, i\nu, \bar{\eta}\} \supseteq \{0, \nu, \bar{\eta}\} = \pi_4^*(2)\bar{\eta} = \{\eta_2^2\eta_3\} = 0$ and $\{2, i\nu, \bar{\eta}\}$ is a coset of $2\pi_5 + \pi_4^*(2)\bar{\eta} = 0$, we have $\{2, \nu_1, \eta_1\} = 0$. From this $(\overline{2})_{\nu_1}(\widetilde{\eta})_{\nu_1} = 0$ for any $(\widetilde{\eta}_1)_{\nu_1} \in \text{Coext}_{\nu_1} \eta_1$.

Now we define $\{\eta_1, 2, \nu_1, \eta_1\} \equiv \{\eta_1, (\overline{2})_{\nu_1}, (\overline{\eta_1})_{\nu_1}\} \mod Q = [S^{n+2}M \cup_{\nu_1} CS^{n+5}M, S^nM](\widetilde{\eta_1})_{\nu_1} + (\overline{\eta_1})_2 [S^{n+6}M, S^nM \cup_2 CS^nM] \text{ for some } (\overline{\eta_1})_2 \in \text{Ext}_2 \eta_1, \text{ where } n \text{ is sufficiently large.}$

It is easy to check the group $Q = \{2\sigma_1, 2\nu_2\}$. So, we have $p\{\eta_1, 2, \nu_1, \eta_1\} = \{p, \eta_1, \overline{(2)}_{\nu_1}\}(\widetilde{\eta_1})_{\nu_1} \subseteq \{\pm \overline{\eta}, \nu_1, \eta_1\} = \{\pm \overline{\eta}, i\nu, \overline{\eta}\} = \{\eta, \nu, \overline{\eta}\}$ since $\{p, \eta_1, \overline{(2)}_{\nu_1}\}i_{\nu_1} \subseteq \{p, \eta_1, 2\} = \pm \overline{\eta}, \overline{\eta}\nu_1 = 0$ and $\{\pm \overline{\eta}, i\nu, \overline{\eta}\}$ is a coset of $(\pm \overline{\eta})\pi_5 + G_5\overline{\eta} = 0$.

It is easy to check $\{\eta, \nu, \overline{\eta}\} = \overline{\nu^2}$ for $\overline{\nu^2} \in \text{Ext } \nu^2$ which satisfies (2.7). Hence we obtain $\{\eta_1, 2, \nu_1, \eta_1\} \equiv \pm \nu_2 \mod 2\sigma_1$.

5. Direct summands of π_k

The object of this section is to improve Theo. 5.1 of [5]. We shall change the notations μ_{ss+1} and j_{ss+3} of §4 of [5] into μ_s and ζ_s respectively.

In [1] Adams proved that $A^s \neq 0$ for $s \ge 1$ and defined $\alpha_s \in (G_{ss-1}; 2)$ as follows:

(5.1)
$$\alpha_s = pA^s i,$$

which is of order 2 and satisfies

(5.2)
$$e_{c}(\alpha_{s}) \equiv \frac{1}{2} \mod 1 \text{ (see [1])}.$$

Choose

(5.3)
$$\overline{\alpha_s} = pA^s = \overline{8\sigma}A^{s-1} \in \operatorname{Ext} \alpha_s \quad and \quad \widetilde{\alpha_s} = A^s i = A^{s-1} \widetilde{8\sigma} \in \operatorname{Coext} \alpha_s.$$

It is clear that

(5.4)
$$\alpha_1 = 8\sigma \quad and \quad \alpha_{s+t} \in \{\alpha_s, 2, \alpha_t\}.$$

By use of α_s Adams defined $\mu_s \in (G_{ss+1}; 2)$ as follows:

(5.5)
$$\mu_s \equiv \bar{\eta} \widetilde{\alpha}_s \mod \eta G_{ss}$$

By (5.5) and i) of (3.9) of [6], we have $\mu_s \equiv \{\eta, 2, \alpha_s\} = \{\alpha_s, 2, \eta\} \equiv \overline{\alpha_s} \tilde{\eta} \mod \eta G_{ss}$. So, we can choose $\overline{\mu_s} \in \text{Ext } \mu_s$ and $\overline{\mu_s} \in \text{Coext } \mu_s$ as follows:

(5.6)
$$\mu_s \equiv \overline{\eta} A^s \mod G_{ss} \overline{\eta} \quad and \quad \mu_s \equiv A^s \overline{\eta} \mod \overline{\eta} G_{ss}.$$

Lemma 5.1.

i) α_s is divisible by 8.

ii) $J(\beta)\alpha_t = 0$ for $\beta \in \pi_{ss-1}(SO)$.

Proof. Let $S^n \cup_{s} e^{n+1}$ be a complex obtained from an *n*-sphere S^n by attaching an (n+1)-dimensional cell e^{n+1} , using a map of degree 8, where *n* is sufficiently large. Let $i': S^n \to S^n \cup_{s} e^{n+1}$ be a natural inclusion.

Obviously α_s is divisible by 8 if and only if $i'\alpha_s=0$.

By induction assume $i'\alpha_{s-1}=0$, then $i'\alpha_s \in i'\{\alpha_{s-1}, 2, 8\}=\{i, \alpha_{s-1}, 2\}8\sigma=8(\{i', \alpha_{s-1}, 2\}\sigma)$. By use of Theo. 4.4 in p. 324 of [7], this consists of 0. Therefore we have i).

Next we shall prove ii). Toda defined the element $\sigma'' \in \pi_{12}(S^5)$ which is of order 2 and satisfies $S^{\infty}\sigma''' = 8\sigma$ (see *p*. 48 of [6]). By use of σ''' , we can define an element $\alpha'_s \in \pi_{ss+4}(S^5)$ for $s \ge 1$ as follows:

(5.7)
$$\alpha'_1 = \sigma'''$$
 and $\alpha'_s \in \{\alpha'_{s-1}, 2, 8\sigma\}$ for $s \ge 2$.

Clealy, α'_s is of order 2 and we can choose α'_s such that

$$(5.8) S^{\infty} \alpha'_s = \alpha_s.$$

Now
$$J(\beta)\alpha_t = J(\beta S^{ss-6}\alpha'_t) = 0$$
 since $\beta S^{ss-6}\alpha'_t \in \pi_{s(s+t)-2}(SO) = 0$.

By use of i) of this lemma,

(5.9)
$$\mu_s \in \{\eta, 2, \alpha_s\} \subseteq \{\eta, \alpha_s, 2\} \mod 2G_{ss+1} + \eta G_{ss}.$$

Therefore, we have

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By use of ii) of Lemma 5.1, we define an element $\rho_s \in (G_{ss+7}; 2) \cap J(\pi_{ss+7}(SO))$ as follows:

(5.11)
$$\rho_0 = \sigma \quad and \quad \rho_s \equiv \{\rho_{s-1}, 8\sigma, 2\} \mod 2G_{ss+7} \quad for \ s \ge 1$$

We can take ρ_s in the *J*-image since $J\{\beta', 8\sigma, 2\} \subseteq \{\rho_{s-1}, 8\sigma, 2\}$ if $J(\beta') = \rho_{s-1}$ for $\beta' \in \pi_{ss-1}$ (SO).

We note the following: Since $\rho_s \sigma = 0$ for $s \ge 1$, we have, by the facts that $\sigma^2 \eta = \sigma \varepsilon = 0$ and $\{\eta \sigma, 2, 8\sigma\} = \{\varepsilon, 2, 8\sigma\}$,

(5.12)
$$\rho_s \mathcal{E} = \rho_s \sigma \eta = 0 \quad \text{for } s \ge 0 .$$

Lemma 5.2.

i) $i\rho_{s+t} = \widetilde{\alpha_t}\rho_s \text{ and } \rho_{s+t}p = \rho_s \overline{\alpha_t}$. ii) $\rho_{s+t} \equiv \{\rho_s, \alpha_t, 2\} \mod \rho_s G_{st} + 2G_{s(s+t)+7}$. This is obvious by use of (5.3), (5.11) and (5.12).

By (5.5) and i) of Lemma 5.2, we have

(5.13)
$$\eta \rho_{s+t} \equiv \mu_t \rho_s \mod \eta \rho_s G_{st}.$$

We can take, by use of i) of Lemma 5.2 and a) of ii) of Prop. 1.8,

(5.14)
$$\sigma_1 A^s \in \operatorname{Coext}(\rho_s p).$$

By (5.6), we have

(5.15)
$$\eta_3 A^s \equiv \tilde{\eta} \mu_s \mod \tilde{\eta} G_{ss} \bar{\eta} .$$

It is clear that

(5.16)
$$2\sigma_1 A^s = i\eta \rho_s p \quad and \quad 2\eta_3 A^s \equiv i\eta^2 \overline{\mu_s} \mod i\eta^2 G_{ss} \overline{\eta}$$
.

By use of Example 12.15 of [1] and (5.13), we can take

(5.17)
$$j_{ss} = \rho_{s-1}\eta \quad and \quad j_{ss+1} = \rho_{s-1}\eta^2 \quad for \ s \ge 1$$
.

Lemma 5.3.

i)
$$\zeta_0 = \nu \text{ and } \zeta_s \equiv \{\zeta_{s-1}, 8, 2\sigma\} \subseteq \{\zeta_{s-1}, 8\sigma, 2\} \mod 2G_{ss+3} + R_s \text{ for } s \ge 1$$
,

ii)
$$\zeta_{s+t} p \equiv \zeta_s \overline{\alpha_t} \mod R_{s+t} p$$
,

- iii) $\zeta_{s+t} \equiv \{\zeta_s, \alpha_t, 2\} \mod 2G_{8(s+t)+3} + R_{s+t},$
- iv) $\zeta_s \equiv \{2, \eta, \eta^2 \rho_{s-1}\} \mod 2G_{s+3} + R_s \text{ for } s \ge 1,$

where R_s consists of the elements $\alpha \in (G_{ss+3}; 2)$ which have the following properties : $e'_R(\alpha) = 0$, $8\alpha = 0$ and $\eta \alpha p = 0$.

Proof. By use of Theo. 11.1 of [1], $e'_{R}(\{\zeta_{s-1}, 8, 2\sigma\}) \equiv -8e'_{R}(2\sigma)e'_{R}(\zeta_{s-1})$ $\equiv -\frac{1}{8} \mod 1 \text{ since } e'_{R}(\sigma) = e_{C}(\sigma) \equiv \frac{1}{16} \mod 1 \text{ and } e'_{R}(\zeta_{s-1}) \equiv \frac{1}{8} \mod 1$. By use of Prop. 3.2. (c) and Prop. 7.1 of [1], $\zeta_{s-1}G_{8} + 2\sigma G_{8s-4} \subseteq \ker e'_{R}$. So, by Theo. 1.5 of [1], we have $-\zeta_{s} \equiv \{\zeta_{s-1}, 8, 2\sigma\} \mod \ker e'_{R}$. We have $8\zeta_{s} = 0$ and $8\{\zeta_{s-1}, 8, 2\sigma\} = \{8, \zeta_{s-1}, 8\}2\sigma = 0$ by use of Cor. 3.7 of [6]. Since $\eta\zeta_{s} = \mu\zeta_{s-1}$ $= J(\pi_{8s+4}(SO)) = 0$ (cf. p. 39 and p. 56 of [6]), we have $\eta\{2\sigma, 8, \zeta_{s-1}\} = \{\eta, 2\sigma, 8\}\zeta_{s-1} = 0$. This leads us to i).

We shall prove that $R_s \overline{\alpha_t} \subseteq R_{s+t} p$. By i) of Lemma 5.1, $\beta \overline{\alpha_t} \in \{\beta, \alpha_t, 2\} p = \{\beta, 8, \frac{1}{4}\alpha_t\} p$ for $\beta \in R_s$. By use of Cor. 3.7 of [6], $8\{\beta, 8, \frac{1}{4}\alpha_t\} = \{8, \beta, 8\} \frac{1}{4}\alpha_t = 0$. By use of Theo. 11.1 of [1]., we have $e'_R(\{\beta, 8, \frac{1}{4}\alpha_t\}) = 0$. Since $\eta\beta$ is divisible by 2, we have $\eta\{\beta, 8, \frac{1}{4}\alpha_t\} p = \eta\beta\overline{\alpha_t} = 0$. Therefore we obtain $\{\beta, 8, \frac{1}{4}\alpha_t\} \subseteq R_{s+t}$. Now we obtain ii) by use of this fact and i).

iii) forllows from ii).

We shall prove iv). First we note that we can define $\{2, \eta, \eta^2 \rho_{s-1}\}$ since $\eta^3 \rho_{s-1} \in J(\pi_{ss+2}(SO)) = 0$.

Since $\zeta_1 = \zeta \equiv \{2, \eta, \eta^2 \sigma\} \mod 2G_{11}$, we have, by use of ii), $\zeta_s p \equiv \zeta \overline{\alpha_{s-1}} = \{2, \eta, \eta^2 \sigma\} \overline{\alpha_{s-1}} \mod R_s p$. By use of i) of Lemma 5.2, we have $\zeta_s p \equiv \{2, \eta, \eta^2 \sigma \overline{\alpha_{s-1}}\} = \{2, \eta, \eta^2 \rho_{s-1} p\} \mod R_s p + 2\pi_{8s+3}(2)$. By Theo. A and Prop. 1.1 of [5] and by Theo. 1.5 of [1], it is clear that $2\pi_{8s+3}(2) \subseteq R_s p$. Therefore we have iv).

By use of ii) of this lemma and a) of ii) of Prop. 1.8, we can take

(5.18)
$$\nu_1 A^s \equiv \operatorname{Coext} (\zeta_s p) \mod \operatorname{Ceoxt} (R_s p) + i\pi_{8s+4}(2) + i\pi_{8s+4}$$

Since $\eta_3 \sigma_1 = \pm \tilde{\eta} \sigma \bar{\eta}$ by vi) of Prop. 2.2, we can take, by (5.6) and (5.13),

(5.19)
$$\eta_{3}\sigma_{1}A^{s} \equiv \tilde{\eta}\rho_{s}\bar{\eta} \mod \tilde{\eta}\sigma G_{ss}\bar{\eta} + \tilde{\eta}G_{ss+9}p$$

By use of Lemma 5.3, we have

(5.20)
$$2\nu_1 A^s \equiv 0 \quad and \quad 2\eta_3 \sigma_1 A^s \equiv i \zeta_{s+1} p \mod i R_{s+1} p.$$

Now we have been ready for improving Theo. 5.1 of [5].

Theorem 5.4. $\pi_k(2)$, $\pi_k^*(2)$ and π_k contain direct summands which are isomorphic to the corresponding groups in the following tables (k>2):

i)
$$k = 8s 8s+1 8s+2 8s+3 8s+4 8s+7$$

 $\pi_{k}(2) \oplus Z_{2}+Z_{2} Z_{4}+Z_{2} Z_{4}+Z_{2} Z_{2}+Z_{2} Z_{2} Z_{2}$
Generators $\overline{\alpha_{s}}, \eta \rho_{s-1} p \rho_{s-1} \overline{\eta}, \mu_{s} p \overline{\mu_{s}}, \rho_{s-1} \eta \overline{\eta}, \eta \overline{\mu_{s}}, \zeta_{s} p \eta^{2} \overline{\mu_{s}} \rho_{s} p$
 $\pi_{k}^{*}(2) \oplus Z_{2}+Z_{2} Z_{4}+Z_{2} Z_{4}+Z_{2} Z_{4}+Z_{2} Z_{2}+Z_{2} Z_{2} Z_{2}$
Generators $\widetilde{\alpha_{s}}, i\eta \rho_{s-1}, \eta \rho_{s-1}, i\mu_{s} \widetilde{\mu_{s}}, \eta \eta \rho_{s-1} \widetilde{\mu_{s}} \eta, i\zeta_{s} \widetilde{\mu_{s}} \eta^{2} i\rho_{s}$
Relations: $2\rho_{s}\overline{\eta}=\rho_{s}\eta^{2}p, 2\overline{\mu_{s}}=\eta \mu_{s}p,$
 $2\overline{\eta}\rho_{s}=i\eta^{2}\rho_{s}, 2\overline{\mu_{s}}=i\eta \mu_{s}.$

ii) $k = \frac{8s}{Z_4 + Z_2 + Z_2} \frac{8s + 1}{Z_2 + Z_2 + Z_2 + Z_2 + Z_2}$ Generators $A^s, \eta_1 \sigma_1 A^{s-1}, \eta_2 \sigma_1 A^{s-1} \eta_1^2 \sigma_1 A^{s-1}, \eta_1^2 \sigma_2 A^{s-1}, \eta_1 A^s, \eta_2 A^s$ $k = \frac{8s + 2}{Z_4 + Z_2} \frac{8s + 3}{8s + 4} \frac{8s + 6}{8s + 6} \frac{8s + 7}{Z_4 + Z_2 + Z_2} \frac{2}{Z_2} \frac{2}{Z_2} \frac{2}{Z_4 + Z_2}$ Generators $\eta_3 \sigma_1 A^{s-1}, \eta_1^2 A^s, \eta_2^2 A^s \eta_3 A^s, \nu_1 A^s \eta_2 \eta_3 A^s \delta \sigma_1 A^s \sigma_1 A^s, \delta A^{s+1}$ Relations: $2A^s \equiv i\mu_s p \mod i\eta G_{ss} p, 2\eta_3 \sigma_1 A^s \equiv i\zeta_{s+1} p \mod iR_{s+1} p, 2\eta_3 A^s \equiv i\eta_1^2 \mu_s \mod i\eta_2^2 G_{ss} \eta_1 2\sigma_1 A^s = i\eta \rho_s p.$

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