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Author(s)	Koiso, Norihito
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Koiso, N.
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CONVERGENCE TO A GEODESIC

Dedicated to Professor Masaru Takeuchi on his 60th birthday

NORIHITO KOISO

(Received April 9, 1992)

0. Introduction

For a closed curve $\gamma(s)$ in a riemannian manifold M we define its energy $E(\gamma)$ by $\|\dot{\gamma}\|^2$. The first variation formula of E is given by $-2\langle \delta\gamma, D_{\dot{\gamma}}\dot{\gamma} \rangle$. Therefore, its Euler-Lagrange equation is $D_{\dot{\gamma}}\dot{\gamma}=0$, the equation of geodesics. We consider the corresponding parabolic equation

$$(EP) \quad \frac{d}{dt}\gamma_t = D_{\dot{\gamma}_t}\dot{\gamma}_t.$$

This is locally expressed as

$$\frac{\partial}{\partial t}\gamma^i = \frac{\partial^2}{\partial s^2}\gamma^i + \Gamma_{j,k}^i \frac{\partial}{\partial s}\gamma^j \frac{\partial}{\partial s}\gamma^k,$$

which is a semi-linear heat equation.

This equation was studied by Eells and Sampson [ES], in higher dimensional case. They proved that if the manifold (M, g) is compact and has non-positive sectional curvature, then a solution γ_t exists for all time, and a subsequence γ_{t_i} converges to a geodesic. And it is not so difficult to show that if the manifold (M, g) has negative sectional curvature, then the solution γ_t itself converges to the geodesic.

Physically, equation (EP) represents the equation of motion of a rubber band in high viscous liquid. Therefore, it seems that the above curvature restriction is not necessary. More precisely, we have the following

Conjecture A. *If the manifold M is compact then Cauchy problem (EP) has a unique solution γ_t for all time.*

Conjecture B. *The solution γ_t converges to a geodesic when $t \rightarrow \infty$.*

In this paper we will show that this conjecture holds “almost always”, with “a few” exceptions.

Theorem A. *If the manifold M is compact then Cauchy problem (EP) with C^∞ initial data has a unique solution γ_t for all time.*

Theorem B. *Moreover, if the riemannian manifold (M, g) is real analytic, then the solution γ_t converges to a geodesic when $t \rightarrow \infty$.*

Theorem C. *There exists a compact riemannian manifold (M, g) such that for certain C^∞ initial data the solution γ_t of Cauchy problem (EP) does not converge.*

1. Preliminaries

Throughout in this paper, we use the following notations. The parameter of a curve γ is denoted by s and the velocity vector $d\gamma/ds$ is denoted by $\dot{\gamma}$ or v . We treat curves γ_t depending on time t and denote by $\dot{\gamma}_t$ or v_t their velocity vectors. But we usually omit the subscript t in them.

The riemannian covariant derivation is denoted by D . The norm $|*|$, the L_2 norm $\|*\|$ and the L_2 inner product $\langle *, * \rangle$ are defined by $|*|^2 = g(*, *)$, $\langle *, * \rangle = \int g(*, *) ds$ and $\|*\|^2 = \langle *, * \rangle$.

We start from results in [ES].

Theorem 1.1. [ES, Theorem 10A, 10B] *For any closed C^1 curve γ_0 , there is a positive constant T depending only on the energy density $|v_0|^2$ such that (EP) has a unique solution γ_t on $0 \leq t \leq T$.*

Let T be the largest number such that a solution with initial data γ_0 exists on $0 \leq t < T$, and suppose that the energy density $|v_t|^2$ is bounded on $\{(s, t)\} = S^1 \times [0, T)$. Then by Theorem 1.1 there exists a positive number T_1 such that any γ_t can be continued as a solution onto the interval $(t, t + T_1)$. This implies that T is infinite. Therefore, the proof of Theorem A is reduced to the following

Proposition 1.2. *Let γ_t be a solution of (EP) on $0 \leq t < T$, where T is a finite positive number. Then the energy density $|v_t|^2$ is bounded from above by a constant C on $\{(s, t)\} = S^1 \times [0, T)$. Here, the constant C depends only on the initial data γ_0 and the time T .*

To prove this, we need some basic inequalities. As usual, we use symbols $D_t = D_{ds/dt}$ and $D_v = D_{ds/ds}$. First, for a solution γ_t on $0 \leq t < T$ we see

$$\frac{d}{dt} \|v\|^2 = 2\langle v, D_t v \rangle = 2\left\langle v, D_s \frac{d}{dt} \gamma \right\rangle = 2\langle v, D_v^2 v \rangle = -2\|D_v v\|^2.$$

It implies that $\|v\|$ is non-increasing. Therefore, we have a positive constant C_1 such that $\|v\| \leq C_1$ on $0 \leq t < T$.

Lemma 1.3. *For any vector field ξ along γ , we have*

$$\max_s |\xi|^2 \leq 2\|\xi\|(\|\xi\| + \|D_v \xi\|)$$

Proof.

$$\begin{aligned} \max_s |\xi|^2 &\leq \min_s |\xi|^2 + \oint \left| d|\xi|^2 \right| ds \leq \frac{1}{2\pi} \|\xi\|^2 + 2 \langle |\xi|, |D_v \xi| \rangle \\ &\leq 2(\|\xi\|^2 + \|\xi\| \|D_v \xi\|) \end{aligned}$$

Q.E.D.

Lemma 1.4. *For any positive integers $p \leq q$, we have a constant C_2 depending only on (the constant C_1 and) p and q such that*

$$\|D_v^p v\| \leq C_2 \|D_v^q v\|^{p/q}.$$

Proof. Since

$$\|D_v^p v\|^2 = -\langle D_v^{p-1} v, D_v^{p+1} v \rangle \leq \|D_v^{p-1} v\| \|D_v^{p+1} v\|,$$

we see that the function $\log \|D_v^p v\|$ is concave with respect to $p \geq 0$. Therefore,

$$\|D_v^p v\| \leq \|v\|^{1-(p/q)} \|D_v^q v\|^{p/q} \leq C_1^{1-(p/q)} \|D_v^q v\|^{p/q}.$$

Q.E.D.

Lemma 1.5. *For any non-negative integers $p < q$, we have a constant C_3 depending only on (C_1 and) p and q such that*

$$\max_s |D_v^p v| \leq C_3 (1 + \|D_v^q v\|^{(2p+1)/(2q)}).$$

Proof. From Lemma 1.3, we know

$$\max_s |D_v^p v| \leq \sqrt{2} \|D_v^p v\|^{1/2} (\|D_v^p v\| + \|D_v^{p+1} v\|)^{1/2}.$$

By Lemma 1.4, the right hand side

$$\begin{aligned} &\leq \text{const} \cdot \|D_v^q v\|^{p/(2q)} (\|D_v^q v\|^{p/q} + \|D_v^{q+1} v\|^{(p+1)/q})^{1/2} \\ &\leq \text{const} \cdot (1 + \|D_v^q v\|^{(2p+1)/(2q)}). \end{aligned}$$

Q.E.D.

2. Proof of Theorem A

Now we have to see more closely equation (EP). For the solution γ_t on $0 \leq t < T$, we see

$$D_t D_v v = R\left(\frac{d}{dt} \gamma, v\right) v + D_v D_t v = D_v^3 v + R(D_v v, v) v.$$

Therefore, by induction, we get for $n \geq 2$,

$$\begin{aligned}
D_t D_v^{n-1} v &= R\left(\frac{d}{dt} \gamma, v\right) D_v^{n-2} v + D_v D_t D_v^{n-1} v \\
&= D_v^{n+1} v + \sum^A A_{ijkl} (D_v^i R) (D_v^j v, D_v^k v) D_v^l v,
\end{aligned}$$

where A 's are universal constants and the sum \sum^A is taken over all $i, k, l \geq 0$, $j \geq 1$ with $i+j+k+l=n-1$. This holds also for $n=1$, taking $A=0$. Thus, we get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|D_v^{n-1} v\|^2 &= \langle D_v^{n-1} v, D_t D_v^{n-1} v \rangle \\
&= \langle D_v^{n-1} v, D_v^{n+1} v + \sum^A A_{ijkl} (D_v^i R) (D_v^j v, D_v^k v) D_v^l v \rangle.
\end{aligned}$$

Here the term $D_v^i R$ is expanded into

$$\sum^B B_{m p_1 \dots p_m} (D^m R) (D_v^{p_1} v, \dots, D_v^{p_m} v),$$

where B 's are universal constants and the sum \sum^B is taken over all $m, p_1, \dots, p_m \geq 0$ with $m + \sum_{1 \leq a \leq m} p_a = i$.

Lemma 2.1. *There is a positive constant C_4 depending only on C_1 and non-negative integer n such that*

$$\frac{d}{dt} \|D_v^n v\|^2 \leq C_4$$

Proof. Let n be a positive integer. From the above equality and Lemmas 1.4, 1.5, we see

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|D_v^{n-1} v\|^2 &= -\|D_v^n v\|^2 + \sum^C \langle D_v^{n-1} v, B_{m p_1 \dots p_m} (D^m R) (D_v^{p_1} v, \dots, D_v^{p_m} v) (D_v^l v, D_v^k v) D_v^j v \rangle \\
&\leq -\|D_v^n v\|^2 + \text{const} \cdot \sum^C \left(\prod_q \max_s |D_v^q v| \right) \|D_v^j v\| \|D_v^{n-1} v\| \\
&\leq -\|D_v^n v\|^2 + \text{const} \cdot \sum^C \left(\prod_q (1 + \|D_v^q v\|^{(2q+1)/(2n)}) \right) \|D_v^n v\|^{j/n} \|D_v^n v\|^{(n-1)/n} \\
&\leq -\|D_v^n v\|^2 + \text{const} \cdot \sum^C (1 + \|D_v^n v\|^{((2q+2q+m+2+2j+2(n-1))/(2n))}) \\
&\leq -\|D_v^n v\|^2 + \text{const} \cdot \sum^C (1 + \|D_v^n v\|^{(4n-2)/(2n)}) \\
&\leq \text{const},
\end{aligned}$$

where \sum^C denotes $\sum^A (\sum^B)$ and q runs in the set $\{p_1, \dots, p_m, k, l\}$. Q.E.D.

Proof of Theorem A. Lemma 2.1 and Lemma 1.3 imply that we can estimate each C^n norm of the solution γ_t only by the initial data γ_0 . This completes the proof of Proposition 1.2, hence Theorem A holds by the remark above Proposition 1.2. Q.E.D.

Before proceeding to Theorem B and C, we derive the following

Lemma 2.2. *For any positive integer n , the integral $\int_0^\infty \|D_v^n v\|^2 dt$ is finite, and $\|D_v^n v\| \rightarrow 0$ when $t \rightarrow \infty$.*

Proof. Since $\frac{1}{2} \frac{d}{dt} \|v\|^2 = -\|D_v v\|^2$, we see

$$\int_0^\infty \|D_v v\|^2 dt = -\frac{1}{2} [\|v\|^2]_0^\infty \leq \frac{1}{2} \|v_0\|^2 < \infty.$$

Combining it with Lemma 2.1, we get the result for $n=1$. Suppose that the assertion holds for any positive integer less than n . Note that $q \leq n-2$ in the third line of the inequality in the proof of Lemma 2.1. Therefore, by Lemma 1.3, all $\max_j |D_v^j v|$ are already bounded by a constant. Thus,

$$\frac{1}{2} \frac{d}{dt} \|D_v^{n-1} v\|^2 \leq -\|D_v^n v\|^2 + \text{const} \cdot \sum \|D_v^j v\| \|D_v^{n-1} v\|,$$

where the sum is taken for $1 \leq j \leq n-1$. By integration, we see

$$\begin{aligned} \frac{1}{2} [\|D_v^{n-1} v\|^2]_0^\infty &\leq - \int_0^\infty \|D_v^n v\|^2 dt + \text{const} \cdot \sum \int_0^\infty \|D_v^j v\| \|D_v^{n-1} v\| dt \\ &\leq - \int_0^\infty \|D_v^n v\|^2 dt + \text{const} \cdot \sum \left(\int_0^\infty \|D_v^j v\|^2 dt \int_0^\infty \|D_v^{n-1} v\|^2 dt \right)^{1/2}. \end{aligned}$$

Thus, $\int_0^\infty \|D_v^n v\|^2 dt$ is finite by the assumption of induction. Combining it with Lemma 2.1, we get the result for n . Q.E.D.

3. Proof of Theorem B

The next Lemma is a direct consequence of a result of [S, Theorem 3].

Lemma 3.1. *Let (M, g) be a real analytic riemannian manifold and η a closed geodesic. Then there are positive constants $\mu \in (0, 1)$, $\theta \in (0, 1/2)$, and a $C^{2+\mu}$ neighbourhood U of η such that if a closed curve γ is in U , then*

$$\|D_v v\| \geq |E(\gamma) - E(\eta)|^{1-\theta}.$$

Again, let γ be a solution of equation (EP). If the manifold M is compact, then γ_t are C^0 bounded and Lemma 2.2 implies that γ_t are C^4 bounded, and so has a C^3 convergent subsequence. Let γ_∞ be its limiting closed curve. Since $\|D_v v_t\| \rightarrow 0$, γ_∞ is a closed geodesic. We apply Lemma 3.1 to $\eta = \gamma_\infty$. Fix a geodesic coordinate system around a point $\gamma_\infty(s_0)$. Take sufficiently large T so that $D_{v_t} v_t$ is sufficiently small for any $t \geq T$. If $t_1 \geq T$ and $\gamma_{t_1}(s_0)$ is close to $\gamma_\infty(s_0)$, then $(\frac{d}{ds})^2 \gamma_{t_1}(s)$ is sufficiently small in the coordinate. It means that if $t_1 \geq T$ and γ_{t_1} is close to γ_∞ in L_2 topology, then they are close in C^3 topology. Thus,

Lemma 3.1 can be rewritten as the following

Lemma 3.2. *Let (M, g) and γ_∞ be as above. Then there are positive constants $\theta \in (0, 1/2)$, T and an L_2 neighbourhood V of γ_∞ such that if $t \geq T$ and $\gamma_t \in V$, then*

$$\|D_v v_t\| \geq (\|v_t\|^2 - \|v_\infty\|^2)^{1-\theta}.$$

Proof of Theorem B. Suppose that on a time interval (t_1, t_2) , γ_t is in V and satisfies the above inequality. Then, for γ_t ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|^2 &= -\|D_v v\|^2 = -\|D_v v\| \left\| \frac{d}{dt} \gamma \right\| \\ &\leq -(\|v\|^2 - \|v_\infty\|^2)^{1-\theta} \left\| \frac{d}{dt} \gamma \right\|. \end{aligned}$$

Therefore,

$$\begin{aligned} -\left\| \frac{d}{dt} \gamma \right\| &\geq \frac{1}{2} (\|v\|^2 - \|v_\infty\|^2)^{\theta-1} \frac{d}{dt} (\|v\|^2 - \|v_\infty\|^2) \\ &= \frac{1}{2\theta} \frac{d}{dt} (\|v\|^2 - \|v_\infty\|^2)^\theta. \end{aligned}$$

Thus, we get

$$\int_{t_1}^{t_2} \left\| \frac{d}{dt} \gamma \right\| dt \leq \frac{1}{2\theta} [(\|v_t\|^2 - \|v_\infty\|^2)^\theta]_{t_1}^{t_2}.$$

Let B_r be the L_2 ball in V centered at γ_∞ with radius r . If γ_t enters in $B_{r/2}$ at $t=t_1$ and leaves from B_r at $t=t_2$, we have $\int_{t_1}^{t_2} \|d\gamma/dt\| dt \geq r/2$. Thus, if γ_t repeats entering and leaving infinitely many times, we get $\int_I \|d\gamma/dt\| dt = \infty$, where $I = \{t; \gamma_t \in B_r\}$. This contradicts to the above inequality. Therefore, there exists a time T so that γ_t stays in B_r on $t \geq T$. Since r can be taken arbitrarily small, we conclude that γ_t converges to γ_∞ in L_2 topology. Thus, γ_t converges to γ_∞ in C^∞ topology by the remark below Lemma 3.1. Q.E.D.

4. A counter example

We recall Theorem 1.1. The uniqueness of the solution implies that if all initial data are invariant under a group action, then so is the solution γ_t .

Let f be a C^∞ function on \mathbf{R}^2 defined by the polar coordinate (r, θ) as

$$f(r, \theta) = \begin{cases} 0 & (r \leq 1) \\ (r-1) \left(2 + \sin \left(\frac{1}{r-1} + \theta \right) \right) e^{-1/(r-1)} & (r > 1) \end{cases}$$

We take a point h_0 outside the circle $r=1$. Then the integral curve h_t of the

gradient vector field $-\text{grad } f$ closes to the circle $r=1$ when $t \rightarrow \infty$, but does not converge. This example is suggested by Professor O. Kobayashi.

We define a C^∞ riemannian metric g on the manifold $S^1 \times \mathbf{R}^2 = \{(u, x, y)\}$ as

$$\begin{cases} g(\partial_u, \partial_x) = g(\partial_u, \partial_y) = g(\partial_x, \partial_y) = 0, \\ g(\partial_x, \partial_x) = g(\partial_y, \partial_y) = 1, \\ g(\partial_u, \partial_u) = 1 + \phi(x, y) \quad (\phi(x, y) = f(r, \theta)). \end{cases}$$

We solve equation (EP) with initial data $\gamma_0(s) = (s, a, b)$, where a and b are constants satisfying $a^2 + b^2 > 1$. Since the initial data are S^1 invariant, so is the solution γ_t . It means that the solution γ_t behaves like the integral curve h_t . In fact we easily compute that the solution $\gamma_t(s) = (s, x(t), y(t))$ is given by a solution of the equation: $\frac{d}{dt}(x, y) = -\frac{1}{2} \text{grad } \phi$. We can easily replace the manifold $S^1 \times \mathbf{R}^2$ by a compact manifold, say $S^1 \times T^2$.

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College of General Education
 Osaka University
 Toyonaka, Osaka 560
 Japan

