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DOUBLY TRANSITIVE GROUPS IN WHICH THE MAXIMAL NUMBER OF FIXED POINTS OF INVOLUTIONS IS FOUR

RYUZABURO NODA

(Received October 19, 1970)

1. Introduction

Doubly transitive groups in which any involution fixes at most three points have been classified by H. Bender [3], C. Hering [14] and J. King [16], [17]. In this paper we shall prove the following results.

Theorem. Let $G$ be a doubly transitive group on $\Omega = \{1, 2, \ldots, n\}$. Assume that the maximal number of fixed points of involutions in $G$ is four. Then, if $n \equiv 0 \pmod{8}$, one of the following holds:

(a) $n = 6$ and $G$ is $S_6$,
(b) $n = 10$ and $G$ is $S_6$ or $P\Gamma L(2, 9)$,
(c) $n = 12$ and $G$ is $M_{11}$ or $M_{12}$ (the Mathieu group of degree 11 or 12),
(d) $n = 28$ and $G$ is $P\Gamma L(2, 8)$,
(e) $n = 28$ and $G$ is $PSU(3, 3^2)$ or $P\Sigma U(3, 3^2)$.

Corollary. Let $G$ be a doubly transitive group on $\Omega = \{1, 2, \ldots, n\}$. If every involution in $G$ fixes four points in $\Omega$, then one of the following holds:

(a) $n = 12$ and $G$ is $M_{12}$,
(b) $n = 28$ and $G$ is $P\Gamma L(2, 8)$,
(c) $n = 28$ and $G$ is $PSU(3, 3^2)$ or $P\Sigma U(3, 3^2)$.

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2. Definitions and notations

A permutation group $G$ on $\Omega$ is called semi-regular if every non-identity element of $G$ has no fixed point, and $G$ is called regular if $G$ is transitive and semi-regular. If a set $X$ of permutations on $\Omega$ fixes a subset $\Delta$ of $\Omega$ the restriction of $X$ on $\Delta$ will be denoted by $X^\Delta$. Further we use the following notations which are standard.

$S_n =$ symmetric group of degree $n$,
$A_n =$ alternating group of degree $n$,
GF(q) = finite field with q elements,
\( P\Gamma L(m, q) = m\)-dimensional projective semi-linear group over GF(q),
\( PSL(m, q) = m\)-dimensional projective special linear group over GF(q),
\( PSU(3, q^2) = 3\)-dimensional projective special unitary group over GF(q^2),
\( P\Sigma U(3, q^2) = \) automorphism group of \( PSU(3, q^2)\),
\( \text{Aut } G = \) automorphism group of a group \( G \),
\( G_{i_1 \cdots i_r} = \) pointwise stabilizer in \( G \) of points \( i, j, \ldots, r \),
\( G_{[i_1 \cdots i_r]} = \) global stabilizer in \( G \) of the set of points \( i, j, \ldots, r \),
\( I(X) = \) totality of points fixed by a set \( X \) of permutations,
\( \alpha_i(x) = \) number of \( i \)-cycles of a permutation \( x \),
\( o(x) = \) order of a permutation \( x \),
\( N_G(X) = \) normalizer of \( X \) in \( G \),
\( C_G(X) = \) centralizer of \( X \) in \( G \),
\( \langle X, Y \rangle = \) subgroup generated by \( X \) and \( Y \),
\( [X, Y] = \) commutator of \( X \) and \( Y \),
\( |X| = \) cardinality of a set \( X \).

3. Proof of Corollary

It suffices by our theorem to prove that \( n \equiv 0 \pmod{8} \) in the case \( G \) satisfies the assumption of Corollary. Assume by way of contradiction that \( n \equiv 0 \pmod{8} \). Then the length of any orbit of an \( S_2 \)-subgroup \( P \) of \( G \) on \( \Omega \) is divisible by eight (see [22], Theorem 3.4'). Then since a central involution of \( P \) fixes a point on \( \Omega \), it fixes more than four points on \( \Omega \), contrary to the assumption.

4. Proof of Theorem

We begin with some lemmas on permutation groups.

Lemma 1 (J. Alperin [1]). Let the group \( G \) be transitive on \( \Omega = \{1, 2, \ldots, n\} \) and let \( H \) be a subgroup of the stabilizer \( G_1 \). If the conjugates \( H^g(g \in G) \) which are contained in \( G \), make up \( k \) different conjugacy classes of subgroups of \( G \), then the normalizer \( N_G(H) \) of \( H \) has exactly \( k \) orbits on \( I(H) \).

We remark that lemma 1 also holds valid if a subgroup \( H \) of \( G \), in the above is replaced by a subset \( K \) of \( G \). In fact, Alperin's proof in [1] does not make use of that \( H \) is a subgroup.

Lemma 2 (H. Nagao). Let \( X \) be a semi-regular permutation group on \( \Omega = \{1, 2, \ldots, n\} \). If a permutation group \( A \) on \( \Omega \) normalizes \( X \) and fixes at least one point, then the order of \( C_X(A) \) is not greater than the number of fixed points of \( A \). If \( X \) is regular, then the order of \( C_X(A) \) equals the number of fixed points of \( A \).

Proof. Suppose \( A \) fixes the point 1. Let \( x \) be an element of \( X \) and \( a \) an ele-
ment of A. If x takes 1 to i and a takes i to j, then $a^{-1}ax$ takes 1 to j. Since $a^{-1}xa \in X$ and X is semi-regular, $x = a^{-1}ax$ if and only if $j = i$, i.e. $i \in I(a)$. Thus we have $|C_X(A)| \leq |I(A)|$. If X is regular, then for any fixed point i of A there is a unique element of X which takes 1 to i. Hence we have $|C_X(A)| = |I(A)|$.

Lemma 3 (D. Livingstone and A. Wagner [18]). Let G be k-fold transitive on $\Omega = \{1, 2, \ldots, n\}$, and let H be the stabilizer of k points in $\Omega$. Assume that an $S_p$-subgroup P of H fixes precisely the given k points. Then for a point $i$ in a minimum P-orbit on $\Omega - I(P)$, $N_G(P)^{(P)}$ is k-fold transitive.

The proof of the above lemma is seen in p. 400-401 of [18].

Lemma 4. Let Y be a cyclic 2-group which acts regularly on $\Omega = \{1, 2, \ldots, n\}$, and assume that Y normalizes a four group U which is semi-regular on $\Omega$. Then $|Y| = |\Omega| = 4$.

Proof. Assume that $n = 2^m \geq 4$, and let $\Delta_i$ ($1 \leq i \leq t$) denote the orbits of U on $\Omega$. Then since Y permutes $\Delta_1, \Delta_2, \ldots, \Delta_t$ transitively we have $|Y : Y_{\Delta_i}| = t \geq 2$, from which it follows that $Y_{\Delta_i}$ centralizes U since $|Y : C_Y(U)| \leq 2$.

Then since $Y_{\Delta_i}^\Delta_i$ is cyclic and $U^\Delta_i$ is self-centralizing, we have $|Y_{\Delta_i}| = |Y_{\Delta_i}^\Delta_i| \leq 2$, which yields that $|Y| = t|Y_{\Delta_i}| \leq 2t = \frac{n}{2}$, a contradiction.

Now to prove our theorem the following two cases will be treated separately.

Case I. $n \equiv 2 \pmod{4}$.

Case II. $n \equiv 0 \pmod{4}$.

Case I. Since an involution in G fixing four points is an odd permutation in this case, $N = G \cap A_n$ is a normal transitive subgroup of index two. Furthermore since $|G : N_i| = 2$ is prime to $n-1$, $N_i$ is transitive on $\Omega - \{1\}$ and hence $N$ is doubly transitive on $\Omega$. Then by a result of C. Hering [14], either of the following holds:

(i) $n = q+1$ and $N$ contains $PSL(2, q)$,
(ii) $n = 6$ and $N$ is $A_6$.

In case(ii), $n=6$ and $G$ is $S_4$. In case(i) an involution in $G - N$ fixing four points acts as a field automorphism. Hence we have $4 = 1+\sqrt{q}$ and hence $q = 9$. Then $n = 10$ and $G$ is $S_6$ or $PGL(2, 9)$.

Case II. We have $n \equiv 4 \pmod{8}$ in this case by our assumption that $n \equiv 0 \pmod{8}$. In particular G contains no regular normal subgroup and hence by a theorem of Burnside ([7], P202) we have

(*) a (unique) minimal normal subgroup H of G is a primitive non abelian simple group.

In what follows we denote by H a (unique) minimal normal subgroup of G throughout.
Lemma 5. We may assume $G$ contains no normal subgroup of index two.

Proof. Let $G$ have a normal subgroup $N$ of index two. Then, as seen in Case I, $N$ is also doubly transitive on $\Omega$. If $N$ contains an involution fixing four points we take $N$ in place of $G$. If $N$ contains no involution fixing four points, the results of H. Bender [3] and C. Hering [14] yield, similarly to Case I, that $n = 6$ or 10, which is not the case.

Lemma 6. An $S_2$-subgroup of $H$ is not dihedral.

Proof. This follows from a result of Gorenstein and Walter [11] and a result of Lüneburg ([19], Satz 1).

Lemma 7. If $H$ has a quasi-dihedral $S_2$-subgroup and $H$ contains an involution $a$ such that $C_H(a)$ is solvable then $n = 12$ and $G$ is $M_{11}$.

Proof. By a result of Alperin, Brauer and Gorenstein [2] $H$ is $PSL(3, 3)$ or $M_{11}$. If $H$ is $PSL(3, 3)$, since $|\text{Aut } PSL(3, 3): PSL(3, 3)| = 2$, we have $G$ is $PSL(3, 3)$ or $\text{Aut } PSL(3, 3)$. But it is easily seen that $PSL(3, 3)$ and $\text{Aut } PSL(3, 3)$ have no doubly transitive representation satisfying our assumption. If $H$ is $M_{11}$, since $\text{Aut } M_{11} \cong M_{11}$, we have $G$ is $M_{11}$ on 12 points.

Lemma 8. If $H$ has an elementary abelian $S_2$-subgroup and $H$ contains an involution $a$ such that $C_H(a)$ is solvable then $n = 28$ and $G$ is $P^T L(2, 8)$.

Proof. By a result of Walter [21] $H$ is $PSL(2, q)$ for a suitable $q$. Then the result of Lüneburg [19] applies.

Lemma 9. If $H$ has a wreathed $S_2$-subgroup of order 32 and $H$ contains an involution $a$ such that $C_H(a)$ is solvable then $n = 28$ and $G$ is $PSU(3, 3^2)$ or $P^S U(3, 3^2)$.

Proof. By a result of P. Fong [8] $H$ is $PSU(3, 3^2)$, and hence $G$ is $PSU(3, 3^2)$ or $P^S U(3, 3^2)$. It is easy to see that $PSU(3, 3^2)$ and $P^S U(3, 3^2)$ have no other doubly transitive representation than the usual one of degree 28.

Now we consider the following two cases.
Subcase 1. An $S_2$-subgroup of $G_{12}$ fixes two points.
Subcase 2. An $S_{12}$-subgroup of $G_{12}$ fixes four points.

Subcase 1. Let $P$ be an $S_2$-subgroup of $G$. Then since $n \equiv 4 \pmod{8}$ $P$ has an orbit of length four, say $\Delta = \{1, 2, 3, 4\}$ (see [22], Theorem 3.4') and then $R = P_1$ is an $S_2$-subgroup of $G$. We may assume $I(R) = \{1, 2\}$. Then $R$ contains an element $b$ of the form $b = (1)(2)(34)\cdots$. We assume first that any element in $R$ of the form $(1)(2)(34)\cdots$ is an involution. Set $S = R_{s_4}$. Then $N_G(S)^{I(S)}$ is doubly transitive by Lemma 3 and hence $N_G(S)^{I(S)} = S_4$. Since any element in
the coset $b S$ is an involution $b$ inverts every element of $S$ and hence $S$ is abelian. On the other hand, since $b$ fixes exactly two points on $\Omega-\Delta$, and since $S$ is semi-regular on $\Omega-\Delta$ Lemma 2 yields $|C_S(b)| = 2$. Thus $S$ must have a unique involution and hence $S$ is cyclic. This implies that $C_G(S)^{t(S)} \geq A_4$, and any element $t$ of $N_G(S)$ with $t^2 \in A_4$ lies in $C_G(S)$. Now since $|P^a| = 8$, $P$ contains an element $y$ of the form $(1324)\cdots$. First assume $o(y) \geq 8$. Then by Lemma 5 $y$ has an odd number of cycles of the same length on $\Omega-\Delta$. Then since the length of any $P$-orbit on $\Omega-\Delta$ is of 2-power $P$ fixes the set of the points of a cycle of $y$, say $(5, 6, \cdots, l)$ on $\Omega-\Delta$ and in particular so does $S$. Then since $y^2 \in S$ we have $o(y) = |S|, 2|S|$ or $4|S|$. If $o(y) = |S|$ either $y$ or the generator $z$ of $S$ is an odd permutation, contrary to Lemma 5. Also if $o(y) = 2|S|$, since $[y^2, S] = 1$, setting $\Gamma = \{5, 6, \cdots, l\}$ we have $(y^2)^\Gamma = (z^m)^\Gamma$ for some odd integer $m$. Then $y^2z^{-m} = (12)(34)\cdots$ fixes every point of $\Gamma$, contrary to the assumption of our theorem. Thus we have $o(y) = 4|S|$ and hence $\langle y \rangle$ is a cyclic subgroup of $P$ of index two. Then the structure of $P$ is known (see [10], Theorem 5.4.4) and since $P$ contains a non normal dihedral subgroup $R$, $P$ is dihedral or quasi-dihedral. We may assume by Lemma 6 that $P$ is quasi-dihedral. Let $a$ be an involution of $S$. Then since $C_G(a)^{t(a)} \leq S$, and since $C_G(a)^{t(a)}$ has a cyclic $S_4$-subgroup it follows that $C_G(a)$ is solvable. Then Lemma 7 yields that $G$ is $M_{12}$ on 12 points.

Now let $o(y) = 4$. Then since $n \equiv 4 \pmod{8}$ $y$ has four fixed points or two 2-cycles on $\Omega-\Delta$ by Lemma 5 and hence $y^2$ fixes four points on $\Omega-\Delta$. On the other hand $y^2 \in A_4$ implies $[y^2, S] = 1$ and hence $|S| \leq 4$ by Lemma 2. Thus we have $|P \cap H| \leq |P| \leq 32$, and by a result of P. Fong $P \cap H$ is dihedral, quasi-dihedral, elementary or wreathed of order 32. Then Lemma 6, Lemma 7, Lemma 8 or Lemma 9 applies, respectively.

Now we assume that $R$ contains an element $x$ of the form $(1)(2)(34)\cdots$ which is not an involution. Then since $n \equiv 4 \pmod{8}$, $x$ has an odd number of cycles of the same length on $\Omega-\Delta$ by Lemma 5. In particular $o(x) \geq 8$, and $P$ fixes the set of the points of a cycle of $x$, say $(56\cdots k)$ on $\Omega-\Delta$. Then since $x^2 \in S$, $\Gamma = \{5, 6, \cdots, k\}$ consists of one or two orbits of $S$. If $\Gamma$ consists of two orbits of $S$ then we have $S = \langle x^2 \rangle$ is cyclic. In this case the similar argument to the above applies. Thus we may assume $\Gamma$ consists of one orbit of $S$. Then $o(x) = |S| = |\Gamma| \geq 8$ and since $\Gamma$ is a $P$-orbit an exponent of $P$ is equal to $o(x)$. In particular $P$ contains a normal four-group $U$(see [10], P215. Exercise 9). Then since $o(x) \geq 8$, Lemma 4 implies that an involution $c$ in $U$ fixes a point on $\Gamma$, and hence we have $|C_S(c)| \leq 4$ by Lemma 2. Then since $|S : C_S(c)| \leq |S : C_S(U)| \leq 2$ it follows that $|S| \leq 8$ and hence $|S| = 8$. If $S \cong Z_4 \times Z_2$, $P$ contains a normal four group which is semi-regular on $\Gamma$, contrary to Lemma 4. Also if $S$ is cyclic $S$ contains an odd permutation. Thus $S$ must be quaternion or dihedral of order eight. Now we shall determine the structure of $P \cap H$. If $|P \cap H| = 32$, $G$ contains a normal subgroup of index two, contrary to Lemma 5. Let $|P \cap H| \leq 16$. Then $P \cap H$ is
dihedral, quasi-dihedral or elementary. Since the transitivity of $H$ on $\Omega$ implies the transitivity of $P \cap H$ on $\Delta$ (see [22], Theorem 3.4'), we have $|S \cap H| \leq |R \cap H| \leq 4$. If $S \cap H$ is a four-group, since it is normal in $P$, Lemma 4 yields a contradiction. Hence $S \cap H$ is cyclic. If $S \cap H = 1$ we have $R \cap H = 1$ and hence $P \cap H \leq 4$. Also if $S \cap H \neq 1$, $H$ contains an involution $a$ such that $C_H(a)$ is solvable. In either case Lemma 6, Lemma 7 or Lemma 8 applies. Thus we may assume $|P \cap H| = 64$. Now let $\Gamma = \{5, 6, \ldots, 12\}$ be a $P$-orbit of length eight and put $T = P_5$. Then $|\Gamma| = 8$, $\Gamma$ is faithful on $\Delta$ and hence $T$ is dihedral of order eight. $T$ also acts faithfully on $S$ and $P$ is a semi-direct product of $S$ by $T$, where, as is seen above, $S$ is quaternion or dihedral. Now we shall determine the action of $T$ on $S$. Let $T = \langle y, b \rangle$ with $o(y) = 4$, $o(b) = 2$ and $y^b = y^{-1}$. We may assume $y$ and $b$ are of the form

$$y = (1324)(5)(6)\cdots,$$

$$b = (1)(2)(34)(5)(6)\cdots.$$  

First let $S$ be quaternion with generators $w$ and $u$. Since $S$ contains three subgroups of order four, $T$ normalizes one of them, say $\langle w \rangle$. Then since $T$ is faithful on $S$ and since $\text{Aut } S \cong S$, we have $w^y = w$ and $w^b = w^{-1}$. We may now assume $w^y = w^{-1}$. Then since $o_i(b) = 2$ implies by Lemma 2 that $b$ centralizes no element of $S$ of order four we have $w^b = wu$ or $w^{-1}u$ and we may assume $u^b = wu$ by taking $b^y$ in place of $b$, if necessary. Next let $S$ be dihedral with generators $z$ and $e$ where $o(z) = 4$, $o(e) = 2$ and $z^e = z^{-1}$. Then we have $z^y = z$, $z^b = z^{-1}$ and we may assume $e^y = e$. If $b$ induces an inner automorphism on $S$ then $bf$ with some $f$ in $S$ centralizes $S$ which is impossible since $(bf)^T$ is an odd permutation. Hence we have $e^b = ez$ or $ez^{-1}$ and we may assume $e^b = ez^{-1}$ by taking $b^y$ in place of $b$, if necessary. We remark the two 2-groups obtained as above are both isomorphic to an $S_2$-subgroup of $M_{12}$. In fact, the two groups are isomorphic by the correspondence; $y \leftrightarrow y$, $b \leftrightarrow b$, $y^b \leftrightarrow e$ and $w \leftrightarrow z$. We claim that $G$ is $M$ on 12 points in this case (and consequently the case $S$ is quaternion occurs). In order to prove this it suffices to show that $G$ is isomorphic to $M_{12}$ since $M_{12}$ has no doubly transitive representation other than the usual one of degree 12. We now assume by way of contradiction

\[ (** ) \] $G$ has an $S_2$-subgroup isomorphic to that of $M_{12}$, but $G$ is not $M_{12}$.  

**Lemma 10.** $P$ contains exactly four involutions of the form $(1)(2)(34)\cdots$, and exactly six or two involutions of the form $(12)(34)\cdots$ according as $S$ is quaternion or dihedral.

Proof. This follows immediately from the action of $T$ on $S$.

**Lemma 11.** $G$ has a single conjugacy class of involutions. The elements of order eight of $G$ with a 2-cycle are all conjugate.

Proof. If $G$ has more than one conjugacy class of involutions then so does
since $|G:H|$ is odd. Then a result of Brauer and Fong ([6], Corollary 6B) yields $H$ is $M_{12}$ and hence $G$ is $M_{12}$, contrary to (**). From the action of $T$ on $S$ we observe that $bu$ (or $be$) is of order eight, $bu$ (or $be$) is conjugate to all its odd power under $<b,y^r>$ and that $G_{12}$ has a quasi-dihedral (or dihedral) $S_4$-subgroup of order sixteen, from which the last half follows.

Lemma 12. $G_{12}$ has a single conjugacy class of involutions. A 2-element of $G$ which has both a 2-cycle and a fixed point is of order two or eight. If $x$ is such an element of order eight $x$ has one 2-cycle and two fixed points and the centralizer $C_G(x)$ of $x$ has $<x>$ as its $S_4$-subgroup so that it has a normal 2-complement.

Proof. By Lemma 11 the involutions of $G_{12}$ are all conjugate under $G$. Now let $a$ be a central involution of $S$. Then since $N_G(S)^a = S$, we have $C_G(a)^b = S$, which implies the first assertion by Lemma 1. If an element of order four in $G$ has a fixed point then it has no 2-cycle by Lemma 5. The last part of Lemma 12 follows from the structure of an $S_4$-subgroup of $M_{12}$.

Lemma 13. Let $a$ be a central involution of $S$. Then an element $d$ of $C_G(a)$ with $d^a = (ijkl)$ or $(ij)(jk)$ fixes at most one orbit of $C_G(a)_\Delta$ on $\Omega-\Delta$ and an element $f$ of $C_G(a)$ with $f^a = (ij)(kl)$ fixes at most three of them.

Proof. Let $T = <y, b>$ be as above. Then $<d, C_G(a)_\Delta>$ is conjugate to $<y, C_G(a)_\Delta>$ or $<b, C_G(a)_\Delta>$ in $C_G(a)$ according as $d^a = (ijkl)$ or $(ij)(jk)$ and $<f, C_G(a)_\Delta>$ is conjugate to $<y^a, C_G(a)_\Delta>$ in $C_G(a)$. Thus to prove Lemma 13 it suffices to show that $b$ and $y$ fix at most one orbit of $S$ on $\Omega-\Delta$ and $y^a$ fixes at most three of them since every orbit of $C_G(a)_\Delta$ on $\Omega-\Delta$ consists of an odd number of $S$-orbits on $\Omega-\Delta$. By Lemma 10 $P$ contains exactly four involutions of the form $(1)(2)(34)...$ and exactly six (or two) involutions of the form $(12)(34)...$. Let $\Gamma_i(1 \leq i \leq l)$ be the orbits of $S$ on $\Omega-\Delta$. Then since $b$ has a fixed point on $\Omega-\Delta$ $b$ fixes some $\Gamma_i$, say $\Gamma_1$. Then $\alpha_i(b^{r_1}) = 2$ and the transitivity of $S$ on $\Gamma_i$ of length eight imply that any of four involutions in $P$ of the form $(1)(2)(34)...$ has its fixed points on $\Gamma_1$ and hence $<b,S>$ is semiregular on $\Omega-\Delta$--$\Gamma_1$. Thus $b$ fixes $\Gamma_i$, only and the similar argument yields that $y^a$ fixes at most three of $\Gamma_i$'s. Now $y$ also fixes $\Gamma_i$ and $y$ has all of its fixed points on $\Gamma_1$. (Note that $y$ has no 2-cycle). Assume $y$ fixes another $\Gamma_i$, say $\Gamma_2$. Then since $y^{r_2}$ and $y^{r_2}$ are even permutations we have $(y^a)^{r_2}$ (or $(ye)^{r_2}$) is of order at most four and hence $(yu)^4$ (or $(ye)^4$) $\neq 1$ fixes eight points, contrary to the assumption of our theorem.

Now since $C_G(a)^a = S$, $C_G(a)_\Delta$ contains a subgroup $L$ of index two such that $L^{a} = A_5$. Put $K = C_G(a)_\Delta$ and let $t$ and $s$ denote the number of orbits of $L$ and $K$ on $\Omega-\Delta$, respectively. Then

Lemma 14. $t = 1$ and $s = 3$.

Proof. We make use of a similar argument to H. Nagao ([20], p. 336-337).
Since $G$ is doubly transitive by a theorem of Frobenius [9] we have

\[ \sum_{g \in G} \alpha_s(g) = \frac{1}{2} |G|. \]

On the other hand we have

\[ \sum_{g \in G} \alpha_s(g) \geq |G: C_G(a)| \sum_{f \in G(a)} \alpha_s(af) + |G: C_G(x)| \sum_{h \in G(x)} \alpha_s(xh), \]

where $a = \omega^x (\text{or } x^2)$, $x = b\omega (\text{or } b\omega^x)$ and the summation in $\sum'$ is taken over all 2-regular elements of $C_G(a)$ or $C_G(x)$. We now calculate the right hand side of (2). First Lemma 12 yields

\[ |G: C_G(x)| \sum_{h \in G(x)} \alpha_s(xh) = \frac{1}{8} |G|. \]

Secondly since a 2-regular element of $C_G(a) - K$ lies in precisely one of four $C_G(a)$-conjugates of $L$, we have

\[ |G: C_G(a)| \sum_{f \in G(a)} \alpha_s(af) = 4 |G: C_G(a) - K| \sum_{f \in K} \alpha_s((af)^2) = 2 \alpha_s(af) \]

Now let $\alpha_i^*$ denote a permutation character of $L$ or $K$ acting on $\Omega - \Delta$. Then we have

\[ \sum_{g \in L} \alpha_i^*(g) = t |L| \quad \text{and} \quad \sum_{g \in K} \alpha_i^*(g) = s |K|. \]

(see [12], Theorem 16.6.13). If $v$ is a 2-singular element of $L$ (or $K$), then $v^{3m}$ with some integer $m$ is an involution fixing 1, 2, 3 and 4. Therefore $\alpha_i^*(v^{3m}) = 0$ and hence $\alpha_i^*(v) = 0$. On the other hand if $f$ is a 2-regular element then $\alpha_i^*((af)^2) = 2 \alpha_s(af)$ since $af$ has no fixed point on $\Omega - \Delta$. Thus by (4) and (5) we get

\[ |G: C_G(a)| \sum_{f \in G(a)} \alpha_s(af) = |G: C_G(a)| \{2t |L| - \frac{3}{2} s |K|\} = \left( \frac{1}{4} t - \frac{1}{16} s \right) |G|. \]

Substituting (3) and (6) into (2) and comparing (2) with (1) we have

\[ 6 \geq 4t - s \geq t, \]
where the last inequality follows from $3t \geq s$. We now show that $t$ is odd. By the theorem of Frobenius we have

\[
\sum_{r \in \mathcal{G}} \left( \begin{array}{c} \alpha_r(g) \\ \alpha_s(g) \end{array} \right) = \frac{1}{2} |G|
\]

for some integer $l$. On the other hand from Lemma 11 and Lemma 12 it follows

\[
\sum_{r \in \mathcal{G}} \left( \begin{array}{c} \alpha_r(g) \\ \alpha_s(g) \end{array} \right) = |G: C_\mathcal{G}(a)| \sum'_{\nu \in \mathcal{H}_\mathcal{G}(x)} \left( \begin{array}{c} \alpha_\nu(xh) \\ \alpha_s(xh) \end{array} \right)
\]

Now by Lemma 12 we have

\[
\sum'_{\nu \in \mathcal{H}_\mathcal{G}(x)} \left( \begin{array}{c} \alpha_\nu(xh) \\ \alpha_s(xh) \end{array} \right) = \frac{1}{4} |G| \quad \text{and}
\]

\[
|G: C_\mathcal{G}(a)| \sum'_{\nu \in \mathcal{H}_\mathcal{G}(x)} \left( \begin{array}{c} \alpha_\nu(xh) \\ \alpha_s(xh) \end{array} \right)
\]

Then by (8)(9)(10) and (11) we have $2l = t + 1$, which implies $t$ is odd. Then by (7) one of the following holds:

(i) $t = 1$ and $s = 1$,
(ii) $t = 1$ and $s = 3$,
(iii) $t = 3$ and $s = 7$,
(iv) $t = 3$ and $s = 9$,
(v) $t = 5$ and $s = 15$.

If $s = 1$, since $a$ and $b$ are conjugate in $G_{13}$ by Lemma 12, $G_{13}$ is transitive on $\Omega - \{1, 2\}$. Then the result of J. King yields that $G$ is $M_{13}$, contrary to (**). We remark that the semi-direct product of elementary abelian group of order $3^3$ by $\text{SL}(3, 3)$ has no transitive extension. This follows from a result of Hering, Kantor and Seitz [15] or a direct calculation of the number of $S_r$-subgroups. Also it will be easily seen that (iii), (iv) or (v) in the above conflicts with Lemma 13. For instance assume that $t = 3$ and $s = 7$. We denote the orbits of $C_\mathcal{G}(a)$ on $\Omega - \Delta$ by $\sum_i (1 \leq i \leq 7)$. If $C_\mathcal{G}(a)$ has three orbits on $\{\Sigma_i's\}$, $b$ would fix at least three of $\Sigma_i$'s, contrary to Lemma 13. Thus the only case to be considered is
that $C_G(a)$ fixes one $\Sigma_i$, say $\Sigma_1$, and permutes $\Sigma_2, \Sigma_3, \ldots, \Sigma_t$ transitively. In this case we consider $C_G(a)(\Sigma_1)\Delta$. Since $|C_G(a):C_G(a)(\Sigma_1)| = |C_G(a):C_G(a)(\Sigma_1)| = 6$, it follows that $C_G(a)(\Sigma_1)\Delta$ is of order four. On the other hand since $C_G(a)(\Sigma_1)$ fixes another $\Sigma_i$, $C_G(a)(\Sigma_1)$ is a regular four group by Lemma 13. Then we have $C_G(a)\Delta > C_G(a)(\Sigma_1)\Delta$ and hence $C_G(a) > C_G(a)(\Sigma_1)$. Then $C_G(a)(\Sigma_1)$ fixes all $\Sigma_i$'s, contrary to Lemma 13. The similar argument eliminates the possibilities of case (iv) and case (v), completing the proof of Lemma 14.

**Lemma 15.** $G$ contains no element which has both a 2-cycle and a 3-cycle.

Proof. Assume $G$ contains an element $z$ of the form $(i j) (klm)\ldots$. Then the 2-part of $z$ is of order two and $d = z^r$ with some odd integer $r$ is an involution. Clearly $z$ is in $C_G(d)$ and $z$ fixes the 2-cycle $(i j)$ of $d$. On the other hand since $a$ and $d$ are conjugate in $G$ by Lemma 11, Lemma 14 implies that any element of $C_G(d)$ with a 3-cycle on $I(d)$ fixes no 2-cycle of $d$. This is a contradiction.

Now set $N = G_{12}$ and $M = G_{123}$. Then

**Lemma 16.** $G_{12}$ and $N$ have two orbits on $\Omega - \{1, 2\}$ and $M$ has one or two orbits on $\Omega - \{1, 2, 3\}$.

Proof. By Lemma 14 $K$ has three orbits $\Sigma_1, \Sigma_2$ and $\Sigma_3$ of the same length on $\Omega - \Delta$ and any orbit of $G_{12}$ on $\Omega - \{1, 2\}$ is a union of some of $\{3, 4\}, \Sigma_1, \Sigma_2$ and $\Sigma_3$. If $\{5, 6\}$ is in $\Sigma_1$, since $a$ and $b$ are conjugate in $G_{12}$ by Lemma 12 $\{3, 4\}$ and $\Sigma_1$ are contained in a $G_{12}$-orbit and $b$ takes $\Sigma_2$ to $\Sigma_3$ by Lemma 13. Thus $G_{12}$ has at most two, hence by (**) precisely two orbits of different length on $\Omega - \{1, 2\}$ and hence so does $N$. The last half is an immediate consequence of Lemma 14.

**Lemma 17.** $N$ has two conjugacy classes of involutions and two classes of elements of order eight. In particular involutions of $N$ with 2-cycle $(12)$ and elements of order eight with 2-cycle $(12)$ are all conjugate in $N$ respectively.

Proof. We regard $G$ as a (transitive) permutation group on the set of the unordered pairs of the points of $\Omega$. Then $N$ is the stabilizer of the pair $\{1, 2\}$ and the involutions in $N$ are all conjugate under $G$ by Lemma 11. On the other hand since $C_G(a)^{(a)} = S_4$ and since $t = 1$ by Lemma 14 $C_G(a)$ has two orbits on the set of the unordered pairs which $a$ fixes. Hence the first half follows from Lemma 1. Also the elements of order eight in $N$ are all conjugate by Lemma 11, and $C_G(a)$ has two orbits on the set of the unordered pairs which $a$ fixes. Thus the last half follows again from Lemma 1 and the remark after Lemma 1.

**Lemma 18.** If $c = (12)(3)(4)\ldots$ is an involution in $C_G(a)$ then $C_M(c)$ is a subgroup of $C_N(c)$ of index four and any 2-regular element of $C_N(c)$ lies in $C_M(c)$.

Proof. Clearly $C_M(c)$ is contained in $N$. (In fact, $C_M(c) = C_N(c) \cap G_3$). Since
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$C_G(a)_{56}$ contains $T$, $C_G(a)_{56}$ is transitive on $\Delta$. Then Lemma 14 implies the transitivity of $C_G(a)_{ij}$ on $\Delta$ for any 2-cycle $(i,j)$ of $a$. Then since $a$ and $c$ are conjugate, $C_G(c)_{12}$ is transitive on $\bar{I}(c)$ and so is $C_N(c)$. This implies $|C_N(c) : C_M(c)| = 4$. Now let $f$ be a 2-regular element of $N$. Then $f^{I(c)} = 1$ by Lemma 15, and hence $f$ is in $M$.

Now we shall give a final contradiction. Let $\alpha_i^*$ and $\beta_i^*$ be permutation characters of $N$ and $M$ acting on $\Omega = \{1, 2\}$ and $\Omega = \{1, 2, 3\}$, respectively. By Lemma 16 we have

\begin{align}
\sum_{g \in N} \alpha_i^*(g) &= 2|N| \\
\sum_{g \in M} \beta_i^*(g) &= 2|M|.
\end{align}

From (12) and Lemma 16 we get

\begin{align}
\sum_{g \in N - a_{12}} \alpha_i^*(g) &= \sum_{g \in N} \alpha_i^*(g) - \sum_{g \in C_G(a)_{12}} \alpha_i^*(g) = 2|N| - |N| = |N|.
\end{align}

On the other hand, it follows from Lemma 12 and Lemma 17

\begin{align}
\sum_{x \in N - a_{12}} \alpha_i^*(x) &= |N : C_N(c)| \sum_{f \in C_N(c)} \alpha_i^*(cf) \\
&+ |N : C_N(v)| \sum_{k \in C_N(v) \in C_N(v)^*} \alpha_i^*(vh)
\end{align}

where $c$ is an involution of $N$ of the form (12) (3) (4)… and $v$ is an element of $N$ of order eight of the form (12) (3) (4)…

Now if we denote the set of 2-regular elements of a group $X$ by $X^*$ Lemma 15 and Lemma 12 imply

\begin{align}
|N : C_N(c)| \sum_{f \in C_N(c)} \alpha_i^*(cf) &= |N : C_N(c)| \times 4|C_N(c)|, \\
\text{and} \\
|N : C_N(v)| \sum_{k \in C_N(v) \in C_N(v)^*} \alpha_i^*(vh) &= |N : C_N(v)| \times 2|C_N(v)| = \frac{|N|}{4},
\end{align}

respectively. Then by (14), (15), (16) and (17) we have $|C_N(c)| = \frac{3}{16} |C_N(c)|$ and hence by Lemma 18

\begin{align}
|C_M(c)|^* &= \frac{3}{4} |C_N(c)|.
\end{align}

On the other hand we have

\begin{align}
\sum_{g \in M} \beta_i^*(g) &= \sum_{g \in M} \beta_i^*(g) \geq |M : C_M(c)| \sum_{f \in C_M(c)} \beta_i^*(cf) \\
&= |M : C_M(c)| \times 3 |C_M(c)|^*,
\end{align}
where the last equality follows from Lemma 15. Then substituting (18) into (19) we have

\[(20) \quad \sum_{\beta \in \mathcal{B}_n} \beta \cdot \mathcal{R}(g) \geq \frac{9}{4} |M|,
\]

which conflicts with (13).

Subcase 2. Let \(P\) and \(R\) be as in Subcase 1. Then \(R\) fixes four points say \(1, 2, 3\) and \(4\) on \(\Omega\) and \(R\) is normal in \(P\). By a theorem of Witt ([22], Theorem 9.4), \(N_G(R)^{\mathcal{R}(R)}\) is doubly transitive and hence we have \(N_G(R)^{\mathcal{R}(R)} = A_4\) since \(N_G(R)^{\mathcal{R}(R)}\) is of odd order. If \(G_{12}\) fixes more than two points it fixes four points on \(\Omega\) and hence by a result of K. Harada [13] and \((\ast)\), \(G\) is \(\text{PGL}(2, 8)\) on \(28\) points. Thus we may assume \(G_{12}\) fixes exactly two points on \(\Omega\). Then the points \(3\) and \(4\) lie in the different \(G_{12}\)-orbits of odd length. We denote them by \(\Gamma_3\) and \(\Gamma_4\), respectively. Set \(W = G_{12}^{\mathcal{R}(R)}\). Then \(W\) is transitive on \(\Gamma_3\) and for any \(i \in \Gamma_3\), \(W_i^{\mathcal{R}(R)}\) is a strongly embedded subgroup of \(W\) in a sense of H. Bender [5] and hence by a result of Bender [5] either of the following cases occurs.

(i) An \(S_2\)-subgroup of \(W\) is cyclic or generalized quaternion,

(ii) \(W\) contains \(\text{PSL}(2, q)\), \(S_4(q)\) or \(\text{PSU}(3, q^2)\) normally with odd index (as a permutation group of usual degree) where \(q\) is a suitable power of 2.

Assume first (i) holds. Then \(R \cong R^{\mathcal{R}(R)}\) is cyclic or generalized quaternion. If \(R\) is cyclic since \(N_G(R)^{\mathcal{R}(R)} = A_4\) we have \(N_G(R) = C_G(R)\). Let \(b = (12)\) be an involution which is conjugate to an involution of \(R\). We may assume \(b\) normalizes, therefore centralizes \(R\). Then since \(R\) is cyclic and since \(C_R(b)^{\mathcal{R}(b)} \leq A_4\) we have \(|C_R(b)| = |C_R(b)^{\mathcal{R}(b)}| = 2\) and hence \(|R| = 2\). Then \(|P \cap H| \leq |P| \leq 8\), and so Lemma 6 or Lemma 8 applies. Now let \(R\) be generalized quaternion. Then involutions of \(G\) fixing four points are all conjugate. Note also that \(P\) contains a normal four group \(U\) in this case, otherwise \(P\) contains a cyclic subgroup \(X = \langle x \rangle\) of index two with \(x^{\mathcal{R}(R)} \in A_4\) which implies \(x^4 \in R\) and hence \(R = \langle x^4 \rangle\), contrary to the assumption. Now since \(R\) is generalized quaternion we have \(U^{\mathcal{R}(R)} = 1\). If an involution \(c\) of \(U\) has a fixed point on \(\Omega - I(R)\), since \(c^{\mathcal{R}(R)} \in A_4\) we have \(|I(c) \cap (\Omega - I(R))| = 4\), \(C_R(c)^{I(c)} \leq A_4\) and hence \(|C_R(c)| = |C_R(c)^{I(c)}| = 2\). Then since \(|R : C_R(c)| \leq |R : C_R(U)| \leq 2\) it follows that \(|R| \leq 4\), a contradiction. Thus any involution of \(U\) has no fixed point on \(\Omega - I(R)\). Now let \(c\) be an involution of \(U\) with \(c^{\mathcal{R}(R)} = 1\), say \(c^{\mathcal{R}(R)} = (12)\) (34) \(\ldots\) and let \(b = (12)\) be an involution fixing four points. We may assume \(b\) normalizes \(R\). Then since \(\langle c, R \rangle\) and \(\langle b, R \rangle\) are \(S_2\)-subgroups of \(G_{12, R}\) they are conjugate and hence \(c\) also fixes an \(R\)-orbit \(\Gamma\) on \(\Omega - I(R)\). Set \(X = \langle c, R \rangle\) and let \(m = |\Gamma|\). Then since \(|X| = 2m\) and since \(X^d \leq A_m\), \(X\) contains an involution \(d\) fixing four points on \(\Gamma\). Then since \(R\) is regular on \(\Gamma\) it follows from Lemma
2 that \( |C_R(d)| = 4 \) and hence \( C_R(d^{f(d)}) \approx C_R(d) \approx Z_4 \), a contradiction.

Now assume (ii) holds. Then \( W_3 \) is 2-closed and \( R \) is transitive on \( \Gamma_3 - \{3\} \). Let \( a \) be an involution of \( R \), assume \( a^{r_3} = (3) (56) \) and let \( b = (5) (6) \cdots \) be an involution commuting with \( a \). Then we have \( b^{f(R)} \in A_4 \). If \( b^{f(R)} = (12) (34) \), then \( b \) normalizes \( G_{12} \) and hence permutes \( \Gamma_3 \) to \( \Gamma_4 \), which is impossible. Thus we may assume \( b^{f(R)} = (14) (23) \). Then since \( b \) normalizes \( G_{124} \) \( b \) also normalizes an \( S_2 \)-subgroup \( Q \) of \( G_{1244} \). Then since \( Q \) is also an \( S_2 \)-subgroup of \( G_{12} \), \( Q^{r_3} \) is an \( S_2 \)-subgroup of \( W_3 \), and hence we have \( Q^{r_3} = R^{r_3} \) since \( W_3 \) is 2-closed. Thus \( \Gamma_3 - \{3\} \) is an orbit of \( Q \) and so \( b \) fixes \( \Gamma_3 - \{3\} \) as a whole. Then \( b \) has two or four fixed points on \( \Gamma_3 - \{3\} \). Suppose first \( b \) fixes two points on \( \Gamma_3 - \{3\} \). Then by a result of Zassenhaus ([23], Satz 5) \( Q \) has a cyclic subgroup of index two. Then by the structure of \( S_2 \)-subgroups of the groups in (ii) we have \( Q \) is a four group, and hence \( |P| = 16 \). If \( |P \cap H| = 16 \), \( P \cap H \) is dihedral, quasi-dihedral or elementary of order sixteen, while as is easily seen, \( P = (P \cap H) \) contains a normal four group and an element of order four. This is a contradiction. Thus we have \( |P \cap H| \leq 8 \) and then Lemma 6 or Lemma 8 applies. Finally suppose \( b \) fixes four points on \( \Gamma_3 - \{3\} \). Put \( \Sigma = \{2\} \cup \Gamma_3 \) and \( L = \langle b, G_{12} \rangle \). Then \( L \leq G_{\{3\}} \) and \( L^x \) is transitive, in particular, \( L_3^x \) and \( L_6^x \) are conjugate. Clearly \( Q^x \) is an \( S_2 \)-subgroup of \( L_3^x \) and \( Q^x \) is semi-regular on \( \Sigma - \{2, 3\} \), while \( L_6^x \) contains an involution \( b^x \) fixing four points, which is a contradiction.

References