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Author(s)	Ikegami, Gikô
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ON CLASSIFICATION OF DYNAMICAL SYSTEMS WITH CROSS-SECTIONS

GIKŌ IKEGAMI

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1. Introduction

Let $(M_i, \mathcal{F}_i)$ be a dynamical system on a manifold M_i with a cross-section X_i, where $\mathcal{F}_i$ is the flow-structure, $i=1, 2$. To $(M_i, \mathcal{F}_i)$ the associated diffeomorphism $f_i: X_i \rightarrow X_i$ is defined.

By S. Smale ([8], [9]), it is shown that if f_1 and f_2 are differentiably or topologically conjugate by a map $h: X_1 \rightarrow X_2$, then $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are differentiably or topologically equivalent respectively.

The main purpose of this paper is to show the converse of the above fact under some conditions, that is; under the assumption that there exists no homomorphism of the fundamental group of one of the two cross-sections onto the infinite cyclic group, $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are differentiably or topologically equivalent if and only if f_1 and f_2 are defferentiably or topologically conjugate respectively (Theorem 4.1).

Furthermore we shall show an example of a pair of dynamical systems $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ with cross-sections X_1 and X_2 respectively such that the fundamental group of X_1 is isomorphic to the infinite cyclic group, and that $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are differentiably equivalent but the associated diffeomorphisms are not conjugate (§4).

As an application of Theorem (4.1), we shall show in §5 that for a given X satisfying the condition stated above concerning its fundamental group, there is a natural correspondence between the equivalence classes of dynamical systems $(M, \mathcal{F})$ with the cross-section X and the equivalence classes of smooth fibre bundles over S^1 with the fibre X (Theorem 5.3).

Another application will be shown in §6; that is classification of dynamical system on $S^n \times S^1 \# \tilde{S}^{n+1}$ with cross-section, where $\tilde{S}^{n+1}$ denotes any homotopy sphere (Theorem 6.6). Here, it is essential that Γ^{n+1} classifies the differentiably conjugate classes of diffeomorphisms on S^n.

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2. Terminology

Throuout this paper, all manifolds considered will be assumed to be compact and differentiable (C^∞).

A dynamical system or a flow $\mathcal{F}$ on a manifold M is a 1-parameter group of transformations φ of M, where φ is a C^∞-map $\varphi: R\times M\to M$ (R; the real numbers) such that if we put $\varphi_t(x)=\varphi(t, x)$, then

(i) $\varphi_0(x)=x$

(ii) $\varphi_{t+s}(x)=\varphi_t\varphi_s(x)$,

and φ_t is a diffeomorphism $(M, \partial M)\to(M, \partial M)$. Here ∂M is the boundary of M.

By a pair $(M, \mathcal{F})$ we mean a dynamical system $\mathcal{F}$ on a manifold M. $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are said to be *differentiably* (*topologically*) *equivalent* if there is a diffeomorphism (homeomorphism) $h: M_1\to M_2$ having the property that h maps every orbit of $\mathcal{F}_1$ onto an orbit of $\mathcal{F}_2$ preserving the orientation. Such a map h will be called an *equivalence*.

Two diffeomorphisms (homeomorphisms) $f_1: M_1\to M_1$ and $f_2: M_2\to M_2$ are said to be *differentiably* (*topologically*) *conjugate* if there exists a diffeomorphism (homeomorphism) $h: M_1\to M_2$ such that $hf_1=f_2h$.

A *cross-section* of a dynamical system $(M, \mathcal{F})$ or (M, φ) is a compact connected submanifold X of codimension 1 of M such that $\partial X\subset\partial M$, and that

(i) X intersects every orbit,

(ii) the intersection of X with each orbit is transversal,

(iii) if $x\in X$, there is a $t>0$ with $\varphi_t(x)\in X$, and

(iv) if $x\in X$, there is a $t<0$ with $\varphi_t(x)\in X$.

There can be no singular point of $(M, \mathcal{F})$ if there is a crosssection. $\partial X\neq\phi$ if $\partial M\neq\phi$, and X is properly imbedded in M, i.e. $\partial X\subset\partial M$ and $\operatorname{Int} X\subset\operatorname{Int} M$. By $(M, \mathcal{F}; X)$ we mean a dynamical system $\mathcal{F}$ on a manifold M having a cross-section X.

For $(M, \mathcal{F}; X)$ we can define a map $f: X\to X$ by $f(x)=\varphi_{t_0}(x)$ where t_0 is the smallest positive t satisfying $\varphi_t(x)\in X$. $f: X\to X$ is a diffeomorphism; we call f the *associated diffeomorphism* of $(M, \mathcal{F}; X)$.

Conversely, suppose that a diffeomorphism f of X onto itself is given. Define a diffeomorphism $\tau: R\times X\to R\times X$ by $\tau(t, x)=(t+1, f^{-1}(x))$. Then the infinite cyclic group $\{\tau^m\}=Z$ operates freely on $R\times X$ and the orbit space $(R\times X)/Z$ is a manifold, say M_0. The flow $\psi_t: R\times X\to R\times X$ defined by $\psi_t(u, x)=(u+t, x)$ induces a flow φ_t on M_0. We call this (M_0, φ_t) the *suspension* of f. M_0 has a cross-section $X_0=q(0\times X)\subset M_0$, where $q: R\times X\to M_0$ is the quotient map.

The following properties are shown by S. Smale ([7] or [8]).

(2.1) *The associated diffeomorphism of* $(M_0, \varphi_t; X_0)$ *is differentiably conjugate to the given* f.

Furthermore

(2.2) *if* $(M', \varphi'_t; X')$ *is the suspension of the associated diffeomorphism of a dynamical system* $(M, \varphi_t; X)$, *then* (M, φ_t) *and* (M', φ'_t) *are differentiably equivalent* (*by an equivalence mapping* X *onto* X').

(2.3) *Let* $(M_0, \mathcal{F}_0)$, $(M_1, \mathcal{F}_1)$ *be the suspensions of* $f_0\colon X_0 \to X_0, f_1 X_1 \to X_1$ *respectively. If* f_0 *and* f_1 *are differentiably* (*topologically*) *conjugate, then* $(M_0, \mathcal{F}_0)$ *and* $(M_1, \mathcal{F}_1)$ *are differentiably* (*topologically*) *equivalent.*

3. Lemmas

Suppose that $h\colon M' \to M$ is a differentiable (topological) equivalence between $(M', \mathcal{F}'; X')$ and $(M, \mathcal{F}; X)$. Let $f\colon X \to X$ be the associated diffeomorphism of $(M, \mathcal{F}; X)$. Let $p\colon R\times X \to X$ be the natural projection and $q\colon R\times X \to M_0$ the quotient map to the suspension M_0 of f. Using (2.2), we consider h to be a differentiable (topological) equivalence: $M' \to M_0$. Put $X_0 = hX' \subset M_0$ and let $\tilde{X}_0$ be a connected component of $q^{-1}(X_0)$. Then we have the following lemmas.

Lemma (3.1). $q|\tilde{X}_0\colon \tilde{X}_0 \to X_0$ *is a covering. If* h *is a differentiable equivalence, then* $q|\tilde{X}_0$ *is a smooth covering map.*

Proof. $q\colon R\times X \to M_0$ is a covering, furthermore q is a smooth covering.

Since X_0 is a properly imbedded submanifold of M_0 and since $q\colon (R\times X, q^{-1}(X_0)) \to (M, X_0)$ is a local homeomorphism, $\tilde{X}_0$ is a proper submanifold of $R\times X$. Since $q(\partial\tilde{X}_0) \subset \partial X_0$, $q(\operatorname{Int}\tilde{X}_0) \subset \operatorname{Int} X_0$, and since q is a local homeomorphism, the image $q(\tilde{X}_0)$ is a proper compact submanifold of X_0 with codimension 0. Therefore, $q|\tilde{X}_0\colon \tilde{X}_0 \to X_0$ is an onto map.

Therefore it is easy to see that $q|\tilde{X}_0\colon \tilde{X}_0 \to X_0$ is a covering and that if h is a differentiable equivalence, then it is a smooth covering.

Lemma (3.2). *If* $\tilde{X}_0$ *is compact, then* $p|\tilde{X}_0\colon \tilde{X}_0 \to X$ *is a covering. If* h *is a differentiable equivalence* $p|\tilde{X}_0$ *is a smooth covering.*

Proof. Since X' has transversal intersection with the flow in M', and since h and q are local homeomorphisms mapping orbit onto orbit, $\tilde{X}_0$ has transversal intersection with the flow of $R\times X$. Hence, $p|\tilde{X}_0\colon \tilde{X}_0 \to X$ is a local homeomorphism. Furthermore, as in (3.1), $\tilde{X}_0$ is a proper submanifold of $R\times X$, and $p\colon R\times X \to X$ maps boundary into boundary and maps interior into interior. Hence, $p(\tilde{X}_0)$ is a proper submanifold of X with codimension 0. Therefore, $p|\tilde{X}_0$ is an onto-map.

For each x in X and each $\tilde{x}_i \in \tilde{X}_0$ in $p^{-1}(x)$ let $\tilde{U}(\tilde{x}) \in \tilde{X}_0$ be a neighbourhood such that $p|\tilde{U}(\tilde{x}_i)$ is a homeomorphism and that if $\tilde{x}_i \neq \tilde{x}_j$, $\tilde{x}_i, \tilde{x}_j \in p^{-1}(x)$, then $\tilde{U}(\tilde{x}_i) \cap \tilde{U}(\tilde{x}_j) = \phi$, and let $U_i(x) \subset X$ be the homeomorphic image of $\tilde{U}(\tilde{x}_j)$ by p. Since $\tilde{X}_0$ is compact, it is clear that $p^{-1}(x)$ is a finite set. Put

$$W(x) = \bigcap_i U_i(x)$$

and

$$\tilde{W}(\tilde{x}_i) = (p^{-1}W(x)) \cap \tilde{U}(\tilde{x}_i) .$$

Then, $W(x)$ and $\tilde{W}(\tilde{x}_i)$ satisfy the usual conditions for covering.

This proves that $p|\tilde{X}_0: \tilde{X}_0 \to X$ is a covering.

Let $h: M' \to M$, X_0, q, and $\tilde{X}_0 \subset R \times X$ be the same as in (3.1). It should be noted that one and only one of the two assumptions in the following lemmas (3.3) and (3.4) holds.

Lemma (3.3). *If there is $t_0 > 0$ such that $(t \times X) \cap \tilde{X}_0 = \phi$ for every t with $|t| > t_0$, then $q|\tilde{X}_0$ is a diffeomorphism or a homeomorphism onto X_0 according that h is a differentiable or topological equivalence.*

Proof. Suppose there are two points $u, v \in \tilde{X}_0$ such that $q(u) = q(v) \in X_0$. Then for some integers i, j, $(i \neq j)$, some $0 \leqq t < 1$, and some $x \in X$, we have

$$u = \tau^i(t, x) \qquad v = \tau^j(t, x) ,$$

where $\tau: R \times X \to R \times X$ is defined, as in §2, by $\tau(t, x) = (t+1, f^{-1}(x))$ (so that $\tau^i(t, x) = (t+i, f^{-i}(x))$). We may suppose $i < j$.

As $\tilde{X}_0$ is connected, there is a simple arc $C_0: I \to R \times X$ such that

$$\begin{aligned} &C_0(s) \in \tilde{X}_0 \qquad \text{for any} \quad 0 \leqq s \leqq 1 , \\ &C_0(0) = u = \tau^i(t, x) , \\ &C_0(1) = v = \tau^j(t, x) . \end{aligned}$$

Next, for any integer r, we can define an arc $C_r: I \to R \times X$ by

$$C_r(s) = \tau^{r(j-i)} C_0(s) .$$

Clearly $C_r(0) = C_{r-1}(1)$ and $qC_r = qC_0$ for any r, whence $C_r(I) \subset q^{-1}X_0$ for any r and $\bigcup_r C_r(I)$ is connected. Hence $\bigcup_r C_r(I) \subset \tilde{X}_0$, where $C_r(0) \in (t+i+r(j-i)) \times X$. Therefore $\tilde{X}_0$ does not satisfy the hypothesis of the lemma. This implies that the covering map $q|\tilde{X}_0: \tilde{X}_0 \to X_0$ is a diffeomorphism or a homeomorphism, by (3.1), according that h is differentiable or topological. This completes the proof of (3.3).

Lemma (3.4). *If for any $t_0 > 0$, there is a $t \in R$ such that $|t| > t_0$ and $(t \times X) \cap \tilde{X}_0 \neq \phi$, then the covering of* (3.1): *$\tilde{X}_0 \to X_0$ is a regular covering with a transformation group isomorphic to Z.*

Proof. By the assumption, the covering is not trivial, since $\tilde{X}_0$ is not compact but X_0 is compact. Let $x_0 \in X_0$ be a base point of X_0. If $\tilde{x}, \tilde{x}' \in \tilde{X}_0 \cap q^{-1}(x_0)$, then, by the definition of q, $\tilde{x}' = \tau^m \tilde{x}$ for some integer m.

Next, we shall show that this τ^m is a covering transformation of the covering: $\tilde{X}_0 \to X_0$. Let $\tilde{y}$ be any point in $\tilde{X}_0$. Since $\tilde{X}_0$ is connected, there is an arc C with the ends $\tilde{x}$ and $\tilde{y}$. $\tau^m C$ is an arc in $q^{-1}X_0$ with the ends $\tilde{x}'$ and $\tau^m\tilde{y}$. Since $\tilde{x}' \in \tilde{X}_0$ and $\tilde{X}_0$ is connected, we have $\tau^m\tilde{y} \in \tilde{X}_0$. Furthermore $q\tau^m y = q\tilde{y}$. Hence τ^m is a covering transformation of this covering.

Therefore $\tilde{X}_0 \to X_0$ is a regular covering.

Let i be the smallest positive integer such that $\tau^i\tilde{x} \in \tilde{X}_0$. We shall show that for any integr k, $\tau^{ki}\tilde{x} \in \tilde{X}_0$. Let C be an arc with the ends $\tilde{x}$ and $\tau^i\tilde{x}$. Then, by repeating the argument above, $\tau^{(k-1)i}C \subset \tilde{X}_0$ if $k>0$ and $\tau^{ki}C \subset \tilde{X}_0$ if $k<0$. Therefore $\tau^{ki}\tilde{x} \in \tilde{X}_0$.

Further we shall show that for any point $\tau^m\tilde{x}$ in the fibre over x_0, $m=ki$ for some integer k. Generally, we put $m=ki+h$, where k, h are integers and $0 \leqq h < i$. If $h \neq 0$, $\tau^h\tilde{x}$ must exist in $\tilde{X}_0$ as above; it is a contradiction to the property of i. Therefore $m=ki$.

Hence the transformation group of the regular covering is isomorphic to Z. This completes the proof of (3.4).

Let $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ be the suspensions of some C^∞-automorphisms (diffeomorphisms) on X and X' respectively. Suppose that there exists a topological equivalence of dynamical systems, $h: M' \to M$. Let $q: R\times X \to M$, $p: R\times X \to X$ and $\tilde{X}_0$ be the same as these in (3.1) and (3.2). And put $q_0 = q\,|\,\tilde{X}_0$.

Lemma (3.5). *Suppose that $\tilde{X}_0$ satisfies the condition of* (3.3), *so that $q_0: \tilde{X}_0 \to h(X')$ is a homeomorphism. Then, $\pi_1(X') \cong \pi_1(X)$ by $i_* = (pq_0^{-1}h\,|\,X')_*$.*

Proof. Since $\tilde{X}_0$ is homeomorphic to X', $p\,|\,\tilde{X}_0$: $\tilde{X}_0 \to X$ is a covering by (3.2). p induces the injection $p_*: \pi_1(\tilde{X}_0) \to \pi_1(X)$. Hence, if $j': X' \to M'$ denotes the including mapping, the composition $i = pq_0^{-1}hj'$: $X' \to X$ induces an inclusion

$$i_*:\ \pi_1(X') \to \pi_1(X)\,.$$

Let $j: X \to M$ denote the inclusion. We shall prove that the following diagram is commutative.

$$\begin{array}{ccc} \pi_1(X') & \xrightarrow{\ j'_*\ } & \pi_1(M') \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ \pi_1(X) & \xrightarrow{\ j_*\ } & \pi_1(M) \end{array}$$

There is a homotopy

$$F_t:\ X' \to R\times X$$

such that

$$F_0 = q_0^{-1}hj'$$
$$F_1 = pq_0^{-1}hj'\,.$$

In fact, F_t can be made easily by sliding $q_0^{-1}h(X')$ along the flow ψ_t of $R\times X$ onto X. Then,

$$G_t = qF_t :\ X' \to M$$

is a homotopy such that

$$G_0 = hj'$$
$$G_1 = jpq_0^{-1}hj' = ji\,.$$

Therefore the diagram above is commutative.

We shall construct a smooth fibering $M\to S^1$ with fibre X from $(M, \mathcal{F}; X)$ (*Cf.* §5). Recall that $M=R\times X/\tau$, as in §2 and that $S^1=R/t\sim t+1$. The mapping: $R\times X\to R$ defined by $(t, x)\to t\ (t\in R,\ x\in X)$ induces a mapping: $R\times X/\tau\to R/t\sim t+1$. This is a fibre map: $M\to S^1$.

In the same way, we construct a fibering: $M'\to S^1$ from $(M', \mathcal{F}'; X')$. Since the fibres X and X' are connected, we have the following commutative diagram, where the horizontal and vertical sequences are exact.

$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \longrightarrow & \pi_1(X') & \xrightarrow{j'_*} & \pi_1(M') & \longrightarrow & Z & \longrightarrow & 1 \\ & & {\scriptstyle i_*}\downarrow & & {\scriptstyle h_*}\downarrow & & & & \\ 1 & \longrightarrow & \pi_1(X) & \xrightarrow{j_*} & \pi_1(M) & \longrightarrow & Z & \longrightarrow & 1 \end{array}$$

Then,

$$Z \cong \pi_1(M)/\pi_1(X)$$
$$\cong (\pi_1(M)/j_*i_*\pi_1(X'))/(\pi_1(X)/i_*\pi_1(X'))\,.$$

But,

$$\pi_1(M)/j_*i_*\pi_1(X') \cong \pi_1(M')/\pi_1(X') \cong Z\,,$$

whence,

$$Z \cong Z/(\pi_1(X)/i_*\pi_1(X'))\,.$$

Hence it is necessary that

$$\pi_1(X)/i_*\pi_1(X') \cong 1\,.$$

Therefore $\pi_1(X')\cong\pi_1(X)$.

This completes the proof of (3.5).

4. Cross-section theorem

The purpose of this section is to prove

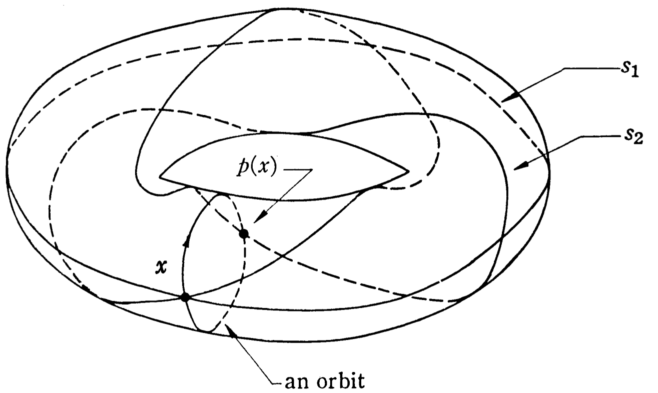
Theorem (4.1). *Let $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ be dynamical systems with cross-sections. (Manifolds may have boundaries.) Suppose that there exists no* *projection of the fundamental group $\pi_1(X)$ of X onto the infinite cyclic group Z.*

Then $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are differentiably (topologically) equivalent if and only if the associated diffeomorphisms of $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ are differentiably (topologically) conjugate.

REMARK (4.2). The assumption in the theorem about the fundamental group is necessary. This is verified by the following.

Let $T=S_0\times S_1$ be a torus, where S_j is a 1–dimensional circle, and let $\mathcal{F}$ be the dynamical system on T such that any orbit of $\mathcal{F}$ is $S_0\times x\subset T$ for $x\in S_1$. Then $S_1=*\times S_1$ in T $(*\in S_0)$ is a cross-section of $(T, \mathcal{F})$ and the associated diffeomorphism of $(T, \mathcal{F}; S_1)$ is the identity map i: $S_1\to S_1$.

Next, we imbed a circle in T in such a way that the imbedded image S_2 is a cross-section of $(T, \mathcal{F})$ and the associated diffeomorphism of $(T, \mathcal{F}; S_2)$ is the antipodal map p: $S_2\to S_2$. This is possible. In fact, the imbedded image S_2 is a clover-knot, if we consider T to be located 3–dimensional euclidean space (see the figure).



Then S_1 and S_2 are two cross-sections of the same (therefore equivalent) dynamical system $(T, \mathcal{F})$, but the associated diffeomorphisms i and g cannot be conjugate. In this case the fundamental groups of the cross-sections are isomorphic to Z.

Corollary (4.3). *Let $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ be dynamical systems with cross-sections. Suppose that the fundamental group $\pi_1(X)$ of X or the 1–dimensional homology group $H_1(X)$ is a finite group.*

Then $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are differentiably (topologically) equivalent if and only if the associated diffeomorphisms of $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ are differentiably (topologically) conjugate.

Proof. This corollary is a direct consequence of (4.1).

In fact, if G is a finite group, there is no projection of G onto Z. If there is a projection of $\pi_1(X)$ onto Z, the projection induces a projection of $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ onto Z, where $[\pi_1(X), \pi_1(X)]$ denotes the commutator subgroup. And $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ is isomorphic to $H_1(X)$.

Corollary (4.4). *Assume that for* $(M_1, \mathcal{F}_1; X_1)$ *and* $(M_2, \mathcal{F}_2; X_2)$, $(M_1, \mathcal{F}_1)$ *and* $(M_2, \mathcal{F}_2)$ *are differentiably (topologically) equivalent and that* M_1 *and* M_2 *are closed (so that* X_1 *and* X_2 *are closed). Suppose that one of the following two conditions is satisfied.*

i) *There exists no projection of* $\pi_1(X_1)$ *onto* Z.

ii) *There exists* $(\tilde{M}_i, \tilde{\mathcal{F}}_i; \tilde{X}_i)$ (i=1, 2) *satisfying the following conditions.*

(a) $\partial\tilde{M}_i = M_i$, $\partial\tilde{X}_i = X_i$.

(b) $\tilde{\mathcal{F}}_i$ *is an extention of* $\mathcal{F}_i$.

(c) $(\tilde{M}_1, \tilde{\mathcal{F}}_1)$ *and* $(\tilde{M}_2, \tilde{\mathcal{F}}_2)$ *are differentiably (topologically) equivalent.*

(d) *There exists no projection of* $\pi_1(\tilde{X}_1)$ *onto* Z.

Then the associated diffeomorphisms of $(M_1, \mathcal{F}_1; X_1)$ *and* $(M_2, \mathcal{F}_2; X_2)$ *are differentiably (topologically) conjugate.*

Proof of Theorem (4.1). If the associated diffeomorphisms of $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ are differentiably (topologically) conjugate, (2.2) and (2.3) imply that $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are differentiably (topologically) equivalent.

Next, we shall prove the converse of this, assuming that there is no projection of $\pi_1(X')$ onto Z. The proof consists of two parts; the first, to fined a diffeomorphism (homeomorphism) of X' onto X, and the second, to prove that the diffeomorphism (homeomorphism) satisfes the condition of conjugacy.

Let $h: M' \to M$ be the differentiable (topological) equivalence of $(M, \mathcal{F})$ and $(M', \mathcal{F}')$. Here, by (2.1) and (2.2) we may regard $(M, \mathcal{F}; X)$ as $(M_0, \mathcal{F}_0; X)$, the suspension of the associated diffeomorphism of $(M, \mathcal{F}; X)$, and may regard $(M', \mathcal{F}'; X')$ similarly.

Part 1. Let $q: R \times X \to M$ be the suspending projection as in §2. Then, by (3.1), $q: \tilde{X}_0 \to X_0$ is a covering map, where $X_0 = h(X')$ and $\tilde{X}_0$ is any connected component of $q^{-1}(X_0)$.

Suppose further that for any $t_0 > 0$ there is a t such that $|t| > t_0$ and $(t \times X) \cap \tilde{X}_0 \neq \phi$. Then by (3.4) this covering $\tilde{X}_0 \to X_0$ is a regular covering with the transformation group isomorphic to Z and we have an exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}_0) \longrightarrow \pi_1(X_0) \longrightarrow Z \longrightarrow 1\,.$$

This is a contradiction to the assumption of $\pi_1(X') \cong \pi_1(X_0)$. Therefore, by (3.3), any component $\tilde{X}_0$ of $q^{-1}(X_0)$ is diffeomorphic (homeomorphic if h is a homeomorphism) to X_0.

Therefore $\tilde{X}_0$ is a compact differentiable (topological) submanifold of $R \times X$. Hence, by (3.2), $p: \tilde{X}_0 \to X$ is a differentiable (topological) covering, where p is the natural projection $R\times X\to X$. Since $\tilde{X}_0$ is diffeomorphic (homeomorphic) to X', we get a differentiable (topological) covering map $i=pq_0^{-1}h\colon X'\to X$, where $q_0=p\,|\,\tilde{X}_0$. Therefore, since i_* is an isomorphism of $\pi_1(X')$ onto $\pi_1(X)$ by (3.5), $pq_0^{-1}h$ is a diffeomorphism (homeomorphism if h is a topological equivalence): $X'\to X$.

Part 2. Since any component of $q^{-1}(X_0)$ is homeomorphic to X_0 by q, any component of $q^{-1}(X_0)$ is $\tau^i\tilde{X}_0$ for some integer i, where $\tilde{X}_0$ is a fixed component and τ is defined, as above, by

$$\tau(t, x) = (t+1, f^{-1}(x))\,, \qquad t\in R,\ x\in X\,.$$

In order to prove Part 2, we shall prove the following lemma.

Lemma (4.5). *Suppose that $\tau^i\tilde{X}_0$ is homeomorphic to X by the map p and that $(s, x)\in\tau^i\tilde{X}_0$, $(t, x)\in\tau^j\tilde{X}_0$. Then, $s<t$ if and only if $i<j$.*

Proof. Since $t=s$ whenever $i=j$, it is sufficient if we can prove that $i<j$ implies $s<t$.

Here, we suppose that there exist integers i, j, real numbers t, $s\in R$ and a point $x\in X$ satisfying the conditions of the lemma, such that $s>t$ and $i<j$.

Since $p\colon \tau^i\tilde{X}_0\to X$ is homeomorphism, $\tau^i\tilde{X}_0$ splits $R\times X$ for any i; that is, if (s_1, x_1), $(s_2, x_2)\in\tau^i\tilde{X}_0$ and $(t_1, x_1)\in\tau^j\tilde{X}_0$, and if $s_1<t_1$, then there is $t_2\in R$ such that $(t_2, x_2)\in\tau^j\tilde{X}_0$ and $s_2<t_2$.

By assumption we have

$$(s+j-i, f^{i-j}(x)) = \tau^{j-i}(s, x)\in\tau^j\tilde{X}_0\,, \qquad j-i>0\,.$$

By the splitting property and by $s>t$, there is a $s_1\in R$ such that

$$(s_1, f^{i-j}(x))\in\tau^i\tilde{X}_0\,, \qquad s_1>s+j-i\,.$$

By repeating this process we get $s_r\in R$ for all r such that

$$(s_r, f^{r(i-j)}(x))\in\tau^i\tilde{X}_0\,, \qquad s_r>s+r(j-i)\,.$$

But this is in contradiction to the fact that $\tau^i\tilde{X}_0$ is compact. This completes the proof of the lemma.

Now, we come back to the proof of Part 2.

Let $(t, x)\in\tilde{X}_0\subset R\times X$. The orbit of $R\times X$ passing through $(t, x)\in\tilde{X}_0$ meets $q^{-1}(X_0)$ at $(s, x)\in\tau\tilde{X}_0$ for the first time. Define a diffeomorphism (homeomorphism) $g\colon \tilde{X}_0\to\tau\tilde{X}_0$ by $g(t, x)=(s, x)$.

Now we can easily see, by (4.5), that the following diagram is comutative.

$$\begin{array}{ccccccc}
X' & \xrightarrow{h} & X_0 & \xrightarrow{q_0^{-1}} & \tilde{X}_0 & \xrightarrow{p} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \tau^{-1}g} & & \downarrow{\scriptstyle f} \\
X' & \xrightarrow{h} & X_0 & \xrightarrow{q_0^{-1}} & \tilde{X}_0 & \xrightarrow{p} & X
\end{array}$$

Here $f_0: X \to X_0$ is a diffeomorphism (homeomorphism) defined as follows; if $x \in X_0$, $f_0(x)$ is the point where the orbit of $(M, \mathcal{F})$ passing through x meet X_0 for the first time.

The commutativity of the diagram implies that f and f' are differentiably or topologically conjugate, according that h is differentiable or topological. This completes the proof of Part 2.

Therefore the proof of Theorem (4.1) is completed.

We deduce the following corollary directly from (4.1).

Corollary (4.6). *Let X and X' be two cross-sections of a dynamical system. If there exists no projection of $\pi_1(X)$ onto Z, then X and X' are diffeomorphic.*

Remark (4.2) is not applicable to this corollary. In a paper in preparation[1], we will show examples of dynamical systems with two cross-sections which are not diffeomorphic or homeomorphic.

5. Relation with fibre bundles

All fibre bundles considered in this section are assumed to be smooth and to have the base spase S^1 and the group $\mathrm{Diff}(X)$, in the sense of [10], where $\mathrm{Diff}(X)$ is the group of diffeomorphisms of the fibre X onto itself. We consider the discrete topology in $\mathrm{Diff}(X)$.

Let X_f be the *mapping torus of* f defined by $X_f = I \times X$ with identification $(0, f(x)) = (1, x)$ for all $x \in X$. If f is a diffeomorphism, X_f is a smooth manifold. For any diffeomorphism f of X onto itself, let ξ_f be the fibre bundle $p: X_f \to I/0 \sim 1 = S^1$ defined by $p(t, x) = t$. Moreover,

(5.1) *for any fibre bundle ξ with fibre X, there exists a diffeomorphism f over X such that ξ_f is equivalent to ξ in $\mathrm{Diff}(X)$, in the sence of* [10].

Let $\mathrm{Hom}(X)$ be the group of homeomorphism of X onto itself. We consider the discrete topology in it. Let $[\xi]_d$ and $[\xi]_t$ be the equivalence classes in $\mathrm{Diff}(X)$ and $\mathrm{Hom}(X)$ respectively containing the fibre bundle ξ. Then,

(5.2) $[\xi_f]_d = [\xi_g]_d$ $([\xi_f]_t = [\xi_f]_t)$ *if and only if f and g are differentiably* (*togologically*) *conjugate.*

Let $[(M, \mathcal{F})]_d$ be the differentiable equivalence class containing a dynamical system $(M, \mathcal{F})$, and $[(M, \mathcal{F})]_t$ the topological equivalence class having the same property. And set

$$\begin{aligned}
\mathcal{F}_d(X) &= \{[(M, \mathcal{F})]_d \,;\ (M, \mathcal{F}) \text{ has a cross-section } X\}\,, \\
\mathcal{F}_t(X) &= \{[(M, \mathcal{F})]_t \,;\ \text{———}\}\,, \\
\boldsymbol{F}_d(X) &= \{[\xi]_d \,;\ \xi \text{ has a fibre } X\}\,, \\
\boldsymbol{F}_t(X) &= \{[\xi]_t \,;\ \text{———}\}\,.
\end{aligned}$$

1) G. Ikegami: *On dynamical systems with cross-sections*, (to appear).

We define a map η_d: $\boldsymbol{F}_d(X)\to\mathcal{F}_d(X)$ (similarly for η_t: $\boldsymbol{F}_t(X)\to\mathcal{F}_t(X)$) as follows; Let $[\xi]_d\in\boldsymbol{F}_d(X)$, then by (5.1) there is f with $\xi_f\in[\xi]_d$. We define $\eta_d[\xi]_d$ by the differentiable equivalence class of the dynamical system wich is the suspension of f. Here, η_d is well defined; in fact by (5.2) and (2.3), η_d is independent of the selection of f.

In order to define the mapping μ_d: $\mathcal{F}_d(X)\to\boldsymbol{F}_d(X)$ (similarly for μ_t: $\mathcal{F}_t(X)\to\boldsymbol{F}_t(X)$), we must suppose that there is no projection of $\pi_1(X)$ onto Z. In this case, for any $[(M,\mathcal{F})]_d\in\mathcal{F}_d(X)$, $\mu_d[(M,\mathcal{F})]_d$ is defined by $[\xi_f]_d\in\boldsymbol{F}_d(X)$, where f is the associated diffeomorphism of $(M,\mathcal{F};X)$. μ_d is well defined; in fact, by (4.1) and (5.2), μ_d is independent of the selection of $(M,\mathcal{F})$ and X.

Here, $\mu_d\eta_d$=identity ($\mu_t\eta_t$=identity) by (2.1) and $\eta_d\mu_d$=identity ($\eta_t\mu_t$= identity) by (2.2).

Therefore we have proved the next theorem;

Theorem (5.3). *If there exist no projection of $\pi_1(X)$ onto Z, then there is a natural one-to-one correspondence between $\mathcal{F}_d(X)$ and $\boldsymbol{F}_d(X)$ ($\mathcal{F}_t(X)$ and $\boldsymbol{F}_t(X)$).*

6. Dynamical systems on tori with cross-section

The purpose of this section is to prove Theorem (6.6).

Define the two fibre maps,

$$p_+:\ S^n\times S^1\to S^1 \qquad \text{by}\quad p_+(x,y)=y\,,$$
$$p_-:\ S^n\times S^1\to S^1 \qquad \text{by}\quad p_-(x,y)=r(y)\,,$$

for any $x\in S^n$, $y\in S^1$, where r: $S^1\to S^1$ denotes an arbitraly given homeomorphism of order -1.

The following lemma is concerned only with topology.

Lemma (6.1). *If $n\geqq 2$, any fibre map p: $S^n\times S^1\to S^1$ with connected fibre is homotopic either to p_+ or to p_-.*

Proof. Denote by $\pi(X;Y)$ the set of homotopy classes of the maps: $X\to Y$ with fixed base points.

First, we shall show $\pi(S^n\times S^1;S^1)\cong Z$. Let i: $S^n\vee S^1\to S^n\times S^1$ be the natural injection, where $\vee$ denotes the union in identifying the base points, and let h: $S^n\times S^1\to(S^n\times S^1)/(S^n\vee S^1)=S^{n+1}$ be the natural projection. We have an exact sequence (Puppe sequence);

$$\pi(S^{n+1};S^1)\xrightarrow{h^*}\pi(S^n\times S^1;S^1)\xrightarrow{i^*}\pi(S^n\vee S^1;S^1)\,.$$

Here,

$$\pi(S^{n+1};S^1)\cong 0$$

and

$$\pi(S^n \vee S^1;\, S^1) \cong \pi(S^n;\, S^1)+\pi(S^1;\, S^1) \cong Z\,.$$

Moreover, we can easily see that i^* is a projection. Therefore,

$$\pi(S^n \times S^1;\, S^1) \cong \pi_1(S^1) \cong Z\,.$$

Let this isomorphism $\pi(S^n\times S^1;\, S^1)\to\pi_1(S^1)$ be denoted by the same i^*.

Next, we shall show that, for a fibre map p with connected fibre F, $i^*([p])$ is a generator of $\pi_1(S^1)$. Where $[p]$ denotes the element of $\pi(S^n\times S^1;\, S^1)$ containing p.

Let

$$\longrightarrow \pi_1(S^n\times S^1) \xrightarrow{p_*} \pi_1(S^1) \longrightarrow \pi_0(F)$$

be the homotopy exact sequence of the fibre space. Since $\pi_0(F)=0$ and since $\pi_1(S^n\times S^1)$ and $\pi_1(S^1)$ are isomorphic to Z, p_* is an isomorphism. If $j\colon S^1\to S^n\times S^1$ is the natural inclusion, $j_*\colon \pi_1(S^1)\to\pi_1(S^n\times S^1)$ is an isomorphism. And if 1 is the generator of $\pi_1(S^1)$ preserving the orientation of S^1, we have

$$p_* j_*(1) = i^*([p])\,.$$

Since $p_*j_*(1)$ is q generator of $\pi_1(S^1)$, $i^*([p])=\pm 1$. But, $i^*([p_+])=1$ and $i^*([p_-])=-1$. Therefore $[p]=[p_+]$ or $[p_-]$.

This completes the proof of (6.1).

Let $\mathrm{Diff}_+(S^n)$ and $\mathrm{Diff}_+(D^{n+1})$ denote the groups of orientation preserving diffeomorphisms on S^n and on a disk D^{n+1} resp., and let $r\colon \mathrm{Diff}_+(D^{n+1})\to \mathrm{Diff}_+(S^n)$ denote the homomorphism obtained by the restriction. Then, the group $\mathcal{D}(S^n)=\mathrm{Diff}_+(S^n)/\mathrm{Image}\, r$ is isomorphic to Γ^{n+1} ([4]). Here Γ^{n+1} denotes the group of differentiable structures on S^{n+1} with usual $p.1.$ structure under the connected sum operation $\#$. $\mathcal{D}(S^n)$ is an abelian group [6]. If $n\geqq 4$ or $n=1$, Γ^{n+1} is the same as θ^{n+1}, which is by definition the group of homotopy $(n+1)$-sphres ([3]).

Let $\mathcal{S}(S^n\times S^1)$ denote the set of all differentiable manifolds homeomorphic to $S^n\times S^1$ classified by diffeomorphisms and let

$$\psi\colon\ \Gamma^{n+1} \longrightarrow \mathcal{S}(S^n\times S^1)$$

be the mapping defined by $\Psi(\tilde{S}^{n+1})=S^n\times S^1\#\tilde{S}^{n+1}$, $\tilde{S}^{n+1}\in\Gamma^{n+1}$. Next, define a mapping

$$\Phi\colon\ \mathrm{Diff}_+(S^n) \longrightarrow \mathcal{S}(S^n\times S^1)$$

by $\Phi(f)=S^n_f$ for any $f\in\mathrm{Diff}_+(S^n)$.

Lemma (6.2). *Φ induces a one-to-one correspondence $\tilde{\Phi}\colon \mathcal{D}(S^n)\to\Psi(\Gamma^{n+1})$. Moreover, if $[f]\in\mathcal{D}(S^n)$ then $\tilde{\Phi}([f])=S^n\times S^1\#\tilde{S}^{n+1}$, where $\tilde{S}^{n+1}\in\Gamma^{n+1}$ is the* *element corresponding to* $[f]$ *under the isomorphism of* $\mathscr{D}(S^n)$ *with* Γ^{n+1}.

Proof. For any $\tilde{S}^{n+1}$ in Γ^{n+1}, $S^n\times S^1 \# \tilde{S}^{n+1}$ is diffeomorphic to S^n_f, where f is any diffeomorphism in the element of $\mathscr{D}(S^n)$ corresponding to $\tilde{S}^{n+1}\in\Gamma^{n+1}$ under the isomorphism. (See [1], Lemma 1.) Therefore $\tilde{\Phi}$ can be well-defined and $\tilde{\Phi}$ maps $\mathscr{D}(S^n)$ onto $\Psi(\Gamma^{n+1})$.

Moreover, for any $\tilde{S}_1^{n+1}$, $\tilde{S}_2^{n+1}\in\Gamma^{n+1}$ with $\tilde{S}_1^{n+1}\neq\tilde{S}_2^{n+1}$,

$$S^n\times S^1\#\tilde{S}_1^{n+1} \neq S^n\times S^1\#\tilde{S}_2^{n+1} \qquad \text{in} \quad \mathcal{S}(S^n\times S^1),$$

because, if

$$S^n\times S^1\#\tilde{S}_1^{n+1} = S^n\times S^1\#\tilde{S}_2^{n+1} \qquad \text{in} \quad \mathcal{S}(S^n\times S^1),$$

then

$$S^n\times S^1\#\tilde{S}_1^{n+1}\#(-\tilde{S}_2^{n+1}) = S^n\times S^1$$

by using an orientation-preserving diffeomorphism. But, the innertia group of $S^n\times S^1$: $I(S^n\times S^1)=\{\tilde{S}^{n+1}\in\Gamma^{n+1};\ S^n\times S^1\#\tilde{S}^{n+1}=S^n\times S^1\}$ is equal to 0 for all n (see [11], [2], [6])[1]. Hence, $\tilde{S}_1^{n+1}\#(-\tilde{S}_2^{n+1})=0$ in Γ^{n+1}. This implies $\tilde{S}_1^{n+1}$ is diffeomorphic to $\tilde{S}_2^{n+1}$. Therefore $\tilde{\Phi}$ is an injection.

These prove the lemma.

Proposition (6.3). *If f, g $Diff_+(S^n)$ are differentiably conjugate*[2], *then f and g are contained in the same element of* $\mathscr{D}(S^n)$[3].

Proof. If f and g are differentiably conjugate, S^n_f and S^n_g are diffeomorphic. Then, (6.2) implies $[f]=[g]$ in $\mathscr{D}(S^n)$.

We denote by $C(\tilde{S}^{n+1})$ the set of differentiably conjugate classes of diffeomorphisms contained in the element of $\mathscr{D}(S^n)$ corresponding to $\tilde{S}^{n+1}\in\Gamma^{n+1}$.

The following property is due to W. Browder ([1], Lemma 2).

(6.4) *Let mapping tori X_f and Y_g be the total spaces of differentiable fibre bundles over S^1 with projection p and q, and with fibres X^n and Y^n which are* 1–*connected closed manifolds of dimeksion $n\geqq 5$. If h: $X_f\to Y_g$ is a diffeomorphism such that qh is homotopic to p, then there is a diffeomorphism h' such that $qh'=p$, so that h' restricts to a diffeomorphism of X with Y.*

If $M^{n+1}\in\mathcal{S}(S^n\times S^1)$ for $n\geqq 5$, any smooth fibre bundle over S^1 with total space M^{n+1} and with connected fibre has fibre with the homotopy groups of sphere, which is homeomorphic to S^n by [7] or [5], p. 109, Prop. B. The following lemma shows a condition for fibre to be diffeomorphic to S^n.

---

1) If $n\geqq 4$, $I(S^n\times S^1)=0$ by the method in [11]; and if $n>4$, $I(S^n\times S^1)=0$, since $\Gamma^{n+1}=0$ by [2] for $n=3$ and by [6] for $n\leqq 2$.

2) If f and g are conjugate, there is a $h\in \mathrm{Diff}(S^n)$ with $hf=gh$. But we should notice that the definition of conjugacy in § 2 does not imply that h preserves orientation. If h preserves orientation, (6.3) is trivial. Because, we can consider $h\in\mathscr{D}(S^n)$, and the group $\mathscr{D}(S^n)$ is abelian.

3) By using J. Cerf's theorem to (6.4) we have; *If $f, g\in \mathrm{Diff}_+(S^n)$ are differentiably conjugate, and $n\geqq 8$, then f and g are isotogic.*

Lemma (6.5). *Suppose M^{n+1} is in $\Psi(\Gamma^{n+1})$, $n\geqq 5$ or $n=2$, then the fibre of any smooth fibre bundle over S^1 with total space M and with connected fibre is diffeomorphic to S^n.*

Proof. By (6.2) there is $f\in \mathrm{Diff}_+(S^n)$ such that S^n_f is diffeomorphic to M^{n+1}. Let $p_+\colon S^n_f\to S^1$ be the fibre bundle defined by $p(t, x)=t$, as in §5. Put $p_-=rp_+$, where $r\colon S^1\to S^1$ is a diffeomorphism with degree -1. And let $q\colon M^{n+1}\to S^1$ be any smooth fibre bundle with connected fibre. q is homotopic either to p_+ or to p_- by (6.1). Hence, if $n\geqq 5$, (6.4) implies that the fibre of q is diffeomorphic to the fibre S^n of p_+ or of p_-. If $n=2$, the Lemma is trivial, since $\theta^2=\Gamma^2=0$.

Theorem (6.6). *If M^{n+1} ($n\geqq 5$ or $n=2$) is diffeomorphic to $S^n\times S^1\#\tilde{S}^{n+1}$ for some $\tilde{S}^{n+1}$ in Γ^{n+1}, then the differentiable equivalence classes of dynamical systems on M with cross-sections have a one-to-one correspondence with $\mathcal{C}(\tilde{S}^{n+1})$. The class corresponding to $f\in\mathcal{C}(\tilde{S}^{n+1})$ is the suspension of f.*

Proof. Let $(M, \mathcal{F}; X)$ denote any dynamical system on M^{n+1} with cross-section. M is diffeomorphic to X_f, where f is the associated diffeomorphism of $(M, \mathcal{F}; X)$. Since there is a smooth fibre bundle $X_f\to S^1$, (6.5) implies that X is diffeomorphic to S^n. Hence, any differentiable equivalence class of dynamical systems on M with cross-sections is in $\mathcal{F}_d(S^n)$, for which (5.3) is a one-to-one correspondence to $\boldsymbol{F}_d(S^n)$. And, by (5.1) and (5.2) there exists a one-to-one correspondence between the differentiably conjugate classes of diffeomorphisms on S^n and $\boldsymbol{F}_d(S^n)$. Therefore (6.2) and (6.3) imply that the differentiable equivalence classes of dynamical systems on M^{n+1} correspond to $\mathcal{C}(\tilde{S}^{n+1})$. This completes the proof of the theorem.

Kobe University

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