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Osaka J. Math.
6 (1969), 63-91

ON THE STABLE HOMOTOPY OF A Z_2-MOORE SPACE

Dedicated to Professor A. Komatu for his 60th birthday

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(Received August 8, 1968)
(Revised January 23, 1969)

Introduction

This paper is a continuation of [5].

Denote by M a Z_2-Moore space. We take $M=S^1\cup_2 e^2$, which is obtained from a 1-sphere S^1 by attaching a 2-cell e^2, using a map $S^1\to S^1$ of degree 2. Let π_k be the k-th group of the stable homotopy of M, i.e., $\pi_k=\operatorname*{Dir\,Lim}_{n\to\infty}[S^{n+k}M, S^nM]$, where the direct limit is taken with respect to suspensions. Put $\pi_*=\sum\pi_k$, then it admits a ring structure with respect to the composition. In fact, it forms an algebra over Z_4.

In [5] we determined the additive structure of π_* in $\dim\leqq 21$. In this paper we shall investigate compositions of elements in π_* and the ring structure of π_* in $\dim\leqq 21$. Our main theorems are Theo. 4.1 and 4.2. Our methods deeply depend on the results and the methods of Toda [6].

In §1 we shall state the general formulas obtained from composing elements of $\pi_j(2)=\operatorname*{Dir\,Lim}_{n\to\infty}[S^{n+j-2}M, S^n]$, $\pi_k^*(2)=\operatorname*{Dir\,Lim}_{n\to\infty}[S^{n+k+1}, S^nM]$ and π_l.

In §2 we fix the generators of the above groups by use of the formulas of §1 and we examine compositions of the generators.

In §3 we prove the theorem in which the relations in the secondary or tertiary compositions are mentioned. They hold the key to the discussions in §2.

Our main theorems are stated in §4.

§5 is devoted to the improvement in Theo. 5.1 of [5].

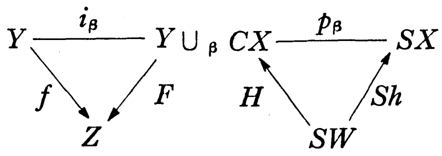
The author wishes to express his sincere gratitude to Professor N. Yamamoto and Professor H. Toda for many advices and kind criticisms given during the preparation of this paper.

Notations and conventions

The notations of [5] are carried over the present work with making a few changes and adding new one.

In [5] we did not distinguish between a representative of a set of the (co-) extensions and the set itself. In this paper this distinction is essential.

Suppose given $\alpha=\{f\}\in[Y, Z]$, $\beta=\{g\}\in[X, Y]$ and $\gamma=\{h\}\in[W, X]$ such that $\alpha\beta=0$ in $[X, Z]$ and $\beta\gamma=0$ in $[W, Y]$. Then we have two maps F: $Y\cup_\beta CX\to Z$ and H: $SW\to Y\cup_\beta CX$ which are characterized by the following homotopy commutative diagram,



where i_β is an inclusion and p_β a map shrinking Y to the base point of SX. F and H are called the extension of f, the coextension of h respectively.

We shall fix the following notations.

i) $\bar{\alpha}_\beta=\{F\}$: the extension of α with respect to β.

ii) $\tilde{\gamma}_\beta=\{H\}$: the coextension of γ with respect to β.

iii) $\mathrm{Ext}_\beta\,\alpha$: the set of $\bar{\alpha}_\beta$.

iv) $\mathrm{Coext}_\beta\,\gamma$: the set of $\tilde{\gamma}_\beta$.

We note that the Toda bracket $\{\alpha, \beta, \gamma\}=(\mathrm{Ext}_\beta\,\alpha)(\mathrm{Coext}_\beta\,\gamma)$ as a double coest of two subgroups $\alpha[SW, Y]$ and $[SX, Z]\,(S\gamma)$ in $[SW, Z]$ (see Prop. 1.7 in p. 13 of [6]).

If $\beta=2\in G_0$, then we put $i_\beta=i$, $p_\beta=p$, $\bar{\alpha}_\beta=\bar{\alpha}$, $\tilde{\gamma}_\beta=\tilde{\gamma}$, $\mathrm{Ext}_\beta\,\alpha=\mathrm{Ext}\,\alpha$ and $\mathrm{Coext}_\beta\,\gamma=\mathrm{Coext}\,\gamma$ respectively.

Assume that $\alpha\in G_k$ is of order 2 and that $\eta\alpha\in G_{k+1}$ is divisible by 2, where η is the generator of G_1. Then we can define $\mathrm{Coext}(\bar{\alpha})$ and $\mathrm{Ext}(\tilde{\alpha})$ by use of Prop. 2.5 of [5]. We take

v) $\widetilde{(\bar{\alpha})}\in\mathrm{Coext}(\bar{\alpha})$ and $\overline{(\tilde{\alpha})}\in\mathrm{Ext}(\tilde{\alpha})$.

We put the following new notation.

vi) $K_s=\ker i^*\cap\ker p_*$, where i^*: $\pi_s\to\pi_s^*(2)$ and p_*: $\pi_s\to\pi_s(2)$ are natural homomorphisms induced by i: $S^1\to S^1\cup_2 e^2$ and p: $S^1\cup_2 e^2\to S^2$.

Our conventions are the following.

In discussions of this paper we often use those properties of Toda brackets which are stated in Prop. 1.2, Prop. 1.4 and i) of (3.9) of [6] and those results about $(G_*; 2)$ which are stated in Theo. B of [5] and Theo. 14.1 of [6] without any reference.

1. Compositions of elements of $\pi_j(2)$, $\pi_k^*(2)$ and π_l

In this section we shall state some general formulas obtained from composing elements of $\pi_j(2)$, $\pi_k^*(2)$ and π_l.

Throughout this section we take α and β in $(G_j; 2)$ and $(G_k; 2)$ respectively.

Proposition 1.1. *If α is neither of order 2 nor divisible by 2, then*

$$\widetilde{\alpha p} \equiv \overline{i\alpha} \bmod i\pi_{j+1}(2)+\pi^*_{j+1}(2)p$$

and

$$\mathrm{Coext}\,(\alpha p)i \equiv i\alpha \bmod i\pi_{j+1}(2)i\,.$$

Proof. By ii) and iii) of Prop. 1.2 of [6], $\widetilde{\alpha p}\in\{i, 2, \alpha p\}\subseteq\{i, 2\alpha, p\}\supseteqq\{i\alpha, 2, p\}\ni\overline{i\alpha}$. Since the bracket $\{i, 2\alpha, p\}$ is a coset of $i\pi_{j+1}(2)+\pi_{j+1}(2)p$, we have the first assertion.

The second assertion is a direct consequence of the first one.

Proposition 1.2. *Assume that α is of order 2 and that $\eta\alpha$ is divisible by 2. Then we have the following.*

i) $$(\widetilde{\overline{\alpha}})\equiv(\overline{\widetilde{\alpha}}) \bmod \sum_{1\leqq s\leqq m}\{\widetilde{\gamma_s p}\}+i\pi_{j+2}(2)+\pi^*_{j+2}(2)p\,,$$

where $\gamma_1, \cdots, \gamma_m$ are the elements of $(\mathrm{G}_{j+1}; 2)$ which are neither of order 2 nor divisible by 2.

ii) *Supose given $\bar{\alpha}$ and $\tilde{\alpha}$ such that $i\eta\bar{\alpha}=\tilde{\alpha}\eta p$. Then*

$$\mathrm{Coext}\,(\bar{\alpha})i\equiv\tilde{\alpha} \bmod \sum_t\{i\gamma_t\}+i\pi_{j+2}(2)i\,,$$

where t runs over the subset of $\{1, 2, \cdots, m\}$ which consists of s satisfying the equation $i\eta\gamma_s p=0$.

Proof. Obviously, $(\widetilde{\overline{\alpha}})\equiv(\overline{\widetilde{\alpha}}) \bmod p_*^{-1}i^{*-1}(0)$. By use of Prop. 1.1, Prop. 1.2 and Prop. 1.3 of [5], it is easy to see that $p_*^{-1}i^{*-1}(0)=p_*^{-1}(G_{j+1}p)$ and that this equals the given subgroup of π_{j+1} in i). So, i) is proved.

By Theo. A of [5], $2(\widetilde{\overline{\alpha}})=i\eta\bar{\alpha}$, $2(\overline{\widetilde{\alpha}})=\tilde{\alpha}\eta p$ and $2\widetilde{\gamma_s p}=i\eta\gamma_s p$. So, i) and the assumption of ii) lead us to the assertion of ii).

Proposition 1.3. *Assume that α is of order 2 and that β is neither of order 2 nor divisible by 2. Then we have the following.*

i) *In case $\alpha\beta\neq 0$:*
   a) $\bar{\alpha}\,\mathrm{Coext}\,(\beta p)\equiv\beta\bar{\alpha} \bmod \alpha\pi_{k+1}(2)+G_{j+k+1}p$,
   b) $\mathrm{Coext}\,(\beta p)(i\bar{\alpha})\equiv i\beta\bar{\alpha} \bmod i\pi_{k+1}(2)i\bar{\alpha}$.

ii) *In case $\alpha\beta=0$:*
   a) $\bar{\alpha}\,\mathrm{Coext}\,(\beta p)\equiv 0 \bmod \alpha\pi_{k+1}(2)+G_{j+k+1}p$,
   b) $\mathrm{Coext}\,(\beta p)(i\bar{\alpha})\equiv i\{\beta, \alpha, 2\}p \bmod i\pi_{k+1}(2)i\bar{\alpha}+i\beta G_{j+1}p$.

Proof. Clearly, $\bar{\alpha}\,\mathrm{Coext}\,(\beta p)\subseteq\{\alpha, 2, \beta p\}\supseteqq\{\alpha, 2\beta, p\}$. This bracket contains $\beta\bar{\alpha}$ or 0 according as $\alpha\beta\neq 0$ or $\alpha\beta=0$ and it is a coset of $\alpha\pi_{k+1}(2)+G_{j+k+1}p$. So, a) of i) and ii) are proved.

By Prop. 1.1, $\mathrm{Coext}\,(\beta p)(i\bar{\alpha})$ contains $i\beta\bar{\alpha} \bmod i\pi_{k+1}(2)i\bar{\alpha}$. So, b) of i) is proved.

If $\alpha\beta=0$, $i\beta\bar{\alpha}\equiv i\beta\{\alpha, 2, p\}=i\{\beta, \alpha, 2\}p \bmod i\beta G_{j+1}p$. This leads us to b) of ii).

Proposition 1.4. *Let α and β be same as the above proposition. Then we have the following.*

i) *In case $\alpha\beta\neq 0$:*

a) $\mathrm{Coext}\,(\beta p)\tilde{\alpha}\equiv\tilde{\alpha}\beta \bmod iG_{j+k+1}$,

b) $(\bar{\alpha}p)\,\mathrm{Coext}\,(\beta p)=\tilde{\alpha}\beta p$.

ii) *In case $\alpha\beta=0$:*

a) $\mathrm{Coext}\,(\beta p)\tilde{\alpha}\equiv 0 \bmod iG_{j+k+1}$,

b) $(\bar{\alpha}p)\,\mathrm{Coext}\,(\beta p)\subseteq i\{2, \alpha, \beta\}p \bmod iG_{j+1}\beta p$.

Proof. If $\alpha\beta\neq 0$, $\mathrm{Coext}\,(\beta p)\tilde{\alpha}=\{i, 2, \beta p\}\tilde{\alpha}\subseteq\{i, 2, \beta\alpha\}\supseteq\{i, 2, \alpha\}\beta\ni\tilde{\alpha}\beta$. Since the bracket $\{i, 2, \beta\alpha\}$ is a coset of iG_{j+k+1}, we have a) of i).

The others are obvious.

Proposition 1.5. *Assume that α and β are of order 2 respectively and that $\eta\alpha$ is divisible by 2. Let $\bar{\alpha}$, $\bar{\beta}$ and $\tilde{\alpha}$ be fixed. Then we have the following.*

i) *In case $\bar{\beta}\tilde{\alpha}\neq 0$:*

a) *If $i\eta\bar{\alpha}=\tilde{\alpha}\eta p$,*

$$\bar{\beta}\,\mathrm{Coext}\,(\bar{\alpha})\equiv(\overline{\bar{\beta}\tilde{\alpha}}) \bmod \sum_t \{\gamma_t\bar{\beta}\}+\beta\pi_{j+2}(2)+G_{j+k+2}p\,,$$

where t runs over the subset of $\{1, 2, \cdots, m\}$ which consists of s satisfying the equation $i\eta\gamma_s p=0$.

b) *If $\alpha\beta=0$ and $\alpha\bar{\beta}=0$ and if there exists $\gamma\in(G_{j+k+1};\, 2)$ which satisfies $\tilde{\alpha}\beta=i\gamma$,*

$$\tilde{\alpha}\bar{\beta}\equiv i\tilde{\gamma} \bmod K_{j+k+1}\,.$$

ii) *In case $\bar{\beta}\tilde{\alpha}=0$:*

a) $\bar{\beta}\,\mathrm{Ext}\,(\tilde{\alpha})=\{\bar{\beta}, \tilde{\alpha}, 2\}p$.

b) *If $\alpha\beta=0$ and $\{\alpha, \beta, 2\}=0$,*

$$\tilde{\alpha}\bar{\beta}\equiv i\{2, \alpha, \beta, 2\}p \bmod iG_{j+1}\,\mathrm{Ext}\,\beta+(\mathrm{Coext}\,\alpha)G_{k+1}p\,.$$

Proof. a) of i) is a direct consequence of ii) of Prop. 1.2.

b) of i) and a) of ii) are obvious.

By use of ii) of (3.9) of [6], $\{\beta, 2, \alpha\}+\{2, \alpha, \beta\}+\{\alpha, \beta, 2\}\ni 0$. So, we can take $\gamma\equiv\bar{\beta}\tilde{\alpha} \bmod \beta G_{j+1}+\alpha G_{k+1}+2G_{j+k+1}$ in b) of i) and we have $\{2, \alpha, \beta\}\ni 0$ under the assumption of b) of ii).

Now we can construct the tertiary composition $\{2, \alpha, \beta, 2\}$ by use of the Mimura's methods (see [2]). Namely, from the fact $\{2, \alpha, \beta\}\ni 0$, we can choose $\bar{2}_\alpha$ and $\tilde{\beta}_\alpha$ such that $\bar{2}_\alpha\tilde{\beta}_\alpha=0$. It is clear that $\tilde{\beta}_\alpha 2\in\{i_\alpha, \alpha, \beta\}2=i_\alpha\{\alpha, \beta, 2\}=0$. So, we can define the Toda bracket $\{\bar{2}_\alpha, \tilde{\beta}_\alpha, 2\}$. We put

$\{2, \alpha, \beta, 2\} \equiv \{\bar{2}_\alpha, \tilde{\beta}_\alpha, 2\} \bmod \bar{2}_\alpha [S^{n+j+k+2}, S^n \cup_\alpha e^{n+j+1}] + [S^{n+j+1} \cup_\beta e^{n+j+k+2}, S^n] \tilde{2}_\beta$, where n is sufficiently large.

It follows that $i\bar{2}_\alpha \in i\{2, \alpha, p_\alpha\} = \{i, 2, \alpha\} p_\alpha = (\text{Coext}\,\alpha) p_\alpha$. Similarly, we obtain $\tilde{2}_\beta p \in i_\beta \,\text{Ext}\,\beta$. Therefore, we have $i\{2, \alpha, \beta, 2\} p \equiv i\{\bar{2}_\alpha, \tilde{\beta}_\alpha, 2\} p = i\bar{2}_\alpha \{\tilde{\beta}_\alpha, 2, p\} \subseteqq (\text{Coext}\,\alpha)(\text{Ext}\,\beta) \bmod iG_{j+1}\,\text{Ext}\,\beta + (\text{Coext}\,\alpha) G_{k+1} p$. This leads us to the assertion of b) of ii).

Proposition 1.6. *α and β are same as the above proposition. Let $\tilde{\beta}$ and $\bar{\alpha}$ be fixed. Then we have the following.*

i) *In case $\bar{\alpha}\tilde{\beta} \neq 0$:*

a) $\text{Coext}\,(\bar{\alpha})\tilde{\beta} \equiv \widetilde{(\bar{\alpha}\tilde{\beta})} \bmod iG_{j+k+2}$.

b) *If $\alpha\beta = 0$ and $\tilde{\beta}\alpha = 0$ and if there exists $\gamma \in (G_{j+k+1}; 2)$ which satisfies $\beta\bar{\alpha} = \gamma p$,*

$$\tilde{\beta}\bar{\alpha} \equiv \tilde{\gamma} p \bmod K_{j+k+1}.$$

ii) *In case $\bar{\alpha}\tilde{\beta} = 0$:*

a) $\text{Coext}\,(\bar{\alpha})\tilde{\beta} = i\{2, \bar{\alpha}, \tilde{\beta}\}$.

b) *If $\alpha\beta = 0$ and $\{2, \beta, \alpha\} = 0$,*

$$\tilde{\beta}\bar{\alpha} \equiv i\{2, \beta, \alpha, 2\} p \bmod iG_{k+1}\,\text{Ext}\,\alpha + (\text{Coext}\,\beta) G_{j+1} p.$$

The proof is quite similar to the one of the above proposition and we omit it.

Proposition 1.7. *Assume that α and β are neither of order 2 nor divisible by 2 respectively.*

i) *If $\alpha\beta$ is neither of order 2 nor divisible by 2,*

$$\widetilde{\alpha p}\,\widetilde{\beta p} \in \text{Coext}\,(\alpha\beta p).$$

ii) *Suppose given $\widetilde{\alpha p}$ and $\widetilde{\beta p}$ such that $\widetilde{\alpha p} i = i\alpha$ and $\widetilde{\beta p} i = \beta i$, then we have the following.*

a) *If $\alpha\beta$ is divisible by 2,*

$$\widetilde{\alpha p}\,\widetilde{\beta p} \equiv 0 \bmod K_{j+k}.$$

b) *If $\alpha\beta$ is not divisible by 2 but of order 2,*

$$\widetilde{\alpha p}\,\widetilde{\beta p} \equiv i\overline{\alpha\beta} + \widetilde{\alpha\beta} p \bmod K_{j+k}.$$

The proof is left to the reader.

Proposition 1.8. *Assume that α is neither of order 2 nor divisible by 2 and that β is of order 2 and $\eta\beta$ is divisible by 2. Let $\bar{\beta}$ be fixed.*

i) *If $\alpha\beta \neq 0$,*

$$\widetilde{\alpha p}(\widetilde{\bar{\beta}}) \in \text{Coext}\,(\alpha\bar{\beta}).$$

ii) *If* $\alpha\beta=0$ *and if there exists* $\gamma\in(G_{j+k+1};2)$ *which satisfies* $\alpha\bar{\beta}=\gamma p$, *we have the following.*

a) $\widetilde{\alpha p}(\widetilde{\beta})\in\mathrm{Coext}\,(\gamma p)$.

b) *If* $\tilde{\beta}$, $\widetilde{\alpha p}$ *and* $(\widetilde{\beta})$ *are fixed such that* $\widetilde{\alpha p}i=i\alpha$ *and* $(\widetilde{\beta})i=\tilde{\beta}$ *and if* $\widetilde{\alpha p}\,\tilde{\beta}=\bar{\beta}\alpha=i\gamma$ *and* $\bar{\beta}\widetilde{\alpha p}=\gamma p$, *we have*

$$(\widetilde{\beta})\widetilde{\alpha p}\equiv\widetilde{\alpha p}(\widetilde{\beta}) \bmod K_{j+k+1}\,.$$

The proof is left to the reader.

In ii) of the above proposition we can take $\gamma\in\{\alpha, \beta, 2\}$.

Proposition 1.9. *Assume that* α *and* β *are of order* 2 *respectively and that* $\eta\alpha$ *and* $\eta\beta$ *are divisible by* 2 *respectively. Let* $\bar{\alpha}$, $\bar{\beta}$, $\tilde{\beta}$ *and* $(\widetilde{\beta})$ *be fixed such that* $(\widetilde{\beta})i=\tilde{\beta}$. *Then we have the following.*

i) *If* $\bar{\alpha}\tilde{\beta}\neq 0$,

$$(\widetilde{\alpha})(\widetilde{\beta})\in\mathrm{Coext}\,(\overline{(\bar{\alpha}\tilde{\beta})})\,.$$

ii) *If* $\bar{\alpha}\tilde{\beta}=0$ *and if* $\{\bar{\alpha}, \tilde{\beta}, 2\}$ *and* $\{2, \bar{\alpha}, \tilde{\beta}\}$ *consist of the elements which are not divisible by* 2 *but of order* 2 *respectively, we have*

$$(\widetilde{\alpha})(\widetilde{\beta})\equiv i\overline{\{2, \bar{\alpha}, \tilde{\beta}\}}+\widetilde{\{\bar{\alpha}, \tilde{\beta}, 2\}}p \bmod K_{j+k+2}\,.$$

Proof. i) is obvious.

ii) follows from a) of ii) of Prop. 1.5 and Prop. 1.6.

2. Generators and relations in $\pi_j(2)$, $\pi_k^*(2)$ and π_l

In this section we shall use the general formulas of § 1 and choose the generators of $\pi_j(2)$, $\pi_k^*(2)$ and π_l. We shall compute compositions of elements of π_*.

The Toda brackets which appear in this section are the following.

Theorem 2.1. (Toda).

i) $\{\eta, 2, \eta\}=\pm 2\nu$, $\{\nu, \eta, 2\}=0$,
$\{\eta, 2, \eta^*\}\equiv\pm 2\nu^* \bmod \eta\bar{\mu}$, $\{\nu^*, \eta, 2\}\equiv 0 \bmod 2G_{20}$.

ii) $\{\eta, 2, \nu^2\}=\{\eta, \nu^2, 2\}\equiv\varepsilon \bmod \eta\sigma$, $\{\nu^2, \eta, 2\}=0$,
$\{\eta, 2, 8\sigma\}=\{\eta, 8\sigma, 2\}\equiv\mu \bmod \{\eta^2\sigma, \eta\varepsilon\}$, $\{8\sigma, \eta, 2\}=0$,
$\{\eta, 2, \sigma^2\}=\{\eta, \sigma^2, 2\}\equiv\eta^* \bmod \eta\rho$, $\{\sigma^2, \eta, 2\}=0$,
$\{\eta, 2, \bar{\sigma}\}=\{\eta, \bar{\sigma}, 2\}\equiv 0 \bmod \eta\bar{\kappa}$, $\{\bar{\sigma}, \eta, 2\}=0$,
$\{\mu, 2, 8\sigma\}=\{\mu, 8\sigma, 2\}\equiv\bar{\mu} \bmod \eta^2\rho$, $\{8\sigma, \mu, 2\}=0$.

iii) $\{\eta, 2, \varepsilon\}\equiv 0 \bmod \eta\mu$, $\{\eta, 2, \kappa\}\equiv 0 \bmod \eta\rho$,
$\{\mu, 2, \varepsilon\}\equiv 0 \bmod \eta\bar{\mu}$.

iv) $\{\sigma, \nu^2, 2\}=\{\sigma, 2\nu, \nu\}\equiv 0 \bmod \sigma^2$,
$\{\nu, \sigma^2, 2\}\equiv\{\nu, 2\sigma, \sigma\}=\nu^* \bmod 2\nu^*$,
$\{\sigma^2, \eta, \nu\}=\{\sigma, \eta\sigma, \nu\}=\bar{\sigma}$, $\{\sigma, \varepsilon, \nu\}=0$.

v) $\{\eta\varepsilon, \eta, 2\}=\{\eta^2\sigma, \eta, 2\}\equiv\zeta \bmod 2G_{11}$,
$\{\eta^2\rho, \eta, 2\}\equiv\bar{\zeta} \bmod 2G_{19}$,
$\{\nu, 8\sigma, 2\}\supset\{\nu, 2\sigma, 8\}\ni\zeta \bmod 2G_{11}$,
$\{\zeta, 8\sigma, 2\}\supset\{\zeta, 2\sigma, 8\}\ni\bar{\zeta} \bmod 2G_{19}$,
$\{\sigma, 8\sigma, 2\}\supset\{\sigma, 2\sigma, 8\}\ni\rho \bmod 2G_{15}$,
$\{\varepsilon, 8\sigma, 2\}=\{\eta\sigma, 8\sigma, 2\}=\eta\rho$, $\{8\sigma, 2, 8\sigma\}=16\rho$.

vi) $\{\eta\kappa, \eta, 2\}=\nu\kappa$, $\{\kappa, 2, \nu^2\}=\eta\bar{\kappa}$.

vii) $\{2, \nu^2, \rho\}=0$, $\{\nu, \eta, \eta^2\sigma\}=0$, $\{\sigma, \nu, \zeta\}=0$.

viii) $\{\kappa, 8\sigma, 2\}\equiv 0 \bmod \eta^2\bar{\kappa}$, $\{\sigma, \kappa, 2\}=\nu\bar{\sigma}$,
$\{\overline{\nu^2}, \widetilde{\nu^2}, 2\}=\{2, \overline{\nu^2}, \widetilde{\nu^2}\}=\kappa$,
$\{\bar{\eta}, \tilde{\kappa}\nu, 2\}=\{2, \nu(\overline{\kappa}), \tilde{\eta}\}\equiv 0 \bmod 2G_{20}$.

ix) $\{2, 4\nu, \eta, 2\}=0$,
$\{2, \bar{\sigma}, \eta, 2\}=\{2, \eta, \bar{\sigma}, 2\}\equiv 0 \bmod \eta^2\bar{\kappa}$.

This theorem will be proved in the next section.

Throughout this section we denote by Roman letters x, y, z, etc. integers 0 or 1.

2.1. First we define $\delta\in\pi_{-1}$ by

$$\delta = ip\,. \tag{2.1}$$

We choose

$$\bar{\eta}\in\operatorname{Ext}\eta \quad \textit{and} \quad \tilde{\eta}\in\operatorname{Coext}\eta \tag{2.2}$$

arbitrarily. Then remark that $\operatorname{Ext}\eta=\{\bar{\eta}, -\bar{\eta}\}$ and $\operatorname{Coext}\eta=\{\tilde{\eta}, -\tilde{\eta}\}$.

We define η_1 and η_2 in π_1 and $\eta_3\in\pi_3$ by

$$\eta_1 = i\bar{\eta}\,, \quad \eta_2 = \tilde{\eta}p \quad \textit{and} \quad \eta_3 = \tilde{\eta}\bar{\eta}\,. \tag{2.3}$$

Take

$$\nu_1\in\operatorname{Coext}(\nu p)\subset\pi_3 \tag{2.4}$$

arbitrarily.

Proposition 2.1.

i) $\delta^2=0$, $\delta\eta_1=\eta_2\delta=0$ *and* $\eta_1\delta=\delta\eta_2=i\eta p=2.1$, *where* 1 *is a generator of* π_0 *and of order* 4.

ii) $\delta\eta_3=\eta_1^2=i\eta\bar{\eta}$, $\eta_3\delta=\eta_2^2=\tilde{\eta}\eta p$ *and* $\delta\eta_3\delta=0$.

iii) $\nu_1\delta=\delta\nu_1=i\nu p$.

iv) $\eta_1\eta_2=\eta_2\eta_1=0$ *and* $\eta_1\eta_3=\eta_3\eta_2=0$.

v) $\eta_3\eta_1=\eta_2\eta_3=\tilde{\eta}\eta\bar{\eta}$.

vi) $\eta_2^2\eta_3=0$.

vii) $\eta_1\nu_1=\nu_1\eta_1=\eta_2\nu_1=\nu_1\eta_2=0$ *and* $\eta_3\nu_1=\nu_1\eta_3=\eta_3^2=0$.

Proof. i), ii) and v) are obvious (see Theo. A of [5]).

By Theo. 3.1 of [5], $\pi_4(2)=\{\eta^2\bar{\eta}\}$. Since $i\eta^2\bar{\eta}i=i\eta^3=i(4\nu)=0$, we have $i\pi_4(2)i=0$. So, we have, by use of Prop. 1.1,

$$\text{Coext}\,(\nu p)i = i\nu\,. \tag{2.5}$$

From this we have the assertion of iii).

iv) follows from the fact $\bar{\eta}\tilde{\eta}=\pm 2\nu$ of i) of Theo. 2.1.

By Theo. 3.1 and Theo. 3.2 of [5], $\pi_5(2)=0$ and $\pi_5^*(2)=0$. So, we have

$$\bar{\eta}\nu_1 = \nu\bar{\eta} = 0 \quad \textit{and} \quad \nu_1\tilde{\eta} = \tilde{\eta}\nu = 0\,. \tag{2.6}$$

From (2.5) and (2.6) we have vii).

Finally, we shall prove vi). By use of b) of ii) of Prop. 1.5, $\eta_2^2\eta_3=\tilde{\eta}\eta^2\bar{\eta}=\widetilde{\eta^3}\bar{\eta}=\widetilde{4\nu}\bar{\eta}\equiv i\{2, 4\nu, \eta, 2\}p$ mod iG_5 Ext $\eta+$Coext $(4\nu)G_2p=0$. By ix) of Theo. 2.1, $\{2, 4\nu, \eta, 2\}$ consists of 0. This leads us to vi).

2.2. By ii) of Theo. 2.1, $\{\nu^2, 2, \eta\}=\{\eta, 2, \nu^2\}\equiv\varepsilon$ mod $\eta\sigma$. So, we can choose $\overline{\nu^2}\in\text{Ext}\,\nu^2$ and $\widetilde{\nu^2}\in\text{Coext}\,\nu^2$ such that

$$\overline{\nu^2}\tilde{\eta} = \bar{\eta}\widetilde{\nu^2} = \varepsilon\,. \tag{2.7}$$

It follows from (2.7) that

$$\eta\overline{\nu^2} = \varepsilon p \quad \textit{and} \quad \widetilde{\nu^2}\eta = i\varepsilon\,. \tag{2.8}$$

We shall prove the first. Since $\{\eta, \nu^2, 2\}\equiv\varepsilon$ mod $\eta\sigma$ by ii) of Theo. 2.1, we have $\eta\overline{\nu^2}\in\eta\{\nu^2, 2, p\}=\{\eta, \nu^2, 2\}p\equiv\varepsilon p$ mod $\eta\sigma p$. So, we can put $\eta\overline{\nu^2}=\varepsilon p+x\eta\sigma p$. Multiply this equality by $\tilde{\eta}$ on the right, then we have $x=0$ by (2.7) and by the result $\eta^2\sigma\neq 0$.

Since $\{8\sigma, 2, \eta\}=\{\eta, 2, 8\sigma\}\equiv\mu$ mod $\{\eta^2\sigma, \eta\varepsilon\}$ by ii) of Theo. 2.1, we can take $\overline{8\sigma}\in\text{Ext}\,(8\sigma)$ and $\widetilde{8\sigma}\in\text{Coext}\,(8\sigma)$ such that

$$\overline{8\sigma}\tilde{\eta} = \bar{\eta}\widetilde{8\sigma} = \mu\,. \tag{2.9}$$

From the results that $\{\varepsilon, 2, \eta\}=\{\eta, 2, \varepsilon\}\equiv 0$ mod $\eta\mu$ of iii) of Theo. 2.1, we can choose $(\bar{\varepsilon})\in\text{Ext}\,\varepsilon$ and $\tilde{\varepsilon}\in\text{Coext}\,\varepsilon$ such that

(2.10) $$\overline{(\varepsilon)}\tilde{\eta}=\bar{\eta}\hat{\varepsilon}=0\,.$$

If follows that

(2.11) $$\eta\overline{(\varepsilon)}=\varepsilon\bar{\eta}\quad and\quad \hat{\varepsilon}\eta=\tilde{\eta}\varepsilon\,.$$

We shall prove the second. Clearly, $\hat{\varepsilon}\eta-\tilde{\eta}\varepsilon\in iG_{10}=\{i\eta\mu\}$. Namely, we can put $\hat{\varepsilon}\eta=\tilde{\eta}\varepsilon+xi\eta\mu$. Multiply this equality by $\bar{\eta}$ on the left, then we have $x=0$. For $\bar{\eta}\hat{\varepsilon}\eta=0$ by (2.10), $\bar{\eta}\tilde{\eta}\varepsilon=2\nu\varepsilon=0$ and $\eta^2\mu\neq 0$.

By Theo. 3.1 of [5], $i\pi_8(2)i=i\{\overline{8\sigma},\ \eta\sigma p,\ \varepsilon p\}i=0$. So, we have, by use of Prop. 1.1,

(2.12) $$\mathrm{Coext}\,(\sigma p)\,i=i\sigma\,.$$

Since $\eta\overline{8\sigma}\in\{\eta,\ 8\sigma,\ 2\}p\equiv\mu p \bmod \{\eta^2\sigma p,\ \eta\varepsilon p\}$ by ii) of Theo. 2.1, we have $\eta\pi_8(2)=G_9 p$. Therefore, we can choose $\sigma_1\in\mathrm{Coext}\,(\sigma p)\subset\pi_7$, by use of a) of i) of Prop. 1.3, such that

(2.13) $$\bar{\eta}\sigma_1\equiv\pm\sigma\bar{\eta}\bmod \eta\varepsilon p\,.$$

By use of a) of i) of Prop. 1.4, we can put $\sigma_1\tilde{\eta}=\pm\tilde{\eta}\sigma+xi\eta\varepsilon+yi\mu$. Multiply this equality by $\bar{\eta}$ on the left, then we have $y=0$. For $\bar{\eta}\sigma_1\tilde{\eta}=\sigma\bar{\eta}\tilde{\eta}=\bar{\eta}\tilde{\eta}\sigma=2\nu\sigma=0$, $\eta^2\varepsilon=0$ and $\eta\mu\neq 0$. So, we have

(2.14) $$\sigma_1\tilde{\eta}\equiv\pm\tilde{\eta}\sigma\bmod i\eta\varepsilon\,.$$

Since $i\pi_8(2)i=0$, $i\eta\sigma p=2\sigma_1\neq 0$ by Theo. 3.3 of [5] and $i\eta\overline{\nu^2}=\widetilde{\nu^2}\eta p=i\varepsilon p$ by (2.8), we have, by use of ii) of Prop. 1.2,

(2.15) $$\mathrm{Coext}\,(\overline{\nu^2})i=\widetilde{\nu^2}\,.$$

Since $\eta\pi_8(2)=G_9 p$ we can choose $\nu_2\in\mathrm{Coext}\,(\overline{\nu^2})\subset\pi_7$, by use of a) of i) of Prop. 1.5, such that

(2.16) $$\bar{\eta}\nu_2\equiv\pm\overline{(\varepsilon)}\bmod \eta^2\sigma p\,.$$

By the similar arguments to (2.14), we have

(2.17) $$\nu_2\tilde{\eta}\equiv\pm\hat{\varepsilon}\bmod i\eta^2\sigma\,.$$

Proposition 2.2.

i) $\sigma_1\delta=\delta\sigma_1=i\sigma p$

ii) $\delta\nu_2=i\overline{\nu^2}$ *and* $\nu_2\delta=\widetilde{\nu^2}p$.

iii) $\nu_1^2=\delta\nu_2+\nu_2\delta$.

iv) $\sigma_1\eta_1=\eta_1\sigma_1=i\sigma\bar{\eta}$ *and* $\sigma_1\eta_2=\eta_2\sigma_1=\tilde{\eta}\sigma p$.

v) $\nu_2\eta_1=\eta_1\nu_2=i\overline{(\varepsilon)}$ *and* $\nu_2\eta_2=\eta_2\nu_2=\hat{\varepsilon}p$.

vi) $\eta_3\sigma_1 = \pm\tilde{\eta}\sigma\bar{\eta}$ *and* $\sigma_1\eta_3 = \pm\eta_3\sigma_1$.

vii) $\eta_3\nu_2 = \pm\tilde{\eta}(\bar{\varepsilon})$ *and* $\nu_2\eta_3 = \pm\eta_3\nu_2$.

Proof. i) and ii) are direct consequences of (2.12) and (2.15) respectively.

By use of b) of ii) of Prop. 1.7, $\nu_1^2 \equiv i\overline{\nu^2} + \widetilde{\nu^2}p$ mod K_6. Since $K_6 = \{i\sigma p\}$ by Theo. 3.3 of [5], we can put $\nu_1^2 = i\overline{\nu^2} + \widetilde{\nu^2}p + xi\sigma p$. Multiply this equality by $\bar{\eta}$ on the left, then we have $x=0$. For $\bar{\eta}\nu_1^2 = 0$ by (2.6), $\eta\overline{\nu^2} = \bar{\eta}\widetilde{\nu^2}p = \varepsilon p$ by (2.7) and (2.8) and $\eta\sigma p \neq 0$ by Theo. 3.1 of [5]. This proves iii).

We shall prove the first assertion of iv). The equality $\sigma_1\eta_1 = i\sigma\bar{\eta}$ is a direct consequence of (2.12). We have $\eta_1\sigma_1 = i\sigma\bar{\eta}$ by (2.13) since $i\eta\varepsilon p = i(2(\bar{\varepsilon})) = 0$.

We shall prove the second assertion of v). By (2.17) we have $\nu_2\eta_2 \equiv \tilde{\varepsilon}p$ mod $i\eta^2\sigma p = 0$.

By use of b) of i) of Prop. 1.6, we have $\eta_2\nu_2 = \tilde{\eta}\overline{\nu^2} \equiv \tilde{\varepsilon}p$ mod K_8 since $\tilde{\eta}\nu^2 = 0$ by (2.6) and $\eta\overline{\nu^2} = \varepsilon p$ by (2.8). It follows from Theo. 3.3 of [5] that $K_8 = \{i\mu p\}$. So, we can put $\tilde{\eta}\overline{\nu^2} = \tilde{\varepsilon}p + xi\mu p$. Multiply this equality by $\bar{\eta}$ on the right, then we have $x=0$. For $\tilde{\eta}\overline{\nu^2}\bar{\eta} = \tilde{\eta}\varepsilon$ by (2.7), $\tilde{\eta}\varepsilon = \tilde{\varepsilon}\eta$ by (2.11) and $i\eta\mu \neq 0$ by Theo. 3.2 of [5].

The first assertions of vi) and vii) are obtained from (2.13) and (2.16) respectively since $2\eta_3\nu_2 = \tilde{\eta}\eta\varepsilon p = i\{2, \eta, \eta\varepsilon\}p = i\zeta p = i\{2, \eta, \eta^2\sigma\}p = \tilde{\eta}\eta^2\sigma p = 2\eta_3\sigma_1$ by v) of Theo. 2.1.

Similarly, we have $i\eta\varepsilon\bar{\eta} = i\eta^2\sigma\bar{\eta} = i\zeta p$. By (2.11) and by Theo. 3.3 of [5], we have $\tilde{\varepsilon}\bar{\eta} \equiv \tilde{\eta}(\bar{\varepsilon})$ mod $K_{11} = \{i\zeta p\}$. Therefore, we obtain the second assertions of vi) and vii) by (2.14) and (2.17) respectively.

2.3. By use of ii) of Prop. 1.2, Coext $(\overline{8\sigma})i \equiv \widetilde{8\sigma}$ mod $i\pi_9(2)i$. So, we can choose $A \in$ Coext $(\overline{8\sigma}) \subset \pi_8$ such that

$$Ai = \widetilde{8\sigma}\,. \tag{2.18}$$

From this and (2.9), we can choose

$$(\bar{\mu}) = \bar{\eta}A \in \mathrm{Ext}\,\mu \quad \textit{and} \quad \tilde{\mu} = A\tilde{\eta} \in \mathrm{Coext}\,\mu\,. \tag{2.19}$$

Since $\{8\sigma, 2, 8\sigma\} = 16\rho$ by v) of Theo. 2.1, we can take

$$\overline{16\rho} = \overline{8\sigma}A \in \mathrm{Ext}\,(16\rho) \quad \textit{and} \quad \widetilde{16\rho} = A\widetilde{8\sigma} \in \mathrm{Coext}\,(16\rho)\,. \tag{2.20}$$

By use of i) of Prop. 1.9, we can take

$$A^2 \in \mathrm{Coext}\,(\overline{16\rho})\pi_{16}\,. \tag{2.21}$$

As $\bar{\eta}A^2 i = (\bar{\mu})\widetilde{8\sigma} \equiv \overline{\mu}$ mod $\eta^2\rho$ and $p(A^2\tilde{\eta}) = \overline{8\sigma}\tilde{\mu} \equiv \overline{\mu}$ mod $\eta^2\rho$ by ii) of Theo. 2.1, we can choose

(2.22) $$(\overline{\overline{\mu}}) \equiv \bar{\eta} A^2 \bmod \eta\rho\bar{\eta} \quad \textit{and} \quad (\widetilde{\widetilde{\mu}}) \equiv A^2\tilde{\eta} \bmod \tilde{\eta}\eta\rho .$$

Proposition 2.3.

i) $\delta A = i\overline{8\sigma}$ *and* $A\delta = \widetilde{8\sigma}p$.

ii) $\delta A = A\delta + \mathrm{x}i\eta\sigma p + \mathrm{y}i\varepsilon p$ *and* $\delta A^2 = A^2\delta$.

iii) $\eta_1 A = i(\overline{\mu})$, $A\eta_1 = \eta_1 A + \mathrm{x}i\eta\sigma\bar{\eta} + \mathrm{y}i\varepsilon\bar{\eta}$ *and* $\eta_1^2 A = A\eta_1^2$.

iv) $A\eta_2 = \tilde{\mu}p$, $\eta_2 A = A\eta_2 + \mathrm{x}\tilde{\eta}\eta\sigma p + \mathrm{y}\tilde{\eta}\varepsilon p$ *and* $\eta_2^2 A = A\eta_2^2$.

v) $\eta_1 A^2 \equiv i(\overline{\overline{\mu}}) \bmod i\eta\rho\bar{\eta}$ *and* $A^2\eta_1 = \eta_1 A^2$.

vi) $A^2\eta_2 \equiv (\widetilde{\widetilde{\mu}})p \bmod \tilde{\eta}\eta\rho p$ *and* $\eta_2 A^2 = A^2\eta_2$.

Proof. i) is obvious.

Since $i\,\mathrm{Ext}\,(8\sigma) = i\{8\sigma, 2, p\} = i\{2, 8\sigma, p\} = \{i, 2, 8\sigma\}p = \mathrm{Coext}\,(8\sigma)p$ and $i\{2, 8\sigma, p\}$ is a coset of $iG_8 p = \{i\eta\sigma p, i\varepsilon p\}$, we can put $\delta A = A\delta + xi\eta\sigma p + yi\varepsilon p$. We have $A(i\eta\sigma p) = \widetilde{8\sigma}\eta\sigma p = i\{2, 8\sigma, \eta\sigma\}p = i\eta\rho p = i\{2, 8\sigma, \varepsilon\}p = A(i\varepsilon p)$ by v) of Theo. 2.1. Similarly, we have $(i\eta\sigma p)A = (i\varepsilon p)A = i\eta\rho p$. So, we obtain $\delta A^2 = A\delta A + (x+y)i\eta\rho p = A^2\delta$.

By the above proof of ii) and (2.9), $\widetilde{8\sigma}\eta = i\mu + xi\eta^2\sigma + yi\eta\varepsilon$. As $K_9 = 0$ by Theo. 3.3 of [5], we have, by use of b) of i) of Prop. 1.5, $A\eta_1 = \widetilde{8\sigma}\bar{\eta} = i(\overline{\mu}) + xi\eta\sigma\bar{\eta} + yi\varepsilon\bar{\eta}$. Therefore, the first assertions of iv) and v) of Prop. 2.2, (2.11) and (2.19) imply iii).

The first of (2.22) implies the first of v).

By the similar arguments to the above proof of ii), $A(i\eta\sigma\bar{\eta}) = A(i\varepsilon\bar{\eta}) = i\eta\rho\bar{\eta}$. On the other hand, $(i\eta\sigma\bar{\eta})A = i\eta\sigma(\overline{\mu}) = i\sigma\mu\bar{\eta} = i\eta\rho\bar{\eta}$ since $\sigma\zeta = 0$ and $\eta(\overline{\mu}) \equiv \mu\bar{\eta} \bmod \zeta p$. We have $(i\varepsilon\bar{\eta})A = i\varepsilon(\overline{\mu}) = i\eta\rho\bar{\eta} + ziv^*p$ since $\varepsilon\mu = \eta^2\rho$ and $i\eta\bar{\mu}p = 0$. Multiply this equality by ν_1 on the left, then we obtain $z = 0$. For $\nu_1(i\varepsilon) = i\nu\varepsilon = 0$, $\nu_1(i\eta) = i\nu\eta = 0$ and $\nu_1(iv^*p) = i\nu v^*p = i\sigma^3 p \neq 0$ by Theo. 3.3 of [5] (see (7.16) and Prop. 7.2 of [3]). Consequently, the second of v) is proved.

The proofs of iv) and vi) are quite similar to the ones of iii) and v) respectively and we omit them.

2.4. From the results that $\{\nu, 8\sigma, 2\} \equiv \zeta \bmod 2G_{11}$, $\{\sigma, 8\sigma, 2\} \equiv \rho \bmod 2G_{15}$ and $\{\zeta, 8\sigma, 2\} \equiv \bar{\zeta} \bmod 2G_{19}$ of v) of Theo. 2.1, we can take, by use of a) of ii) of Prop. 1.8,

(2.23) $$\nu_1 A \in \mathrm{Coext}\,(\zeta p) \subset \pi_{11} ,$$

(2.24) $$\sigma_1 A \in \mathrm{Coext}\,(\rho p) \subset \pi_{15}$$

and

(2.25) $$\nu_1 A^2 \in \mathrm{Coext}\,(\bar{\zeta} p) \subset \pi_{19} .$$

Proposition 2.4.

i) $A\sigma_1=\pm\sigma_1 A$.

ii) $A\nu_1=\nu_1 A$.

iii) $\eta_3 A=\tilde{\eta}\overline{(\mu)}$ *and* $A\eta_3\equiv\pm\eta_3 A \bmod \nu_1 A$.

iv) $\eta_3 A^2\equiv\pm\tilde{\eta}\overline{(\bar{\mu})} \bmod \nu_1 A^2$ *and* $A^2\eta_3=\eta_3 A^2$.

Proof. Sinc $\rho e\in\{\sigma, 2\sigma, 8\}\subset\{\sigma, 8\sigma, 2\}\bmod 2G_{15}$ by v) of Theo. 2.1, $p(A\sigma_1)=\overline{8\sigma}\sigma_1\in\{8\sigma, 2, \sigma p\}=\{8, 2\sigma, \sigma p\}=\{8, 2\sigma, \sigma\}p=\rho p$ and similarly $\sigma_1 Ai=A\sigma_1 i=i\rho$. Therefore, we have, by use of b) of ii) of Prop. 1.8, $A\sigma_1\equiv\sigma_1 A \bmod K_{15}=\{i\eta\rho p, i\eta^* p, i\overline{16\rho}\}$. Namely, we can put $A\sigma_1=\pm\sigma_1 A+xi\eta^* p+yi\overline{16\rho}$. Multiply this equality by $\bar{\eta}$ on the left and by $\tilde{\eta}$ on the right at the same time, then we have $x=y=0$. For it is clear that $\bar{\eta}A\sigma_1\tilde{\eta}=\bar{\eta}\sigma_1 A\tilde{\eta}=0$ and that $\eta^2\eta^*$ and $\eta\bar{\mu}$ are linearly independent in $(G_{18}; 2)$.

By the similar arguments to the above, we obtain $A\nu_1=\nu_1 A+zi\eta\mu\bar{\eta}$. By i) $\nu_1 A\sigma_1=\nu_1\sigma_1 A$ and this equals $A\nu_1\sigma_1$ since $\nu_1\sigma_1\in K_{10}=\{i\zeta p\}$ and $(i\zeta p)A=A(i\zeta p)=i\bar{\zeta}p$. On the other hand, $i\eta\mu\bar{\eta}\sigma_1=i\eta\mu\sigma\bar{\eta}=i\eta^2\rho\bar{\eta}=i\bar{\zeta}p$ since $\{\eta^2\rho, \eta, 2\}\equiv\bar{\zeta} \bmod 2G_{19}$ by v) of Theo. 2.1. This leads us to the assertion $z=0$.

It is clear that $2\eta_3 A=2A\eta_3=i\eta\mu\bar{\eta}$. So, iii) follows from Theo. 3.3 of [5]. The proof of iv) is left to the reader.

2.5. From the results that $\{\kappa, 2, \eta\}=\{\eta, 2, \kappa\}\equiv 0 \bmod \eta\rho$ of iii) of Theo. 2.1, we can choose $\overline{(\kappa)}\in\text{Ext}\,\kappa$ and $\tilde{\kappa}\in\text{Coext}\,\kappa$ such that

$$(2.26)\qquad \overline{(\kappa)}\tilde{\eta}=\bar{\eta}\tilde{\kappa}=0\,.$$

By the similar arguments to (2.11), we obtain

$$(2.27)\qquad \eta\overline{(\kappa)}=\kappa\bar{\eta}\quad\textit{and}\quad \tilde{\kappa}\eta=\tilde{\eta}\kappa\,.$$

We define $\kappa_1\in\pi_{14}$ and $\kappa_2\in\pi_{16}$ by

$$(2.28)\qquad \kappa_1=i\overline{(\kappa)}\quad\textit{and}\quad \kappa_2=\tilde{\eta}\overline{(\kappa)}\,.$$

Proposition 2.5.

i) $\delta\kappa_1=0$ *and* $\kappa_1\delta=i\kappa p$.

ii) $\kappa_1\eta_2=\eta_2\kappa_1=0$ *and* $\delta\kappa_2=\kappa_1\eta_1=\eta_1\kappa_1=i\eta\overline{(\kappa)}$.

iii) $\eta_1\kappa_2=\kappa_2\eta_2=\kappa_1\eta_3=0$ *and* $\eta_3\kappa_2=\kappa_2\eta_3=0$.

iv) $\nu_2^2+\kappa_1=\tilde{\kappa}p$.

v) $\kappa_2\eta_1=\eta_2\kappa_2=\eta_3\kappa_1=\tilde{\eta}\kappa\bar{\eta}\equiv i\nu\overline{(\kappa)}+\tilde{\kappa}\nu p \bmod i\nu^* p$.

vi) $\nu_1\kappa_1=i\nu\overline{(\kappa)}$ *and* $\kappa_1\nu_1\equiv\nu_1\kappa_1 \bmod i\nu^* p$.

vii) $\kappa_2\nu_1=\nu_1\kappa_2=0$.

viii) $\nu_2\kappa_1=\widetilde{\nu^2}(\overline{\kappa})$ *and* $\kappa_1\nu_2\equiv i\bar{\kappa}\bar{\eta}$ mod $i\nu\bar{\sigma}p$.

ix) $\delta\nu_2^2=\nu_2^2\delta=\nu_2\delta\nu_2=\kappa_1\delta$ *and* $\kappa_2\delta=\eta_2\nu_2^2$.

Proof. i), ii) and iii) are obvious.

By viii) of Theo. 2.1, $\{\overline{\nu^2}, \widetilde{\nu^2}, 2\}=\{2, \overline{\nu^2}, \widetilde{\nu^2}\}=\kappa$. So, we obtain, by use of ii) of Prop. 1.9 and Theo. 3.3 of [5], $\nu_2^2=\kappa_1+\tilde{\kappa}p+xi\rho p$. Multiply this equality by $\bar{\eta}$ on the left and by η_2 on the right at the same time, then we have $x=0$. For $\bar{\eta}\nu_2^2\eta_2=\bar{\eta}\eta_2\nu_2^2=0$, $\bar{\eta}\kappa_1\eta_2=0$, $\bar{\eta}\tilde{\kappa}p\eta_2=0$ and $\eta^2\rho p\neq 0$ in $\pi_{17}(2)$. Thus, iv) is proved.

By iv) of Theo. 2.1, $p(\tilde{\eta}\kappa\bar{\eta})=\eta\kappa\bar{\eta}=\{\eta\kappa, \eta, 2\}p=\nu\kappa p$ and similarly $(\tilde{\eta}\kappa\bar{\eta})i=i\nu\kappa$. So, we have $\tilde{\eta}\kappa\bar{\eta}\equiv i\nu(\overline{\kappa})+\tilde{\kappa}\nu p$ mod $K_{17}=\{i\nu^*p\}$. From this and (2.27), we obtain v).

The first of vi) is obvious.

By use of a) of i) of Prop. 1.3, $(\overline{\kappa})\nu_1\equiv\nu(\overline{\kappa})$ mod $\kappa\pi_4(2)+G_{18}p=\{\nu^*p, \eta\bar{\mu}p\}$. By the similar arguments to (2.14), we have

$$(\overline{\kappa})\nu_1\equiv\nu(\overline{\kappa}) \bmod \nu^*p\ . \tag{2.29}$$

From this we have the second of vi).

By (2.6), $\nu_1\kappa_2=\nu_1\tilde{\eta}(\overline{\kappa})=0$. By (2.29), (2.6) and i) of Theo. 2.1, $\kappa_2\nu_1=\tilde{\eta}(\overline{\kappa})\nu_1\equiv\tilde{\eta}\nu(\overline{\kappa})=0$ mod $\tilde{\eta}\nu^*p=i\{2, \eta, \nu^*\}p=0$. So, vii) is proved.

The first of viii) is obvious.

By use of a) of i) of Prop. 1.5, $\kappa_1\nu_2\equiv i\bar{\kappa}\bar{\eta}$ mod $i\kappa\pi_8(2)+iG_{22}p=\{i\nu\bar{\sigma}p\}$ since $(\overline{\kappa})\widetilde{\nu^2}=\bar{\kappa}\eta$ and $i\kappa\overline{8\sigma}\equiv 0$ mod $i\kappa\varepsilon p=0$ by vi) and viii) of Theo. 2.1.

i), ii) and iv) imply ix) except for the relation $\nu_2\delta\nu_2=\delta\nu_2^2$. This will be proved in Prop. 2.9.

2.6. We have the relations

$$(\overline{\varepsilon})\tilde{\mu}=(\overline{\mu})\tilde{\varepsilon}=0\ . \tag{2.30}$$

We shall prove the first. By iii) of Theo. 2.1, $(\overline{\varepsilon})\tilde{\mu}=x\eta\bar{\mu}$. Multiply this by η on the left, then we have $x=0$ since $\eta(\overline{\varepsilon})\tilde{\mu}=\varepsilon\tilde{\eta}\tilde{\mu}\in\varepsilon G_{11}=0$ and $\eta^2\bar{\mu}\neq 0$.

Proposition 2.6.

i) $A\nu_2\equiv\nu_2A\equiv\pm\sigma_1A$ mod $\{i\kappa\bar{\eta}, \tilde{\eta}\kappa p\}$.

ii) $\delta A\nu_2=A\delta\nu_2=A\nu_2\delta=\delta\nu_2A=\nu_2\delta A=\nu_2A\delta=\delta\sigma_1A$.

Proof. By v) of Theo. 2.1, $2A\nu_2=2\nu_2A=i\{\varepsilon, 8\sigma, 2\}p=i\eta\rho p=i\{\eta\sigma, 8\sigma, 2\}p=2\sigma_1A$. So, we have, by Theo. 3.3 of [5], $A\nu_2\equiv\nu_2A\equiv\sigma_1A$ mod $\{i\eta\rho p, i\eta^*p, i\overline{16\rho}, i\kappa\bar{\eta}, \tilde{\eta}\kappa p\}$. Multiply these by $\bar{\eta}$ on the left and by $\tilde{\eta}$ on the right at the same time, then we have i) by (2.30).

By use of iii) of Prop. 2.2, ii) of Prop. 2.3 and Prop. 2.4 and i), we obtain ii).

2.7. Lemma 2.7 $\sigma\overline{\nu^2}=\overline{\nu^2}\sigma_1=0$ *and* $\widetilde{\nu^2}\sigma=\sigma_1\widetilde{\nu^2}=0$.

Proof. By iv) of Theo. 2.1, $\sigma\overline{\nu^2}\in\{\sigma, \nu^2, 2\}p\equiv 0 \bmod \sigma^2 p$. Assume that $\sigma\overline{\nu^2}=\sigma^2 p$, then we have, by the definition of $\bar{\sigma}\in(G_{19}; 2)$ (see iv) of Theo. 2.1) and by the relation $\tilde{\eta}\nu=0$ of (2.6),

$$\begin{aligned}\bar{\sigma} &=\{\sigma^2, \eta, \nu\}\\ &\supseteqq\{\sigma^2 p, \tilde{\eta}, \nu\}=\{\sigma\overline{\nu^2}, \tilde{\eta}, \nu\}\\ &\subseteqq\{\sigma, \overline{\nu^2}\tilde{\eta}, \nu\}\\ &=\{\sigma, \varepsilon, \nu\} && \text{by (2.7)}\\ &=0 && \text{by iv) of Theo. 2.1.}\end{aligned}$$

This contradicts to the result that $\bar{\sigma}\neq 0$ in G_{19}. Thus the first relation is proved.

By Theo. 3.1 of [5], $\nu\pi_{11}(2)=\nu\{\eta(\overline{\mu}), \zeta p\}=0$. So, we have $\overline{\nu^2}\sigma_1\in\{\nu^2, 2, \sigma p\}$ $=\{\nu, 2\nu, \sigma p\}=\{\nu, 2\nu, \sigma\}p\equiv 0 \bmod \sigma^2 p$ by iv) of Theo. 2.1. Namely, we can put $\overline{\nu^2}\sigma_1=x\sigma^2 p$. Multiply this by σ on the left, then we have $x=0$ since $\sigma\overline{\nu^2}\sigma_1=0$ by the first relation and $\sigma^3 p\neq 0$ by Theo. 3.1 of [5].

By the quite similar arguments to the above, we obtain the other relations.

As an immediate consequence of this lemma we have

Corollary. $\sigma_1\nu_2=\nu_2\sigma_1=0$.

2.8. Proposition 2.8. $\kappa_1\sigma_1=\sigma_1\kappa_1=i\nu\bar{\sigma}p$.

Proof. By viii) of Theo. 2.1, $\sigma_1\kappa_1=i\sigma(\overline{\kappa})=i\{\sigma, \kappa, 2\}p=i\nu\bar{\sigma}p$.

On the other hand, we have $\kappa_1\sigma_1\in i\{\kappa, 2, \sigma p\}$. The bracket $\{\kappa, 2, \sigma p\}$ is a coset of $\kappa\pi_8(2)+G_{15}\sigma p=\{\eta^2\bar{\kappa}p\}$ since $\sigma\rho=\eta\sigma\kappa=0$, $\varepsilon\kappa=\eta^2\bar{\kappa}$ and $\kappa\overline{8\sigma}\equiv 0 \bmod \varepsilon\kappa p$. By use of Prop. 1.5 of [6], $\{\{2, \overline{\nu^2}, \widetilde{\nu^2}\}, 2, \sigma p\}+\{2, \{\overline{\nu^2}, \widetilde{\nu^2}, 2\}, \sigma p\}+\{2, \overline{\nu^2}, \{\widetilde{\nu^2}, 2, \sigma p\}\}\ni 0$. By (2.15) and Corollary of 2.7, we have $\{\widetilde{\nu^2}, 2, \sigma p\}\equiv \nu_2\sigma_1=0 \bmod \widetilde{\nu^2}\pi_8(2)+\pi_8^*(2)\sigma p=\{i\rho p\}$. By vii) of Theo. 2.1, $\{2, \overline{\nu^2}, i\rho p\}\subseteqq \{2, \nu^2, \rho p\}\supseteqq\{2, \nu^2, \rho\}p=0 \bmod 2\pi_{22}(2)=\{\eta^2\bar{\kappa}p\}$. Therefore, we have, by viii) of Theo. 2.1, $\{\kappa, 2, \sigma p\}=\{2, \kappa, \sigma p\}\supseteqq\{2, \kappa, \sigma\}p=\nu\bar{\sigma}p \bmod \eta^2\bar{\kappa}p$. This leads us to the first relation.

2.9. By ii) of (1.4) of [6], we have $(1\#\nu)(1\#\sigma)=1\#\nu\sigma=0$ and $(1\#\sigma)(1\#\nu)=1\#\sigma\nu=0$, where 1 is the generator of π_0 and $\alpha\#\beta$ is the reduced join (see p. 6 of [6]). Clearly, we have $1\#\nu\in\text{Coext}(\nu p)$ and $1\#\sigma\in\text{Coext}(\sigma p)$. Since $\text{Coext}(\nu p)$ is a coset of $i\pi_4(2)=\{2\eta_3\}$ and $\text{Coext}(\sigma p)$ is a coset of $i\pi_8(2)=\{\delta A, 2\sigma_1, 2\nu_2\}$, we have $1\#\nu=\nu_1+2x\eta_3$ and $1\#\sigma=\pm\sigma_1+2y\nu_2+z\delta A$. So, by the above two relations and the ones that $\nu_1\delta A=$

$\delta A\nu_1=2\eta_3\sigma_1$, we have $\nu_1\sigma_1=\sigma_1\nu_1=2(x+y)\eta_3\sigma_1$.

Now we change the definition of ν_1. We replace ν_1 by $\nu_1+2(x+y)\eta_3$. Then we have

$$\nu_1\sigma_1=\sigma_1\nu_1=0\,. \tag{2.31}$$

We note that $\nu_1+2(x+y)\eta_3$ is contained in Coext(νp) since this is a coset of $\{2\eta_3\}$.

Proposition 2.9.

i) $\nu_1\nu_2=\nu_2\nu_1=\eta_3(\nu_2\pm\sigma_1)$

ii) $\eta_2\eta_3\nu_2=\eta_2\eta_3\sigma_1=\nu_1 A$.

iii) $\nu_1\nu_2^2=\eta_3\nu_2^2=\eta_3\kappa_1$.

iv) $\nu_2\delta\nu_2=\delta\nu_2^2$.

Proof. Since $p\nu_1\nu_2 i=p\nu_2\nu_1 i=\nu^3=\eta(\varepsilon+\sigma\eta)=p\eta_3(\nu_2+\sigma_1)i$, we have, by use of Theo. 3.3 of [5], $\nu_1\nu_2\equiv\nu_2\nu_1\equiv\eta_3(\nu_2+\sigma_1)$ mod $\{2\,\eta_3\sigma_1,\ \eta_1^2A,\ \eta_2^2A\}$. Multiply these by η_1 on the left and by η_2 on the right respectively, then we have $\nu_1\nu_2\equiv\nu_2\nu_1\equiv\eta_3(\nu_2+\sigma_1)$ mod $2\eta_3\sigma_1$. Furthermore, multiply the equality $\nu_1\nu_2=\nu_2\nu_1+2x\eta_3\sigma_1$ by A, then we have $x=0$ by i) of Prop, 2.6. This leads us to i).

By vii) of Prop. 2.1 and iii) of Prop. 2.2 and by i), $0=\nu_1^2\nu_2=\delta\nu_2^2+\nu_2\delta\nu_2$. Namely, iv) is proved.

The proofs of ii) and iii) are left to the reader.

2.10. From the results that $\{\sigma^2, 2, \eta\}=\{\eta, 2, \sigma^2\}\equiv\eta^*$ mod $\eta\rho$ of ii) of Theo. 2.1, we can choose $\overline{\sigma^2}\in$ Ext σ^2 and $\widetilde{\sigma^2}\in$ Coext σ^2 such that

$$\overline{\sigma^2}\tilde{\eta}=\bar{\eta}\widetilde{\sigma^2}=\eta^*. \tag{2.32}$$

By the similar arguments to (2.8), we have

$$\eta\overline{\sigma^2}=\eta^* p \quad \textit{and} \quad \widetilde{\sigma^2}\eta=i\eta^*. \tag{2.33}$$

From the results that $\{\eta^*, 2, \eta\}=-\{\eta, 2, \eta^*\}\equiv\pm 2\nu^*$ mod $\eta\bar{\mu}$ of i) of Theo. 2.1, we can choose $\overline{\eta^*}\in$ Ext η^* and $\widetilde{\eta^*}\in$ Coext η^* such that

$$\overline{\eta^*}\tilde{\eta}=\pm 2\nu^* \quad \textit{and} \quad \bar{\eta}\widetilde{\eta^*}=\pm 2\nu^*\,. \tag{2.34}$$

It is clear that

$$\eta\overline{\eta^*}=\eta^*\bar{\eta} \quad \textit{and} \quad \widetilde{\eta^*}\eta=\tilde{\eta}\eta^*. \tag{2.35}$$

By Theo. 3.1 of [5], it is clear that $i\pi_{16}(2)i=\{i\eta\kappa\}$ and $\eta\pi_{16}(2)=G_{17}p$. By use of ii) of Prop. 1.2 and a) of i) of Prop. 1.5, we can choose $\sigma_2\in$ Coext$(\overline{\sigma^2})\subset\pi_{15}$ such that

(2.36) $$\sigma_2 i = \widetilde{\sigma^2}$$

and

(2.37) $$\bar{\eta}\sigma_2 \equiv \pm \overline{\eta^*} \bmod \{\nu\kappa p, \eta^2\rho p\}.$$

Obviously, we have

(2.38) $$\sigma_2\tilde{\eta} \equiv \pm \widetilde{\eta^*} \bmod \{i\nu\kappa, i\eta^2\rho\}.$$

Since $\{\nu, \sigma^2, 2\} \equiv \nu^* \bmod 2\nu^*$ by iv) of Theo. 2.1, we can choose, by a) of ii) of Prop. 1.8,

(2.39) $$\nu_1\sigma_2 \in \text{Coext}(\nu^* p) \subset \pi_{18}.$$

Proposition 2.10.

i) $\delta\sigma_2 = i\overline{\sigma^2}$ *and* $\sigma_2\delta = \widetilde{\sigma^2}p$.

ii) $\sigma_1^2 = \delta\sigma_2 + \sigma_2\delta$.

iii) $\eta_1\sigma_2 \equiv i\overline{\eta^*} \bmod i\nu\kappa p$ *and* $\sigma_2\eta_1 \equiv \eta_1\sigma_2 \bmod i\nu\kappa p$.

iv) $\sigma_2\eta_2 \equiv \widetilde{\eta^*}p \bmod i\nu\kappa p$ *and* $\eta_2\sigma_2 \equiv \sigma_2\eta_2 \bmod i\nu\kappa p$.

v) $\eta_3\sigma_2 \equiv \pm \widetilde{\eta}\overline{\eta^*} \bmod i\bar{\zeta}p$ *and* $\sigma_2\eta_3 \equiv \pm \eta_3\sigma_2 \bmod i\bar{\zeta}p$.

vi) $\sigma_2\nu_1 \equiv \nu_1\sigma_2 \bmod \{2\eta_3\sigma_2, i\bar{\zeta}p, i\bar{\sigma}p\}$.

The proof is similar to the one of Prop. 2.2 and we omit it.

We note that the following relations hold.

(2.40) $$\overline{\nu^2}\widetilde{\sigma^2} = \overline{\sigma^2}\widetilde{\nu^2} = \sigma^3.$$

We shall prove the second relation. By Lemma 2.7 and ii) of Prop. 2.10, $0 = \sigma_1^2\widetilde{\nu^2} = \delta\sigma_2\widetilde{\nu^2} + \sigma_2\delta\widetilde{\nu^2}$. So, we have $i\overline{\sigma^2}\widetilde{\nu^2} = \widetilde{\sigma^2}\nu^2 = i\{2, \sigma^2, \nu\}\nu = i\nu^*\nu = i\sigma^3$. Therefore we obtain $\overline{\sigma^2}\widetilde{\nu^2} - \sigma^3 \in 2G_{21} = 0$ (see [3]).

2.11 By use of ii) of Prop. 1.2, we can choose $\kappa_3 \in \text{Coext}(\nu(\bar{\kappa})) \subset \pi_{18}$ such that

(2.41) $$\kappa_3 i = \tilde{\kappa}\nu + x i\nu^*.$$

Since $\nu_1\sigma_2 i = i\nu^*$, we have

(2.42) $$\kappa_3 + x\nu_1\sigma_2 \in \text{Ext}(\tilde{\kappa}\nu).$$

Since $\{\bar{\eta}, \tilde{\kappa}\nu, 2\} = \{2, \nu(\bar{\kappa}), \tilde{\eta}\} \equiv 0 \bmod 2G_{20}$ by viii) of Theo. 2.1, we have, by sue of a) of ii) of Prop. 1.5 and 1.6 and by (2.6),

(2.43) $$\bar{\eta}\kappa_3 = 0 \quad \textit{and} \quad \kappa_3\tilde{\eta} = 0.$$

Proposition 2.11.

i) $\delta\kappa_3 = i\nu(\bar{\kappa})$ *and* $\kappa_3\delta \equiv \tilde{\kappa}\nu p \bmod i\nu^* p$.

ii) $\eta_1\kappa_3=\kappa_3\eta_1=\eta_2\kappa_3=\kappa_3\eta_2=0$ *and* $\eta_3\kappa_3=\kappa_3\eta_3=0$.

iii) $\nu_1\kappa_3\equiv\nu_2\kappa_1+\kappa_1\nu_2 \bmod \{i\sigma\overline{\sigma^2},\ i\nu\bar{\sigma}p\}$ *and* $\kappa_3\nu_1\equiv\nu_1\kappa_3 \bmod \{i\sigma\overline{\sigma^2},\ \widetilde{\sigma^2}\sigma p,\ i\nu\bar{\sigma}p\}$.

The proof is easy and left to the reader.

2.12. From the results that $\{\bar{\sigma}, 2, \eta\}=\{\eta, 2, \bar{\sigma}\}\equiv 0 \bmod \eta\bar{\kappa}$ of ii) of Theo. 2.1, we can choose $\overline{(\bar{\sigma})}\in\mathrm{Ext}\,(\bar{\sigma})$ and $\widetilde{(\bar{\sigma})}\in\mathrm{Coext}\,(\bar{\sigma})$ such that

(2.44) $$\overline{(\bar{\sigma})}\tilde{\eta} = \bar{\eta}\widetilde{(\bar{\sigma})} = 0\,.$$

It is clear that

(2.45) $$\eta\overline{(\bar{\sigma})} = 0 \quad \textit{and} \quad \widetilde{(\bar{\sigma})}\eta = 0\,.$$

Since $i\pi_{21}(2)i=0$ by Theo. 3.1 of [5], we have, by ii) of Prop. 1.2,

(2.46) $$\mathrm{Coext}\left(\overline{(\bar{\sigma})}\right)i = \widetilde{(\bar{\sigma})}\,.$$

Choose $\bar{\sigma}_1\in\mathrm{Coext}\left(\overline{(\bar{\sigma})}\right)\subset\pi_{20}$ arbitrarily, then we have

(2.47) $$\bar{\eta}\bar{\sigma}_1\equiv 0 \bmod \{\nu\bar{\sigma}p,\ \eta^2\bar{\kappa}p\} \quad \textit{and} \quad \bar{\sigma}_1\tilde{\eta}\equiv 0 \bmod \{i\nu\bar{\sigma},\ i\eta^2\bar{\kappa}\}\,.$$

Proposition 2.12. $\eta_1\bar{\sigma}_1\equiv\bar{\sigma}_1\eta_2\equiv\eta_2\bar{\sigma}_1=\bar{\sigma}_1\eta_1=0 \bmod i\nu\bar{\sigma}p$.

Proof. The first two relations are obivous.

For the proofs that $\eta_2\bar{\sigma}_1=\bar{\sigma}_1\eta_1=0$, we use the facts that $\{2, \eta, \bar{\sigma}, 2\}=\{2, \bar{\sigma}, \eta, 2\}\equiv 0 \bmod \eta^2\bar{\kappa}$ of ix) of Theo. 2.1. The details are left to the reader.

2.13. Since $i\pi_{21}(2)i=0$ by Theo. 3.1 of [5], we have, by use of Prop. 1.1,

(2.48) $$\mathrm{Coext}\,(\bar{\kappa}p)i = i\bar{\kappa}\,.$$

Let $\bar{\kappa}_1$ be a representative of $\mathrm{Coext}\,(\bar{\kappa}p)\subset\pi_{20}$, then we have, by use of a) of i) of Prop. 1.3 and 1.4,

(2.49) $$\bar{\eta}\bar{\kappa}_1\equiv\pm\bar{\kappa}\bar{\eta} \bmod \nu\bar{\sigma}p \quad \textit{and} \quad \bar{\kappa}_1\tilde{\eta}\equiv\pm\tilde{\eta}\bar{\kappa} \bmod i\nu\bar{\sigma}\,.$$

Proposition 2.13.

i) $\bar{\kappa}_1\delta=\delta\bar{\kappa}_1=i\bar{\kappa}p$.

ii) $\bar{\kappa}_1\eta_1=i\bar{\kappa}\bar{\eta}=$ *and* $\eta_1\bar{\kappa}_1\equiv\bar{\kappa}_1\eta_1 \bmod i\nu\bar{\sigma}p$.

iii) $\eta_2\bar{\kappa}_1=\tilde{\eta}\bar{\kappa}p$ *and* $\bar{\kappa}_1\eta_2\equiv\eta_2\bar{\kappa}_1 \bmod i\nu\bar{\sigma}p$.

iv) $\nu_2^3\equiv\nu_2\kappa_1+\eta_2\bar{\kappa}_1 \bmod i\nu\bar{\sigma}p$.

The proof is obvious.

3. Proof of Theorem 2.1

In this section we shall prove Theorem 2.1 which holds the key to our computations in the previous section.

We can find almost all of the results of Theo. 2.1 in [3], [4] and [6]. The ones which we can not find there will be proved by use of the methods and the results of [6].

3.1. Proof of i)

$\{\eta, 2, \eta\}=\pm 2\nu$ by (5,4) in p. 41 of [6].

$\{\nu, \eta, 2\}=0$ since $G_5=0$.

$\{\eta, 2, \eta^*\}\equiv\pm 2\nu^*$ mod $\eta\bar{\mu}$ since $2\{\eta, 2, \eta^*\}=\{2, \eta, 2\}\eta^*=\eta^2\eta^*=4\nu^*$ by Cor. 3.7 in p. 31 of [6].

$\{\nu^*, \eta, 2\}\equiv 0$ mod $2G_{20}$ since $\eta\bar{\kappa}\neq 0$ and $\{\nu^*, \eta, 2\}\eta=\nu^*\{\eta, 2, \eta\}=2\nu\nu^*=0$.

3.2. Proof of ii)

$\{\eta, 2, \nu^2\}=\{\eta, \nu^2, 2\}\equiv\varepsilon$ mod $\eta\sigma$ by (6.1) in p. 51 of [6].

$\{\nu^2, \eta, 2\}=\nu\{\nu, \eta, 2\}=0$ by i).

$\{\eta, 2, 8\sigma\}=\{\eta, 8\sigma, 2\}\equiv\mu$ mod $\{\eta^2\sigma, \eta\varepsilon\}$. See p. 189 of [6].

$\{8\sigma, \eta, 2\}=0$ since $\{8\sigma, \eta, 2\}\subseteqq\{2, 0, 2\}=2G_9=0$.

$\{\eta, 2, \sigma^2\}=\{\eta, \sigma^2, 2\}\equiv\eta^*$ mod $\eta\rho$ and $\{\sigma^2, \eta, 2\}=0$. See the proof of (2) of Lemma 4.2 in p. 279 of [5].

$\{\eta, 2, \bar{\sigma}\}=\{\eta, \bar{\sigma}, 2\}\equiv 0$ mod $\eta\bar{\kappa}$ and $\{\bar{\sigma}, \eta, 2\}=0$. See the proof of (4) of Lemma 4.2 in p. 280 of [5].

$\{\mu, 2, 8\sigma\}=\{\mu, 8\sigma, 2\}\equiv\bar{\mu}$ mod $\eta^2\rho$. See p. 189 of [6].

$\{8\sigma, \mu, 2\}\subseteqq\{2, 0, 2\}=2G_{17}=0$

3.3. Proof of iii)

$\{\eta, 2, \varepsilon\}\equiv 0$ mod $\eta\mu$. We know that $\{\eta, 2, \bar{\nu}\}\equiv 0$ mod $\eta\mu$ by (10.1) in p. 95 of [6]. Since $\bar{\nu}=\eta\sigma+\varepsilon$ and $\{\eta, 2, \eta\sigma\}=\{\eta, 2, \eta\}\sigma=2\nu\sigma=0$, we have the assertion.

$\{\eta, 2, \kappa\}\equiv 0$ mod $\eta\rho$ by Lemma 15.2 in p. 39 of [4].

$\{\mu, 2, \varepsilon\}\equiv 0$ mod $\eta\bar{\mu}$ since $(G_{18}; 2)=\{\nu^*, \eta\bar{\mu}\}$, $\nu\{\mu, 2, \varepsilon\}=\{\nu, \mu, 2\}\varepsilon\in G_{13}\varepsilon=0$ and $\nu\nu^*=\sigma^3\neq 0$.

3.4. Proof of iv)

$\{\sigma, \nu^2, 2\}=\{\sigma, \nu, 2\nu\}\equiv 0$ mod σ^2 by the fact $\{\nu, \sigma, \nu\}=\sigma^2$ (see Exsmple 4 in p. 85 of [6]) and by (3.10) in p. 33 of [6].

$\{\nu, \sigma^2, 2\}\supset\{\nu, \sigma, 2\sigma\}=\{\nu, 2\sigma, \sigma\}=\nu^*$ mod $2\nu^*$. See p. 153 of [6].

$\{\sigma^2, \eta, \nu\}=\{\sigma, \eta\sigma, \nu\}=\bar{\sigma}$ by the definition of $\bar{\sigma}$ (see p. 189 of [6]).

$\{\sigma, \varepsilon, \nu\}=0$. It is sufficient to prove $\{\sigma, \bar{\nu}, \nu\}=\bar{\sigma}$. By use of (3.7) in p. 33 of [6], $\{\{\nu, \sigma, \nu\}, \eta, \nu\}-\{\nu, \{\sigma, \nu, \eta\}, \nu\}+\{\nu, \sigma, \{\nu, \eta, \nu\}\}\ni 0$. Since $\{\nu, \sigma, \nu\}=\sigma^2$, $\{\sigma, \nu, \eta\}\subseteqq(G_{12}; 2)=0$ and $\{\nu, \eta, \nu\}=\bar{\nu}$ (see p. 53 of [6]), we have $\{\nu, \sigma, \bar{\nu}\}=\{\sigma^2, \eta, \nu\}=\bar{\sigma}$. By use of ii) of (3.9) in p. 33 of [6], $\{\nu, \sigma, \bar{\nu}\}-\{\sigma, \bar{\nu}, \nu\}+\{\bar{\nu}, \nu, \sigma\}\ni 0$. Since $\{\nu^2, 2, \eta\}\equiv\bar{\nu}$ mod $\eta\sigma$ and $\{\eta\sigma, \nu, \sigma\}=\sigma\{\eta, \nu, \sigma\}=0$, we can put $\{\bar{\nu}, \nu, \sigma\}=\{\{\nu^2, 2, \eta\}, \nu, \sigma\}$. By use of (3.7) of [6], $\{\{\nu^2, 2, \eta\}, \nu, \sigma\}+\{\nu^2, \{2, \eta, \nu\}, \sigma\}+\{\nu^2, 2, \{\eta, \nu, \sigma\}\}\ni 0$. So, we have $\{\bar{\nu}, \nu, \sigma\}=0$. From this and the above, we have $\{\sigma, \bar{\nu}, \nu\}=\{\nu, \sigma, \bar{\nu}\}=\bar{\sigma}$.

3.5. Proof of v)

$\{\eta\varepsilon, \eta, 2\}=\{\eta^2\sigma, \eta, 2\}\equiv\zeta \bmod 2G_{11}$ by Lemma 9.1 in p. 91 of [6].

$\{\eta^2\rho, \eta, 2\}\equiv\bar{\zeta} \bmod 2G_{19}$. See (3) of Lemma 4.2 in p. 278 of [5].

$\{\nu, 8\sigma, 2\}\supset\{\nu, 8, 2\sigma\}=\zeta \bmod 2G_{11}$. See p. 189 of [6].

$\{\zeta, 8\sigma, 2\}\supset\{\zeta, 2, 8\sigma\}=\bar{\zeta} \bmod 2G_{19}$. See p. 189 of [6].

$\{\sigma, 8\sigma, 2\}\supset\{\sigma, 2\sigma, 8\}\ni\rho \bmod 2G_{15}$ by Lemma 10.9 in p. 110 of [6].

$\{\varepsilon, 8\sigma, 2\}=\{\eta\sigma, 8\sigma, 2\}=\eta\rho$. We have $\{\bar{\nu}, 8\sigma, 2\}\eta=\bar{\nu}\{8\sigma, 2, \eta\}\equiv\bar{\nu}\mu=0 \bmod \{\bar{\nu}\eta^2\sigma, \bar{\nu}\eta\varepsilon\}=0$. Since $G_{16}=\{\eta\rho, \eta^*\}=Z_2+Z_2$ and $\eta^2\rho$ and $\eta\eta^*$ are linearly independent in G_{17}, we obtain $\{\bar{\nu}, 8\sigma, 2\}\equiv 0 \bmod \{\bar{\nu}\eta\sigma, \bar{\nu}\varepsilon\}+2G_{17}=0$. On the other hand, $\{\bar{\nu}, 8\sigma, 2\}=\{\varepsilon, 8\sigma, 2\}+\{\eta\sigma, 8\sigma, 2\}$ and $\{\eta\sigma, 8\sigma, 2\}=\eta\{\sigma, 8\sigma, 2\}=\eta\rho$. This leads us to the assertion.

$\{8\sigma, 2, 8\sigma\}=16\rho$. See p. 103 of [6].

3.6. Proof of vi)

$\{\eta\kappa, \eta, 2\}=\nu\kappa$. By Lemma 15.1 in p. 39 of [4], $\{\eta\kappa, \eta, 2\}\subset\{\eta, \eta\kappa, 2\}\equiv\nu\kappa \bmod \{\eta\eta^*, \eta^2\rho\}$. Since $\{\eta\kappa, \eta, 2\}$ is a coset of 0, we can put $\{\eta\kappa, \eta, 2\}=\nu\kappa+x\eta\eta^*+y\eta^2\rho$, where x and y are 0 or 1 respectively. Multiply this equality by η, then we have $x=0$ since $\{\eta\kappa, \eta, 2\}\eta=\eta\kappa\{\eta, 2, \eta\}=2\nu\eta\kappa=0$, $\eta\nu\kappa=\eta^3\rho=0$ and $\eta^2\eta^*\neq 0$ in G_{18}. Multiply it by $\bar{\eta}$ on the right, then we have $y=0$. For $\kappa\nu\bar{\eta}=\kappa\{\nu, \eta, 2\}p=0$ by i), $\eta^2\rho\bar{\eta}=\{\eta^2\rho, \eta, 2\}p=\bar{\zeta}p\neq 0$ by v) and Theo. 3.1 of [5] and $\{\eta\kappa, \eta, 2\}\bar{\eta}\subseteq\{\eta\kappa, \eta, \eta^2 p\}\subseteq\{\kappa, 4\nu, \eta p\}\supseteq\{\kappa, 4, 0\}\equiv 0 \bmod \kappa\pi_5(2)+G_{18}\eta p=0$.

$\{\kappa, 2, \nu^2\}=\eta\bar{\kappa}$. By the definition of $\bar{\kappa}$ (see p. 44 of [4]) and by the fact $\nu^2=\{\eta, \nu, \eta\}$ (see Example 4 in p. 85 of [6]), we have $\eta\bar{\kappa}=\eta\{\nu, \bar{\eta}, \tilde{\kappa}\}=\{\eta, \nu, \bar{\eta}\}\tilde{\kappa}=\{\nu^2, 2, \kappa\}$.

3.7. Proof of vii)

$\{2, \nu^2, \rho\}=0$. By use of (3.7) in p. 33 of [6], $\{2, \nu^2, \{\sigma, 2\sigma, 8\}\}+\{2, \{\nu^2, \sigma, 2\sigma\}, 8\}+\{\{2, \nu^2, \sigma\}, 2\sigma, 8\}\ni 0$. Since $\{\nu^2, \sigma, 2\sigma\}=\nu\{\nu, \sigma, 2\sigma\}=\nu\nu^*=\sigma^3$, $\{2, \nu^2, \sigma\}\equiv 0 \bmod \sigma^2$ and $\{\sigma^2, 2\sigma, 8\}=\sigma\{\sigma, 2\sigma, 8\}=\sigma\rho=0$, we have $\{2, \{\nu^2, \sigma, 2\sigma\}, 8\}=\{2, \sigma^3, 8\}=4\{2, \sigma^3, 2\}=0$ and $\{\{2, \nu^2, \sigma\}, 2\sigma, 8\}=0$. This leads us to the assertion.

$\{\nu, \eta, \eta^2\sigma\}=\{\nu, 4\nu, \sigma\}=2\{\nu, 2\nu, \sigma\}=0$.

$\{\sigma, \nu, \zeta\}=0$. By use of (3.7) in p. 33 of [6], $\{\sigma, \nu, \{\eta, \eta^2\sigma, 2\}\}-\{\sigma, \{\nu, \eta, \eta^2\sigma\}, 2\}+\{\{\sigma, \nu, \eta\}, \eta^2\sigma, 2\}\ni 0$. We have $\{\sigma, \{\nu, \eta, \eta^2\sigma\}, 2\}=\{\sigma, 0, 2\}=\sigma G_{15}+2G_{22}=0$ and $\{\{\sigma, \nu, \eta\}, \eta^2\sigma, 2\}=\{0, \eta^2\sigma, 2\}=2G_{22}=0$. This leads us to the assertion.

3.8. Proof. of viii)

$\{\bar{\eta}, \bar{\kappa}\nu, 2\}\equiv 0 \bmod 2G_{20}$. By the proof of Lemma 15.3 in p. 43 and Lemma 15.4 in p. 44 of [4], $\{\eta\kappa, \eta, \nu\}=\pm 2\bar{\kappa}$. On the other hand $\{\eta\kappa, \eta, \nu\}=\{\bar{\eta}, i\eta\kappa, \nu\}=\{\bar{\eta}, 2\bar{\kappa}, \nu\}\supseteq\{\bar{\eta}, \bar{\kappa}, 2\nu\}\subseteq\{\bar{\eta}, \bar{\kappa}\nu, 2\}$. Therefore we have the assertion.

$\{2, \nu(\overline{\kappa}), \tilde{\eta}\} \equiv 0 \bmod 2G_{20}$. The proof is quite similar to the above and we omit it.

$\{\overline{\nu^2}, \widetilde{\nu^2}, 2\} = \{2, \overline{\nu^2}, \widetilde{\nu^2}\} = \kappa$. The proof that $\{\overline{\nu^2}, \widetilde{\nu^2}, 2\} \equiv \{2, \overline{\nu^2}, \widetilde{\nu^2}\} \equiv \kappa \bmod \sigma^2$ is quite similar to the dicussions in p. 40 of [4] and we omit it. By Lemma of §2, $\sigma\{\overline{\nu^2}, \widetilde{\nu^2}, 2\} = 2\{\sigma, \overline{\nu^2}, \widetilde{\nu^2}\} = 2G_{21} = 0$ and $\{2, \overline{\nu^2}, \widetilde{\nu^2}\}\sigma = 2\{\overline{\nu^2}, \widetilde{\nu^2}, \sigma\} = 0$. So, the fact $\sigma^3 \neq 0$ leads us to the assertion.

$\{\kappa, 8\sigma, 2\} = \{\kappa, 2, 8\sigma\} \equiv 0 \bmod \eta^2\bar{\kappa}$. By use of Prop. 1.5 in p. 12 of [6], $\{\{2, \overline{\nu^2}, \widetilde{\nu^2}\}, 2, 8\sigma\} + \{2, \{\overline{\nu^2}, \widetilde{\nu^2}, 2\}, 8\sigma\} + \{2, \overline{\nu^2}, \{\overline{\nu^2}, 2, 8\sigma\}\} \ni 0$. We have $\{\overline{\nu^2}, 2, 8\sigma\} \subseteqq iG_{15} = \{i\rho, i\eta\kappa\}$ since $\{\nu^2, 2, 8\sigma\} = \{\nu, 2\nu, 8\sigma\} = 8\{\nu, 2\nu, \sigma\} = 0$. It follows that $\{2, \overline{\nu^2}, i\rho\} \subseteqq \{2, \nu^2, \rho\} = 0$ and $\{2, \overline{\nu^2}, i\eta\kappa\} \subseteqq \{2, \nu^2, \eta\kappa\} = \{2, \nu^2, \eta\}\kappa = \varepsilon\kappa = \eta^2\bar{\kappa}$. Therefore, we have $\{\kappa, 2, 8\sigma\} \equiv \{2, \kappa, 8\sigma\} = 8\{2, \kappa, \sigma\} = 8G_{22} = 0 \bmod \eta^2\bar{\kappa}$.

$\{\sigma, \kappa, 2\} = \nu\bar{\sigma}$. Since $\{\nu, \eta, \eta^2\sigma\} = 0$ and $\{\nu, \eta, \nu^3\} = \{\nu, \eta, \nu\}\nu^2 = \bar{\nu}\nu^2 = 0$, the tertial composition $\{\nu, \eta, 2, \bar{\nu}\}$ is a coset of 0. Obviously, $\sigma\{\nu, \eta, 2, \bar{\nu}\} = \sigma\{(\overline{\nu})_\eta, \tilde{2}_\eta, \bar{\nu}\} = \{\sigma, (\overline{\nu})_\eta, \tilde{2}_\eta\}\bar{\nu} = G_{13}\bar{\nu} = 0$. So, by the definition of κ, we can take $\kappa = \{(\overline{\nu})_\eta, \tilde{2}_\eta, \bar{\nu}\}$ (see p. 96 of [6]).

By use of Prop. 1.5 in p. 12 of [6], $\{\sigma, \{(\overline{\nu})_\eta, \tilde{2}_\eta, \bar{\nu}\}, 2\} + \{\sigma, (\overline{\nu})_\eta, \{\tilde{2}_\eta, \bar{\nu}, 2\}\} + \{\{\sigma, (\overline{\nu})_\eta, \tilde{2}_\eta\}, \bar{\nu}, 2\} \ni 0$. Since $\{\sigma, (\overline{\nu})_\eta, \tilde{2}_\eta\} \subseteqq G_{13} \approx Z_3$, we have $\{\sigma, \kappa, 2\} = \{\sigma, (\overline{\nu})_\eta, \{\tilde{2}_\eta, \bar{\nu}, 2\}\}$. Since $\{2, \bar{\nu}, 2\} = \eta\bar{\nu} = \nu^3$ by Cor. 3.7 in p. 31 of [6], we can take $(\tilde{\bar{\nu}}_\eta)\nu^2 \equiv \{\tilde{2}_\eta, \bar{\nu}, 2\} \bmod i_\eta\zeta$. Since $\{\sigma, (\overline{\nu})_\eta, i_\eta\zeta\} \subseteqq \{\sigma, \nu, \zeta\} = 0$, we have $\{\sigma, \kappa, 2\} = \{\sigma, (\overline{\nu})_\eta, \tilde{\bar{\nu}}_\eta\nu^2\} = \{\sigma, (\overline{\nu})_\eta\tilde{\bar{\nu}}_\eta, \nu^2\} = \{\sigma, \bar{\nu}, \nu^2\} = \{\sigma, \bar{\nu}, \nu\}\nu = \nu\bar{\sigma}$.

3.9. Proof of ix)

$\{2, 4\nu, \eta, 2\} = 0$. By the definition of $\bar{\kappa}$, we have $0 = 8\bar{\kappa} = 8\{(\overline{\nu})_\eta, \tilde{2}_\eta, \kappa\} \subseteqq 2\{4(\nu)_\eta, \tilde{2}_\eta, \kappa\} = \{2, 4(\overline{\nu})_\eta, \tilde{2}_\eta\}\kappa = \{2, 4\nu, \eta, 2\}\kappa$. It is clear that $\{2, 4\nu, \eta, 2\}$ is a coset of 0. So, we have $\{2, 4\nu, \eta, 2\}\kappa = 0$. Since $G_6 = \{\nu^2\}$ and $\nu^2\kappa = 4\bar{\kappa} \neq 0$ (see Lemma 15.4 in p. 44 of [4]), we have the assertion.

$\{2, \bar{\sigma}, \eta, 2\} \equiv 0 \bmod \eta^2\bar{\kappa}$. The proof that $\{2, \bar{\sigma}, \eta, 2\}$ is a coset of $\eta^2\bar{\kappa}$ is left to the reader.

Since $\bar{\sigma} = \{\eta\sigma, \sigma, \nu\}$ and $\{\sigma, \nu, \eta\} = 0$, we can choose $(\overline{\bar{\sigma}})_\eta \in \{\eta\sigma, \sigma, (\overline{\nu})_\eta\}$. So, we can put $\{2, \bar{\sigma}, \eta, 2\} \equiv \{2, (\overline{\bar{\sigma}})_\eta, \tilde{2}_\eta\} = \{2, \{\eta\sigma, \sigma, (\overline{\nu})_\eta\}, \tilde{2}_\eta\} \bmod \eta^2\bar{\kappa}$. By use of Prop. 1.5 of [6], $\{2, \{\eta\sigma, \sigma, (\overline{\nu})_\eta\}, \tilde{2}_\eta\} + \{2, \eta\sigma, \{\sigma, (\overline{\nu})_\eta, \tilde{2}_\eta\}\} + \{\{2, \eta\sigma, \sigma\}, (\overline{\nu})_\eta, \tilde{2}_\eta\} \ni 0$. Since $\{\sigma, (\overline{\nu})_\eta, \tilde{2}_\eta\} \subseteqq G_{13} \approx Z_3$, $\{2, \eta\sigma, \sigma\} \equiv \{2, \eta, \sigma^2\} = 0 \bmod \mu\sigma$ and $\{\mu\sigma, (\overline{\nu})_\eta, \tilde{2}_\eta\} = \mu\{\sigma, (\overline{\nu})_\eta, \tilde{2}_\eta\} = 0$, we have the assertion.

$\{2, \eta, \bar{\sigma}, 2\} \equiv 0 \bmod \eta^2\bar{\kappa}$. The proof is quite similar to the above and we omit it.

4. The ring structure of π_*

In this section we shall state our main theorems.

By use of the discussions in §2 and Theo. 3.3 of [5], we obtain the following

Theorem 4.1. *A set of additive generators for* π_* *is as follows in* dim $\leqq 21$:

δ, 1, η_1, η_2, η_1^2, η_2^2, $\delta\nu_1$, η_3, ν_1, $\eta_2\eta_3$, $\delta\nu_2\delta$, $\delta\nu_2$, $\nu_2\delta$, $\delta\sigma_1$, ν_2, σ_1, δA, A, $\eta_1\nu_2$, $\eta_2\nu_2$, $\eta_1\sigma_1$, $\eta_2\sigma_1$, $\eta_1^2\nu_2$, $\eta_2^2\nu_2$, $\eta_1^2\sigma_1$, $\eta_2^2\sigma_1$, $\eta_1 A$, $\eta_2 A$, $\eta_1^2 A$, $\eta_2^2 A$, $\eta_3\sigma_1$, $\eta_3(\nu_2+\sigma_1)$, $\eta_3 A$, $\nu_1 A$, $\eta_2\eta_3 A$, $\delta\sigma_2\delta$, $\kappa_1\delta$, $\delta\sigma_2$, $\sigma_2\delta$, κ_1, ν_2^2, $\delta\sigma_1 A$, σ_2, $\sigma_1 A$, $\eta_1\kappa_1$, $\eta_2\nu_2^2$, δA^2, A^2, κ_2, $\eta_1\sigma_2$, $\eta_2\sigma_2$, $\eta_1\sigma_1 A$, $\eta_2\sigma_1 A$, $\eta_1 A^2$, $\eta_2 A^2$, $\delta\kappa_3$, $\kappa_3\delta$, $\eta_1^2\sigma_2$, $\eta_2^2\sigma_2$, $\eta_1^2\sigma_1 A$, $\eta_2^2\sigma_1 A$, $\delta\nu_1\sigma_2$, κ_3, $\nu_1\sigma_2$, $\eta_1^2 A^2$, $\eta_2^2 A^2$, $\eta_3\sigma_2$, $\eta_3\sigma_1 A$, $\delta\bar{\sigma}_1\delta$, $\eta_2\eta_3\sigma_2$, $\eta_3 A^2$, $\nu_1 A^2$, $\delta\bar{\sigma}_1$, $\bar{\sigma}_1\delta$, $\delta\bar{\kappa}_1$, $\bar{\sigma}_1$, $\bar{\kappa}_1$, $\eta_2\eta_3 A^2$, $\delta\nu_2\kappa_1$, $\delta\sigma_1\sigma_2\delta$, $\nu_2\kappa_1$, $\eta_1\bar{\kappa}_1$, $\eta_2\bar{\kappa}_1$, $\delta\sigma_1\sigma_2$, $\sigma_1\sigma_2\delta$, $\delta\nu_1\bar{\sigma}_1\delta$.

The ring structure of π_*, in dim$\leqq 21$, is given by the following

Theorem 4.2. *The ring* π_*, *in* dim$\leqq 21$, *is generated by* δ, η_1, η_2, η_3, ν_1, ν_2, σ_1, A, κ_1, σ_2, κ_2, κ_3, $\bar{\sigma}_1$, $\bar{\kappa}_1$, *with the following relations*:

$\delta^2=0$,

$\delta\eta_1=\eta_2\delta=0$, $\delta\eta_2=\eta_1\delta=2.1$,

$\eta_1\eta_2=\eta_2\eta_1=0$, $\delta\eta_3=\eta_1^2$, $\eta_3\delta=\eta_2^2$, $\nu_1\delta=\delta\nu_1$,

$\eta_1\eta_3=\eta_3\eta_2=\eta_1\nu_1=\nu_1\eta_1=\eta_2\nu_1=\nu_1\eta_2=0$, $\eta_3\eta_1=\eta_2\eta_3$,

$\eta_2^2\eta_3=0$,

$\eta_3^2=\eta_3\nu_1=\nu_1\eta_3=0$, $\nu_1^2=\delta\nu_2+\nu_2\delta$, $\sigma_1\delta=\delta\sigma_1$,

$A\delta=\delta A+2x\sigma_1+2y\nu_2$,

$\nu_2\eta_1=\eta_1\nu_2$, $\nu_2\eta_2=\eta_2\nu_2$, $\sigma_1\eta_1=\eta_1\sigma_1$, $\sigma_1\eta_2=\eta_2\sigma_1$,

$A\eta_1=\eta_1 A+x\eta_1^2\sigma_1+y\eta_1^2\nu_2$, $A\eta_2=\eta_2 A+x\eta_2^2\sigma_1+y\eta_2^2\nu_2$,

$\nu_2\eta_3=\pm\eta_3\nu_2$, $\sigma_1\eta_3=\pm\eta_3\sigma_1$, $\nu_2\nu_1=\nu_1\nu_2=\eta_3(\nu_2\pm\sigma_1)$, $\nu_1\sigma_1=\sigma_1\nu_1=0$.

$\eta_2\eta_3\sigma_1=A\nu_1=\nu_1 A$, $A\eta_3=\pm\eta_3 A+(x+y)\nu_1 A$,

$\delta\kappa_1=0$, $\delta\nu_2^2=\kappa_1\delta$,

$\sigma_1\nu_2=\nu_2\sigma_1=0$, $\sigma_1^2=\delta\sigma_2+\sigma_2\delta$,

$\eta_2\kappa_1=\kappa_1\eta_2=0$, $\eta_1\nu_2^2=\delta\kappa_2=\kappa_1\eta_1=\eta_1\kappa_1$, $\kappa_2\delta=\eta_2\nu_2^2$, $A\sigma_1=\pm\sigma_1 A$, $A\nu_2\equiv\nu_2 A\equiv \sigma_1 A \bmod \{2\sigma_1 A, \eta_1\kappa_1, \eta_2\nu_2^2\}$,

$\sigma_2\eta_1\equiv\eta_1\sigma_2 \bmod 2\kappa_2$, $\sigma_2\eta_2\equiv\eta_2\sigma_2 \bmod 2\kappa_2$,

$\eta_1\kappa_2=\kappa_2\eta_2=\kappa_1\eta_3=0$, $\kappa_1\nu_1\equiv\nu_1\kappa_1=\delta\kappa_3 \bmod \delta\nu_1\sigma_2$, $\eta_3\nu_2^2=\eta_3\kappa_1=\eta_2\kappa_2=\kappa_2\eta_1\equiv\delta\kappa_3+\kappa_3\delta \bmod \delta\nu_1\sigma_2$,

$\sigma_2\eta_3\equiv\pm\eta_3\sigma_2 \bmod 2\eta_3\sigma_1 A$, $\sigma_2\nu_1\equiv\nu_1\sigma_2 \bmod \{2\eta_3\sigma_2, 2\eta_3\sigma_1 A, \delta\bar{\sigma}_1\delta\}$,

$\eta_1\kappa_3=\kappa_3\eta_1=\eta_2\kappa_3=\kappa_3\eta_2=\eta_3\kappa_2=\kappa_2\eta_3=\nu_1\kappa_2=\kappa_2\nu_1=0$, $\bar{\kappa}_1\delta=\delta\bar{\kappa}_1$,

$\eta_3\kappa_3=\kappa_3\eta_3=\eta_2\bar{\sigma}_1=\bar{\sigma}_1\eta_1=0$, $\eta_1\bar{\sigma}_1\equiv\bar{\sigma}_1\eta_2\equiv 0 \bmod \delta\nu_1\bar{\sigma}_1\delta$, $\sigma_1\kappa_1=\kappa_1\sigma_1=\delta\nu_1\bar{\sigma}_1\delta$, $\kappa_1\nu_2\equiv\bar{\kappa}_1\eta_1\equiv\eta_1\bar{\kappa}_1 \bmod \delta\nu_1\bar{\sigma}_1\delta$, $\bar{\kappa}_1\eta_2\equiv\eta_2\bar{\kappa}_1 \bmod \delta\nu_1\bar{\sigma}_1\delta$, $\nu_2^3\equiv\nu_2\kappa_1+\eta_2\bar{\kappa}_1 \bmod \delta\nu_1\bar{\sigma}_1\delta$, $\nu_1\kappa_3\equiv\nu_2\kappa_1+\eta_1\bar{\kappa}_1 \bmod \{\delta\sigma_1\sigma_2, \delta\nu_1\bar{\sigma}_1\delta\}$, $\kappa_3\nu_1\equiv\nu_1\kappa_3 \bmod \{\delta\sigma_1\sigma_2, \sigma_1\sigma_2\delta, \delta\nu_1\bar{\sigma}_1\delta\}$, *where* x *and* y *are fixed integers* 0 *or* 1 *respectively*.

Proof. The relations hold by use of our propositions in §2.

To complete the proof of this theorem, we must construct the table obtained

Table of relations, I.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ_3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
δ	0	0	2	η_1^2	$\delta\nu_1$	$\delta\nu_2$	$\delta\sigma_1$	δA	0	$\delta\sigma_2$	$\eta_1\kappa_1$	$\delta\kappa_3$	$\delta\bar{\sigma}_1$	$\delta\bar{\kappa}_1$
η_1	2	η_1^2	0	0	0	$\eta_1\nu_2$	$\eta_1\sigma_1$	$\eta_1 A$	$\eta_1\kappa_1$	$\eta_1\sigma_2$	0	0	0 mod $\delta\nu_1\bar{\sigma}_1\delta$	$\eta_1\bar{\kappa}_1$
η_2	0	0	η_2^2	$\eta_2\eta_3$	0	$\eta_2\nu_2$	$\eta_2\sigma_1$	$\eta_2 A$	0	$\eta_2\sigma_2$	$\delta\kappa_3+\kappa_3\delta$ mod $\delta\nu_1\sigma_2$	0	0	$\eta_2\bar{\kappa}_1$
η_1^2	0	$2\eta_3$	0	0	0	$\eta_1^2\nu_2$	$\eta_1^2\sigma_1$	$\eta_1^2 A$	$2\kappa_2$	$\eta_1^2\sigma_2$	0	0	0	$\eta_1^2\bar{\kappa}_1$
η_2^2	0	0	$2\eta_3$	0	0	$\eta_2^2\nu_2$	$\eta_2^2\sigma_1$	$\eta_2^2 A$	0	$\eta_2^2\sigma_2$	0	0	0	$\eta_2^2\bar{\kappa}_1$
$\delta\nu_1$	0	0	0	0	$\delta\nu_2\delta$	$\eta_1^2(\nu_2+\sigma_1)$	0	$2\eta_3\sigma_1$	0	$\delta\nu_1\sigma_2$	0	$\delta\nu_2\kappa_1$	$\delta\nu_1\bar{\sigma}_1$	
η_3	η_2^2	$\eta_2\eta_3$	0	0	0	$\eta_3\nu_2$	$\eta_3\sigma_1$	$\eta_3 A$	$\eta_2\kappa_2$	$\eta_3\sigma_2$	0	0		$\eta_3\bar{\kappa}_1$
ν_1	$\delta\nu_1$	0	0	0	$\delta\nu_2+\nu_2\delta$	$\eta_3(\nu_2\pm\sigma_1)$	0	$\nu_1 A$	$\delta\kappa_3$	$\nu_1\sigma_2$	0	$\nu_2\kappa_1+\eta_1\bar{\kappa}_1$ mod $\{\delta\sigma_1\sigma_2, \delta\nu_1\bar{\sigma}_1\delta\}$	$\nu_1\bar{\sigma}_1$	
$\eta_2\eta_3$	$2\eta_3$	0	0	0	0	$\nu_1 A$	$\nu_1 A$	$\eta_2\eta_3 A$	0	$\eta_2\eta_3\sigma_2$	0	0	0	
$\delta\nu_2\delta$	0	0	0	0	0	0	0	0	0	$\delta\sigma_1\sigma_2\delta$	0			
$\delta\nu_2$	$\delta\nu_2\delta$	0	$2\nu_2$	$\eta_1^2\nu_2$	$\eta_1^2(\nu_2+\sigma_1)$	$\kappa_1\delta$	0	$\delta\sigma_1 A$	$\delta\nu_2\kappa_1$	$\delta\sigma_1\sigma_2$ mod $\delta\nu_1\bar{\sigma}_1\delta$	$\eta_1^2\bar{\kappa}_1$			
$\nu_2\delta$	0	0	$2\nu_2$	$\eta_1^2\nu_2$	$\eta_2^2(\nu_2+\sigma_1)$	$\kappa_1\delta$	0	$\delta\sigma_1 A$	0	$\sigma_1\sigma_2\delta$ mod $\delta\nu_1\bar{\sigma}_1\delta$	$\eta_1^2\bar{\kappa}_1$			
$\delta\sigma_1$	0	0	$2\sigma_1$	$\eta_1^2\sigma_1$	0	0	$\delta\sigma_2\delta$	$\delta\sigma_1 A$	0	$\delta\sigma_1\sigma_2$	0			
ν_2	$\nu_2\delta$	$\eta_1\nu_2$	$\eta_2\nu_2$	$\pm\eta_3\nu_2$	$\nu_1\nu_2$	ν_2^2	0	$\sigma_1 A$	$\nu_2\kappa_1$					

Table of rleatiotions, II.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ_3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
σ_1	$\delta\sigma_1$	$\eta_1\sigma_1$	$\eta_2\sigma_1$	$\pm\eta_3\sigma_1$	0	0	$\delta\sigma_2+\sigma_2\delta$	$\sigma_1 A$	$\delta\nu_1\bar{\sigma}_1\delta$	$\sigma_1\sigma_2$				
δA	0	0	$2A$	$\eta_1^2 A+2(x+y)\eta_3\sigma_1$	$2\eta_3\sigma_1$	$\delta\sigma_1 A$	$\delta\sigma_1 A$	δA^2	0					
A	$\delta A+2x\sigma_1+2y\nu_2$	$\eta_1 A+x\eta_1^2\sigma_1+y\eta_1^2\nu_2$	$\eta_2 A+x\eta_2^2\sigma_1+y\eta_2^2\nu_2$	$\pm\eta_3 A+(x+y)\nu_1 A$	$\nu_1 A$	$\pm\sigma_1 A$ mod $\{\eta_1\kappa_1,\eta_2\nu_2^2\}$	$\pm\sigma_1 A$	A^2						
$\eta_1\nu_2$	$2\nu_2$	$\eta_1^2\nu_2$	0	0	0	$\eta_1\kappa_1$	0	$\eta_1\sigma_1 A$ mod $2\kappa_2$	$\eta_1^2\bar{\kappa}_1$					
$\eta_2\nu_2$	0	0	$\eta_2^2\nu_2$	$\nu_1 A$	0	$\eta_2\nu_2^2$	0	$\eta_2\sigma_1 A$ mod $2\kappa_2$	0					
$\eta_1\sigma_1$	$2\sigma_1$	$\eta_1^2\sigma_1$	0	0	0	0	0	$\eta_1\sigma_1 A$	0					
$\eta_2\sigma_1$	0	0	$\eta_2^2\sigma_1$	$\nu_1 A$	0	0	0	$\eta_2\sigma_1 A$	0					
$\eta_1^2\nu_2$	0	$2\eta_3\sigma_1$	0	0	0	$2\kappa_2$	0	$\eta_1^2\sigma_1 A$						
$\eta_2^2\nu_2$	0	0	$2\eta_3\sigma_1$	0	0	$2\kappa_2$	0	$\eta_2^2\sigma_1 A$	0					
$\eta_1^2\sigma_1$	0	$2\eta_3\sigma_1$	0	0	0	0	0	$\eta_1^2\sigma_1 A$	0					
$\eta_2^2\sigma_1$	0	0	$2\eta_3\sigma_1$	0	0	0	0	$\eta_2^2\sigma_1 A$	0					
$\eta_1 A$	$2A$	$\eta_1^2 A+2\eta_3(x\sigma_1+y\nu_2)$	0	0	0	$\eta_1\sigma_1 A$ mod $2\kappa_2$	$\eta_1\sigma_1 A$	$\eta_1 A^2$						
$\eta_2 A$	0	0	$\eta_2^2 A+2\eta_3(x\sigma_1+y\nu_2)$	$\eta_2\eta_3 A$	0	$\eta_2\sigma_1 A$ mod $2\kappa_2$	$\eta_2\sigma_1 A$	$\eta_2 A^2$						
$\eta_1^2 A$	0	$2\eta_3 A$	0	0	0	$\eta_1^2\sigma_1 A$	$\eta_1^2\sigma_1 A$	$\eta_1^2 A^2$						

Table of relations, III.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ_3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
$\eta_2^2 A$	0	0	$2\eta_3 A$	0	0	$\eta_2^2\sigma_1 A$	$\eta_2^2\sigma_1 A$	$\eta_2^2 A$						
$\eta_3\sigma_1$	$\eta_2^2\sigma_1$	$\nu_1 A$	0	0	0	0	0	$\eta_3\sigma_1 A$	0					
$\eta_3(\nu_2+\sigma_1)$	$\eta_2^2(\nu_2+\sigma_1)$	0	0	0	0	$\eta_2\kappa_2$	0	0 mod $2\eta_3\sigma_1 A$						
$\eta_3 A$	$\eta_2^2 A+2\eta_3(x\sigma_1+y\nu_2)$	$\eta_2\eta_3 A$	0	0	0	$\pm\eta_3\sigma_1 A$	$\pm\eta_3\sigma_1 A$	$\eta_3 A^2$						
$\nu_1 A$	$2\eta_3\sigma_1$	0	0	0	0	0	0	$\nu_1 A^2$						
$\eta_2\eta_3 A$	$2\eta_3 A$	0	0	0	0	$\nu_1 A^2$	$\nu_1 A^2$	$\eta_2\eta_3 A^2$						
$\delta\sigma_2\delta$	0	0	0	0	0	0	$\delta\sigma_1\sigma_2\delta$							
$\kappa_1\delta$	0	0	0	$2\kappa_2$	$2\kappa_2$	$\delta\nu_2\kappa_1+2\bar{\kappa}_1$	0							
$\delta\sigma_2$	$\delta\sigma_2\delta$	0	$2\sigma_2$	$\eta_1^2\sigma_2$	$\delta\nu_1\sigma_2$	$\delta\sigma_1\sigma_2$ mod $\delta\nu_1\bar{\sigma}_1\delta$	$\delta\sigma_1\sigma_2$ mod $\delta\nu_1\bar{\sigma}_1\delta$							
$\sigma_2\delta$	0	0	$2\sigma_2$	$\eta_1^2\sigma_2$	$\delta\nu_1\sigma_2$	$\delta\sigma_1\sigma_2$ mod $\delta\nu_1\bar{\sigma}_1\delta$	$\sigma_1\sigma_2\delta$ mod $\delta\nu_1\bar{\sigma}_1\delta$							
κ_1	$\kappa_1\delta$	$\eta_1\kappa_1$	0	0	$\delta\kappa_3$ mod $\delta\nu_1\sigma_2$	$\eta_1\bar{\kappa}_1$ mod $\delta\nu_1\bar{\sigma}_1\delta$	$\delta\nu_1\bar{\sigma}_1\delta$							
ν_2^2	$\kappa_1\delta$	$\eta_1\kappa_1$	$\eta_2\nu_2^2$	$\eta_2\kappa_2$	$\eta_2\kappa_2$	$\nu_2\kappa_1+\eta_2\bar{\kappa}_1$ mod $\delta\nu_1\bar{\sigma}_1\delta$	0							
$\delta\sigma_1 A$	0	0	$2\sigma_1 A$	$\eta_1^2\sigma_1 A$	0	0								
σ_2	$\sigma_2\delta$	$\eta_1\sigma_2$ mod $2\kappa_2$	$\eta_2\sigma_2$ mod $2\kappa_2$	$\eta_3\kappa_2$ mod $2\eta_3\sigma_1 A$	$\nu_1\sigma_2$ mod $\{2\eta_3\sigma_2, 2\eta_3\sigma_1 A, \delta\bar{\sigma}_1\delta\}$									

Table of relations, IV.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ_3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
$\sigma_1 A$	$\delta\sigma_1 A$	$\eta_1\sigma_1 A$	$\eta_2\sigma_1 A$	$\pm\eta_3\sigma_1 A$	0	0								
$\eta_1\kappa_1$	0	$2\kappa_2$	0	0	0	$\eta_1^2\bar{\kappa}_1$	0							
$\eta_2\nu_2^2$	0	0	$2\kappa_2$	0	0	$\eta_2^2\bar{\kappa}_1$	0							
δA^2	0	0	$2A^2$	$\eta_1^2 A^2$	$2\eta_3\sigma_1 A$									
A^2	δA^2	$\eta_1 A^2$	$\eta_2 A^2$	$\eta_3 A^2$	$\nu_1 A^2$									
κ_2	$\eta_2\nu_2^2$	$\eta_2\kappa_2$	0	0	0									
$\eta_1\sigma_2$	$2\sigma_2$	$\eta_1^2\sigma_2$	0	0	0									
$\eta_2\sigma_2$	0	0	$\eta_2^2\sigma_2$	$\eta_2\eta_3\sigma_2$	0									
$\eta_1\sigma_1 A$	$2\sigma_1 A$	$\eta_1^2\sigma_1 A$	0	0	0	0								
$\eta_2\sigma_1 A$	0	0	$\eta_2^2\sigma_1 A$	$\nu_1 A^2$	0	0								
$\eta_1 A^2$	$2A^2$	$\eta_1^2 A^2$	0	0	0									
$\eta_2 A^2$	0	0	$\eta_2^2 A^2$	$\eta_2\eta_3 A^2$	0									
$\delta\kappa_3$	$2\kappa_2$	0	0	0	$\delta\nu_2\kappa_1$ mod $\delta\sigma_1\sigma_2\delta$									
$\kappa_3\delta$	0	0	0	0	$\delta\nu_2\kappa_1$ mod $\delta\sigma_1\sigma_2\delta$									

Table of relations, V.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ_3	$\bar{\sigma}_1$	$\bar{\kappa}_1$
$\eta_1^2\sigma_2$	0	$2\eta_3\sigma_2$	0	0	0									
$\eta_2^2\sigma_2$	0	0	$2\eta_3\sigma_2$	0	0									
$\eta_1^2\sigma_1 A$	0	$2\eta_3\sigma_1 A$	0	0	0									
$\eta_2^2\sigma_1 A$	0	0	$2\eta_3\sigma_1 A$	0	0									
$\delta\nu_1\sigma_2$	0	0	0	0	$\delta\sigma_1\sigma_2\delta$									
κ_3	$\kappa_3\delta$	0	0	0	$\nu_2\kappa_1+\eta_1\bar{\kappa}_1$ mod $\{\delta\sigma_1\sigma_2,$ $\sigma_1\sigma_2\delta,\ \delta\nu_1\bar{\sigma}_1\delta\}$									
$\nu_1\sigma_2$	$\delta\nu_1\sigma_2$	0	0	0	0 mod $\delta\nu_1\bar{\sigma}\delta$									
$\eta_1^2 A^2$	0	$2\eta_3 A^2$	0	0	0									
$\eta_2^2 A^2$	0	0	$2\eta_3 A^2$	0	0									
$\eta_3\sigma_2$	$\eta_2^2\sigma_2$	$\eta_2\eta_3\sigma_2$	0	0	0									
$\eta_3\sigma_1 A$	$\eta_2^2\sigma_1 A$	$\nu_1 A^2$	0	0	0									
$\delta\bar{\sigma}_1\delta$	0	0	0	0	$\delta\nu_1\bar{\sigma}_1\delta$									
$\eta_2\eta_3\sigma_2$	$2\eta_3\sigma_2$	0	0	0	0									
$\eta_3 A^2$	$\eta_2^2 A^2$	$\eta_2\eta_3 A^2$	0	0	0									

Table of relations, VI.

	δ	η_1	η_2	η_3	ν_1	ν_2	σ_1	A	κ_1	σ_2	κ_2	κ_3	$\bar{\sigma}_1$	$\bar{\kappa}_1$.
$\nu_1 A^2$	$2\eta_3\sigma_1 A$	0	0	0	0	0	0							
$\delta\bar{\sigma}_1$	$\delta\bar{\sigma}_1$	0	0	0										
$\bar{\sigma}_1\delta$	0	0	0	0										
$\delta\bar{\kappa}_1$	0	0	$2\bar{\kappa}_1$	$\eta_1^2\bar{\kappa}_1$										
$\bar{\sigma}_1$	$\bar{\sigma}_1\delta$	0	0 mod $\delta\nu_1\bar{\sigma}_1\delta$											
$\bar{\kappa}_1$	$\delta\bar{\kappa}_1$	$\eta_1\bar{\kappa}_1$ mod $\delta\nu_1\bar{\sigma}_1\delta$	$\eta_2\bar{\kappa}_1$ mod $\delta\nu_1\bar{\sigma}_1\delta$											
$\eta_2\eta_3 A^2$	$2\eta_3 A^2$	0	0	0	0									
$\delta\nu_2\kappa_1$	0	0	0	0										
$\delta\sigma_1\sigma_2\delta$	0	0	0	0	0	0								
$\nu_2\kappa_1$	$\delta\nu_2\kappa_1+2\bar{\kappa}_1$	$\eta_1^2\bar{\kappa}_1$	0	0										
$\eta_1\bar{\kappa}_1$	$2\bar{\kappa}_1$	$\eta_1^2\bar{\kappa}_1$	0	0										
$\eta_2\bar{\kappa}_1$	0	0	$\eta_2^2\bar{\kappa}_1$	0										
$\delta\sigma_1\sigma_2$	$\delta\sigma_1\sigma_2\delta$	0												
$\sigma_1\sigma_2\delta$	0	0												
$\delta\nu_1\bar{\sigma}_1\delta$	0	0	0	0										

from multiplying all of the additive generators of π_* except for the unit by themselves. But this whole work is too long and tedious. So, we write down a part of the table which plays an essential role for this work. Really, it finishes the proof.

REMARK. In Theo. 4.2 we can take $y=0$ by use of the results that $4\nu\bar{\kappa}=\eta^3\bar{\kappa}\neq 0$ in G_{23}.

Finally we mention the following relations in secondary or tertiary compositions.

Proposition 4.3.

$$\eta_1=\{\delta, \delta, \eta_2^2\}, \qquad \eta_2=\{\eta_1^2, \delta, \delta\},$$
$$\pm\eta_3=\{\eta_2, \eta_1, 2\}, \qquad \nu_2\in\{\eta_1, 2, \nu_1, \eta_1\},$$
$$\kappa_1\in\{\delta, \delta, \eta_2\nu_2^2\}, \qquad \kappa_2\in\{\eta_2, \kappa_1, 2\},$$
$$\kappa_3\in\{\nu_1, 2, \nu_2^2\}, \qquad \bar{\kappa}_1\in\{\nu_1, \eta_1, 2, \nu_2^2+\kappa_1\}.$$

Proof. We shall prove the first and the fourth relation.

Clearly, $\{\delta, \delta, \eta_2^2\}=\{ip, ip, \eta_2^2\}\supseteqq i\{p, i, \eta^2 p\}=i\{p, i, 2\bar{\eta}\}\supseteqq i\{p, i, 2\}\bar{\eta}=\eta_1$ since $\{p, i, 2\}\equiv 1 \bmod 2G_0$. As $\{\delta, \delta, \eta_2^2\}$ is a coset of $\delta\pi_2+\pi_{-1}\eta_2^2=0$, we obtain the first.

Since $\{\eta_1, 2, \nu_1\}$ is a coset of $\eta_1\pi_4+\pi_2\nu_1=\{\delta\nu_1^2\}=\pi_5$, we have $\eta_1\overline{(2)}_{\nu_1}\in\{\eta_1, 2, \nu_1\}p_{\nu_1}=\{\delta\nu_1^2 p_{\nu_1}\}=0$ for any $\overline{(2)}_{\nu_1}\in \mathrm{Ext}_{\nu_1} 2$.

As $\{2, \nu_1, \eta_1\}=\{2, \nu_1, i\bar{\eta}\}\subseteqq\{2, i\nu, \bar{\eta}\}\supseteqq\{0, \nu, \bar{\eta}\}=\pi_4^*(2)\bar{\eta}=\{\eta_2^2\eta_3\}=0$ and $\{2, i\nu, \bar{\eta}\}$ is a coset of $2\pi_5+\pi_4^*(2)\bar{\eta}=0$, we have $\{2, \nu_1, \eta_1\}=0$. From this $\overline{(2)}_{\nu_1}\widetilde{(\eta_1)}_{\nu_1}=0$ for any $\widetilde{(\eta_1)}_{\nu_1}\in \mathrm{Coext}_{\nu_1}\eta_1$.

Now we define $\{\eta_1, 2, \nu_1, \eta_1\}\equiv\{\eta_1, \overline{(2)}_{\nu_1}, \widetilde{(\eta_1)}_{\nu_1}\} \bmod Q=[S^{n+2}M\cup_{\nu_1}CS^{n+5}M, S^nM]\widetilde{(\eta_1)}_{\nu_1}+\overline{(\eta_1)}_2[S^{n+6}M, S^nM\cup_2 CS^nM]$ for some $\overline{(\eta_1)}_2\in \mathrm{Ext}_2\,\eta_1$, where n is sufficiently large.

It is easy to check the group $Q=\{2\sigma_1, 2\nu_2\}$. So, we have $p\{\eta_1, 2, \nu_1, \eta_1\}=\{p, \eta_1, \overline{(2)}_{\nu_1}\}\widetilde{(\eta_1)}_{\nu_1}\subseteqq\{\pm\bar{\eta}, \nu_1, \eta_1\}=\{\pm\bar{\eta}, i\nu, \bar{\eta}\}=\{\eta, \nu, \bar{\eta}\}$ since $\{p, \eta_1, \overline{(2)}_{\nu_1}\}i_{\nu_1}\subseteqq\{p, \eta_1, 2\}=\pm\bar{\eta}$, $\bar{\eta}\nu_1=0$ and $\{\pm\bar{\eta}, i\nu, \bar{\eta}\}$ is a coset of $(\pm\bar{\eta})\pi_5+G_5\bar{\eta}=0$.

It is easy to check $\{\eta, \nu, \bar{\eta}\}=\overline{\nu^2}$ for $\overline{\nu^2}\in \mathrm{Ext}\,\nu^2$ which satisfies (2.7). Hence we obtain $\{\eta_1, 2, \nu_1, \eta_1\}\equiv\pm\nu_2 \bmod 2\sigma_1$.

5. Direct summands of π_k

The object of this section is to improve Theo. 5.1 of [5]. We shall change the notations μ_{8s+1} and j_{8s+3} of §4 of [5] into μ_s and ζ_s respectively.

In [1] Adams proved that $A^s\neq 0$ for $s\geqq 1$ and defined $\alpha_s\in(G_{8s-1}; 2)$ as follows:

(5.1) $$\alpha_s=pA^s i,$$

which is of order 2 and satisfies

(5.2) $$e_C(\alpha_s)\equiv\tfrac{1}{2} \bmod 1 \text{ (see [1])}.$$

Choose

(5.3) $$\overline{\alpha_s} = pA^s = \overline{8\sigma}A^{s-1}\in \mathrm{Ext}\,\alpha_s \quad \textit{and} \quad \widetilde{\alpha_s} = A^s i = A^{s-1}\widetilde{8\sigma}\in \mathrm{Coext}\,\alpha_s .$$

It is clear that

(5.4) $$\alpha_1 = 8\sigma \quad \textit{and} \quad \alpha_{s+t}\in\{\alpha_s, 2, \alpha_t\} .$$

By use of α_s Adams defined $\mu_s\in(G_{8s+1}; 2)$ as follows:

(5.5) $$\mu_s\equiv\bar{\eta}\widetilde{\alpha_s} \bmod \eta G_{8s} .$$

By (5.5) and i) of (3.9) of [6], we have $\mu_s\equiv\{\eta, 2, \alpha_s\}=\{\alpha_s, 2, \eta\}\equiv \overline{\alpha_s}\tilde{\eta} \bmod \eta G_{8s}$. So, we can choose $\overline{\mu_s}\in \mathrm{Ext}\,\mu_s$ and $\widetilde{\mu_s}\in \mathrm{Coext}\,\mu_s$ as follows:

(5.6) $$\overline{\mu_s}\equiv\bar{\eta}A^s \bmod G_{8s}\bar{\eta} \quad \textit{and} \quad \widetilde{\mu_s}\equiv A^s\tilde{\eta} \bmod \tilde{\eta}G_{8s} .$$

Lemma 5.1.

i) *α_s is divisible by* 8.

ii) *$J(\beta)\alpha_t=0$ for $\beta\in\pi_{8s-1}(SO)$.*

Proof. Let $S^n\cup_8 e^{n+1}$ be a complex obtained from an n-sphere S^n by attaching an $(n+1)$-dimensional cell e^{n+1}, using a map of degree 8, where n is sufficiently large. Let i': $S^n\to S^n\cup_8 e^{n+1}$ be a natural inclusion.

Obviously α_s is divisible by 8 if and only if $i'\alpha_s=0$.

By induction assume $i'\alpha_{s-1}=0$, then $i'\alpha_s\in i'\{\alpha_{s-1}, 2, 8\}=\{i, \alpha_{s-1}, 2\}8\sigma= 8(\{i', \alpha_{s-1}, 2\}\sigma)$. By use of Theo. 4.4 in p. 324 of [7], this consists of 0. Therefore we have i).

Next we shall prove ii). Toda defined the element $\sigma'''\in\pi_{12}(S^5)$ which is of order 2 and satisfies $S^\infty\sigma'''=8\sigma$ (see *p*. 48 of [6]). By use of σ''', we can define an element $\alpha'_s\in\pi_{8s+4}(S^5)$ for $s\geqq 1$ as follows:

(5.7) $$\alpha'_1 = \sigma''' \quad \text{and} \quad \alpha'_s\in\{\alpha'_{s-1}, 2, 8\sigma\} \qquad \text{for } s\geqq 2 .$$

Clealy, α'_s is of order 2 and we can choose α'_s such that

(5.8) $$S^\infty\alpha'_s = \alpha_s .$$

Now $J(\beta)\alpha_t=J(\beta S^{8s-6}\alpha'_t)=0$ since $\beta S^{8s-6}\, \alpha'_t\in\pi_{8(s+t)-2}(SO)=0$.

By use of i) of this lemma,

(5.9) $$\mu_s\in\{\eta, 2, \alpha_s\}\subseteqq\{\eta, \alpha_s, 2\} \bmod 2G_{8s+1}+\eta G_{8s} .$$

Therefore, we have

(5.10) $$2A^s \equiv i\mu_s p \bmod i\eta G_{8s}p .$$

By use of ii) of Lemma 5.1, we define an element $\rho_s \in (G_{8s+7}; 2) \cap J(\pi_{8s+7}(SO))$ as follows:

(5.11) $$\rho_0 = \sigma \quad \textit{and} \quad \rho_s \equiv \{\rho_{s-1}, 8\sigma, 2\} \bmod 2G_{8s+7} \qquad \textit{for } s \geqq 1 .$$

We can take ρ_s in the J-image since $J\{\beta', 8\sigma, 2\} \subseteqq \{\rho_{s-1}, 8\sigma, 2\}$ if $J(\beta')=\rho_{s-1}$ for $\beta' \in \pi_{8s-1}(SO)$.

We note the following: Since $\rho_s\sigma=0$ for $s\geqq 1$, we have, by the facts that $\sigma^2\eta=\sigma\varepsilon=0$ and $\{\eta\sigma, 2, 8\sigma\}=\{\varepsilon, 2, 8\sigma\}$,

(5.12) $$\rho_s\varepsilon = \rho_s\sigma\eta = 0 \qquad \textit{for } s\geqq 0 .$$

Lemma 5.2.

i) $i\rho_{s+t}=\widetilde{\alpha}_t\rho_s$ *and* $\rho_{s+t}p=\rho_s\overline{\alpha}_t$.

ii) $\rho_{s+t}\equiv\{\rho_s, \alpha_t, 2\} \bmod \rho_s G_{8t}+2G_{8(s+t)+7}$.

This is obvious by use of (5.3), (5.11) and (5.12).

By (5.5) and i) of Lemma 5.2, we have

(5.13) $$\eta\rho_{s+t}\equiv \mu_t\rho_s \bmod \eta\rho_s G_{8t} .$$

We can take, by use of i) of Lemma 5.2 and a) of ii) of Prop. 1.8,

(5.14) $$\sigma_1 A^s \in \mathrm{Coext}\,(\rho_s p) .$$

By (5.6), we have

(5.15) $$\eta_3 A^s \equiv \tilde{\eta}\overline{\mu_s} \bmod \tilde{\eta} G_{8s}\bar{\eta} .$$

It is clear that

(5.16) $$2\sigma_1 A^s = i\eta\rho_s p \quad \textit{and} \quad 2\eta_3 A^s \equiv i\eta^2\overline{\mu_s} \bmod i\eta^2 G_{8s}\bar{\eta} .$$

By use of Example 12.15 of [1] and (5.13), we can take

(5.17) $$j_{8s} = \rho_{s-1}\eta \quad \textit{and} \quad j_{8s+1}=\rho_{s-1}\eta^2 \qquad \textit{for } s\geqq 1 .$$

Lemma 5.3.

i) $\zeta_0=\nu$ *and* $\zeta_s\equiv\{\zeta_{s-1}, 8, 2\sigma\}\subseteqq\{\zeta_{s-1}, 8\sigma, 2\} \bmod 2G_{8s+3}+R_s$ *for* $s\geqq 1$,

ii) $\zeta_{s+t}p\equiv\zeta_s\overline{\alpha_t} \bmod R_{s+t}p$,

iii) $\zeta_{s+t}\equiv\{\zeta_s, \alpha_t, 2\} \bmod 2G_{8(s+t)+3}+R_{s+t}$,

iv) $\zeta_s\equiv\{2, \eta, \eta^2\rho_{s-1}\} \bmod 2G_{8s+3}+R_s$ *for* $s\geqq 1$,

where R_s *consists of the elements* $\alpha\in(G_{8s+3}; 2)$ *which have the following properties:* $e'_R(\alpha)=0$, $8\alpha=0$ *and* $\eta\alpha p=0$.

Proof. By use of Theo. 11.1 of [1], $e'_R(\{\zeta_{s-1}\, 8, 2\sigma\})\equiv -8e'_R(2\sigma)e'_R(\zeta_{s-1})$ $\equiv -\frac{1}{8} \bmod 1$ since $e'_R(\sigma)=e_C(\sigma)\equiv\frac{1}{16} \bmod 1$ and $e'_R(\zeta_{s-1})\equiv\frac{1}{8} \bmod 1$. By use of Prop. 3.2. (c) and Prop. 7.1 of [1], $\zeta_{s-1}G_8+2\sigma G_{8s-4}\subseteqq \ker e'_R$. So, by Theo. 1.5 of [1], we have $-\zeta_s\equiv\{\zeta_{s-1}, 8, 2\sigma\} \bmod \ker e'_R$. We have $8\zeta_s=0$ and $8\{\zeta_{s-1}, 8, 2\sigma\}=\{8, \zeta_{s-1}, 8\}2\sigma=0$ by use of Cor. 3.7 of [6]. Since $\eta\zeta_s=\mu\zeta_{s-1}$ $=J(\pi_{8s+4}(SO))=0$ (*cf.* *p.* 39 and *p.* 56 of [6]), we have $\eta\{2\sigma, 8, \zeta_{s-1}\}=$ $\{\eta, 2\sigma, 8\}\zeta_{s-1}=0$. This leads us to i).

We shall prove that $R_s\overline{\alpha_t}\subseteqq R_{s+t}p$. By i) of Lemma 5.1, $\beta\overline{\alpha_t}\in\{\beta, \alpha_t, 2\}p=$ $\{\beta, 8, \frac{1}{4}\alpha_t\}p$ for $\beta\in R_s$. By use of Cor. 3.7 of [6], $8\{\beta, 8, \frac{1}{4}\alpha_t\}=\{8, \beta, 8\}\frac{1}{4}\alpha_t=0$. By use of Theo. 11.1 of [1]., we have $e'_R(\{\beta, 8, \frac{1}{4}\alpha_t\})=0$. Since $\eta\beta$ is divisible by 2, we have $\eta\{\beta, 8, \frac{1}{4}\alpha_t\}p=\eta\beta\overline{\alpha_t}=0$. Therefore we obtain $\{\beta, 8, \frac{1}{4}\alpha_t\}\subseteqq R_{s+t}$.

Now we obtain ii) by use of this fact and i).

iii) forllows from ii).

We shall prove iv). First we note that we can define $\{2, \eta, \eta^2\rho_{s-1}\}$ since $\eta^3\rho_{s-1}\in J(\pi_{8s+2}(SO))=0$.

Since $\zeta_1=\zeta\equiv\{2, \eta, \eta^2\sigma\} \bmod 2G_{11}$, we have, by use of ii), $\zeta_s p\equiv$ $\zeta\overline{\alpha_{s-1}}=\{2, \eta, \eta^2\sigma\}\overline{\alpha_{s-1}} \bmod R_s p$. By use of i) of Lemma 5.2, we have $\zeta_s p\equiv$ $\{2, \eta, \eta^2\sigma\overline{\alpha_{s-1}}\}=\{2, \eta, \eta^2\rho_{s-1}p\} \bmod R_s p+2\pi_{8s+3}(2)$. By Theo. A and Prop. 1.1 of [5] and by Theo. 1.5 of [1], it is clear that $2\pi_{8s+3}(2)\subseteqq R_s p$. Therefore we have iv).

By use of ii) of this lemma and a) of ii) of Prop. 1.8, we can take

(5.18) $$\nu_1 A^s\equiv \mathrm{Coext}\,(\zeta_s p) \bmod \mathrm{Ceoxt}\,(R_s p)+i\pi_{8s+4}(2)\,.$$

Since $\eta_3\sigma_1=\pm\tilde{\eta}\sigma\bar{\eta}$ by vi) of Prop. 2.2, we can take, by (5.6) and (5.13),

(5.19) $$\eta_3\sigma_1 A^s\equiv\tilde{\eta}\rho_s\bar{\eta} \bmod \tilde{\eta}\sigma G_{8s}\bar{\eta}+\tilde{\eta}G_{8s+9}p\,.$$

By use of Lemma 5.3, we have

(5.20) $$2\nu_1 A^s=0 \quad \textit{and} \quad 2\eta_3\sigma_1 A^s\equiv i\zeta_{s+1}p \bmod iR_{s+1}p\,.$$

Now we have been ready for improving Theo. 5.1 of [5].

Theorem 5.4. *$\pi_k(2)$, $\pi_k^*(2)$ and π_k contain direct summands which are isomorphic to the corresponding groups in the following tables ($k>2$):*

i)

$k=$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+7$
$\pi_k(2)\underset{+}{\supset}$	Z_2+Z_2	Z_4+Z_2	Z_4+Z_2	Z_2+Z_2	Z_2	Z_2
Generators	$\overline{\alpha_s}$, $\eta\rho_{s-1}p$	$\rho_{s-1}\bar{\eta}$, $\mu_s p$	$\overline{\mu_s}$, $\rho_{s-1}\eta\bar{\eta}$	$\eta\overline{\mu_s}$, $\zeta_s p$	$\eta^2\overline{\mu_s}$	$\rho_s p$
$\pi_k^*(2)\underset{+}{\supset}$	Z_2+Z_2	Z_4+Z_2	Z_4+Z_2	Z_2+Z_2	Z_2	Z_2
Generators	$\widetilde{\alpha_s}$, $i\eta\rho_{s-1}$	$\tilde{\eta}\rho_{s-1}$, $i\mu_s$	$\widetilde{\mu_s}$, $\tilde{\eta}\eta\rho_{s-1}$	$\widetilde{\mu_s}\eta$, $i\zeta_s$	$\widetilde{\mu_s}\eta^2$	$i\rho_s$

Relations: $2\rho_s\bar{\eta}=\rho_s\eta^2 p$, $2\overline{\mu_s}=\eta\mu_s p$,

$2\tilde{\eta}\rho_s=i\eta^2\rho_s$, $2\widetilde{\mu_s}=i\eta\mu_s$.

ii)

$k=$	$8s$	$8s+1$
$\pi_k \supset$	$Z_4+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$
Generators	A^s, $\eta_1\sigma_1A^{s-1}$, $\eta_2\sigma_1A^{s-1}$	$\eta_1^2\sigma_1A^{s-1}$, $\eta_1^2\sigma_2A^{s-1}$, η_1A^s, η_2A^s

$k=$	$8s+2$	$8s+3$	$8s+4$	$8s+6$	$8s+7$
	$Z_4+Z_2+Z_2$	Z_4+Z_2	Z_2	Z_2	Z_4+Z_2
Generators	$\eta_3\sigma_1A^{s-1}$, $\eta_1^2A^s$, $\eta_2^2A^s$	η_3A^s, ν_1A^s	$\eta_2\eta_3A^s$	$\delta\sigma_1A^s$	σ_1A^s, δA^{s+1}

Relations: $2A^s \equiv i\mu_s p \bmod i\eta G_{8s}p$, $2\eta_3\sigma_1A^s \equiv i\zeta_{s+1}p \bmod iR_{s+1}p$,
$2\eta_3A^s \equiv i\eta^2\overline{\mu_s} \bmod i\eta^2G_{8s}\bar{\eta}$, $2\sigma_1A^s = i\eta\rho_s p$.

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