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Author(s)	Sakai, Fumio; Tono, Keita
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## RATIONAL CUSPIDAL CURVES OF TYPE $(d, d - 2)$ WITH ONE OR TWO CUSPS

FUMIO SAKAI and KEITA TONO

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### 1. Introduction

A plane curve  $C \subset \mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$  is said to be of *type*  $(d, \nu)$  if the degree of  $C$  is  $d$  and the maximal multiplicity of  $C$  is  $\nu$ . In case  $C$  is rational and cuspidal, we have proved the inequality:  $d < 3\nu$  ([4]). A *cusp* means a unibranch (i.e., locally irreducible) singular point. Rational cuspidal curves of type  $(d, d - 2)$  are classified in [2] under the assumption that  $C$  has at least three cusps. The type  $(d, d - 3)$  case with at least three cusps is also discussed in [3]. In this note, we classify rational cuspidal curves of type  $(d, d - 2)$  having at most two cusps. For a cusp  $P \in C$ , let  $\underline{m}_P = (m_{0,P}, \dots, m_{n+1,P})$  denote the *multiplicity sequence* ([2, 3, 4]), where  $m_{0,P}$  is the multiplicity  $m_P = m_P(C)$  of  $C$  at  $P$ . The notation  $(2_4)$  means the sequence  $(2, 2, 2, 2, 1, 1)$ . We prove the following

**Theorem.** *Let  $C$  be a rational cuspidal curve of type  $(d, d - 2)$ . Let  $Q$  be the cusp with multiplicity  $d - 2$ .*

- (i) *If  $C$  has the unique cusp  $Q$ , then  $d$  is even and  $\underline{m}_Q = (d - 2, 2_{d-2})$ . Furthermore, the equation of  $C$  has the form (up to projective equivalence):*

$$\left( y^k z + \sum_{i=1}^{k+1} a_i x^i y^{k+1-i} \right)^2 - x y^{2k+1} = 0 \quad (a_{k+1} \neq 0),$$

where  $d = 2k + 2$ .

- (ii) *If  $C$  has two cusps  $Q$  and  $P$ , then one of the following occurs.*

- (a)  *$\underline{m}_Q = (d - 2)$ ,  $\underline{m}_P = (2_{d-2})$ . In this case,  $C$  is unique up to projective equivalence. For small  $d$ , the equations can be written as follows:*

$$\begin{aligned} (yz + x^2)^2 - x^3 z &= 0, \\ (yz + x^2)^2 y + 2(yz + x^2)x^3 - x^4 z &= 0, \\ (yz + 2x^2)^2 y^2 + 2(yz + 2x^2)yx^3 + (2yz + 5x^2)x^4 - x^5 z &= 0. \end{aligned}$$

- (b)  *$d$  is odd,  $\underline{m}_Q = (d - 2, 2_{(d-3)/2})$ ,  $\underline{m}_P = (2_{(d-1)/2})$ . The equation of  $C$  has*

the form (up to projective equivalence):

$$\left( y^{k-1}z + \sum_{i=2}^k a_i x^i y^{k-i} \right)^2 y - x^{2k+1} = 0,$$

where  $d = 2k + 1$ .

- (c)  $d$  is even,  $\underline{m}_Q = (d - 2, 2_{d-2-j})$ ,  $\underline{m}_P = (2_j)$ ,  $1 \leq j \leq (d - 2)/2$ . The equation of  $C$  has the form (up to projective equivalence):

$$\left( y^{k+j}z + \sum_{i=2}^{k+j+1} a_i x^i y^{k+j+1-i} \right)^2 - x^{2j+1} y^{2k+1} = 0 \quad (a_{k+j+1} \neq 0),$$

where  $d = 2k + 2j + 2$  and  $k \geq 0$ ,  $j \geq 1$ .

By construction, any of the above rational cuspidal curves of type  $(d, d - 2)$  with one or two cusps can be transformed into a line by a Cremona transformation. Combining this with the result of [2], we obtain

**Corollary.** Any rational cuspidal curve of type  $(d, d - 2)$  is transformable into a line by a Cremona transformation.

## 2. Quadratic transformation

Let  $C$  be a plane curve. Choose a Cremona transformation  $\psi$  (i.e., a birational transformation of  $\mathbf{P}^2$ ). In certain circumstances, the strict transform  $C'$  of  $C$  via  $\psi$  is a rational cuspidal curve. We successively use the Cremona transformation  $\psi(O, A, \ell)$ , which was used in [3]. We discuss this process in detail.

Let  $O, A$  be two points in  $\mathbf{P}^2$ . Let  $t$  be the line passing through  $O$  and  $A$ , and let  $\ell$  be another line passing through  $O$ . Let  $\pi : X \rightarrow \mathbf{P}^2$  be the blowing up at  $O$ . We denote by  $E$  the exceptional curve. Note that  $X$  is the geometric ruled surface  $\mathbf{F}_1$ . Put  $\tilde{O} = \tilde{\ell} \cap E$ , where  $\tilde{\ell}, \tilde{t}$  are the strict transforms of  $\ell, t$ . Let  $\varphi : X \rightsquigarrow X'$  be the result of elementary transformations at  $\tilde{O}$  and  $\tilde{A} = \pi^{-1}(A)$ . More precisely, let  $\eta : Z \rightarrow X$  be the blowing up at  $\tilde{O}$  and  $\tilde{A}$ . Let  $\tilde{\ell}', \tilde{t}'$  denote the exceptional curves over  $\tilde{O}$  and  $\tilde{A}$ , respectively. Let  $\check{\ell}, \check{t}$  and  $\check{E}$  be the strict transforms of  $\tilde{\ell}, \tilde{t}$  and  $E$ . Let  $\eta' : Z \rightarrow X'$  be the contraction of  $\check{\ell}$  and  $\check{t}$ . Let  $\tilde{\ell}', \tilde{t}'$  be the images of  $\check{\ell}', \check{t}'$ . Since  $\tilde{O} \in E$  and  $\tilde{A} \notin E$ ,  $E' = \eta'(\check{E})$  is again a  $(-1)$  curve. So  $X'$  is the surface  $\mathbf{F}_1$ . Set  $\tilde{O}' = \tilde{t}' \cap E'$ . Let  $\pi' : X' \rightarrow \mathbf{P}^2$  be the contraction of  $E'$ . Let  $\ell', t'$  denote the images of  $\tilde{\ell}', \tilde{t}'$ . Set  $O' = \eta'(\tilde{O}')$ . Then  $\ell'$  and  $t'$  are lines with  $O' = \ell' \cap t'$ .

**DEFINITION.** The Cremona transformation  $\psi = \pi' \circ \varphi \circ \pi^{-1}$  is denoted by  $\psi(O, A, \ell)$ . Note that  $\psi : \mathbf{P}^2 \setminus \{\ell \cup t\} \rightarrow \mathbf{P}^2 \setminus \{\ell' \cup t'\}$  is an isomorphism and that  $\psi : \mathbf{P}^2 \setminus \{O, A\} \rightarrow \mathbf{P}^2$  is a morphism. Let  $C$  be an irreducible plane curve ( $\neq \ell, t$ ). Then

there exists an irreducible plane curve  $C'$  such that  $\psi$  gives an isomorphism  $C \setminus \{\ell \cup t\} \cong C' \setminus \{\ell' \cup t'\}$ . The curve  $C'$  is called the *strict transform* of  $C$  via  $\psi$ . Symbolically, we write as  $C' = \psi[C]$ .

Set  $\check{B}' = \check{\ell} \cap \check{\ell}'$ ,  $\bar{B}' = \eta'(\check{B}')$ ,  $B' = \pi'(\bar{B}')$ . The inverse Cremona transformation  $\pi \circ \varphi^{-1} \circ \pi'^{-1}$  can be written as  $\psi(O', B', t')$ .

NOTATION. As a convention, we use the following notation:

$$O' = \psi(O), B' = \psi(A), \ell' = \psi(\ell), t' = \psi(t).$$

It is straightforward to prove the following

**Lemma 1.** *Let  $(x, y, z)$  be the homogeneous coordinates of  $\mathbb{P}^2$  so that the line  $\ell$  is defined by  $x = 0$ , the line  $t$  is defined by  $y = 0$  and the point  $A$  has the coordinates  $(1, 0, a)$ . Suppose that the line  $\ell'$  is defined by  $x = 0$ , the line  $t'$  is defined by  $y = 0$  and the point  $B'$  has the coordinates  $(0, 1, 0)$ . Then the Cremona transformation  $\psi(O, A, \ell)$  is given by the quadratic transformation  $(xy, y^2, x(z - ax))$ . The inverse quadratic transformation has the form  $(x^2, xy, yz + ax^2)$ .*

So if a plane curve  $C$  has the defining equation  $f(x, y, z) = 0$ , then the equation of the strict transform of  $C$  via  $\psi$  is obtained from the irreducible factor of the polynomial  $f(x^2, xy, yz + ax^2)$ .

**Lemma 2.** *Let  $C$  be an irreducible plane curve. We choose  $O, A, \ell, t$  as  $C \neq \ell, t$ . Set  $C' = \psi(O, A, \ell)[C]$ . Then*

$$\deg(C') - m_{O'}(C') = \deg(C) - m_O(C).$$

Proof. We have the relations:

$$\pi^*(\ell) = \check{\ell} + E, \quad \pi'^*(\ell') = \check{\ell}' + E', \quad \eta^*(\check{\ell}) = \check{\ell} + \check{\ell}' = \eta'^*(\check{\ell}').$$

We denote by  $\tilde{C}$ ,  $\check{C}$  and  $\tilde{C}'$  the strict transforms of  $C$  in  $X$ ,  $Z$  and  $X'$ , respectively. Then we have

$$\pi^*(C) = \tilde{C} + m_O(C)E, \quad \pi'^*(C') = \tilde{C}' + m_{O'}(C')E'.$$

We also have

$$\begin{aligned} \eta^*(\tilde{C}) &= \check{C} + m_{\check{O}}(\tilde{C})\check{\ell}' + m_{\check{A}}(\tilde{C})\check{t}', \\ \eta'^*(\tilde{C}') &= \check{C}' + m_{\check{O}'}(\tilde{C}')\check{t} + m_{\check{B}'}(\tilde{C}')\check{\ell}. \end{aligned}$$

It follows that  $\deg(C) = \ell \cdot C = \tilde{\ell} \cdot \tilde{C} + m_O(C)$  and  $\deg(C') = \ell' \cdot C' = \tilde{\ell}' \cdot \tilde{C}' + m_{O'}(C')$ . Since  $\tilde{\ell} \cdot \tilde{C} = (\tilde{\ell} + \tilde{\ell}')\tilde{C} = \tilde{\ell}' \cdot \tilde{C}'$ , we obtain the desired formula.  $\square$

Let  $(\gamma, P)$  be an irreducible curve germ in  $\mathbf{P}^2$ . For such a germ  $(\gamma, P)$ , the multiplicity sequence  $\underline{m}_P = (m_0, m_1, m_2, \dots)$  is also defined. We employ the convention:  $\underline{m}_P^* = (m_1, m_2, \dots)$ ,  $\underline{m}_P^{**} = (m_2, m_3, \dots)$ . Sometimes, we simply write as  $P = (\gamma, P)$ .

If  $(\gamma, P) \neq (\ell, P), (t, P)$ , then we can define the strict transform  $(\gamma', P')$  of  $(\gamma, P)$  via the Cremona transformation  $\psi = \psi(O, A, \ell)$ . Symbolically, we write as  $(\gamma', P') = \psi[(\gamma, P)]$ , or simply  $P' = \psi[P]$ . We denote by  $i(\ell)$  (resp.  $i(t)$ ) the contact order  $(\ell \cdot \gamma)_P$  (resp.  $(t \cdot \gamma)_P$ ). Note that  $i(\ell) = m_0$  or  $i(\ell) = m_0 + m_1 + \dots + m_s$  for some  $s > 0$  with  $m_0 = \dots = m_{s-1}$ . The same thing holds for  $i(t)$ . Cf. [2], Lemma 1.3. According to the position of  $P$  and the contact orders with  $\ell$  and  $t$ , we divide the germs  $(\gamma, P)$  into the following types. If  $(\gamma, P)$  is of type  $T$  (resp.  $T^*$ ) with respect to  $\{O, A, \ell\}$ , then  $(\gamma', P')$  is of type  $T^*$  (resp.  $T$ ) with respect to  $\{O', B', t'\}$ . Type  $I_a$  is self dual.

type	$P$	$i(\ell)$	$i(t)$
$I_a$	$O$	$m_0$	$m_0$
$I_b$	$O$	$m_1 + m_0$ ( $m_0 > m_1$ )	$m_0$
$I_b^*$	$O$	$m_0$	$> m_0$
$II$	$O$	$> 2m_0$ ( $m_0 = m_1$ )	$m_0$
$II^*$	$A$	$0$	$> m_0$
$III$	$O$	$2m_0$ ( $m_0 = m_1$ )	$m_0$
$III^*$	$\in t \setminus \{O, A\}$	$0$	$\geq m_0$
$IV$	$A$	$0$	$m_0$
$IV^*$	$\in \ell \setminus \{O\}$	$\geq m_0$	$0$

**Lemma 3.** *The strict transform  $(\gamma', P')$  has the following invariants.*

type	$\underline{m}_{P'}$	$P'$	$i(t')$	$i(\ell')$
$I_a$	$\underline{m}_P$	$O'$	$m_0$	$m_0$
$I_b$	$(m_0 - m_1, \underline{m}_P^{**})$	$O'$	$m_0 - m_1$	$m_0$
$I_b^*$	$(i(t), i(t) - m_0, \underline{m}_P^*)$	$O'$	$2i(t) - m_0$	$i(t)$
$II$	$(i(\ell) - 2m_0, \underline{m}_P^{**})$	$B'$	$0$	$i(\ell) - m_0$
$II^*$	$(i(t) - m_0, i(t) - m_0, \underline{m}_P^*)$	$O'$	$2i(t) - m_0$	$i(t) - m_0$
$III$	$\underline{m}_P^{**} \in \ell' \setminus \{O', B'\}$		$0$	$m_0$
$III^*$	$(i(t), i(t), \underline{m}_P)$	$O'$	$2i(t)$	$i(t)$
$IV$	$\underline{m}_P^* \in t' \setminus \{O'\}$		$m_0$	$0$
$IV^*$	$(i(\ell), \underline{m}_P)$	$B'$	$0$	$i(\ell)$

**Proof.** We here prove the case in which  $P = O$ . We can similarly deal with the

other cases. Write  $\underline{m}_{P'} = (m'_0, m'_1, m'_2, \dots)$ . Let  $(\tilde{\gamma}, \tilde{P})$ ,  $(\check{\gamma}, \check{P})$ ,  $(\tilde{\gamma}', \tilde{P}')$  be the strict transforms of  $(\gamma, P)$  in  $X, Z, X'$ , respectively. Note that  $\tilde{P} = \tilde{O}$  if and only if  $i(\ell) > m_0$ . If  $\tilde{P} = \tilde{O}$ , then

$$m_0 = (E\tilde{\gamma})_{\tilde{P}} = (\check{E}\check{\gamma})_{\check{P}} + m_1, \quad m_1 = (\check{\gamma}\check{\ell}')_{\check{P}}, \quad i(\ell) = m_0 + m_1 + (\check{\gamma}\check{\ell})_{\check{P}}.$$

If  $\tilde{P}' = \tilde{O}'$ , or equivalently if  $i(t) > m_0$ , then

$$m'_0 = (E'\tilde{\gamma}')_{\tilde{P}'} = (\check{E}\check{\gamma})_{\check{P}} + m'_1, \quad m'_1 = (\check{\gamma}\check{t})_{\check{P}}, \quad i(t') = m'_0 + m'_1 + (\check{\gamma}\check{t}')_{\check{P}}.$$

Type  $I_a$ . Neither  $\ell$  nor  $t$  is tangent to  $(\gamma, P)$ . We infer that  $\tilde{P} \in E \setminus \{\tilde{\ell}, \tilde{t}\}$ . It follows that neither  $t'$  nor  $\ell'$  is tangent to  $(\gamma', P')$ .

Type  $I_b$ . Since  $m_0 > m_1$ , the exceptional curve  $E$  is tangent to  $\tilde{\gamma}$  at  $\tilde{P}$ , hence  $\tilde{P} \in \check{E}$ . We infer that  $\check{P}$  is the infinitely near point of  $P'$ . We have  $\underline{m}_{P'} = (m'_0, \underline{m}_P^{**})$ . Since  $\tilde{P}' \neq \tilde{O}'$ ,  $m'_0 = (E'\tilde{\gamma}')_{\tilde{P}'} = (\check{E}\check{\gamma})_{\check{P}}$ , hence  $m_0 = m'_0 + m_1$ . Thus  $m'_0 = m_0 - m_1$ . We have

$$i(\ell') = (\gamma'\ell')_{P'} = (\tilde{\gamma}'\tilde{\ell}')_{\tilde{P}'} + m'_0 = m_1 + m'_0 = m_0.$$

Type II. In this case,  $\tilde{\ell}$  is tangent to  $\tilde{\gamma}$  at  $\tilde{P}$ . It follows that  $\check{P} = \check{B}'$ , which is the infinitely near point of  $P' = B'$ , hence  $\underline{m}_{P'} = (m'_0, \underline{m}_P^{**})$ . We have  $m'_0 = (\check{\gamma}\check{\ell})_{\check{P}} = i(\ell) - m_0 - m_1 = i(\ell) - 2m_0$ . Now we have

$$i(\ell') = (\gamma'\ell')_{P'} = (\tilde{\gamma}'\tilde{\ell}')_{\tilde{P}'} = (\check{\gamma}\check{\ell}')_{\check{P}} + m'_0 = m_1 + m'_0 = i(\ell) - m_0.$$

Type III. In this case,  $\check{P} \neq \check{B}'$ ,  $\check{P} \notin \check{E}$ . We conclude that  $P' \in \ell' \setminus \{O', B'\}$ . We also have  $P' \cong \check{P}$ . It follows that  $\underline{m}_{P'} = \underline{m}_P^{**}$ ,  $i(\ell') = (\check{\gamma}\check{\ell}')_{\check{P}} = m_1 = m_0$ .

Type  $I_b^*$ . In this case,  $\tilde{P} = \check{P}$  and  $\tilde{P}' = \tilde{O}'$ . So  $\underline{m}_{P'} = (m'_0, m'_1, \underline{m}_P^{**})$ . Since  $\tilde{P} \neq \tilde{O}$ , we have  $m_0 = (E\tilde{\gamma})_{\tilde{P}} = (\check{E}\check{\gamma})_{\check{P}}$ . Thus  $m'_0 = m_0 + m'_1$ . On the other hand,  $i(t) = (\check{\gamma}\check{t})_{\check{P}} + m_0$ ,  $m'_1 = (\check{\gamma}\check{t})_{\check{P}}$ . Combining these, we get  $m'_1 = i(t) - m_0$ ,  $m'_0 = i(t)$ . Since  $(\check{\gamma}\check{t}')_{\check{P}} = 0$ , we find that  $i(t') = m'_0 + m'_1 = 2i(t) - m_0$ .  $\square$

### 3. Proof of Theorem

Let  $C$  be a rational cuspidal curve of type  $(d, d-2)$ . Let  $Q$  be the cusp with  $m_Q = d-2$ . The possible multiple sequences of cusps are given by the following

#### Lemma 4.

- (i) If  $C$  has the unique cusp  $Q$ , then  $d$  is even and  $\underline{m}_Q = (d-2, 2_{d-2})$ .
- (ii) If  $C$  has two cusps  $Q$  and  $P$ , then one of the following occurs
  - (a)  $\underline{m}_Q = (d-2)$ ,  $\underline{m}_P = (2_{d-2})$ ,
  - (b)  $d$  is odd,  $\underline{m}_Q = (d-2, 2_{(d-3)/2})$ ,  $\underline{m}_P = (2_{(d-1)/2})$ ,
  - (c)  $d$  is even,  $\underline{m}_Q = (d-2, 2_{d-2-j})$ ,  $\underline{m}_P = (2_j)$ ,  $1 \leq j \leq (d-2)/2$ .

**Proof.** Since  $m_{0,Q} + m_{1,Q} \leq d$ , we find that the multiplicity sequence  $\underline{m}_Q$  can take one of the following types: (i)  $(d-2)$ , (ii)  $(d-2, 2k)$ . In the second case, if  $d$  is odd, then we must have  $d-2 = 2k+1$ , hence  $k = (d-3)/2$ , and if  $d$  is even, then  $d-2 \leq 2k$  (Cf. [2], Proposition 1.2). The genus formula implies that if  $C$  is unicuspidal, then  $d$  is even and  $\underline{m}_Q = (d-2, 2_{d-2})$ . Suppose  $C$  is bicuspidal. Since  $m_Q + m_P \leq d$ , we have  $\underline{m}_P = (2_j)$  for some  $j$ . If  $\underline{m}_Q = (d-2)$ , then we have  $\underline{m}_P = (2_{d-2})$ . Consider the case  $\underline{m}_Q = (d-2, 2_k)$ . If  $d$  is odd, then  $k = (d-3)/2$ , hence we have  $\underline{m}_P = (2_{(d-1)/2})$ . If  $d$  is even, then we have  $j = d-2-k$ . Since  $d-2 \leq 2k$ , we infer that  $j \leq (d-2)/2$ .  $\square$

**REMARK.** The characteristic sequence  $\text{Ch}(P)$  (Cf. [4]) of the cusp  $P$  with  $\underline{m}_P = (d-2)$  is  $(d-2, d-1)$ . In case  $\underline{m}_P = (d-2, 2_k)$ , we have

$$\text{Ch}(P) = \begin{cases} (d-2, d) & (\text{if } d \text{ is odd}), \\ (d-2, d, 2k+3) & (\text{if } d \text{ is even}). \end{cases}$$

**3.1. The bicuspidal case (a)** We first construct rational bicuspidal plane curves in the class (a). For  $k \geq 1$ , we inductively construct a rational plane curve  $C_k$  of degree  $k+2$  having two cusps  $Q_k$  and  $P_k$  with  $\underline{m}_{Q_k} = (k)$ ,  $\underline{m}_{P_k} = (2_k)$ . If such a curve  $C_k$  exists, then the tangent line  $t_k$  to  $C_k$  at  $Q_k$  and the line  $\ell_k$  passing through  $Q_k$  and  $P_k$  satisfies the intersection property:

$$\begin{aligned} \ell_k \cdot C_k &= kQ_k + 2P_k, \\ t_k \cdot C_k &= (k+1)Q_k + R_k. \end{aligned}$$

where  $R_k$  is a non-singular point of  $C_k$ . We begin with a cuspidal cubic  $C_1$  with a cusp  $P_1$  and a non-singular point  $Q_1$ , which is not the flex of  $C_1$ . Let  $\ell_1$  be the line passing through  $Q_1$  and  $P_1$ . Let  $t_1$  be the tangent line to  $C_1$  at  $Q_1$ . Then  $t_1$  intersects  $C_1$  at another point  $R_1$ . Using  $\psi_k = \psi(Q_k, R_k, \ell_k)$ , we construct  $C_{k+1}$  from  $C_k$ . Set  $C_{k+1} = \psi_k[C_k]$ ,  $\ell_{k+1} = \psi_k(\ell_k)$ ,  $t_{k+1} = \psi_k(t_k)$ ,  $Q_{k+1} = \psi_k(Q_k)$ ,  $P_{k+1} = \psi_k(P_k)$ . By Lemma 3, we see that  $\underline{m}_{Q_{k+1}} = (k+1)$ ,  $\underline{m}_{P_{k+1}} = (2_{k+1})$ . In this case,  $Q_k$  is of type  $I_b^*$  and  $P_k$  is of type  $IV^*$  with respect to  $\{Q_k, R_k, \ell_k\}$ . By Lemma 2, we have  $\deg(C_{k+1}) = k+2$ . The line  $t_{k+1}$  is tangent to  $C_{k+1}$  at  $Q_{k+1}$  and  $\ell_{k+1} \cap \Gamma_{k+1} = \{Q_{k+1}, P_{k+1}\}$ .

In order to find the equation of  $C_k$ , we begin with the cuspidal cubic:

$$\begin{aligned} C_1 : f_1(x, y, z) &= yz^2 - x^2z + x^3 = 0, \\ \ell_1 : x &= 0, \quad t_1 : y = 0, \\ Q_1 : (0, 0, 1), \quad P_1 : (0, 1, 0), \quad R_1 : (1, 0, 1). \end{aligned}$$

By Lemma 1,  $\psi_1$  is given by the quadratic transformation  $(xy, y^2, x(z-x))$ . Substitut-

ing  $(x^2, xy, yz + x^2)$ , we obtain

$$f_1(x^2, xy, yz + x^2) = xy\{(yz + x^2)^2 - x^3z\}.$$

We infer from this that the defining equation of  $C_2$  is  $(yz + x^2)^2 - x^3z = 0$ . Continuing in this way, we arrive at the equation of  $C_k$ .

**3.2. The bicuspidal case (b)** We now construct rational bicuspidal plane curves of class (b). We inductively construct a plane curve  $\Gamma_k$  of odd degree  $2k + 1$  having a cusp  $Q_k$  of type  $(2k - 1, 2_{k-1})$  and a cusp  $P_k$  of type  $(2_k)$ . If such a curve exists, then the tangent line  $t_k$  to  $\Gamma_k$  at  $Q_k$  and the line  $\ell_k$  passing through  $Q_k$  and  $P_k$  satisfies the intersection property:

$$\begin{aligned}\ell_k \cdot \Gamma_k &= (2k - 1)Q_k + 2P_k, \\ t_k \cdot \Gamma_k &= (2k + 1)Q_k.\end{aligned}$$

We first take a cuspidal cubic  $\Gamma_1$  with a cusp  $P_1$  and a flex  $Q_1$ . Let  $\ell_1$  be the line passing through  $P_1$  and  $Q_1$  and let  $t_1$  be the flex tangent line to  $\Gamma_1$  at  $Q_1$ . We construct  $\Gamma_{k+1}$  from  $\Gamma_k$  by using  $\psi_k = \psi(Q_k, A_k, \ell_k)$ , where the point  $A_k \in t_k \setminus \{Q_k\}$  can be arbitrarily chosen. As before, set  $\Gamma_{k+1} = \psi_k[\Gamma_k]$ ,  $\ell_{k+1} = \psi_k(\ell_k)$ ,  $t_{k+1} = \psi_k(t_k)$ ,  $Q_{k+1} = \psi_k(Q_k)$ ,  $P_{k+1} = \psi_k(P_k)$ . The curve  $\Gamma_{k+1}$  is a rational bicuspidal curve. By Lemma 3, we see that  $\underline{m}_{Q_{k+1}} = (2k + 1, 2_k)$  and  $\underline{m}_{P_{k+1}} = (2_{k+1})$ . By Lemma 2, we have  $\deg(\Gamma_{k+1}) = 2k + 3$ . We also know that  $t_{k+1}$  is the tangent line to  $\Gamma_{k+1}$  at  $Q_{k+1}$  and  $\ell_{k+1} \cap \Gamma_{k+1} = \{Q_{k+1}, P_{k+1}\}$ . So  $\Gamma_{k+1}$  is the desired rational bicuspidal curve.

Choose the equation of  $\Gamma_1$  as follows:

$$\begin{aligned}\Gamma_1 : f_1(x, y, z) &= yz^2 - x^3 = 0, \\ \ell_1 : x &= 0, \quad t_1 : y = 0, \\ Q_1 : (0, 0, 1), \quad P_1 : (0, 1, 0), \quad A_1 : (1, 0, a).\end{aligned}$$

In this case, substituting the inverse quadratic transformation of  $\psi_1$ , we obtain

$$f_1(x^2, xy, yz + ax^2) = x\{(yz + ax^2)^2y - x^5\}.$$

Thus, the equation of  $\Gamma_2$  is given by  $(yz + ax^2)^2y - x^5 = 0$ . Continuing this process  $k - 1$  times, we get the equation of  $\Gamma_k$  given in Theorem.

**REMARK.** For the quintic  $\Gamma_2$ , we can further put  $a = 0$  or  $a = 1$ . Cf. [9, 10].

**3.3. The unicuspidal case** We pass to the unicuspidal case. We construct a rational unicuspidal curve  $D_k$  of degree  $2k + 2$  with a cusp  $Q_k$  of type  $(2k, 2_{2k})$ . Let  $\Gamma_k$  be the rational bicuspidal curve in the class (b) constructed as above. Namely,  $\Gamma_k$  has



a cusp  $Q_k$  of type  $(2k-1, 2_{k-1})$  and a cusp  $P_k$  of type  $(2_k)$ . Let  $\ell_k$  be the line joining  $Q_k$  and  $P_k$  and let  $t_k$  be the tangent line to  $\Gamma_k$  at  $Q_k$ . We construct  $D_k$  from  $\Gamma_k$  by performing quadratic transformations  $k$ -times.

Put  $\Gamma_{k,0} = \Gamma_k$ ,  $O_{k,0} = Q_{k,0} = Q_k$ ,  $P_{k,0} = P_k$ . Here, by  $Q_{k,0}$  and  $P_{k,0}$  we also mean the germ  $(\Gamma_k, Q_k)$  and  $(\Gamma_k, P_k)$ . Interchanging the roles of  $\ell_k$  and  $t_k$ , we set  $\ell_{k,0} = t_k$ ,  $t_{k,0} = \ell_k$ . For  $0 \leq j \leq k$ , we inductively construct plane curves  $\Gamma_{k,j}$ . Set  $\psi_j = \psi(O_{k,j}, A_{k,j}, \ell_{k,j})$ , where  $A_{k,0} \in \ell_{k,0} \setminus \{O_{k,0}, P_{k,0}\}$  and  $A_{k,j} \in \ell_{k,j} \setminus \{O_{k,j}\}$  for  $0 < j \leq k$ . We define  $\Gamma_{k,j+1} = \psi_j[\Gamma_{k,j}]$ ,  $O_{k,j+1} = \psi_j(O_{k,j})$ ,  $\ell_{k,j+1} = \psi_j(\ell_{k,j})$ ,  $t_{k,j+1} = \psi_j(t_{k,j})$ ,  $Q_{k,j+1} = \psi_j[Q_{k,j}]$ ,  $P_{k,j+1} = \psi_j[P_{k,j}]$ . In case  $k = 1$ , we easily see that  $D_1 = \Gamma_{1,1}$  is a quartic curve having a cusp  $Q_1 = P_{1,1}$  with  $\underline{m}_{Q_1} = (2_3)$ . Now we assume that  $k \geq 2$ . For  $1 \leq j < k$ , we see that both  $Q_{k,j}$  and  $P_{k,j}$  are locally irreducible branches centered at  $O_{k,j}$  with  $\underline{m}_{Q_{k,j}} = (2k-2j-1, 2_{k-j-1})$  and  $\underline{m}_{P_{k,j}} = (2j, 2_{k+j})$ . Indeed,  $Q_{k,0}$  is of type  $I_b$  and  $P_{k,0}$  is of type  $III^*$ . By Lemma 3, we find that as points  $Q_{k,1} = P_{k,1} = O_{k,1}$  and  $\underline{m}_{Q_{k,1}} = (2k-3, 2_{k-2})$  and  $\underline{m}_{P_{k,1}} = (2_{k+2})$ . Furthermore,  $t_{k,1}$  (resp.  $\ell_{k,1}$ ) is the tangent line to  $Q_{k,1}$  (resp.  $P_{k,1}$ ) with  $\ell_{k,1} \cdot Q_{k,1} = 2k-1$  and  $t_{k,1} \cdot P_{k,1} = 4$ . Thus  $Q_{k,1}$  is of type  $I_b$  and  $P_{k,1}$  is of type  $I_b^*$  with respect to  $\{O_{k,1}, A_{k,1}, \ell_{k,1}\}$ . Again by Lemma 3, we see that  $\underline{m}_{Q_{k,2}} = (2k-5, 2_{k-3})$  and  $\underline{m}_{P_{k,2}} = (4, 2_{k+2})$ . By induction, we can prove the claim for  $Q_{k,j}$  and  $P_{k,j}$ . Since  $m_{O_{k,j}}(\Gamma_{k,j}) = 2k-1$ , it follows from Lemma 2 that  $\deg(\Gamma_{k,j}) = 2k+1$ . Finally, the curve  $\Gamma_{k,k-1}$  has two irreducible branches  $Q_{k,k-1}$ ,  $P_{k,k-1}$  at  $O_{k,k-1}$  with  $\underline{m}_{Q_{k,k-1}} = (1)$ ,  $\underline{m}_{P_{k,k-1}} = (2k-2, 2_{2k-1})$ . Namely,  $Q_{k,k-1}$  is a non-singular branch at  $O_{k,k-1}$  with  $\ell_{k,k-1} \cdot Q_{k,k-1} = 3$ . So  $Q_{k,k-1}$  is of type II with respect to  $\{Q_{k,k-1}, A_{k,k-1}, \ell_{k,k-1}\}$ . It follows that  $Q_{k,k} = \psi_k[Q_{k,k-1}]$  is a non-singular branch whose center  $\neq O_{k,k} = \psi_k(O_{k,k-1})$ . We conclude that the rational curve  $D_k = \Gamma_{k,k}$  has only one cusp  $Q_k = P_k = \psi_k[P_{k,k-1}]$  with  $\underline{m}_{Q_k} = (2k, 2_{2k})$  as desired. By Lemma 2, we see that  $\deg(D_k) = 2k+2$ .

By interchanging  $x$  and  $y$ , the equation of the curve  $\Gamma_k$  has the form:

$$\begin{aligned} \Gamma_k : f_0(x, y, z) &= \left( x^{k-1}z + \sum_{i=2}^k a_i y^i x^{k-i} \right)^2 x - y^{2k+1} = 0, \\ \ell_k : x &= 0, \quad t_k : y = 0, \\ O_{k,0} : (0, 0, 1), \quad P_{k,0} : (1, 0, 0), \quad A_{k,0} : (1, 0, b), \quad b \neq 0. \end{aligned}$$

The inverse quadratic transformation of  $\psi_0$  has the form  $(x^2, xy, yz+bx^2)$ . The inverse transformation of the successive quadratic transformations  $\psi_{k-1} \circ \cdots \circ \psi_0$  has the form  $(x^{k+1}, x^k y, y^k z + \sum_{j=2}^{k+1} b_j x^j y^{k+1-j})$  with  $b_{k+1} \neq 0$ . Substituting this, we obtain

$$\begin{aligned} & f_0 \left( x^{k+1}, y^k x, y^k z + \sum_{j=2}^{k+1} b_j x^j y^{k+1-j} \right) \\ &= x^{2k^2+k-1} \left\{ \left( y^k z + \sum_{j=2}^{k+1} b_j x^j y^{k+1-j} + \sum_{i=2}^k a_i y^i x^{k+1-i} \right)^2 - xy^{2k+1} \right\}. \end{aligned}$$

Putting the same terms together, we get the equation in Theorem.

**3.4. The bicuspidal case (c)** We construct rational bicuspidal plane curves in the class (c). For  $k \geq 0$ , We inductively construct a plane curve  $D_{k,j}$  of degree  $2k + 2j + 2$  having a cusp  $Q_{k,j}$  of type  $(2k + 2j, 2_{2k+j})$  and a cusp  $P_{k,j}$  of type  $(2_j)$ . In this case, the tangent line  $t_{k,j}$  to  $D_{k,j}$  at  $Q_{k,j}$  and the line  $\ell_{k,j}$  passing through  $Q_{k,j}$  and  $P_{k,j}$  satisfies the following intersection property:

$$\begin{aligned}\ell_{k,j} \cdot D_{k,j} &= (2k + 2j)Q_{k,j} + 2P_{k,j}, \\ t_{k,j} \cdot D_{k,j} &= (2k + 2j + 2)Q_{k,j}.\end{aligned}$$

If  $k \geq 1$ , we start with the rational unicuspidal curve  $D_k$ . Let  $t_k$  be the tangent line to  $D_k$  at  $Q_k$ . Then we have  $t_k \cdot D_k = (2k + 2)Q_k$ . By considering the projection from  $Q_k$ , we can find a unique non-singular point  $P_k \in D_k$  such that the line  $\ell_k$  passing through  $Q_k$  and  $P_k$  satisfies the relation:

$$\ell_k \cdot D_k = (2k)Q_k + 2P_k.$$

For the case  $k = 0$ , take a non-singular conic  $D_0$  and two distinct points  $P_0, Q_0$ . Let  $\ell_0, t_0$  be the tangent lines to  $C_0$  at  $P_0$  and  $Q_0$ . Put  $D_{k,0} = D_k$ . By induction, using  $\psi_j = \psi(Q_{k,j}, A_{k,j}, \ell_{k,j})$  with  $A_{k,j} \in t_{k,j} \setminus \{Q_{k,j}\}$ , we construct  $D_{k,j+1}$  from  $D_{k,j}$ . Set  $D_{k,j+1} = \psi_j[D_{k,j}]$ ,  $Q_{k,j+1} = \psi_j(Q_{k,j})$ ,  $P_{k,j+1} = \psi_j(P_{k,j})$ . Since  $Q_{k,j}$  is of type  $I_b^*$  and  $P_{k,j}$  is of type  $IV^*$  with respect to  $\{Q_{k,j}, A_{k,j}, \ell_{k,j}\}$ , we see from Lemma 3 that  $\underline{m}_{Q_{k,j+1}} = (2k + 2j + 2, 2_{2k+j+1})$  and  $\underline{m}_{P_{k,j+1}} = (2_{j+1})$  as desired. By Lemma 2, we have  $\deg(D_{k,j}) = 2k + 2j + 2$ .

The equation of  $D_k$  is given by

$$\left( y^k z + \sum_{i=1}^{k+1} a_i x^i y^{k+1-i} \right)^2 - xy^{2k+1} = 0 \quad (a_{k+1} \neq 0).$$

The inverse of the successive quadratic transformation  $\psi_{j-1} \circ \cdots \circ \psi_0$  has the form  $(x^{j+1}, x^j y, y^j z + \sum_{h=2}^{j+1} c_h x^h y^{j+1-h})$ . Substituting this in the above equation and putting the same terms together, we obtain the equation in Theorem.

**3.5. Projective equivalence** It remains to prove that a rational cuspidal curve of type  $(d, d - 2)$  is projectively equivalent to the plane curve defined by one of the equations given in Theorem. We here prove this for rational bicuspidal curves in the class (b). We essentially follow the argument in [3]. Let  $C$  be a rational bicuspidal curve of odd degree  $d$  with a cusp  $Q$  and  $P$  such that  $\underline{m}_Q = (2d - 2, 2_{(d-2)/2})$ ,  $\underline{m}_P = (2_{(d-1)/2})$ . Let  $t$  be the tangent line to  $C$  at  $Q$  and let  $\ell$  be the line passing through  $Q$  and  $P$ . Then

$$\ell \cdot C = (2d - 2)Q + 2P,$$

$$t \cdot C = dQ.$$

Set  $d = 2k + 1$ . Inductively, we construct rational bicuspidal curves  $C^{(s)}$  of degree  $2k + 1 - 2(s - 1)$  having cusps  $Q^{(s)}, P^{(s)}$  with  $\underline{m}_P^{(s)} = (2k - 2s + 1, 2k - s)$  and  $\underline{m}_{P^{(s)}} = (2k + 1 - s)$ , for  $1 \leq s \leq k$ . Set  $C^{(1)} = C$ . Assuming that  $C^{(s)}$  is defined, let  $t^{(s)}$  be the tangent line to  $C^{(s)}$  at  $Q^{(s)}$  and let  $\ell^{(s)}$  be the line passing through  $Q^{(s)}$  and  $P^{(s)}$ . Set  $\psi^{(s)} = \psi(Q^{(s)}, P^{(s)}, t^{(s)})$ ,  $Q^{(s+1)} = \psi^{(s)}(Q^{(s)})$ ,  $P^{(s+1)} = \psi^{(s)}[P^{(s)}]$ . By definition,  $P^{(s+1)}$  is the strict transform of the germ  $(C^{(s)}, P^{(s)})$ . As a point, we easily see that  $P^{(s+1)} \in \ell^{(s+1)} \setminus \{O^{(s+1)}\}$ . We also set  $B^{(s+1)} = \psi^{(s)}(P^{(s)}) \in t^{(s+1)} \setminus \{O^{(s+1)}\}$ . Since  $Q^{(s)}$  (resp.  $P^{(s)}$ ) is of type  $I_b$  (resp. IV) with respect to  $\{Q^{(s)}, P^{(s)}, t^{(s)}\}$ , we see that  $\underline{m}_Q^{(s+1)} = (2k - 2s - 1, 2k - s - 1)$  (resp.  $\underline{m}_{P^{(s+1)}} = (2k - s)$ ). It turns out that  $C^{(k)}$  is a cuspidal cubic and  $Q^{(k)}$  is a flex and  $P^{(k)}$  is the cusp. Let  $\Gamma_1$  be the cuspidal cubic defined by  $yz^2 - x^3 = 0$ . Set  $O_1 = (0, 0, 1)$ ,  $\ell_1 = \{x = 0\}$ ,  $t_1 = \{y = 0\}$ . Then there is a projective transformation  $\varphi_1$  with the property:

$$\varphi_1(C^{(k)}) = \Gamma_1, \quad \varphi_1(Q^{(k)}) = O_1, \quad \varphi_1(\ell^{(k)}) = \ell_1, \quad \varphi_1(t^{(k)}) = t_1.$$

Put  $A_1 = \varphi_1(B^{(k)})$ . Letting  $\psi_1 = \psi(O_1, A_1, \ell_1)$ , we set  $\Gamma_2 = \psi_1[\Gamma_1]$ . We easily see that  $\varphi_2 = \psi_1^{-1} \circ \varphi_1 \circ \psi^{(k-1)}$  is a projective transformation which transforms  $C^{(s-1)}$  to  $\Gamma_2$ . Inductively, we can construct a rational bicuspidal curve  $\Gamma_k$  and a projective transformation  $\varphi_k$ , which transforms  $C$  to  $\Gamma_k$ . Namely, if  $\Gamma_i$  is defined, then we set  $\Gamma_{i+1} = \psi_i[\Gamma_i]$  and  $\varphi_{i+1} = \psi_i^{-1} \circ \varphi_i \circ \psi^{(k-i)}$ , where  $\psi_i = \psi(O_i, A_i, \ell_i)$  with  $A_i = \varphi_i(B^{(k+1-i)})$ . As we have seen before,  $\Gamma_k$  is defined by the equation in Theorem.

REMARK. We can similarly deal with the other cases. In the bicuspidal case (a), the quadratic transformation  $\psi_i = \psi(O_i, R_i, \ell_i)$  is uniquely determined by  $C_i$ , so the equation of  $C_k$  has no parameter.

REMARK. The techniques of constructing new curves by Cremona transformations are also discussed in [1, 5, 8].

REMARK. Prof. M. Zaidenberg informed us that T. Fenske independently classified rational cuspidal curves of type  $(d, d-2)$  with at most two cusps, as well as those of type  $(d, d-3)$  with at most two cusps. The second author discussed the defining equations of rational cuspidal curves of type  $(d, d-3)$  with at most two cusps [7].

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F. Sakai  
 Department of Mathematics  
 Faculty of Science  
 Saitama University  
 Urawa 338-8570  
 Japan  
 e-mail: fsakai@rimath.saitama-u.ac.jp

K. Tono  
 Department of Mathematics  
 Faculty of Science  
 Saitama University  
 Urawa 338-8570  
 Japan  
 e-mail: ktono@rimath.saitama-u.ac.jp

