Title	On classification of dynamical systems with cross-sections
Author(s)	Ikegami, Gikô
Citation	Osaka Journal of Mathematics. 1969, 6(2), p. 419-433
Version Type	VoR
URL	https://doi.org/10.18910/4104
rights	
Note	

Ikegami, G.
Osaka J. Math.
6 (1969), 419-433

ON CLASSIFICATION OF DYNAMICAL SYSTEMS WITH CROSS-SECTIONS

GIKŌ IKEGAMI

(Received October 26, 1968)
(Revised March 18, 1969)

1. Introduction

Let $(M_i, \mathcal{F}_i)$ be a dynamical system on a manifold M_i with a cross-section X_i, where $\mathcal{F}_i$ is the flow-structure, $i=1, 2$. To $(M_i, \mathcal{F}_i)$ the associated diffeomorphism $f_i: X_i \to X_i$ is defined.

By S. Smale ([8], [9]), it is shown that if f_1 and f_2 are differentiably or topologically conjugate by a map $h: X_1 \to X_2$, then $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are differentiably or topologically equivalent respectively.

The main purpose of this paper is to show the converse of the above fact under some conditions, that is; under the assumption that there exists no homomorphism of the fundamental group of one of the two cross-sections onto the infinite cyclic group, $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are differentiably or topologically equivalent if and only if f_1 and f_2 are defferentiably or topologically conjugate respectively (Theorem 4.1).

Furthermore we shall show an example of a pair of dynamical systems $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ with cross-sections X_1 and X_2 respectively such that the fundamental group of X_1 is isomorphic to the infinite cyclic group, and that $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are differentiably equivalent but the associated diffeomorphisms are not conjugate (§4).

As an application of Theorem (4.1), we shall show in §5 that for a given X satisfying the condition stated above concerning its fundamental group, there is a natural correspondence between the equivalence classes of dynamical systems $(M, \mathcal{F})$ with the cross-section X and the equivalence classes of smooth fibre bundles over S^1 with the fibre X (Theorem 5.3).

Another application will be shown in §6; that is classification of dynamical system on $S^n \times S^1 \# \tilde{S}^{n+1}$ with cross-section, where $\tilde{S}^{n+1}$ denotes any homotopy sphere (Theorem 6.6). Here, it is essential that Γ^{n+1} classifies the differentiably conjugate classes of diffeomorphisms on S^n.

The author wishes to express his sincere gratitude to Professors Y. Saito, Y. Shikata and T. Ura who offered helpful advices.

2. Terminology

Throuout this paper, all manifolds considered will be assumed to be compact and differentiable (C^∞).

A dynamical system or a flow $\mathcal{F}$ on a manifold M is a 1-parameter group of transformations φ of M, where φ is a C^∞-map $\varphi: R\times M\to M$ (R; the real numbers) such that if we put $\varphi_t(x)=\varphi(t, x)$, then

(i) $\varphi_0(x)=x$

(ii) $\varphi_{t+s}(x)=\varphi_t\varphi_s(x)$,

and φ_t is a diffeomorphism $(M, \partial M)\to(M, \partial M)$. Here ∂M is the boundary of M.

By a pair $(M, \mathcal{F})$ we mean a dynamical system $\mathcal{F}$ on a manifold M. $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are said to be *differentiably* (*topologically*) *equivalent* if there is a diffeomorphism (homeomorphism) $h: M_1\to M_2$ having the property that h maps every orbit of $\mathcal{F}_1$ onto an orbit of $\mathcal{F}_2$ preserving the orientation. Such a map h will be called an *equivalence*.

Two diffeomorphisms (homeomorphisms) $f_1: M_1\to M_1$ and $f_2: M_2\to M_2$ are said to be *differentiably* (*topologically*) *conjugate* if there exists a diffeomorphism (homeomorphism) $h: M_1\to M_2$ such that $hf_1=f_2h$.

A *cross-section* of a dynamical system $(M, \mathcal{F})$ or (M, φ) is a compact connected submanifold X of codimension 1 of M such that $\partial X\subset\partial M$, and that

(i) X intersects every orbit,

(ii) the intersection of X with each orbit is transversal,

(iii) if $x\in X$, there is a $t>0$ with $\varphi_t(x)\in X$, and

(iv) if $x\in X$, there is a $t<0$ with $\varphi_t(x)\in X$.

There can be no singular point of $(M, \mathcal{F})$ if there is a crosssection. $\partial X\neq\phi$ if $\partial M\neq\phi$, and X is properly imbedded in M, i.e. $\partial X\subset\partial M$ and $\operatorname{Int} X\subset\operatorname{Int} M$. By $(M, \mathcal{F}; X)$ we mean a dynamical system $\mathcal{F}$ on a manifold M having a cross-section X.

For $(M, \mathcal{F}; X)$ we can define a map $f: X\to X$ by $f(x)=\varphi_{t_0}(x)$ where t_0 is the smallest positive t satisfying $\varphi_t(x)\in X$. $f: X\to X$ is a diffeomorphism; we call f the *associated diffeomorphism* of $(M, \mathcal{F}; X)$.

Conversely, suppose that a diffeomorphism f of X onto itself is given. Define a diffeomorphism $\tau: R\times X\to R\times X$ by $\tau(t, x)=(t+1, f^{-1}(x))$. Then the infinite cyclic group $\{\tau^m\}=Z$ operates freely on $R\times X$ and the orbit space $(R\times X)/Z$ is a manifold, say M_0. The flow $\psi_t: R\times X\to R\times X$ defined by $\psi_t(u, x)=(u+t, x)$ induces a flow φ_t on M_0. We call this (M_0, φ_t) the *suspension* of f. M_0 has a cross-section $X_0=q(0\times X)\subset M_0$, where $q: R\times X\to M_0$ is the quotient map.

The following properties are shown by S. Smale ([7] or [8]).

(2.1) *The associated diffeomorphism of* $(M_0, \varphi_t; X_0)$ *is differentiably conjugate to the given* f.

Furthermore

(2.2) *if* $(M', \varphi'_t; X')$ *is the suspension of the associated diffeomorphism of a dynamical system* $(M, \varphi_t; X)$, *then* (M, φ_t) *and* (M', φ'_t) *are differentiably equivalent (by an equivalence mapping* X *onto* X').

(2.3) *Let* $(M_0, \mathcal{F}_0)$, $(M_1, \mathcal{F}_1)$ *be the suspensions of* $f_0: X_0 \to X_0, f_1 X_1 \to X_1$ *respectively. If* f_0 *and* f_1 *are differentiably (topologically) conjugate, then* $(M_0, \mathcal{F}_0)$ *and* $(M_1, \mathcal{F}_1)$ *are differentiably (topologically) equivalent.*

3. Lemmas

Suppose that $h: M' \to M$ is a differentiable (topological) equivalence between $(M', \mathcal{F}'; X')$ and $(M, \mathcal{F}; X)$. Let $f: X \to X$ be the associated diffeomorphism of $(M, \mathcal{F}; X)$. Let $p: R \times X \to X$ be the natural projection and $q: R \times X \to M_0$ the quotient map to the suspension M_0 of f. Using (2.2), we consider h to be a differentiable (topological) equivalence: $M' \to M_0$. Put $X_0 = hX' \subset M_0$ and let $\tilde{X}_0$ be a connected component of $q^{-1}(X_0)$. Then we have the following lemmas.

Lemma (3.1). $q | \tilde{X}_0: \tilde{X}_0 \to X_0$ *is a covering. If* h *is a differentiable equivalence, then* $q | \tilde{X}_0$ *is a smooth covering map.*

Proof. $q: R \times X \to M_0$ is a covering, furthermore q is a smooth covering. Since X_0 is a properly imbedded submanifold of M_0 and since $q: (R \times X, q^{-1}(X_0)) \to (M, X_0)$ is a local homeomorphism, $\tilde{X}_0$ is a proper submanifold of $R \times X$. Since $q(\partial \tilde{X}_0) \subset \partial X_0$, $q(\operatorname{Int} \tilde{X}_0) \subset \operatorname{Int} X_0$, and since q is a local homeomorphism, the image $q(\tilde{X}_0)$ is a proper compact submanifold of X_0 with codimension 0. Therefore, $q | \tilde{X}_0: \tilde{X}_0 \to X_0$ is an onto map.

Therefore it is easy to see that $q | \tilde{X}_0: \tilde{X}_0 \to X_0$ is a covering and that if h is a differentiable equivalence, then it is a smooth covering.

Lemma (3.2). *If* $\tilde{X}_0$ *is compact, then* $p | \tilde{X}_0: \tilde{X}_0 \to X$ *is a covering. If* h *is a differentiable equivalence* $p | \tilde{X}_0$ *is a smooth covering.*

Proof. Since X' has transversal intersection with the flow in M', and since h and q are local homeomorphisms mapping orbit onto orbit, $\tilde{X}_0$ has transversal intersection with the flow of $R \times X$. Hence, $p | \tilde{X}_0: \tilde{X}_0 \to X$ is a local homeomorphism. Furthermore, as in (3.1), $\tilde{X}_0$ is a proper submanifold of $R \times X$, and $p: R \times X \to X$ maps boundary into boundary and maps interior into interior. Hence, $p(\tilde{X}_0)$ is a proper submanifold of X with codimension 0. Therefore, $p | \tilde{X}_0$ is an onto-map.

For each x in X and each $\tilde{x}_i \in \tilde{X}_0$ in $p^{-1}(x)$ let $\tilde{U}(\tilde{x}) \in \tilde{X}_0$ be a neighbourhood such that $p | \tilde{U}(\tilde{x}_i)$ is a homeomorphism and that if $\tilde{x}_i \neq \tilde{x}_j$, $\tilde{x}_i, \tilde{x}_j \in p^{-1}(x)$, then $\tilde{U}(\tilde{x}_i) \cap \tilde{U}(\tilde{x}_j) = \phi$, and let $U_i(x) \subset X$ be the homeomorphic image of $\tilde{U}(\tilde{x}_j)$ by p. Since $\tilde{X}_0$ is compact, it is clear that $p^{-1}(x)$ is a finite set. Put

$$W(x) = \bigcap_i U_i(x)$$

and

$$\tilde{W}(\tilde{x}_i) = (p^{-1}W(x)) \cap \tilde{U}(\tilde{x}_i) .$$

Then, $W(x)$ and $\tilde{W}(\tilde{x}_i)$ satisfy the usual conditions for covering.

This proves that $p|\tilde{X}_0: \tilde{X}_0 \to X$ is a covering.

Let $h: M' \to M$, X_0, q, and $\tilde{X}_0 \subset R \times X$ be the same as in (3.1). It should be noted that one and only one of the two assumptions in the following lemmas (3.3) and (3.4) holds.

Lemma (3.3). *If there is $t_0 > 0$ such that $(t \times X) \cap \tilde{X}_0 = \phi$ for every t with $|t| > t_0$, then $q|\tilde{X}_0$ is a diffeomorphism or a homeomorphism onto X_0 according that h is a differentiable or topological equivalence.*

Proof. Suppose there are two points u, $v \in \tilde{X}_0$ such that $q(u) = q(v) \in X_0$. Then for some integers i, j, $(i \neq j)$, some $0 \leqq t < 1$, and some $x \in X$, we have

$$u = \tau^i(t, x) \qquad v = \tau^j(t, x) ,$$

where $\tau: R \times X \to R \times X$ is defined, as in §2, by $\tau(t, x) = (t+1, f^{-1}(x))$ (so that $\tau^i(t, x) = (t+i, f^{-i}(x))$). We may suppose $i < j$.

As $\tilde{X}_0$ is connected, there is a simple arc $C_0: I \to R \times X$ such that

$$C_0(s) \in \tilde{X}_0 \qquad \text{for any} \quad 0 \leqq s \leqq 1 ,$$
$$C_0(0) = u = \tau^i(t, x) ,$$
$$C_0(1) = v = \tau^j(t, x) .$$

Next, for any integer r, we can define an arc $C_r: I \to R \times X$ by

$$C_r(s) = \tau^{r(j-i)} C_0(s) .$$

Clearly $C_r(0) = C_{r-1}(1)$ and $qC_r = qC_0$ for any r, whence $C_r(I) \subset q^{-1}X_0$ for any r and $\bigcup_r C_r(I)$ is connected. Hence $\bigcup_r C_r(I) \subset \tilde{X}_0$, where $C_r(0) \in (t+i+r(j-i)) \times X$. Therefore $\tilde{X}_0$ does not satisfy the hypothesis of the lemma. This implies that the covering map $q|\tilde{X}_0: \tilde{X}_0 \to X_0$ is a diffeomorphism or a homeomorphism, by (3.1), according that h is differentiable or topological. This completes the proof of (3.3).

Lemma (3.4). *If for any $t_0 > 0$, there is a $t \in R$ such that $|t| > t_0$ and $(t \times X) \cap \tilde{X}_0 \neq \phi$, then the covering of* (3.1)*: $\tilde{X}_0 \to X_0$ is a regular covering with a transformation group isomorphic to Z.*

Proof. By the assumption, the covering is not trivial, since $\tilde{X}_0$ is not compact but X_0 is compact. Let $x_0 \in X_0$ be a base point of X_0. If $\tilde{x}, \tilde{x}' \in \tilde{X}_0 \cap q^{-1}(x_0)$, then, by the definition of q, $\tilde{x}' = \tau^m \tilde{x}$ for some integer m.

Next, we shall show that this τ^m is a covering transformation of the covering: $\tilde{X}_0 \to X_0$. Let $\tilde{y}$ be any point in $\tilde{X}_0$. Since $\tilde{X}_0$ is connected, there is an arc C with the ends $\tilde{x}$ and $\tilde{y}$. $\tau^m C$ is an arc in $q^{-1}X_0$ with the ends $\tilde{x}'$ and $\tau^m\tilde{y}$. Since $\tilde{x}' \in \tilde{X}_0$ and $\tilde{X}_0$ is connected, we have $\tau^m\tilde{y} \in \tilde{X}_0$. Furthermore $q\tau^m y = q\tilde{y}$. Hence τ^m is a covering transformation of this covering.

Therefore $\tilde{X}_0 \to X_0$ is a regular covering.

Let i be the smallest positive integer such that $\tau^i\tilde{x} \in \tilde{X}_0$. We shall show that for any integr k, $\tau^{ki}\tilde{x} \in \tilde{X}_0$. Let C be an arc with the ends $\tilde{x}$ and $\tau^i\tilde{x}$. Then, by repeating the argument above, $\tau^{(k-1)i}C \subset \tilde{X}_0$ if $k>0$ and $\tau^{ki}C \subset \tilde{X}_0$ if $k<0$. Therefore $\tau^{ki}\tilde{x} \in \tilde{X}_0$.

Further we shall show that for any point $\tau^m\tilde{x}$ in the fibre over x_0, $m=ki$ for some integer k. Generally, we put $m=ki+h$, where k, h are integers and $0 \leqq h < i$. If $h \neq 0$, $\tau^h\tilde{x}$ must exist in $\tilde{X}_0$ as above; it is a contradiction to the property of i. Therefore $m=ki$.

Hence the transformation group of the regular covering is isomorphic to Z. This completes the proof of (3.4).

Let $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ be the suspensions of some C^∞-automorphisms (diffeomorphisms) on X and X' respectively. Suppose that there exists a topological equivalence of dynamical systems, $h: M' \to M$. Let $q: R \times X \to M$, $p: R \times X \to X$ and $\tilde{X}_0$ be the same as these in (3.1) and (3.2). And put $q_0 = q \mid \tilde{X}_0$.

Lemma (3.5). *Suppose that* $\tilde{X}_0$ *satisfies the condition of* (3.3), *so that* $q_0: \tilde{X}_0 \to h(X')$ *is a homeomorphism. Then,* $\pi_1(X') \cong \pi_1(X)$ *by* $i_* = (pq_0^{-1}h \mid X')_*$.

Proof. Since $\tilde{X}_0$ is homeomorphic to X', $p \mid \tilde{X}_0: \tilde{X}_0 \to X$ is a covering by (3.2). p induces the injection $p_*: \pi_1(\tilde{X}_0) \to \pi_1(X)$. Hence, if $j': X' \to M'$ denotes the including mapping, the composition $i = pq_0^{-1}hj': X' \to X$ induces an inclusion

$$i_*: \pi_1(X') \to \pi_1(X).$$

Let $j: X \to M$ denote the inclusion. We shall prove that the following diagram is commutative.

$$\begin{array}{ccc} \pi_1(X') & \xrightarrow{j'_*} & \pi_1(M') \\ i_* \downarrow & & \downarrow h_* \\ \pi_1(X) & \xrightarrow{j_*} & \pi_1(M) \end{array}$$

There is a homotopy

$$F_t: X' \to R \times X$$

such that

$$F_0 = q_0^{-1}hj'$$
$$F_1 = pq_0^{-1}hj'.$$

In fact, F_t can be made easily by sliding $q_0^{-1}h(X')$ along the flow ψ_t of $R\times X$ onto X. Then,

$$G_t = qF_t : X' \to M$$

is a homotopy such that

$$G_0 = hj'$$
$$G_1 = jpq_0^{-1}hj' = ji\,.$$

Therefore the diagram above is commutative.

We shall construct a smooth fibering $M\to S^1$ with fibre X from $(M, \mathcal{F}; X)$ (*Cf.* §5). Recall that $M=R\times X/\tau$, as in §2 and that $S^1=R/t\sim t+1$. The mapping: $R\times X\to R$ defined by $(t, x)\to t\,(t\in R,\ x\in X)$ induces a mapping: $R\times X/\tau\to R/t\sim t+1$. This is a fibre map: $M\to S^1$.

In the same way, we construct a fibering: $M'\to S^1$ from $(M', \mathcal{F}'; X')$. Since the fibres X and X' are connected, we have the following commutative diagram, where the horizontal and vertical sequences are exact.

$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & \downarrow & & & & & & \\ 1 & \longrightarrow & \pi_1(X') & \xrightarrow{j'_*} & \pi_1(M') & \longrightarrow & Z & \longrightarrow & 1 \\ & & {\scriptstyle i_*}\downarrow & & {\scriptstyle h_*}\downarrow & & & & \\ 1 & \longrightarrow & \pi_1(X) & \xrightarrow{j_*} & \pi_1(M) & \longrightarrow & Z & \longrightarrow & 1 \end{array}$$

Then,

$$Z \cong \pi_1(M)/\pi_1(X)$$
$$\cong (\pi_1(M)/j_*i_*\pi_1(X'))/(\pi_1(X)/i_*\pi_1(X'))\,.$$

But,

$$\pi_1(M)/j_*i_*\pi_1(X') \cong \pi_1(M')/\pi_1(X') \cong Z\,,$$

whence,

$$Z \cong Z/(\pi_1(X)/i_*\pi_1(X'))\,.$$

Hence it is necessary that

$$\pi_1(X)/i_*\pi_1(X') \cong 1\,.$$

Therefore $\pi_1(X')\cong\pi_1(X)$.

This completes the proof of (3.5).

4. Cross-section theorem

The purpose of this section is to prove

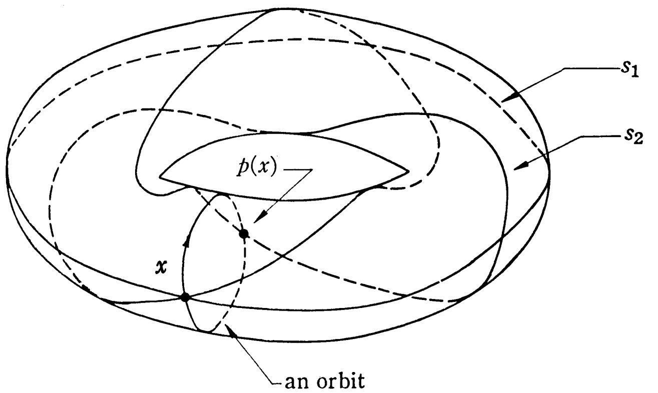
Theorem (4.1). *Let $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ be dynamical systems with cross-sections. (Manifolds may have boundaries.) Suppose that there exists no* *projection of the fundamental group $\pi_1(X)$ of X onto the infinite cyclic group Z.*

Then $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are differentiably (topologically) equivalent if and only if the associated diffeomorphisms of $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ are differentiably (topologically) conjugate.

REMARK (4.2). The assumption in the theorem about the fundamental group is necessary. This is verified by the following.

Let $T=S_0\times S_1$ be a torus, where S_j is a 1-dimensional circle, and let $\mathcal{F}$ be the dynamical system on T such that any orbit of $\mathcal{F}$ is $S_0\times x\subset T$ for $x\in S_1$. Then $S_1=*\times S_1$ in T $(*\in S_0)$ is a cross-section of $(T, \mathcal{F})$ and the associated diffeomorphism of $(T, \mathcal{F}; S_1)$ is the identity map i: $S_1\rightarrow S_1$.

Next, we imbed a circle in T in such a way that the imbedded image S_2 is a cross-section of $(T, \mathcal{F})$ and the associated diffeomorphism of $(T, \mathcal{F}; S_2)$ is the antipodal map p: $S_2\rightarrow S_2$. This is possible. In fact, the imbedded image S_2 is a clover-knot, if we consider T to be located 3-dimensional euclidean space (see the figure).



Then S_1 and S_2 are two cross-sections of the same (therefore equivalent) dynamical system $(T, \mathcal{F})$, but the associated diffeomorphisms i and g cannot be conjugate. In this case the fundamental groups of the cross-sections are isomorphic to Z.

Corollary (4.3). *Let $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ be dynamical systems with cross-sections. Suppose that the fundamental group $\pi_1(X)$ of X or the 1-dimensional homology group $H_1(X)$ is a finite group.*

Then $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are differentiably (topologically) equivalent if and only if the associated diffeomorphisms of $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ are differentiably (topologically) conjugate.

Proof. This corollary is a direct consequence of (4.1).

In fact, if G is a finite group, there is no projection of G onto Z. If there is a projection of $\pi_1(X)$ onto Z, the projection induces a projection of $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ onto Z, where $[\pi_1(X), \pi_1(X)]$ denotes the commutator subgroup. And $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ is isomorphic to $H_1(X)$.

Corollary (4.4). *Assume that for* $(M_1, \mathcal{F}_1; X_1)$ *and* $(M_2, \mathcal{F}_2; X_2)$, $(M_1, \mathcal{F}_1)$ *and* $(M_2, \mathcal{F}_2)$ *are differentiably (topologically) equivalent and that* M_1 *and* M_2 *are closed (so that* X_1 *and* X_2 *are closed). Suppose that one of the following two conditions is satisfied.*

i) *There exists no projection of* $\pi_1(X_1)$ *onto* Z.

ii) *There exists* $(\tilde{M}_i, \tilde{\mathcal{F}}_i; \tilde{X}_i)$ (i=1, 2) *satisfying the following conditions.*

(a) $\partial\tilde{M}_i=M_i$, $\partial\tilde{X}_i=X_i$.

(b) $\tilde{\mathcal{F}}_i$ *is an extention of* $\mathcal{F}_i$.

(c) $(\tilde{M}_1, \tilde{\mathcal{F}}_1)$ *and* $(\tilde{M}_2, \tilde{\mathcal{F}}_2)$ *are differentiably (topologically) equivalent.*

(d) *There exists no projection of* $\pi_1(\tilde{X}_1)$ *onto* Z.

Then the associated diffeomorphisms of $(M_1, \mathcal{F}_1; X_1)$ *and* $(M_2, \mathcal{F}_2; X_2)$ *are differentiably (topologically) conjugate.*

Proof of Theorem (4.1). If the associated diffeomorphisms of $(M, \mathcal{F}; X)$ and $(M', \mathcal{F}'; X')$ are differentiably (topologically) conjugate, (2.2) and (2.3) imply that $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are differentiably (topologically) equivalent.

Next, we shall prove the converse of this, assuming that there is no projection of $\pi_1(X')$ onto Z. The proof consists of two parts; the first, to fined a diffeomorphism (homeomorphism) of X' onto X, and the second, to prove that the diffeomorphism (homeomorphism) satisfes the condition of conjugacy.

Let $h: M'\to M$ be the differentiable (topological) equivalence of $(M, \mathcal{F})$ and $(M', \mathcal{F}')$. Here, by (2.1) and (2.2) we may regard $(M, \mathcal{F}; X)$ as $(M_0, \mathcal{F}_0; X)$, the suspension of the associated diffeomorphism of $(M, \mathcal{F}; X)$, and may regard $(M', \mathcal{F}'; X')$ similarly.

Part 1. Let $q: R\times X\to M$ be the suspending projection as in §2. Then, by (3.1), $q: \tilde{X}_0\to X_0$ is a covering map, where $X_0=h(X')$ and $\tilde{X}_0$ is any connected component of $q^{-1}(X_0)$.

Suppose further that for any $t_0>0$ there is a t such that $|t|>t_0$ and $(t\times X)\cap\tilde{X}_0\neq\phi$. Then by (3.4) this covering $\tilde{X}_0\to X_0$ is a regular covering with the transformation group isomorphic to Z and we have an exact sequence

$$1\longrightarrow\pi_1(\tilde{X}_0)\longrightarrow\pi_1(X_0)\longrightarrow Z\longrightarrow 1\,.$$

This is a contradiction to the assumption of $\pi_1(X')\simeq\pi_1(X_0)$. Therefore, by (3.3), any component $\tilde{X}_0$ of $q^{-1}(X_0)$ is diffeomorphic (homeomorphic if h is a homeomorphism) to X_0.

Therefore $\tilde{X}_0$ is a compact differentiable (topological) submanifold of $R\times X$. Hence, by (3.2), $p: \tilde{X}_0\to X$ is a differentiable (topological) covering, where p is the natural projection $R\times X\to X$. Since $\tilde{X}_0$ is diffeomorphic (homeomorphic) to X', we get a differentiable (topological) covering map $i=pq_0^{-1}h\colon X'\to X$, where $q_0=p\,|\,\tilde{X}_0$. Therefore, since i_* is an isomorphism of $\pi_1(X')$ onto $\pi_1(X)$ by (3.5), $pq_0^{-1}h$ is a diffeomorphism (homeomorphism if h is a topological equivalence): $X'\to X$.

Part 2. Since any component of $q^{-1}(X_0)$ is homeomorphic to X_0 by q, any component of $q^{-1}(X_0)$ is $\tau^i\tilde{X}_0$ for some integer i, where $\tilde{X}_0$ is a fixed component and τ is defined, as above, by

$$\tau(t,\, x) = (t+1,\, f^{-1}(x))\,, \qquad t\in R,\ x\in X\,.$$

In order to prove Part 2, we shall prove the following lemma.

Lemma (4.5). *Suppose that $\tau^i\tilde{X}_0$ is homeomorphic to X by the map p and that $(s,\, x)\in\tau^i\tilde{X}_0$, $(t,\, x)\in\tau^j\tilde{X}_0$. Then, $s<t$ if and only if $i<j$.*

Proof. Since $t=s$ whenever $i=j$, it is sufficient if we can prove that $i<j$ implies $s<t$.

Here, we suppose that there exist integers i, j, real numbers t, $s\in R$ and a point $x\in X$ satisfying the conditions of the lemma, such that $s>t$ and $i<j$.

Since $p\colon \tau^i\tilde{X}_0\to X$ is homeomorphism, $\tau^i\tilde{X}_0$ splits $R\times X$ for any i; that is, if $(s_1,\, x_1)$, $(s_2,\, x_2)\in\tau^i\tilde{X}_0$ and $(t_1,\, x_1)\in\tau^j\tilde{X}_0$, and if $s_1<t_1$, then there is $t_2\in R$ such that $(t_2,\, x_2)\in\tau^j\tilde{X}_0$ and $s_2<t_2$.

By assumption we have

$$(s+j-i,\, f^{i-j}(x)) = \tau^{j-i}(s,\, x)\in\tau^j\tilde{X}_0\,, \qquad j-i>0\,.$$

By the splitting property and by $s>t$, there is a $s_1\in R$ such that

$$(s_1,\, f^{i-j}(x))\in\tau^i\tilde{X}_0\,, \qquad s_1>s+j-i\,.$$

By repeating this process we get $s_r\in R$ for all r such that

$$(s_r,\, f^{r(i-j)}(x))\in\tau^i\tilde{X}_0\,, \qquad s_r>s+r(j-i)\,.$$

But this is in contradiction to the fact that $\tau^i\tilde{X}_0$ is compact. This completes the proof of the lemma.

Now, we come back to the proof of Part 2.

Let $(t,\, x)\in\tilde{X}_0\subset R\times X$. The orbit of $R\times X$ passing through $(t,\, x)\in\tilde{X}_0$ meets $q^{-1}(X_0)$ at $(s,\, x)\in\tau\tilde{X}_0$ for the first time. Define a diffeomorphism (homeomorphism) $g\colon \tilde{X}_0\to\tau\tilde{X}_0$ by $g(t,\, x)=(s,\, x)$.

Now we can easily see, by (4.5), that the following diagram is comutative.

$$\begin{array}{ccccccc}
X' & \xrightarrow{h} & X_0 & \xrightarrow{q_0^{-1}} & \tilde{X}_0 & \xrightarrow{p} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \tau^{-1}g} & & \downarrow{\scriptstyle f} \\
X' & \xrightarrow{h} & X_0 & \xrightarrow{q_0^{-1}} & \tilde{X}_0 & \xrightarrow{p} & X
\end{array}$$

Here $f_0: X \to X_0$ is a diffeomorphism (homeomorphism) defined as follows; if $x \in X_0$, $f_0(x)$ is the point where the orbit of $(M, \mathcal{F})$ passing through x meet X_0 for the first time.

The commutativity of the diagram implies that f and f' are differentiably or topologically conjugate, according that h is differentiable or topological. This completes the proof of Part 2.

Therefore the proof of Theorem (4.1) is completed.

We deduce the following corollary directly from (4.1).

Corollary (4.6). *Let X and X' be two cross-sections of a dynamical system. If there exists no projection of $\pi_1(X)$ onto Z, then X and X' are diffeomorphic.*

Remark (4.2) is not applicable to this corollary. In a paper in preparation[1], we will show examples of dynamical systems with two cross-sections which are not diffeomorphic or homeomorphic.

5. Relation with fibre bundles

All fibre bundles considered in this section are assumed to be smooth and to have the base spase S^1 and the group $\mathrm{Diff}(X)$, in the sense of [10], where $\mathrm{Diff}(X)$ is the group of diffeomorphisms of the fibre X onto itself. We consider the discrete topology in $\mathrm{Diff}(X)$.

Let X_f be the *mapping torus of* f defined by $X_f = I \times X$ with identification $(0, f(x)) = (1, x)$ for all $x \in X$. If f is a diffeomorphism, X_f is a smooth manifold. For any diffeomorphism f of X onto itself, let ξ_f be the fibre bundle $p: X_f \to I/0 \sim 1 = S^1$ defined by $p(t, x) = t$. Moreover,

(5.1) *for any fibre bundle ξ with fibre X, there exists a diffeomorphism f over X such that ξ_f is equivalent to ξ in $\mathrm{Diff}(X)$, in the sence of* [10].

Let $\mathrm{Hom}(X)$ be the group of homeomorphism of X onto itself. We consider the discrete topology in it. Let $[\xi]_d$ and $[\xi]_t$ be the equivalence classes in $\mathrm{Diff}(X)$ and $\mathrm{Hom}(X)$ respectively containing the fibre bundle ξ. Then,

(5.2) $[\xi_f]_d = [\xi_g]_d$ $([\xi_f]_t = [\xi_f]_t)$ *if and only if f and g are differentiably (togologically) conjugate.*

Let $[(M, \mathcal{F})]_d$ be the differentiable equivalence class containing a dynamical system $(M, \mathcal{F})$, and $[(M, \mathcal{F})]_t$ the topological equivalence class having the same property. And set

$$\mathcal{F}_d(X) = \{[(M, \mathcal{F})]_d\,;\ (M, \mathcal{F}) \text{ has a cross-section } X\}\,,$$
$$\mathcal{F}_t(X) = \{[(M, \mathcal{F})]_t\,;\ \text{———}\ \}\,,$$
$$\boldsymbol{F}_d(X) = \{[\xi]_d\,;\ \xi \text{ has a fibre } X\}\,,$$
$$\boldsymbol{F}_t(X) = \{[\xi]_t\,;\ \text{———}\ \}\,.$$

1) G. Ikegami: *On dynamical systems with cross-sections*, (to appear).

We define a map η_d: $\boldsymbol{F}_d(X) \to \mathcal{F}_d(X)$ (similarly for η_t: $\boldsymbol{F}_t(X) \to \mathcal{F}_t(X)$) as follows; Let $[\xi]_d \in \boldsymbol{F}_d(X)$, then by (5.1) there is f with $\xi_f \in [\xi]_d$. We define $\eta_d[\xi]_d$ by the differentiable equivalence class of the dynamical system wich is the suspension of f. Here, η_d is well defined; in fact by (5.2) and (2.3), η_d is independent of the selection of f.

In order to define the mapping μ_d: $\mathcal{F}_d(X) \to \boldsymbol{F}_d(X)$ (similarly for μ_t: $\mathcal{F}_t(X) \to \boldsymbol{F}_t(X)$), we must suppose that there is no projection of $\pi_1(X)$ onto Z. In this case, for any $[(M, \mathcal{F})]_d \in \mathcal{F}_d(X)$, $\mu_d[(M, \mathcal{F})]_d$ is defined by $[\xi_f]_d \in \boldsymbol{F}_d(X)$, where f is the associated diffeomorphism of $(M, \mathcal{F}; X)$. μ_d is well defined; in fact, by (4.1) and (5.2), μ_d is independent of the selection of $(M, \mathcal{F})$ and X.

Here, $\mu_d \eta_d$=identity ($\mu_t \eta_t$=identity) by (2.1) and $\eta_d \mu_d$=identity ($\eta_t \mu_t$= identity) by (2.2).

Therefore we have proved the next theorem;

Theorem (5.3). *If there exist no projection of $\pi_1(X)$ onto Z, then there is a natural one-to-one correspondence between $\mathcal{F}_d(X)$ and $\boldsymbol{F}_d(X)$ ($\mathcal{F}_t(X)$ and $\boldsymbol{F}_t(X)$).*

6. Dynamical systems on tori with cross-section

The purpose of this section is to prove Theorem (6.6).

Define the two fibre maps,

$$p_+ : S^n \times S^1 \to S^1 \quad \text{by} \quad p_+(x, y) = y\,,$$
$$p_- : S^n \times S^1 \to S^1 \quad \text{by} \quad p_-(x, y) = r(y)\,,$$

for any $x \in S^n$, $y \in S^1$, where r: $S^1 \to S^1$ denotes an arbitraly given homeomorphism of order -1.

The following lemma is concerned only with topology.

Lemma (6.1). *If $n \geqq 2$, any fibre map p: $S^n \times S^1 \to S^1$ with connected fibre is homotopic either to p_+ or to p_-.*

Proof. Denote by $\pi(X; Y)$ the set of homotopy classes of the maps: $X \to Y$ with fixed base points.

First, we shall show $\pi(S^n \times S^1; S^1) \cong Z$. Let i: $S^n \vee S^1 \to S^n \times S^1$ be the natural injection, where $\vee$ denotes the union in identifying the base points, and let h: $S^n \times S^1 \to (S^n \times S^1)/(S^n \vee S^1) = S^{n+1}$ be the natural projection. We have an exact sequence (Puppe sequence);

$$\pi(S^{n+1}; S^1) \xrightarrow{h^*} \pi(S^n \times S^1; S^1) \xrightarrow{i^*} \pi(S^n \vee S^1; S^1)\,.$$

Here,

$$\pi(S^{n+1}; S^1) \cong 0$$

and

$$\pi(S^n \vee S^1;\, S^1) \cong \pi(S^n;\, S^1) + \pi(S^1;\, S^1) \cong Z\,.$$

Moreover, we can easily see that i^* is a projection. Therefore,

$$\pi(S^n \times S^1;\, S^1) \cong \pi_1(S^1) \cong Z\,.$$

Let this isomorphism $\pi(S^n \times S^1;\, S^1) \to \pi_1(S^1)$ be denoted by the same i^*.

Next, we shall show that, for a fibre map p with connected fibre F, $i^*([p])$ is a generator of $\pi_1(S^1)$. Where $[p]$ denotes the element of $\pi(S^n \times S^1;\, S^1)$ containing p.

Let

$$\longrightarrow \pi_1(S^n \times S^1) \xrightarrow{p_*} \pi_1(S^1) \longrightarrow \pi_0(F)$$

be the homotopy exact sequence of the fibre space. Since $\pi_0(F)=0$ and since $\pi_1(S^n \times S^1)$ and $\pi_1(S^1)$ are isomorphic to Z, p_* is an isomorphism. If $j\colon S^1 \to S^n \times S^1$ is the natural inclusion, $j_*\colon \pi_1(S^1) \to \pi_1(S^n \times S^1)$ is an isomorphism. And if 1 is the generator of $\pi_1(S^1)$ preserving the orientation of S^1, we have

$$p_* j_*(1) = i^*([p])\,.$$

Since $p_* j_*(1)$ is q generator of $\pi_1(S^1)$, $i^*([p])=\pm 1$. But, $i^*([p_+])=1$ and $i^*([p_-])=-1$. Therefore $[p]=[p_+]$ or $[p_-]$.

This completes the proof of (6.1).

Let $\mathrm{Diff}_+(S^n)$ and $\mathrm{Diff}_+(D^{n+1})$ denote the groups of orientation preserving diffeomorphisms on S^n and on a disk D^{n+1} resp., and let $r\colon \mathrm{Diff}_+(D^{n+1}) \to \mathrm{Diff}_+(S^n)$ denote the homomorphism obtained by the restriction. Then, the group $\mathscr{D}(S^n)=\mathrm{Diff}_+(S^n)/\mathrm{Image}\, r$ is isomorphic to Γ^{n+1} ([4]). Here Γ^{n+1} denotes the group of differentiable structures on S^{n+1} with usual $p.1.$ structure under the connected sum operation $\#$. $\mathscr{D}(S^n)$ is an abelian group [6]. If $n \geqq 4$ or $n=1$, Γ^{n+1} is the same as θ^{n+1}, which is by definition the group of homotopy $(n+1)$-sphres ([3]).

Let $\mathcal{S}(S^n \times S^1)$ denote the set of all differentiable manifolds homeomorphic to $S^n \times S^1$ classified by diffeomorphisms and let

$$\psi:\ \Gamma^{n+1} \longrightarrow \mathcal{S}(S^n \times S^1)$$

be the mapping defined by $\Psi(\tilde{S}^{n+1})=S^n \times S^1 \# \tilde{S}^{n+1}$, $\tilde{S}^{n+1} \in \Gamma^{n+1}$. Next, define a mapping

$$\Phi:\ \mathrm{Diff}_+(S^n) \longrightarrow \mathcal{S}(S^n \times S^1)$$

by $\Phi(f)=S^n_f$ for any $f \in \mathrm{Diff}_+(S^n)$.

Lemma (6.2). *Φ induces a one-to-one correspondence $\tilde{\Phi}\colon \mathscr{D}(S^n) \to \Psi(\Gamma^{n+1})$. Moreover, if $[f] \in \mathscr{D}(S^n)$ then $\tilde{\Phi}([f])=S^n \times S^1 \# \tilde{S}^{n+1}$, where $\tilde{S}^{n+1} \in \Gamma^{n+1}$ is the* *element corresponding to* $[f]$ *under the isomorphism of* $\mathcal{D}(S^n)$ *with* Γ^{n+1}.

Proof. For any $\tilde{S}^{n+1}$ in Γ^{n+1}, $S^n\times S^1 \# \tilde{S}^{n+1}$ is diffeomorphic to S^n_f, where f is any diffeomorphism in the element of $\mathcal{D}(S^n)$ corresponding to $\tilde{S}^{n+1}\in\Gamma^{n+1}$ under the isomorphism. (See [1], Lemma 1.) Therefore $\tilde{\Phi}$ can be well-defined and $\tilde{\Phi}$ maps $\mathcal{D}(S^n)$ onto $\Psi(\Gamma^{n+1})$.

Moreover, for any $\tilde{S}^{n+1}_1$, $\tilde{S}^{n+1}_2\in\Gamma^{n+1}$ with $\tilde{S}^{n+1}_1 \neq \tilde{S}^{n+1}_2$,

$$S^n\times S^1 \# \tilde{S}^{n+1}_1 \neq S^n\times S^1 \# \tilde{S}^{n+1}_2 \qquad \text{in} \quad \mathcal{S}(S^n\times S^1)\,,$$

because, if

$$S^n\times S^1 \# \tilde{S}^{n+1}_1 = S^n\times S^1 \# \tilde{S}^{n+1}_2 \qquad \text{in} \quad \mathcal{S}(S^n\times S^1)\,,$$

then

$$S^n\times S^1 \# \tilde{S}^{n+1}_1 \# (-\tilde{S}^{n+1}_2) = S^n\times S^1$$

by using an orientation-preserving diffeomorphism. But, the innertia group of $S^n\times S^1$: $I(S^n\times S^1)=\{\tilde{S}^{n+1}\in\Gamma^{n+1};\ S^n\times S^1\#\tilde{S}^{n+1}=S^n\times S^1\}$ is equal to 0 for all n (see [11], [2], [6])[1]. Hence, $\tilde{S}^{n+1}_1\#(-\tilde{S}^{n+1}_2)=0$ in Γ^{n+1}. This implies $\tilde{S}^{n+1}_1$ is diffeomorphic to $\tilde{S}^{n+1}_2$. Therefore $\tilde{\Phi}$ is an injection.

These prove the lemma.

Proposition (6.3). *If* f, g *Diff*$_+(S^n)$ *are differentiably conjugate*[2], *then* f *and* g *are contained in the same element of* $\mathcal{D}(S^n)$[3].

Proof. If f and g are differentiably conjugate, S^n_f and S^n_g are diffeomorphic. Then, (6.2) implies $[f]=[g]$ in $\mathcal{D}(S^n)$.

We denote by $\mathcal{C}(\tilde{S}^{n+1})$ the set of differentiably conjugate classes of diffeomorphisms contained in the element of $\mathcal{D}(S^n)$ corresponding to $\tilde{S}^{n+1}\in\Gamma^{n+1}$.

The following property is due to W. Browder ([1], Lemma 2).

(6.4) *Let mapping tori* X_f *and* Y_g *be the total spaces of differentiable fibre bundles over* S^1 *with projection* p *and* q, *and with fibres* X^n *and* Y^n *which are* 1-*connected closed manifolds of dimeksion* $n\geqq 5$. *If* $h\colon X_f\to Y_g$ *is a diffeomorphism such that* qh *is homotopic to* p, *then there is a diffeomorphism* h' *such that* $qh'=p$, *so that* h' *restricts to a diffeomorphism of* X *with* Y.

If $M^{n+1}\in\mathcal{S}(S^n\times S^1)$ for $n\geqq 5$, any smooth fibre bundle over S^1 with total space M^{n+1} and with connected fibre has fibre with the homotopy groups of sphere, which is homeomorphic to S^n by [7] or [5], p. 109, Prop. B. The following lemma shows a condition for fibre to be diffeomorphic to S^n.

---

1) If $n\geqq 4$, $I(S^n\times S^1)=0$ by the method in [11]; and if $n>4$, $I(S^n\times S^1)=0$, since $\Gamma^{n+1}=0$ by [2] for $n=3$ and by [6] for $n\leqq 2$.

2) If f and g are conjugate, there is a $h\in\mathrm{Diff}(S^n)$ with $hf=gh$. But we should notice that the definition of conjugacy in § 2 does not imply that h preserves orientation. If h preserves orientation, (6.3) is trivial. Because, we can consider $h\in\mathcal{D}(S^n)$, and the group $\mathcal{D}(S^n)$ is abelian.

3) By using J. Cerf's theorem to (6.4) we have; *If* $f, g\in\mathrm{Diff}_+(S^n)$ *are differentiably conjugate, and* $n\geqq 8$, *then* f *and* g *are isotogic.*

Lemma (6.5). *Suppose M^{n+1} is in $\Psi(\Gamma^{n+1})$, $n\geqq 5$ or $n=2$, then the fibre of any smooth fibre bundle over S^1 with total space M and with connected fibre is diffeomorphic to S^n.*

Proof. By (6.2) there is $f\in \mathrm{Diff}_+(S^n)$ such that S^n_f is diffeomorphic to M^{n+1}. Let $p_+: S^n_f\to S^1$ be the fibre bundle defined by $p(t, x)=t$, as in §5. Put $p_-=rp_+$, where $r: S^1\to S^1$ is a diffeomorphism with degree -1. And let $q: M^{n+1}\to S^1$ be any smooth fibre bundle with connected fibre. q is homotopic either to p_+ or to p_- by (6.1). Hence, if $n\geqq 5$, (6.4) implies that the fibre of q is diffeomorphic to the fibre S^n of p_+ or of p_-. If $n=2$, the Lemma is trivial, since $\theta^2=\Gamma^2=0$.

Theorem (6.6). *If M^{n+1} ($n\geqq 5$ or $n=2$) is diffeomorphic to $S^n\times S^1\# \tilde{S}^{n+1}$ for some $\tilde{S}^{n+1}$ in Γ^{n+1}, then the differentiable equivalence classes of dynamical systems on M with cross-sections have a one-to-one correspondence with $C(\tilde{S}^{n+1})$. The class corresponding to $f\in C(\tilde{S}^{n+1})$ is the suspension of f.*

Proof. Let $(M, \mathcal{F}; X)$ denote any dynamical system on M^{n+1} with cross-section. M is diffeomorphic to X_f, where f is the associated diffeomorphism of $(M, \mathcal{F}; X)$. Since there is a smooth fibre bundle $X_f\to S^1$, (6.5) implies that X is diffeomorphic to S^n. Hence, any differentiable equivalence class of dynamical systems on M with cross-sections is in $\mathcal{F}_d(S^n)$, for which (5.3) is a one-to-one correspondence to $\boldsymbol{F}_d(S^n)$. And, by (5.1) and (5.2) there exists a one-to-one correspondence between the differentiably conjugate classes of diffeomorphisms on S^n and $\boldsymbol{F}_d(S^n)$. Therefore (6.2) and (6.3) imply that the differentiable equivalence classes of dynamical systems on M^{n+1} correspond to $C(\tilde{S}^{n+1})$. This completes the proof of the theorem.

KOBE UNIVERSITY

References

[1] W. Browber: *Diffeomorphisms of 1-connected manifolds*, Trans. Amer. Math. Soc. **128** (1967), 155–163.

[2] J. Ceff: *La nullité de π_0(Diff S^3), 1: Position du probléme,* Séminaire H. Cartan: Topologie differentielle, t. 15, 1962/63, n^0 9, 27 p.

[3] M. Kervaire and J. Milnor: *Groups of homotopy spheres*: I, Ann. of Math (2) **77** (1963), 504–537.

[4] J. Milnor: *On manifolds homeomorphic to the 7-sphere,* Ann. of Math. (2) **64** (1956), 399–405.

[5] J. Milnor: Lectures on the *h*-cobordism Theorem, Preliminary informal notes, Princeton Univ. Press, 1965.

[6] J. Munkres: *Obstructions to the smoothing of piecewise-differentiable homeomorphisms,* Ann. of Math. (2) **72** (1960), 521–554.

[7] S. Smale: *Generalized Poincaré conjecture in dimension greater than four,* Ann.

of Math. (2) **74** (1961), 391–406.

[8] S. Smale: *Stable manifolds for differential equations and diffeomorphisms*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 97–116.

[9] S. Smale: *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[10] N. Steenrod: The Topology of Fibre Bundles, Princeton Math. Series 14, Princeton Univ. Press, 1961.

[11] I. Tamura: *Classification des variétés différentiables*, $(n-1)$*-connexes, ans torsion, de dimension* $2n-1$, Séminaire H. Cartan: Topologie differentielle, t. 15 1962/63, *n*^{0} 16, 27 p.