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DOUBLY TRANSITIVE GROUPS IN WHICH THE MAXIMAL NUMBER OF FIXED POINTS OF INVOLUTIONS IS FOUR

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## 1. Introduction

Doubly transitive groups in which any involution fixes at most three points have been classified by H. Bender [3], C. Hering [14] and J. King [16], [17]. In this paper we shall prove the following results.

Theorem. Let $G$ be a doubly transitive group on $\Omega=\{1,2, \cdots, n\}$. Assume that the maximal number of fixed points of involutions in $G$ is four. Then, if $n \neq 0$ $(\bmod 8)$, one of the following holds:
(a) $n=6$ and $G$ is $S_{6}$,
(b) $n=10$ and $G$ is $S_{6}$ or $P \Gamma L(2,9)$,
(c) $n=12$ and $G$ is $M_{11}$ or $M_{12}($ the Mathieu group of degree 11 or 12),
(d) $n=28$ and $G$ is $P \Gamma L(2,8)$,
(e) $n=28$ and $G$ is $P S U\left(3,3^{2}\right)$ or $P \Sigma U\left(3,3^{2}\right)$.

Corollary. Let $G$ be a doubly transitive group on $\Omega=\{1,2, \cdots, n\}$. If every involution in $G$ fixes four points in $\Omega$, then one of the following holds:
(a) $n=12$ and $G$ is $M_{11}$,
(b) $n=28$ and $G$ is $P \Gamma L(2,8)$,
(c) $n=28$ and $G$ is $\operatorname{PSU}\left(3,3^{2}\right)$ or $P \Sigma U\left(3,3^{2}\right)$.

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## 2. Definitions and notations

A permutation group $G$ on $\Omega$ is called semi-regular if every non-identity element of $G$ has no fixed point, and $G$ is called regular if $G$ is transitive and semiregular. If a set $X$ of permutations on $\Omega$ fixes a subset $\Delta$ of $\Omega$ the restriction of $X$ on $\Delta$ will be denoted by $X^{\Delta}$. Further we use the following notations which are standard.
$S_{n}=$ symmetric group of degree $n$,
$A_{n}=$ alternating group of degree $n$,
$G F(q)=$ finite field with $q$ elements,
$P \Gamma L(m, q)=m$-dimensional projective semi-linear group over $G F(q)$,
$\operatorname{PSL}(m, q)=m$-dimensional projective special linear group over $G F(q)$,
$\operatorname{PSU}\left(3, q^{2}\right)=$ 3-dimensional projective special unitary group over $G F\left(q^{2}\right)$,
$P \Sigma U\left(3, q^{2}\right)=$ automorphism group of $\operatorname{PSU}\left(3, q^{2}\right)$,
Aut $G=$ automorphism group of a group $G$,
$G_{i j \ldots r}=$ pointwise stabilizer in $G$ of points $i, j \cdots, r$,
$G_{(i j \ldots r)}=$ global stsbilizer in $G$ of the set of points $i, j, \cdots, r$,
$I(X)=$ totality of points fixed by a set $X$ of permutations,
$\alpha_{i}(x)=$ number of $i$-cycles of a permutation $x$,
$\mathrm{o}(x)=$ order of a permutation $x$,
$N_{G}(X)=$ normalizer of $X$ in $G$,
$C_{G}(X)=$ centralizer of $X$ in $G$,
$\langle X, Y\rangle=$ subgroup generated by $X$ and $Y$,
$[X, Y]=$ commutator of $X$ and $Y$,
$|X|=$ cardinality of a set $X$.

## 3. Proof of Corollary

It suffices by our theorem to prove that $n \neq 0(\bmod 8)$ in the case $G$ satisfies the assumption of Corollary. Assume by way of contradiction that $n \equiv 0(\bmod 8)$. Then the length of any orbit of an $S_{2}$-subgroup $P$ of $G$ on $\Omega$ is divisible by eight (see [22], Theorem 3. 4'). Then since a central involution of $P$ fixes a point on $\Omega$, it fixes more than four points on $\Omega$, contrary to the assumption.

## 4. Proof of Theorem

We begin with some lemmas on permutation groups.
Lemma 1 (J. Alperin [1]). Let the group $G$ be transitive on $\Omega=\{1,2, \cdots, n\}$ and let $H$ be a subgroup of the stabilizer $G_{1}$. If the conjugates $H^{g}(g \in G)$ which are contained in $G_{1}$ make up $k$ different conjugacy classes of subgroups of $G$, then the normalizer $N_{G}(H)$ of $H$ has exactly $k$ orbits on $I(H)$.

We remark that lemma 1 also holds valid if a subgroup $H$ of $G_{1}$ in the above is replaced by a subset $K$ of $G_{1}$. In fact, Alperin's proof in [1] does not make use of that $H$ is a subgroup.

Lemma 2 (H. Nagao). Let $X$ be a semi-regular permutation group on $\Omega$ $=\{1,2, \cdots, n\}$. If a permutation group $A$ on $\Omega$ normalizes $X$ and fixes at least one point, then the order of $C_{X}(A)$ is not greater than the number of fixed points of $A$. If $X$ is regular, then the order of $C_{X}(A)$ equals the number of fixed points of $A$.

Proof. Suppose $A$ fixes the point 1. Let $x$ be an element of $X$ and $a$ an ele-
ment of A. If $x$ takes 1 to $i$ and $a$ takes $i$ to $j$, then $a^{-1} a x$ takes 1 to $j$. Since $a^{-1}$ $x a \in X$ and $X$ is semi-regular, $x=a^{-1} x a$ if and only if $j=i$, i.e. $i \in I(a)$. Thus we have $\left|C_{X}(A)\right| \leq|I(A)|$. If $X$ is regular, then for any fixed point $i$ of $A$ there is a unique element of $X$ which takes 1 to $i$. Hence we have $\left|C_{X}(A)\right|=|I(A)|$.

Lamma 3 (D. Livingstone and A. Wagner [18]). Let $G$ be $k$-fold transitive on $\Omega=\{1,2, \cdots, n\}$, and let $H$ be the stabilizer of $k$ points in $\Omega$. Assume that an $S_{p^{-}}$ subgroup $P$ of $H$ fixes precisely the given $k$ points. Then for a point in a minimum $P$ orbit on $\Omega-I(P), N_{G}\left(P_{i}\right)^{I\left(P_{i}\right)}$ is $k$-fold transitive.

The proof of the above lemma is seen in p. 400-401 of [18].
Lemma 4. Let $Y$ be a cyclic 2-group which acts regularly on $\Omega$ $=\{1,2, \cdots, n\}$, and assume that $Y$ normalizes a four group $U$ which is semi-regular on $\Omega$. Then $|Y|=|\Omega|=4$.

Proof. Assume that $n=2^{m} \geq 4$, and let $\Delta_{i}(1 \leq i \leq t)$ denote the orbits of of $U$ on $\Omega$. Then since $Y$ permutes $\Delta_{1}, \Delta_{2}, \cdots, \Delta_{t}$ transitively we have $\left|Y: Y_{\left\{\Delta_{1}\right\}}\right|$ $=t \geq 2$, from which it follows that $Y_{\left(\Delta_{1}\right)}$ centralizes $U$ since $\left|Y: C_{Y}(U)\right| \leq 2$.
Then since $Y_{\left\{\Delta_{1}\right\}} \Delta_{1}$ is cyclic and $\boldsymbol{U}^{\Delta_{1}}$ is self-centralizing, we have $\left|Y_{\left\{\Delta_{1}\right\}}\right|=\left|Y_{\left\{\Delta_{1}\right\}}{ }^{\Delta_{1}}\right|$ $\leqq 2$, which yields that $|Y|=t\left|Y_{\left\{\Delta_{1} \mid\right.}\right| \left\lvert\, \leq 2 t=\frac{n}{2}\right.$, a contradiction.

Now to prove our theorem the following two cases will be treated separately.
Case I. $\quad n \equiv 2(\bmod 4)$.
Case II. $\quad n \equiv 0(\bmod 4)$.
Case I. Since an involution in $G$ fixing four points is an odd permutation in this case, $N=G \cap A_{n}$ is a normal transitive subgroup of index two. Furthermore since $\left|G_{1}: N_{1}\right|=2$ is prime to $n-1, N_{1}$ is transitive on $\Omega-\{1\}$ and hence $N$ is doubly transitive on $\Omega$. Then by a result of $C$. Hering [14], either of the following holds:
(i) $\quad n=q+1$ and $N$ contains PSL (2.q),
(ii) $n=6$ and $N$ is $A_{6}$.

In case(ii), $n=6$ and $G$ is $S_{6}$. In case(i) an involution in $G-N$ fixing four points acts as a field automorphism. Hence we have $4=1+\sqrt{q}$ and hence $q=9$. Then $n=10$ and $G$ is $S_{6}$ or $P \Gamma L(2,9)$.

Case II. We have $n \equiv 4(\bmod 8)$ in this case by our assumption that $n \neq$ $0(\bmod 8)$. In particular $G$ contains no regular normal subgroup and hence by a theorem of Burnside ([7], P202) we have
(*) a (unique) minimal normal subgroup $H$ of $G$ is a primitive non abelian simple group.

In what follows we denote by $H$ a (unique) minimal normal subgroup of $G$ throughout.

## Lemma 5. We may assume $G$ contains no normal subgroup of index two.

Proof. Let $G$ have a normal subgroup $N$ of index two. Then, as seen in Case I, $N$ is also doubly transitive on $\Omega$. If $N$ contains an involution fixing four points we take $N$ in place of $G$. If $N$ contains no involution fixing four points, the results of H. Bender [3] and C. Hering [14] yield, similarly to Case I, that $n=6$ or 10 , which is not the case.

Lemma 6. An $S_{2}$-subgroup of $H$ is not dihedral.
Proof. This follows from a result of Gorenstein and Walter [11] and a result of Lüneburg ([19], Satz 1).

Lemma 7. If $H$ has a quasi-dihedral $S_{2}$-subgroup and $H$ contains an involution a such that $C_{H}(a)$ is solvable then $n=12$ and $G$ is $M_{11}$.

Proof. By a result of Alperin, Brauer and Gorenstein [2] $H$ is $\operatorname{PSL}(3,3)$ or $M_{11}$. If $H$ is $\operatorname{PSL}(3,3)$, since $\mid$ Aut $\operatorname{PSL}(3,3): \operatorname{PSL}(3,3) \mid=2$, we have $G$ is $P S L$ $(3,3)$ or Aut $\operatorname{PSL}(3,3)$. But it is easily seen that $\operatorname{PSL}(3,3)$ and Aut $\operatorname{PSL}(3,3)$ have no doubly transitive representation satisfying our assumption. If $H$ is $M_{11}$, since Aut $M_{11} \simeq M_{11}$, we have $G$ is $M_{11}$ on 12 points.

Lemma 8. If $H$ has an elementary abelian $S_{2}$-subgroup and $H$ contains an involution a such that $C_{H}(a)$ is solvable then $n=28$ and $G$ is $P \Gamma L(2,8)$.

Proof. By a result of Walter [21] $H$ is $P S L(2, q)$ for a suitable $q$. Then the result of Luneburg [19] applies.

Lemma 9. If $H$ has a wreathed $S_{2}$-subgroup of order 32 and $H$ contains an involution a such that $C_{H}(a)$ is solvable then $n=28$ and $G$ is $P S U\left(3,3^{2}\right)$ or $P \Sigma U$ $\left(3,3^{2}\right)$.

Proof. By a result of P. Fong [8] $H$ is $\operatorname{PSU}\left(3,3^{2}\right)$, and hence $G$ is $P S U$ $\left(3,3^{2}\right)$ or $P \Sigma U\left(3,3^{2}\right)$. It is easy to see that $P S U\left(3,3^{2}\right)$ and $P \Sigma U\left(3,3^{2}\right)$ have no other doubly transitive representation than the usual one of degree 28 .

Now we consider the following two cases.
Subcase 1. an $S_{2}$-subgroup of $G_{12}$ fixes two points.
Subcase 2. an $S_{2}$-subgroup of $G_{12}$ fixes four points.
Subcase 1. Let $P$ be an $S_{2}$-subgroup of $G$. Then since $n \equiv 4(\bmod 8) P$ has an orbit of length four, say $\Delta=\{1,2,3,4\}$ (see [22], Theorem 3.4') and then $R=P_{1}$ is an $S_{2}$-subgrou of $G_{1}$. We may assume $I(R)=\{1,2\}$. Then $R$ contains an element $b$ of the form $b=(1)(2)(34) \cdots$. We assume first that any element in $R$ of the form (1)(2)(34) $\cdots$ is an involution. Set $S=R_{34}$. Then $N_{G}(S)^{I(S)}$ is doubly transitive by Lemma 3 and hence $N_{G}(S)^{I(S)}=S_{4}$. Since any element in
the coset $b S$ is an involution $b$ inverts every element of $S$ and hence $S$ is abelian. On the other hand, since $b$ fixes exactly two points on $\Omega-\Delta$, and since $S$ is semiregular on $\Omega-\Delta$ Lemma 2 yields $\left|C_{S}(b)\right|=2$. Thus $S$ must have a unique involution and hence $S$ is cyclic. This implies that $C_{G}(S)^{I(S)} \geqq A_{4}$ and any element $t$ of $N_{G}(S)$ with $t^{\Delta} \in A_{4}$ lies in $C_{G}(S)$. Now since $\left|P^{\Delta}\right|=8, P$ contains an element $y$ of the form (1324) $\cdots$. First assume $o(y) \geqq 8$. Then by Lemma $5 y$ has an odd number of cycles of the same length on $\Omega-\Delta$. Then since the length of any $P$-orbit on $\Omega-\Delta$ is of 2 -power $P$ fixes the set of the points of a cycle of $y$, say $(5,6$, $\cdots, l)$ on $\Omega-\Delta$ and in particular so does $S$. Then since $y^{4} \in S$ we have $o(y)=$ $|S|, 2|S|$ or $4|S|$. If $o(y)=|S|$ either $y$ or the generator $z$ of $S$ is an odd permutation, contrary to Lemma 5 . Also if $o(y)=2|S|$, since $\left[y^{2}, S\right]=1$, setting $\Gamma=\{5,6, \cdots l$,$\} we have \left(y^{2}\right)^{\Gamma}=\left(z^{m}\right)^{\Gamma}$ for some odd integer $m$. Then $y^{2} z^{-m}$ $=(12)(34) \cdots$ fixes every point of $\Gamma$, contrary to the assumption of our theorem. Thus we have $o(y)=4|S|$ and hence $\langle y\rangle$ is a cyclic subgroup of $P$ of index two. Then the structure of $P$ is known (see [10], Theorem 5.4.4) and since $P$ contains a non normal dihedral subgroup $R, P$ is dihedral or quasi-dihedral. We may assume by Lemma 6 that $P$ is quasi-dihedral. Let $a$ be an involution of $S$. Then since $C_{G}(a)^{I(a)} \leqq S_{4}$ and since $C_{G}(a)_{I(a)}$ hsa a cyclic $S_{2}$-subgroup it follows that $C_{G}(a)$ is solvable. Then Lemma 7 yields that $G$ is $M_{11}$ on 12 points.

Now let $o(y)=4$. Then since $n \equiv 4(\bmod 8) y$ has four fixed points or two 2 -cycles on $\Omega-\Delta$ by Lemma 5 and hence $y^{2}$ fixes four points on $\Omega-\Delta$. On the other hand $y^{2} \in A_{4}$ implies $\left[y^{2}, S\right]=1$ and hence $|S| \leq 4$ by Lemma 2. Thus we have $|P \cap H| \leq|P| \leq 32$, and by a result of $P$. Fong $P \cap H$ is dihedral, quasidihedral, elementary or wreathed of order 32. Then Lemma 6, Lemma 7, Lemma 8 or Lemma 9 applies, respectively.

Now we assume that $R$ contains an element $x$ of the form (1)(2)(34) $\cdots$ which is not an involution. Then since $n \equiv 4(\bmod 8), x$ has an odd number of cycles of the same length on $\Omega-\Delta$ by Lemma 5. In particular $o(x) \geq 8$, and $P$ fixes the set of the points of a cycle of $x$, say $(56 \cdots k)$ on $\Omega-\Delta$. Then since $x^{2} \in S, \Gamma=\{5,6, \cdots k\}$ consists of one or two orbits of $S$. If $\Gamma$ consists of two orbits of $S$ then we have $S=\left\langle x^{2}\right\rangle$ is cyclic. In this case the similar argument to the above applies. Thus we may assume $\Gamma$ consists of one orbit of $S$. Then $o(x)=|S|=|\Gamma| \geq 8$ and since $\Gamma$ is a $P$-orbit an exponent of $P$ is equal to $o(x)$. In particular $P$ contains a normal four-group $U$ (see [10], P215. Exercise 9). Then since $o(x) \geq 8$, Lemma 4 implies that an involution $c$ in $U$ fixes a point on $\Gamma$, and hence we have $\left|C_{S}(c)\right|$ $\leq 4$ by Lemma 2. Then since $\left|S: C_{S}(c)\right| \leq\left|S: C_{S}(U)\right| \leq 2$ it follows that $|S|$ $\leq 8$ and hence $|S|=8$. If $S \cong Z_{4} \times Z_{2}, P$ contains a normal four group which is semi-regular on $\Gamma$, contrary to Lemma 4 . Also if $S$ is cyclic $S$ contains an odd permutation. Thus $S$ must be quarternion or dihedral of order eight. Now we shall determine the structure of $P \cap H$. If $|P \cap H|=32, G$ contains a normal subgroup of index two, contrary to Lemma 5. Let $|P \cap H| \leq 16$. Then $P \cap H$ is
dihedral, quasi-dihedral or elementary. Since the transitivity of $H$ on $\Omega$ implies the transitivity of $P \cap H$ on $\Delta$ (see [22], Theorem 3.4') we have $|S \cap H| \leq \mid R \cap$ $H \mid \leq 4$. If $S \cap H$ is a four-group, since it is normal in P, Lemma 4 yields a contradiction. Hence $S \cap H$ is cyclic. If $S \cap H=1$ we have $R \cap H=1$ and hence $P \cap H \leq 4$. Also if $S \cap H \neq 1, H$ contains an involution $a$ such that $C_{H}(a)$ is solvable. In either case Lemma 6, Lemma 7 or Lemma 8 applies. Thus we may assume $|P|=|P \cap H|=64$. Now let $\Gamma=\{5,6, \cdots, 12\}$ be a $P$-orbit of length eight and put $T=P_{5}$. Then $|T|=8, T$ is faithful on $\Delta$ and hence $T^{\Delta}$ is dihedral of order eight. $T$ also acts faithfully on $S$ and $P$ is a semi-direct product of $S$ by $T$, where, as is seen above, $S$ is quarternion or dihedral. Now we shall determine the action of $T$ on $S$. Let $T=\langle y, b\rangle$ with $o(y)=4, o(b)=2$ and $y^{b}=y^{-1}$. We may assume $y$ and $b$ are of the form

$$
y=(1324)(5)(6) \cdots
$$

$$
b=(1)(2)(34)(5)(6) \cdots
$$

First let $S$ be quarternion with generators $w$ and $u$. Since $S$ contains three subgroups of order four, $T$ normalizes one of them, say $\langle w\rangle$. Then since $T$ is faithful on $S$ and since Aut $S \cong S_{4}$ we have $w^{\mathbf{y}}=w$ and $w^{b}=w^{-1}$. We may now assume $u^{y}=w u^{-1}$. Then since $\alpha_{1}\left(b^{0-\Delta}\right)=2$ implies by Lemma 2 that $b$ centralizes no element of $S$ of order four we have $u^{b}=w u$ or $w^{-1} u$ and we may assume $u^{b}$ $=w u$ by taking $b^{y}$ in place of $b$, if necessary. Next let $S$ be dihedral with generators $z$ and $e$ where $o(z)=4, o(e)=2$ and $z^{e}=z^{-1}$. Then we have $z^{y}=z, z^{b}=z^{-1}$ and we may assume $e^{y}=e z$. If $b$ induces an inner automorphism on $S$ then $b f$ with some $f$ in $S$ centralizes $S$ which is impossible since ( $b f)^{\Gamma}$ is an odd permutation. Hence we have $e^{b}=e z$ or $e z^{-1}$ and we may assume $e^{b}=e z^{-1}$ by taking $b^{y}$ in place of $b$, if necessary. We remark the two 2-groups obtained as above are both isomorphic to an $S_{2}$-subgroup of $M_{12}$. In fact, the two groups are isomorphic by the correspondence; $y \leftrightarrow y, b \leftrightarrow b, y^{2} u \leftrightarrow e$ and $w \leftrightarrow z$. We claim that $G$ is $M_{12}$ on 12 points in this case (and consequently the case $S$ is quarternion occurs). In order to prove this it suffices to show that $G$ is isormorphic to $M_{12}$ since $M_{12}$ has no doubly transitive representation other than the usual one of degree $12 . \mathrm{We}$ now assume by way of contradiction
$(* *) G$ has an $S_{2}$-subgroup isomorphic to that of $M_{12}$, but $G$ is not $M_{12}$.
Lemma 10. $P$ contains exactly four involutions of the form (1)(2)(34) $\cdots$, and exactly six or two involutions of the form (12)(34)…according as $S$ is quarternion or dihedral.

Proof. This follows immediately from the action of $T$ on $S$.
Lemma 11. G has a single conjugacy class of involutions. The elements of order eight of $G$ with a 2 -cycle are all conjugate.

Proof. If $G$ has more than one conjugacy class of involutions then so does
$H$ since $|G: H|$ is odd. Then a result of Brauer and Fong ([6], Corollary $6 B$ ) yields $H$ is $M_{12}$ and hence $G$ is $M_{12}$, contrary to (**). From the action of $T$ on $S$ we observe that $b u$ (or $b e$ ) is of order eight, $b u$ (or $b e$ ) is conjugate to all its odd power under $\left\langle b, y^{2}\right\rangle$ and that $G_{12}$ has a quasi-dihedral (or dihedral) $S_{2}$-subgroup of order sixteen, from which the last half follows.

Lemma 12. $G_{12}$ has a single conjugacy class of involutions. A 2-element of $G$ which has both a 2-cycle and a fixed point is of order two or eight. If $x$ is such an element of order eight $x$ has one 2-cycle and two fixed points and the centralizer $C_{G}(x)$ of $x$ has $\langle x\rangle$ as its $S_{2}$-subgroup so that it has a normal 2-complement.

Proof. By Lemma 11 the involutions of $G_{12}$ are all conjugate under $G$. Now let $a$ be a central involution of $S$. Then since $N_{G}(S)^{\Delta}=S_{4}$ we have $C_{G}(a)^{\Delta}=S_{4}$, which implies the first assertion by Lemma 1. If an element of order four in $G$ has a fixed point then it has no 2 -cycle by Lemma 5. The last part of Lemma 12 follows from the structure of an $S_{2}$-subgroup of $M_{12}$.

Lemma 13. Let a be a central involution of $S$. Then an element $d$ of $C_{G}(a)$ with $d^{\Delta}=(i j k l)$ or $(i)(j)(k l)$ fixes at most one orbit of $C_{G}(a)_{\Delta}$ on $\Omega-\Delta$ and an element $f$ of $C_{G}(a)$ with $f^{\Delta}=(i j)(k l)$ fixes at most three of them.

Proof. Let $T=\langle y, b\rangle$ be as avobe. Then $\left\langle d, C_{G}(a)_{\Delta}\right\rangle$ is conjugate to $\left\langle y, C_{G}(a)_{\Delta}\right\rangle$ or $\left\langle b, C_{G}(a)_{\Lambda}\right\rangle$ in $C_{G}(a)$ according as $d^{\Delta}=(i j k l)$ or $(i)(j)(k l)$ and $\left\langle f, C_{G}(a)_{\Lambda}\right\rangle$ is conjugate to $\left\langle y^{2}, C_{G}(a)_{\Delta}\right\rangle$ in $C_{G}(a)$. Thus to prove Lemma 13 it suffices to show that $b$ and $y$ fix at most one orbit of $S$ on $\Omega-\Delta$ and $y^{2}$ fixes at most three of them since every orbit of $C_{G}(a)_{\Delta}$ on $\Omega-\Delta$ consists of an odd number of $S$-orbits on $\Omega-\Delta$. By Lemma $10 P$ contains exactly four involutions of the form (1)(2)(34) $\cdots$ and exactly six (or two) involutions of the form (12)(34) $\cdots$. Let $\Gamma_{i}(1 \leq i \leq l)$ be the orbits of $S$ on $\Omega-\Delta$. Then since $b$ has a fixed point on $\Omega-\Delta b$ fixes some $\Gamma_{i}$, say $\Gamma_{1}$. Then $\alpha_{1}\left(b^{\Gamma_{1}}\right)=2$ and the transitivity of $S$ on $\Gamma_{1}$ of length eight imply that any of four involutions in $P$ of the form (1)(2)(34) $\ldots$ has its fixed points on $\Gamma_{1}$ and hence $\langle b, S\rangle$ is semiregular on $\Omega-\Delta-\Gamma_{1}$. Thus $b$ fixes $\Gamma_{1}$ only and the similar argument yields that $y^{2}$ fixes at most three of $\Gamma_{i}$ 's. Now $y$ also fixes $\Gamma_{1}$ and $y$ has all of its fixed points on $\Gamma_{1}$. (Note that $y$ has no 2-cycle). Assume $y$ fixes another $\Gamma_{i}$, say $\Gamma_{2}$. Then since $y^{\Gamma_{2}}$ and $u^{\Gamma_{2}}$ (or $e^{\Gamma_{2}}$ ) are even permutations we have $(y u)^{\Gamma_{2}}\left(\operatorname{or}(y e)^{\Gamma_{2}}\right)$ is of order at most four and hence $(y u)^{4}\left(\right.$ or $\left.(y e)^{4}\right) \neq 1$ fixes eight points, contrary to the assumption of our theorem.

Now since $C_{G}(a)_{1}{ }^{\Delta}=S_{3}, C_{G}(a)_{1}$ contains a subgroup $L$ of index two such that $L^{\Delta}=A_{3}$. Put $K=C_{G}(a)_{\Delta}$ and let $t$ and $s$ denote the number of orbits of $L$ and $K$ on $\Omega-\Delta$, respectively. Then

Lemma 14. $t=1$ and $s=3$.
Proof. We make use of a similar argument to H. Nagao ([20], p. 336-337).

Since $G$ is doubly transitive by a theorem of Frobenius [9] we have

$$
\begin{equation*}
\sum_{8 \in \theta} \alpha_{2}(g)=\frac{1}{2}|G| \tag{1}
\end{equation*}
$$

On the other hand we have

$$
\begin{equation*}
\sum_{\delta \in G} \alpha_{2}(g) \geq\left|G: C_{G}(a)\right| \sum_{f \in C_{G}(a)}^{\prime} \alpha_{2}(a f)+\left|G: C_{G}(x)\right| \sum_{h \in \sigma_{G}\left(x^{x}\right)}^{\prime} \alpha_{2}(x h), \tag{2}
\end{equation*}
$$

where $a=w^{2}$ (or $x^{2}$ ), $x=b u$ (or $b e$ ) and the summation in $\Sigma^{\prime}$ is taken over all 2-regular elements of $C_{G}(a)$ or $C_{G}(x)$. We now calculate the right hand side of (2). First Lemma 12 yields

$$
\begin{equation*}
\left|G: C_{G}(x)\right| \sum_{h \in C_{G^{(x)}}}^{\prime} \alpha_{2}(x h)=\frac{1}{8}|G| \tag{3}
\end{equation*}
$$

Secondly since a 2-regular element of $C_{G}(a)-K$ lies in precisely one of four $C_{G}(a)$-conjugates of $L$, we have

$$
\begin{align*}
& \left|G: C_{G}(a)\right| \sum_{f \in \sigma_{G}(a)}^{\prime} \alpha_{2}(a f)  \tag{4}\\
= & \left|G: C_{G}(a)\right|\left\{4 \sum_{f \in L^{-K}}^{\prime} \alpha_{2}(a f)+\sum_{f \in K} \alpha_{2}(a f)\right\} \\
= & \left|G: C_{G}(a)\right|\left\{4 \sum_{f \in L}^{\prime} \alpha_{2}(a f)-3 \sum_{f \in K}^{\prime} \alpha_{2}(a f)\right\}
\end{align*}
$$

Now let $\alpha_{1}^{*}$ denote a permutation character of $L$ or $K$ acting on $\Omega-\Delta$. Then we have

$$
\begin{align*}
& \sum_{g \in L} \alpha_{1}^{*}(g)=t|L| \text { and }  \tag{5}\\
& \sum_{g \in K} \alpha_{1}^{*}(g)=s|K|
\end{align*}
$$

(see [12], Theorem 16.6.13). If $v$ is a 2 -singular element of $L$ (or $K$ ), then $v^{3 m}$ with some integer $m$ is an involution fixing $1,2,3$ and 4. Therefore $\alpha_{1}^{*}\left(v^{3 m}\right)=0$ and hence $\alpha_{1}{ }^{*}(v)=0$. On the other hand if $f$ is a 2-regular element then $\alpha_{1}{ }^{*}(f)$ $=\alpha_{1}{ }^{*}\left((a f)^{2}\right)=2 \alpha_{2}(a f)$ since af has no fixed point on $\Omega-\Delta$. Thus by (4) and (5) we get

$$
\begin{align*}
\left|G: C_{G}(a)\right| \sum_{f \in \sigma_{\left.G^{( }\right)}}^{\prime} \alpha_{2}(a f) & =\left|G: C_{G}(a)\right|\left\{2 t|L|-\frac{3}{2} s|K|\right\}  \tag{6}\\
& =\left(\frac{1}{4} t-\frac{1}{16} s\right)|G|
\end{align*}
$$

Substituting (3) and (6) into (2) and comparing (2) with (1) we have

$$
\begin{equation*}
6 \geq 4 t-s \geq \mathrm{t} \tag{7}
\end{equation*}
$$

where the last inequality follows from $3 t \geq s$. We now show that $t$ is odd. By the theorem of Frobenius we have

$$
\begin{equation*}
\sum_{q \in G}\binom{\alpha_{2}(g)}{1}\binom{\alpha_{1}(g)}{1}=\frac{l}{2}|G| \tag{8}
\end{equation*}
$$

for some integer $l$. On the other hand from Lemma 11 and Lemma 12 it follows

$$
\begin{align*}
\sum_{g \in G}\binom{\alpha_{2}(g)}{1}\binom{\alpha_{1}(g)}{1} & =\left|G: C_{G}(a)\right| \sum_{f \in G_{G^{(a)}}^{\prime}}\binom{\alpha_{2}(a f)}{1}\binom{\alpha_{1}(a f)}{1}  \tag{9}\\
& +\left|G: C_{G}(x)\right| \sum_{h \in G_{G^{(x)}}^{\prime}}^{\prime}\binom{\alpha_{2}(x h)}{1}\binom{\alpha_{1}(x h)}{1} .
\end{align*}
$$

Now by Lemma 12 we have

$$
\begin{align*}
& \left|G: C_{G}(x)\right| \sum_{n \in C_{G}(*)}^{\prime}\binom{\alpha_{2}(x h)}{1}\binom{\alpha_{1}(x h)}{1}=\frac{1}{4}|G| \text { and }  \tag{10}\\
& \mid G: C_{G}(a) \sum_{f \in C_{G}(a)}^{\prime}\binom{\alpha_{2}(a f)}{1}\binom{\alpha_{1}(a f)}{1}  \tag{11}\\
= & \left.\left|G: C_{G}(a)\right|\left\{4 \sum_{f \in K-L}^{\prime}\binom{\alpha_{2}(a f)}{1}\binom{\alpha_{2}(a f)}{1}+\sum_{f \in K}^{\prime}\binom{\alpha_{1}(a f)}{1} \begin{array}{c}
\alpha_{1}(a f) \\
1
\end{array}\right)\right\} \\
= & \left|G: C_{G}(a)\right|\left\{4 \sum_{f \in L_{-K}^{\prime}}^{\prime} \frac{1}{2} \alpha_{1}^{*}(f)+4 \sum_{f \in K}^{\prime} \frac{1}{2} \alpha_{1}^{*}(f)\right\} \\
= & 2\left|G: C_{G}(a)\right| \sum_{f \in L}^{\prime} \alpha_{1}^{*}(f) \\
= & t|G|
\end{align*}
$$

Then by (8)(9)(10) and (11) we have $2 l=t+1$, which implies $t$ is odd. Then by (7) one of the following holds:
(i) $t=1$ and $s=1$,
(ii) $t=1$ and $s=3$,
(iii) $t=3$ and $s=7$,
(iv) $t=3$ and $s=9$,
(v) $t=5$ and $s=15$.

If $s=1$, since a and $b$ are conjugate in $G_{12}$ by Lemma $12, G_{12}$ is transitive on $\Omega-\{1,2\}$. Then the result of J. King yiedls that $G$ is $M_{12}$, contrary to (**). We remark that the semi-direct product of elementary abelian grou of order $3^{3}$ by SL $(3,3)$ has no transitive extension. This follows from a result of Hering, Kantor and Seitz [15] or a direct calculation of the number of $S_{7}$-subgroups. Also it will be easily seen that (iii), (iv) or (v) in the above conflicts with Lemma 13. For instance assume that $t=3$ and $s=7$. We denote the orbits of $C_{G}(a)_{\Delta}$ on $\Omega-\Delta$ by $\sum_{i}(1 \leqq i \leqq 7)$. If $C_{G}(a)$ has three orbits on $\left\{\Sigma_{i}\right.$ 's $\}, b$ would fix at least three of $\Sigma_{i}$ 's, contrary to Lemma 13. Thus the only case to be considered is
that $C_{G}(a)$ fixes one $\Sigma_{i}$, say $\Sigma_{7}$ and permutes $\Sigma_{1}, \Sigma_{2}, \cdots \Sigma_{6}$, transitively. In this case we consider $C_{G}(a)_{\left\{\Sigma_{1}\right\}}{ }^{\Delta}$. Since $\left|C_{G}(a)^{\Delta}: C_{G}(a)_{\left\{\Sigma_{1}\right\}}{ }^{\Delta}\right|=\left|C_{G}(a): C_{G}(a)_{\left\{\Sigma_{1}\right\}}\right|=$ 6 , it follows that $C_{G}(a)_{\left\{\Sigma_{1}\right\}}{ }^{\Delta}$ is of order four. On the other hand since $C_{G}(a)_{\left\{\Sigma_{1}\right\}}$ fixes another $\Sigma_{i}\left(\neq \Sigma_{1}\right), C_{G}(a)_{\left(\Sigma_{1}\right)}{ }^{\Delta}$ is a regular four group by Lemma 13. Then we have $C_{G}(a)^{\Delta} \triangleright C_{G}(a)_{\left\{\Sigma_{1}\right\}}{ }^{\Delta}$ and hence $C_{G}(a) \triangleright C_{G}(a)_{\left\{\Sigma_{1}\right\}}$. Then $C_{G}(a)_{\left\{\Sigma_{1}\right\}}$ fixes all $\Sigma_{i}$ 's, contrary to Lemma 13. The similar argument eliminates the possiblilities of case (iv) and case (v), completing the proof of Lemma 14.

Lemma 15. G contans no element which has both a 2-cycle and a 3-cycle.
Proof. Assume $G$ contains an element $z$ of the form (ij) (klm) $\cdots$. Then the 2-part of $z$ is of order two and $d=z^{3 r}$ with some odd integer $r$ is an involution. Clearly $z$ is in $C_{G}(d)$ and $z$ fixes the 2-cycle (ij) of $d$. On the other hand since $a$ and $d$ are conjugate in $G$ by Lemma 11, Lemma 14 implies that any element of $C_{G}(d)$ with a 3 -cycle on $I(d)$ fixes no 2 -cycle of $d$. This is a contradiction.

Now set $N=G_{\{12\}}$ and $M=G_{\{123\}}$. Then
Lemma 16. $G_{12}$ and $N$ have two orbits on $\Omega-\{1,2\}$ and $M$ has one or two orbits on $\Omega-\{1,2,3\}$.

Proof. By Lemma $14 K$ has three orbits $\Sigma_{1}, \Sigma_{2}$ and $\Sigma_{3}$ of the same length on $\Omega-\Delta$ and any orbit of $G_{12}$ on $\Omega-\{1,2\}$ is a union of some of $\{3,4\}, \Sigma_{1}, \Sigma_{2}$ and $\Sigma_{3}$. If $\{5,6\}$ is in $\Sigma_{1}$, since $a$ and $b$ are conjugate in $G_{12}$ by Lemma $12\{3,4\}$ and $\Sigma_{1}$ are contained in a $G_{12}$-orbit and $b$ takes $\Sigma_{2}$ to $\Sigma_{3}$ by Lemma 13. Thus $G_{12}$ has at most two, hence by ( $* *$ ) precisely two orbits of different length on $\Omega-\{1,2\}$ and hence so does $N$. The last half is an immediate consequence of Lemma 14.

Lemma 17. $N$ has two conjugacy classes of involutions and two classes of elements of order eight. In particular involutions of $N$ with 2-cycle (12) and elements of order eight with 2-cycle (12) are all conjugae in $N$ respectively.

Proof. We regard $G$ as a (transitive) permutation group on the set of the unordered pairs of the points of $\Omega$. Then $N$ is the stabilizer of the pair $\{1,2\}$ and the involutions in $N$ are all conjugate under $G$ by Lemma 11. On the other hand since $C_{G}(a)^{I(a)}=S_{4}$ and since $t=1$ by Lemma $14 C_{G}(a)$ has two orbits on the set of the unordered pairs which $a$ fixes. Hence the first half follows from Lemma 1. Also the elements of order eight in $N$ are all conjugate by Lemma 11, and $C_{G}(x)$ has two orbits on the set of the unordered pairs which $x$ fixes. Thus the last half follows again from Lemma 1 and the remark after Lemma 1.

Lemma 18. If $c=(12)$ (3) (4) $\cdots$ is an involution in $\mathrm{C}_{G}(a)$ then $\mathrm{C}_{M}(c)$ is a subgroup of $\mathrm{C}_{N}(c)$ of index four and any 2-regular element of $\mathrm{C}_{N}(c)$ lies in $\mathrm{C}_{M}(c)$.

Proof. Clearly $\mathrm{C}_{M}(c)$ is contained in $N$. (In fact, $\mathrm{C}_{M}(c)=\mathrm{C}_{N}(c) \cap G_{3}$ ). Since
$C_{G}(a)_{56}$ contains $T, C_{G}(a)_{56}$ is transitive on $\Delta$. Then Lemma 14 implies the transitivity of $\mathrm{C}_{G}(a)_{i j}$ on $\Delta$ for any 2-cycle (ij) of $a$. Then since $a$ and $c$ are conjugate, $C_{G}(c)_{12}$ is transitive on $I(c)$ and so is $C_{N}(c)$. This implies $\mid C_{N}(c)$ : $C_{M}(c) \mid=4$. Now let $f$ be a 2-regular element of $N$. Then $f^{I(c)}=1$ by Lemma 15, and hence $f$ is in $M$.

Now we shall give a final contradiction. Let $\alpha_{1}^{*}$ and $\beta_{1} *$ be permutation characters of $N$ and $M$ acting on $\Omega-\{1,2\}$ and $\Omega-\{1,2,3\}$, respectively. By Lemma 16 we have

$$
\begin{align*}
& \sum_{g \in N} \alpha_{1}^{*}(g)=2|N| \quad \text { and }  \tag{12}\\
& \sum_{g \in \boldsymbol{M}} \beta_{1}^{*}(g) \leqq 2|M| \tag{13}
\end{align*}
$$

From (12) and Lemma 16 we get

$$
\begin{equation*}
\sum_{g \in N-\theta_{12}} \alpha_{1} *(g)=\sum_{g \in N} \alpha_{1} *(g)-\sum_{g \in G_{12}} \alpha_{1}^{*}(g)=2|N|-|N|=|N| . \tag{14}
\end{equation*}
$$

On the other hand, it follows from Lemma 12 and Lemma 17

$$
\begin{align*}
\sum_{x \in N-G_{12}} \alpha_{1}^{*} *(x) & =\left|N: C_{N}(c)\right| \sum_{f \in \sigma_{N}(c)}^{\prime} \alpha_{1}^{*} *(c f)  \tag{15}\\
& +\left|N: C_{N}(v)\right| \sum_{h \in \sigma_{N}(u)}^{\prime} \alpha_{1}^{*} *(v h)
\end{align*}
$$

where $c$ is an involution of $N$ of the form (12) (3) (4) $\cdots$ and $v$ is an element of $N$ of order eight of the form (12) (3) (4)... .

Now if we denote the set of 2-regular elemtns of a group $X$ by $X^{*}$ Lemma 15 and Lemma 12 imply

$$
\begin{equation*}
\left|N: C_{N}(c)\right| \sum_{f \in C_{N}(c)}^{\prime} \alpha_{1}^{*}(c f)=\left|N: C_{N}(c)\right| \times 4\left|C_{N}(c) *\right| \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|N: C_{N}(v)\right| \sum_{h \in C_{N}(v)}^{\prime} \alpha_{1}^{*}(v h)=\left|N: C_{N}(v)\right| \times 2\left|C_{N}(v)^{*}\right|=\frac{|N|}{4}, \tag{17}
\end{equation*}
$$

respectively. Then by (14), (15), (16) and (17) we have $\left|C_{N}(c)^{*}\right|=\frac{3}{16}\left|C_{N}(c)\right|$ and hence by Lemma 18

$$
\begin{equation*}
\left|C_{M}(c)^{*}\right|=\frac{3}{4}\left|C_{N}(c)\right| \tag{18}
\end{equation*}
$$

On the other hand we have

$$
\begin{align*}
& \sum_{g \in \boldsymbol{\mu}} \beta_{1} *(g) \geq \underset{\substack{g \in \boldsymbol{M} \\
g=(12) \ldots}}{ } \beta_{1} *(g) \geq\left|M: C_{M}(c)\right| \sum_{f \in \sigma_{\left.M^{( }\right)}^{(c)}}^{\prime} \beta_{1} *(c f)  \tag{19}\\
& \quad=\left|M: C_{M}(c)\right| \times 3\left|C_{M}(c)^{*}\right|,
\end{align*}
$$

where the last equality follows from Lemma 15 . Then substituting (18) into (19) we have

$$
\begin{equation*}
\sum_{g \in \boldsymbol{M}} \beta_{2}^{*}(g) \geq \frac{9}{4}|M| \tag{20}
\end{equation*}
$$

which conficts with (13).
Subcase 2. Let $P$ and $R$ be as in Subcase 1. Then $R$ fixes four points say $1,2,3$ and 4 on $\Omega$ and $R$ is normal in $P$. By a theorem of Witt ([22], Theorem 9.4), $N_{G}(R)^{I(R)}$ is doubly transitive and hence we have $N_{G}(R)^{I(R)}=A_{4}$ since $N_{G}$ $(R)^{I(R)}$ is of odd order. If $G_{12}$ fixes more than two points it fixes four points on $\Omega$ and hence by a result of K. Harda [13] and ( $*), G$ is $P \Gamma L(2,8)$ on 28 points. Thus we may assume $G_{12}$ fixes exactly two points on $\Omega$. Then the points 3 and 4 lie in the different $G_{12}$-orbits of odd length. We denote them by $\Gamma_{3}$ and $\Gamma_{4}$ respectively. Set $W=G_{12}{ }^{\Gamma 3}$. Then $W$ is transitive on $\Gamma_{3}$ and for any $i \in \Gamma_{3} W_{i}{ }^{\Gamma 3}$ is a strongly embedded subgroup of $W$ in a sense of H. Bender [5] and hence by a result of Bender [5] either of the following cases occurs.
(i) An $S_{2}$-subgroup of $W$ is cyclic or generalized quarternion,
(ii) $W$ contains $\operatorname{PSL}(2, q), S_{z}(q)$ or $\operatorname{PSU}\left(3, q^{2}\right)$ normally with odd index (as a permutation group of usual degree) where $q$ is a suiitable power of 2 .

Assume first (i) holds. Then $R \cong R^{\Gamma 3}$ is cyclic or generalized quarternion. If $R$ is cyclic since $N_{G}(R)^{I(R)}=A_{4}$ we have $N_{G}(R)=C_{G}(R)$. Let $b=(12) \cdots$ be an involution which is conjugate to an involution of $R$. We may assume $b$ normalizes, therefore centralizes $R$. Then since $R$ is cyclic and since $C_{R}(b)^{I(b)}$ $\leq A_{4}$ we have $\left|C_{R}(b)\right|=\left|C_{R}(b)^{I(b)}\right|=2$ and hence $|R|=2$. Then $|P \cap H|$ $\leq|P| \leq 8$, and so Lemma 6 or Lemma 8 applies. Now let $R$ be generalized quarternion. Then involutions of $G$ fixing four points are all conjugate. Note also that $P$ contains a normal four group $U$ in this case, for oteherwise $P$ contains a cyclic subgroup $X=\langle x\rangle$ of index two with $x^{I(R)} \in A_{4}$ which implies $x^{2} \in$ $R$ and hence $R=\left\langle x^{2}\right\rangle$, contrary to the assumption. Now since $R$ is generalized quarternion we have $U^{I(R)} \neq 1$. If an involution $c$ of $U$ has a fixed point on $\Omega-I(R)$, since $c^{I(R)} \in A_{4}$ we have $|I(c) \cap(\Omega-I(R))|=4, \mathrm{C}_{R}(c)^{I(c)} \leq A_{4}$ and hence $\left|C_{R}(c)\right|=\left|C_{R}(c)^{I(c)}\right|=2$. Then since $\left|R: C_{R}(c)\right| \leq\left|R: C_{R}(U)\right| \leq 2$ it follows that $|R| \leq 4$, a contradiction. Thus any involution of $U$ has no fixed point on $\Omega-I(R)$. Now let $c$ be an involution of $U$ with $c^{I(R)} \neq 1$, say $c^{I(R)}=(12)(34) \cdots$ and let $b=(12) \cdots$ be an involution fixing four points. We may assume $b$ normalizes $R$. Then since $\langle c, R\rangle$ and $\langle b, R\rangle$ are $S_{2}$-subgroups of $G_{\{1,2\}}$ they are conjugate and hence $c$ also fixes an R-orbit $\Gamma$ on $\Omega-I(R)$. Set $X=\langle c, R\rangle$ and let $m=$ $|\Gamma|$. Then since $|X|=2 m$ and since $X^{\Gamma} \leq A_{m} X$ contains an involution $d$ fixing four points on $\Gamma$. Then since $R$ is regular on $\Gamma$ it follows from Lemma

2 that $\left|C_{R}(d)\right|=4$ and hence $C_{R}(d)^{I(d)} \cong \mathrm{C}_{R}(d) \cong Z_{4}$, a contradiction.
Now assume (ii) holds. Then $W_{3}$ is 2-closed and $R$ is transitive on $\Gamma_{3}-\{3\}$. Let $a$ be an involuiton of $R$, assume $a^{\Gamma 3}=(3)(56) \cdots$ and let $b=(5)(6) \cdots$ be an involution commuting with $a$. Then we have $b^{I(R)} \in A_{4}$. If $b^{I(R)}=$ (12) (34), then $b$ normalizes $G_{12}$ and hence permutes $\Gamma_{3}$ to $\Gamma_{4}$, which is impossible. Thus we may assume $b^{I(R)}=(14)(23)$. Then since $b$ normalizes $G_{1234} b$ also normalizes an $S_{2}$-subgroup $Q$ of $G_{1234}$. Then since $Q$ is also an $S_{2}$-subgroup of $G_{12}, Q^{\Gamma 3}$ is an $S_{2}$-subgroup of $W_{3}$, and hence we have $Q^{\Gamma 3}=R^{\Gamma 3}$ since $W_{3}$ is 2 -closed. Thus $\Gamma_{3}-\{3\}$ is an orbit of $Q$ and so $b$ fixes $\Gamma_{3}-\{3\}$ as a whole. Then $b$ has two or four fixed points on $\Gamma_{3}-\{3\}$. Suppose first $b$ fixes two points on $\Gamma_{3}-\{3\}$. Then by a result of Zassenhaus ([23], Satz 5) $Q$ has an cylic subgroup of index two. Then by the structure of $\mathrm{S}_{2}$-subgroups of the groups in (ii) we have $Q$ is a four group, and hence $|P|=16$. If $|P \cap H|=16, P \cap H$ is dihedral, quasi-dihedral or elementary of order sixteen, while as is easily seen, $P(=P \cap H)$ contains a normal four group and an element of order four. This is a contradiction. Thus we have $|P \cap H| \leq 8$ and then Lemma 6 or Lemma 8 applies. Finally supose $b$ fixes four points on $\Gamma_{3}-\{3\}$. Put $\Sigma=\{2\} \cup \Gamma_{3}$ and $L=\left\langle b, G_{12}\right\rangle$. Then $L$ $\leq G_{\{\Sigma]}$ and $L^{\Sigma}$ is transitive, in particular, $L_{2}{ }^{\Sigma}$ and $L_{5}{ }^{\Sigma}$ are conjugate. Clearly $Q^{\Sigma}$ is an $S_{2}$-subgroup of $L_{2}{ }^{\Sigma}$ and $Q^{\Sigma}$ is semi-regular on $\Sigma-\{2,3\}$, while $L_{5}{ }^{\Sigma}$ contains an involution $b^{\Sigma}$ fixing four points, which is a contradiction.

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