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Yokogawa, Kenji

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Osaka University
AN APPLICATION OF THE THEORY OF DESCENT
TO THE $S \otimes_R S$-MODULE STRUCTURE OF
$S/R$-AZUMAYA ALGEBRAS

KENJI YOKOGAWA

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Introduction. Let $R$ be a commutative ring and $S$ a commutative $R$-algebra which is a finitely generated faithful projective $R$-module. An $R$-Azumaya algebra $A$ is called an $S/R$-Azumaya algebra if $A$ contains $S$ as a maximal commutative subalgebra and is left $S$-projective. $S$-$S$-bimodule structure (for which we shall call $S \otimes_R S$-module structure) of $S/R$-Azumaya algebras is determined in [5] when $S/R$ is a separable Galois extension and in [8] when $S/R$ is a Hopf Galois extension, both are connected with one which is so called seven terms exact sequence due to Chase, Harrison and Rosenberg [3].

In this paper we shall investigate the $S \otimes_R S$-module structure of $S/R$-Azumaya algebras assuming only that $S$ is a finitely generated faithful projective $R$-module. So $S/R$-Azumaya algebras are not necessarily $S \otimes_R S$-projective (c.f. [8] Th. 2.1). But in §1 we shall show for any $S/R$-Azumaya algebra $A$, there exists a unique finitely generated projective $S \otimes_R S$-module $P$ of rank one with certain cohomological properties such that $A$ is $S \otimes_R S$-isomorphic to $P \otimes_{S \otimes_R S} \text{End}_R(S)$. In §2, we shall investigate $S/R$-Azumaya algebras resulting from Amitsur’s 2-cocycles. Finally we shall deal with the seven terms exact sequence in §3.

Throughout $R$ will be a fixed commutative ring with unit, a commutative $R$-algebra $S$ is a finitely generated faithful projective as $R$-module, each $\otimes$, $\text{End}$, etc. is taken over $R$ unless otherwise stated. Repeated tensor products of $S$ are denoted by exponents, $S^q=S \otimes \cdots \otimes S$ with $q$-factors. We shall consider $S^1$ as an $S$-algebra on first term. To indicate module structure, we write if necessary, $S_1 \otimes S_2$ instead of $S^2=S \otimes S$, $S_1 \otimes M_2$ instead of $S^2=S_1 \otimes S_2$-module $M$ etc.. $H^q(S/R, U)$ and $H^q(S/R, \text{Pic})$ denote the $q$-th Amitsur’s cohomology groups of the extension $S/R$ with respect to the unit functor $U$ and Picard group functor $\text{Pic}$ respectively.

1. $S/R$-Azumaya algebras and $H^1(S/R, \text{Pic})$

First we prove the following, which clarify the $S^2$-module structure of
Lemma 1.1. Let $M$ be a finitely generated projective $S$-module of rank one, then $\text{End}(M)$ is isomorphic to $(M \otimes S) \otimes_S (S \otimes M^*) \otimes_S \text{End}(S)$ as $S^2$-modules, where $M^* = \text{Hom}_S(M, S)$.

Proof. We define $\psi: (M \otimes S) \otimes_S (S \otimes M^*) \otimes_S \text{End}(S) \rightarrow \text{End}(M)$ as follows:

$$\psi((m \otimes s) \otimes (t \otimes f) \otimes g)(n) = tg(f(sm))m$$

$m, n \in M, s, t \in S, f \in M^*, g \in \text{End}(S)$. Then $\psi$ is a well-defined $S^2$-homomorphism and by localization we get $\psi$ is an isomorphism.

Remark. By $\psi$, the multiplication of $(M \otimes S) \otimes_S (S \otimes M^*) \otimes_S \text{End}(S)$ is given by the formula

$$((m \otimes s) \otimes (t \otimes f) \otimes g) \cdot ((n \otimes u) \otimes (v \otimes p) \otimes q) = (m \otimes u) \otimes (t \otimes p) \otimes g \cdot f(n) \cdot s \cdot vq.$$

Now let $A$ be an $S/R$-Azumaya algebra then $A$ is split by $S$. Hence there exists a finitely generated faithful projective $S$-module $M$ such that $S \otimes A$ is isomorphic to $\text{End}_S(M)$ as $S$-algebras. As is well known, $M$ inherits the $S^2$-module structure and is $S^2$-projective of rank one. By Lemma 1.1, $S \otimes A \cong \text{End}_S(M) \cong (M \otimes_S S^2) \otimes_S (S^2 \otimes_M S^2) \otimes_S \text{End}_S(S^2) = (S^2_M \otimes_S S^2) \otimes_S (S^2_M \otimes_S S^2) \otimes_S \text{End}_S(S^2)$, where we regard $S^2$ (resp. $S^3$) as $S^2$-modules. Define the $S^2$-algebra isomorphism $\Phi: \text{End}_S(M) \cong \text{End}_S(S^2) = \text{End}_S(S^2_M \otimes_S S^2) \cong \text{End}_S(S^2_M \otimes_S S^2)$ by the composite of the isomorphisms $\text{End}_S(S^2_M \otimes_S S^2) \cong (S^2_M \otimes_S S^2) \cong (S^2_M \otimes_S S^2)$, where the middle isomorphism is the one induced from the twisting homomorphism $A \otimes S^2 \rightarrow A \otimes S^2$ (and the others are induced from $S \otimes A \cong \text{End}_S(M)$. Then from Morita theory there exists a finitely generated projective $S^2$-module $Q$ of rank one such that $(S^2_M \otimes_S S^2) \otimes_S \text{End}_S(S^2_M \otimes_S S^2)$ is $\text{End}_S(S^2_M \otimes_S S^2)$-modules, hence as $S^3$-modules. Tensoring with $S^2$ over $S^3$ (regarding $S^2$ as an $S^3$-module by $1 \otimes \mu: S^3 \rightarrow S^3$), we get an $S^2$-isomorphism

$$S^2 \otimes \text{End}_S(S^2_M \otimes_S S^2) \cong S^2 \otimes (S \otimes S^2).$$

Therefore,

$$S \otimes P = (S \otimes (M \otimes S) \otimes S) \otimes (S \otimes M^*) \cong (S \otimes (M \otimes S^2) \otimes S) \otimes (S \otimes M^*) \cong (M \otimes S) \otimes (S \otimes M^*)$$

split $S/R$-Azumaya algebras.
\[ \simeq (M \otimes S) \otimes s^3((M^* \otimes s^2 S) \otimes S^2) \otimes s^3(S^2 \otimes s M^*) \otimes s^3((M \otimes s^2 S) \otimes S^2) \]

\[ = (M \otimes S) \otimes s^3((M^* \otimes s^2 S) \otimes S^2) \otimes s^3(S^2 \otimes s(M^* \otimes s^2 S) \otimes s^3(S^2 \otimes s M^*)) \]

\[ = (P^* \otimes S) \otimes s^3(S^2 \otimes s P), P^* = \text{Hom}_S(P, S^3). \]

This means \( P \) is a 1-cocycle of the extension \( S/R \) with respect to the functor \( \text{Pic} \) (we call simply 1-cocycle). Since \( P^* = ((M^* \otimes S^2) \otimes S) \otimes s^3(M^* \otimes S^2 S) \otimes s^3(S^2 \otimes s M^*) \), \( P^* = \text{End}_S(M) \) as \( S \)-algebras.

If \( S \otimes A \simeq \text{End}_S(N) \) for another \( N \), then \( \text{End}_S(M) \simeq \text{End}_S(N) \) as \( S \)-algebras. So there exists a finitely generated projective \( S \)-module \( Q' \) of rank one such that \( s_1 M_{s_2} \otimes s_3 Q' \simeq N \) as \( S^2 \)-modules. Easy calculation shows that the 1-cocycles obtained from \( M \) and \( N \) are \( S^2 \)-isomorphic.

To prove the uniqueness of 1-cocycle \( P \), we prepare the following

**Lemma 1.2.** Let \( T \) be a commutative \( R \)-algebra, which is a finitely generated faithful projective \( R \)-module. And let \( P, Q \) be finitely generated projective \( T \)-modules of rank one. Then

\[ \text{Hom}_T(P \otimes Q, Q \otimes P) \simeq \text{Hom}_T(\text{End}(P), \text{End}(Q)) \]

Especially, \( \text{Iso}_{\otimes T}(P \otimes Q, Q \otimes P) \) corresponds to \( \text{Iso}_{\otimes T}(\text{End}(P), \text{Ecd}(Q)) \).

Proof. For any \( T \)-module \( M_i, N_i (i = 1, 2) \), we have the following isomorphism \( \rho: \text{Hom}_{\otimes T}(M_1 \otimes M_2, \text{Hom}(N_2, N_1)) \simeq \text{Hom}_{\otimes T}(M_1 \otimes N_2, \text{Hom}(M_2, N_1)) \) given by \( (\rho(\varphi)(m_1 \otimes n_2))(m_2) = (\varphi(m_1 \otimes m_2))(n_2), m_i \in M_i, n_i \in N_i, \varphi \in \text{Hom}_{\otimes T}(M_1 \otimes M_2, \text{Hom}(N_2, N_1)), \) ([6] I.4.2). Put \( M_1 = P, M_2 = Q, N_1 = \text{Hom}(P, R), \) then we get easily. Further assertion follows easily by localization.

Let \( P, P' \) be 1-cocycles such that \( P \otimes s^3 \text{End}(S) \simeq P' \otimes s^3 \text{End}(S) \simeq A \) as \( S^2 \)-modules. Then \( \text{End}_S(P^*) \simeq \text{End}_S(P'^*) \) as \( S^3 \)-modules by Lemma 1.1 and the cocycle condition of \( P, P' \). From Lemma 1.2 we get an \( S^3 \)-isomorphism \( P^* \otimes s^3 P'^* = (s_1 P^* s_2 \otimes S_3) \otimes s^3(s_1 P'^* s_3 \otimes S_2) \simeq P^* \otimes s^3 P'^* = (s_1 P^* s_2 \otimes S_3) \otimes s^3(s_1 P'^* s_3 \otimes S_2) \).

Thus \( s_1 P^* s_2 \otimes S_3) \otimes s^3(s_1 P^* s_3 \otimes S_2) \simeq (s_1 P'^* s_2 \otimes S_3) \otimes s^3(s_1 P'^* s_3 \otimes S_2) \), the left side is isomorphic to \( S_1 \otimes s^2 P^* s_3 \) and the right side is isomorphic to \( S_1 \otimes s^2 P'^* s_3 \) by the cocycle condition of \( P, P' \). Tensoring with \( S^2 \) over \( S^3 \) (regarding \( S^2 \) as an \( S^3 \)-module by \( \mu \otimes 1: S^3 \rightarrow S^3 \)), we get \( P \simeq P' \).

Summing up we get

**Theorem 1.3.** Let \( A \) be an \( S/R \)-Azumaya algebra, then there exists a unique 1-cocycle \( P \) such that \( A \) is isomorphic to \( P \otimes s^3 \text{End}(S) \) as \( S^2 \)-modules and \( S \otimes A \) is isomorphic to \( \text{End}_S(P^*) \) as \( S \)-algebras, where \( P^* = \text{Hom}_S(P, S^3) \).

**Remark.** In proving the above theorem, we used the \( S \)-algebra isomorphism
If we assume this isomorphism is only an $S^3$-module isomorphism, then by using Lemma 1.2 in suitable situations we shall get Theorem 1.3 only replacing "$S$-algebras" to "$S^3$-modules" in the last statement. So Theorem 1.3 does not fully characterize $S/R$-Azumaya algebras.

**Proposition 1.4.** Let $A, B$ be $S/R$-Azumaya algebras, $P, Q$ be 1-cocycles obtained from $A, B$ respectively. Then the 1-cocycle obtained from $A \cdot B = \text{End}_{A \otimes B}(S \otimes (A \otimes B))$ is $P \otimes Q$.

**Proof.** $S \otimes A = \text{End}_S(P*)$ and $S \otimes B = \text{End}_S(Q*)$, so $S \otimes (A \cdot B) = (S \otimes A) \cdot (S \otimes B) = \text{End}_S(P* \otimes Q*)$, (c.f. [3] 2.13.). Thus the 1-cocycle obtained from $A \cdot B$ equals to $P \otimes Q$.

Next we shall start from a 1-cocycle $P$ and an $S^3$-isomorphism $\phi: S^2 \otimes P = S^3 \otimes S_2 \otimes S_3 \simeq (S \otimes S_2 \otimes S_3) \otimes s^1(S_1 \otimes S_2 \otimes S_3)$, identity on $S_1$

$\phi_2: S_1^* \otimes S_2 \otimes S_3 \simeq (S_1 \otimes S_2 \otimes S_3) \otimes s^1(S_1 \otimes S_2 \otimes S_3)$

identity on $S_2$

$\phi_3: S_1 \otimes S_2 \otimes S_3 \simeq (S_1 \otimes S_2 \otimes S_3) \otimes s^1(S_1 \otimes S_2 \otimes S_3)$

identity on $S_3$.

Further we define $u(\phi) \in \text{End}_{s^1}(S_1 \otimes S_2 \otimes S_3)$ by the composite

$$
\begin{align*}
&\phi_3^{-1} \circ \phi_1 \circ (1 \otimes s^1) \circ \phi_2 \quad (S_1 \otimes S_2 \otimes S_3) \otimes s^1(S_1 \otimes S_2 \otimes S_3) \\
&\phi_3^{-1} \circ \phi_1 \circ (1 \otimes s^1) \circ \phi_2 \quad (S_1 \otimes S_2 \otimes S_3) \otimes s^1(S_1 \otimes S_2 \otimes S_3) \\
&\phi_3^{-1} \circ \phi_1 \circ (1 \otimes s^1) \circ \phi_2 \quad (S_1 \otimes S_2 \otimes S_3) \otimes s^1(S_1 \otimes S_2 \otimes S_3)
\end{align*}
$$

Then we may think $u(\phi)$ is a unit of $S^4$ by homothety. As easily checked, $u(\alpha \phi) = \delta(\alpha^{-1})u(\phi)$ for a unit $\alpha \in S^3$, where $\delta$ is the coboundary operator in Amitsur's complex with respect to the unit functor $U$.

**Lemma 1.5.** $u(\phi)$ is a 3-cocycle.

**Proof.** By localization it follows readily.

**Theorem 1.6.** Let $P$ be a 1-cocycle with an $S^3$-isomorphism $\phi: S_1^* \otimes S_2 \otimes S_3 \simeq (S_1 \otimes S_2 \otimes S_3) \otimes s^1(S_1 \otimes S_2 \otimes S_3)$. Then $A = P \otimes s^1 \text{End}(S)$ has an $S/R$-Azumaya algebra structure, if and only if, $u(\phi)$ is a coboundary. If $u(\phi) = \delta(\beta)$ where $\beta$ is a
unit of $S^3$, then $(\beta \phi)^*$ induces a $S$-algebra isomorphism $S \otimes A \cong \text{End}_S(P^*)$, where $(\beta \phi)^*$ is the isomorphism $S \otimes P \cong (P^* \otimes S) \otimes S'(s_1 P^* s_2 \otimes S_2)$ induced from $\beta \phi$.

Proof. First we assume $A = P \otimes S^2 \text{End}(S)$ is an $S/R$-Azumaya algebra, then $S \otimes A \cong \text{End}_S(P^*)$ as $S$-algebras from the uniqueness of 1-cocycle. Define the $S^2$-algebra isomorphism

$$\Phi : \text{End}_{S^2} (s_1 P^* s_2 \otimes S_2) = S_1 \otimes A \otimes S_2 \rightarrow S_1 \otimes S_2 \otimes A = \text{End}_{S^2} (s_1 \otimes s_2)^*$$

by the twisting homomorphism $A \otimes S_2 \rightarrow S_2 \otimes A$. $\Phi$ is a descent homomorphism, that is if we put $\Phi_1 = 1 \otimes \Phi : S_1 \otimes \text{End}_S(P^*) \otimes S \rightarrow S_1 \otimes S \otimes \text{End}_S(P^*)$ identity on $S_1$, $\Phi_2 : \text{End}_S(P^*) \otimes S_2 \otimes S \rightarrow S \otimes S_2 \otimes \text{End}_S(P^*)$ identity on $S_2$, $\Phi_3 = \Phi \otimes 1 : \text{End}_S(P^*) \otimes S \otimes S_2 \rightarrow S \otimes \text{End}_S(P^*) \otimes S_1$ identity on $S_1$, then $\Phi_2 = \Phi_1 \cdot \Phi_3$. Since $\Phi$ is an $S^2$-algebra isomorphism, there exists a finitely generated projective $S^2$-module $Q$ of rank one such that $s_1 P^* s_2 \otimes S_2$ is isomorphic to $(s_1 \otimes s_2 P^* s_2) \otimes s_1 \otimes s_2 s_1 Q_{s_2} = (s_1 \otimes s_2 P^* s_2) \otimes s_2'(s_2 Q_{s_2} \otimes S_2)$ as $S^2$-modules and $\Phi$ is induced by this isomorphism $\phi'$. From the cocycle condition of $P$, $Q$ is isomorphic to $P^*$. From the definition of $\Phi_1$, $\Phi_2$, $\Phi_3$, the following diagram is commutative for any $f \in \text{End}_{S^2} (s_1 \otimes s_2) (s_1 P^* s_2 \otimes S_3)$.

Thus $(1 \otimes s'(\phi' \otimes 1)) \cdot \phi' \cdot f \cdot \phi'^{-1} \cdot (1 \otimes s'(\phi'^{-1} \otimes 1)) = (\phi' \otimes s') \cdot f \cdot \phi'^{-1} \cdot (\phi'^{-1} \otimes s')$. Hence $f \cdot u(\phi') = u(\phi') \cdot f$ for any $f \in \text{End}_{S^2} (s_1 \otimes s_2) (s_1 P^* s_2 \otimes S_3)$. Therefore 3-cocycle $u(\phi')$ is contained in the center of $\text{End}_{S^2} (s_1 \otimes s_2) (s_1 P^* s_2 \otimes S_3)$, which is $S_1 \otimes S_2 \otimes S_3$. Easily we get $u(\phi')$ is a coboundary. Thus $u(\phi') = \ldots$
$u(\alpha^{-1}\phi') = \delta(\alpha)u(\phi')$ is a coboundary.

Conversely let $u(\phi)$ be a coboundary then we may assume $u(\phi) = 1 \otimes 1 \otimes 1 \otimes 1$. Let $\phi^*$ be the isomorphism $S \otimes P = (P^* \otimes S) \otimes S^*(S_1 \otimes P_2 \otimes S_3)$ induced from $\phi$ by duality pairing. We consider $S \otimes A = (S \otimes P) \otimes S^*(S^* \otimes S_3 \otimes P_2 \otimes S_3)$ equals $\text{End}_A(P^*) = (P^* \otimes S) \otimes S^*(S_1 \otimes P_2 \otimes S_3)$ by $\phi^* \otimes S$. Thus $S \otimes A$ has an $S$-algebra structure. Define $\Phi : S \otimes A \simeq S \otimes S \otimes A$ by the twisting homomorphism $A \otimes S \rightarrow S \otimes A$. Clearly $\Phi_2 = \Phi_1 \cdot \Phi_3$. From the theory of faithfully flat descent, if $\Phi$ is an $S^2$-algebra isomorphism, then the descended module $A$ has an $R$-algebra structure (necessarily an $S/R$-Azumaya algebra structure) such that the induced $S$-algebra structure of $S \otimes A$ coincides the original one of $S \otimes A$. Therefore all is settled if we show $\Phi$ is an $S^2$-algebra homomorphism. So we may assume $R$ is a local ring. Thus $P = S^2, A = \text{End}(S)$ and $\phi^*$ is the homothety by $\sum x_i \otimes y_i \otimes z_i$. Since $u(\phi) = 1 \otimes 1 \otimes 1 \otimes 1$, $\sum x_i \otimes y_i \otimes z_i$ is a 2-cocycle. The multiplication in $S \otimes \text{End}(S) \otimes S$ is given by $(s \otimes f \otimes t) \cdot (u \otimes g \otimes v) = (\sum x_i \otimes y_i \otimes z_i \otimes 1)^{-1} \cdot (\sum x_i x_j s u \otimes y_i y_j g \otimes z_i z_j \otimes t v)$, $s \otimes f \otimes t, u \otimes g \otimes v \in S \otimes \text{End}(S) \otimes S$, which is equal to $\sum s u \otimes f y \otimes g z \otimes t v$ since $\sum x_i \otimes y_i \otimes z_i$ is a 2-cocycle. The multiplication in $S \otimes S \otimes \text{End}(S)$ is given similarly. As easily checked, $\Phi$ is an $S^2$-algebra homomorphism. This completes the proof.

**Proposition 1.7.** If $P$ is a 1-coboundary then $u(\phi)$ is a 3-coboundary.

**Proof.** Since $P = (Q \otimes S) \otimes S^*(S \otimes Q^*)$ for some finitely generated projective $S$-module $Q$ of rank one, $Q^* = \text{Hom}_S(Q, S)$, $A = P \otimes \text{End}(S) \simeq \text{End}(Q)$ has an algebra structure. Hence $u(\phi)$ is a coboundary by Theorem 1.6.

Let $Br(S/R)$ denotes the Brauer group of $R$-Azumaya algebras split by $S$. For an element of $Br(S/R)$, we can choose an $S/R$-Azumaya algebra as its representative, and this representative is uniquely determined modulo $\{\text{End}(Q) \mid Q \text{ is a finitely generated projective } S \text{-module of rank one}\}$ (c.f. [3] 2.13). Thus summing up the results of this section, we get

**Corollary 1.8.** The following sequence is exact

$$Br(S/R) \xrightarrow{\theta_5} H^1(S/R, \text{Pic}) \xrightarrow{\theta_6} H^2(S/R, U)$$

where $\theta_5$ is the homomorphism induced from the one which carries $S/R$-Azumaya algebras to 1-cocycles determined by Theorem 1.3, $\theta_6$ is the one induced by Lemma 1.5.

### 2. $S/R$-Azumaya algebras and $H^2(S/R, U)$

Let $\sigma = \sum x_i \otimes y_i \otimes z_i$ be an Amitsur's 2-cocycle (of the extension $S/R$ with respect to the unit functor $U$). We shall define a new multiplication "*"
on $\text{End}(S)$ by setting
\[(f \ast g)(s) = \sum x_i f(y_i g(z_i s))\]
for all $f, g \in \text{End}(S)$, $s \in S$. Then Sweedler [7] proved this algebra $A(\sigma)$ is isomorphic to the Rosenberg Zelinsky central separable algebra coming from the 2-cocycle $\sigma^{-1}$.

We shall call that a 2-cocycle $\sigma$ is normal if $\sum x_i y_i \otimes z_i = \sum x_i \otimes y_i z_i = 1 \otimes 1$.

As can be easily proved, every 2-cocycle $\sigma$ is cohomologeous to a normal 2-cocycle $\sigma'$ and $A(\sigma) \simeq A(\sigma')$. For a normal 2-cocycle $\sigma'$, the $S/R$-Azumaya algebra $A(\sigma')$ is isomorphic to $\text{End}(S)$ as $S^2$-modules. The following asserts the converse is true.

**Proposition 2.1.** An $S/R$-Azumaya algebra $A$ is obtained from a normal 2-cocycle, if and only if, $A$ is isomorphic to $\text{End}(S)$ as $S^2$-modules.

Proof. If $A$ is isomorphic to $\text{End}(S)$, then the 1-cocycle $P$ obtained from $A$ is isomorphic to $S^2$. The method of the proof of the well-known fact that \[\text{H}^2(S/R, U) \simeq \text{Br}(S/R)\text{ if Pic}(S \otimes S) = 0\] can be applied in this case (c.f. [6] V.2.1).

**Corollary 2.2.** The sequence $\text{H}^2(S/R, U) \xrightarrow{\theta_4} \text{Br}(S/R) \xrightarrow{\theta_5} \text{H}^1(S/R, \text{Pic})$, where $\theta_4$ is induced from the homomorphism which carries a 2-cocycle $\sigma$ to $A(\sigma)$, is exact.

**Lemma 2.3.** The homomorphisms $\rho: S \otimes \text{End}(S) \rightarrow \text{End}_S(\text{End}(S))$, $\rho': S \otimes S \otimes \text{End}(S) \rightarrow \text{Hom}_S(\text{End}(S) \otimes_S \text{End}(S), \text{End}(S))$ defined by setting $(\rho(s \otimes f))(g) = sg \cdot f$, $(\rho'(s \otimes t \otimes f))(g \otimes h) = sg \cdot t \cdot f$, $f, g, h \in \text{End}(S)$, $s, t \in S$, are isomorphisms.

Proof. $\sigma$ is nothing else the well-known isomorphism $S \otimes \text{End}(S)^\text{op} \simeq \text{End}_S(\text{End}(S))$. The composite of the isomorphisms $S \otimes S \otimes \text{End}(S) \simeq S \otimes \text{End}_S(\text{End}(S)) \simeq \text{Hom}_S(\text{End}(S), S \otimes \text{End}(S)) \simeq \text{Hom}_S(\text{End}(S), S \otimes \text{End}(S)) \simeq \text{Hom}_S(S \otimes_S \text{End}(S), \text{End}(S))$ is $\rho'$.

**Proposition 2.4.** Let $\sigma = \sum x_i \otimes y_i \otimes z_i$, $\tau = \sum x_i^j \otimes y_i^j \otimes z_i^j$ be normal 2-cocycles, then $A(\sigma) \simeq A(\tau)$ as $S/R$-Azumaya algebras (that is isomorphic as $R$-algebras and compatible with the maximal commutative imbeddings of $S$), if and only only if, $\sigma$ is cohomologeous to $\tau$.

Proof. "If part" is trivial. Let $\Psi: A(\sigma) \simeq A(\tau)$ be the given isomorphism, then by Lemma 1.2 with $T = P = Q = S$, $\Psi$ corresponds to the homothety by the unit $\sum_i u_i \otimes v_i \in S^2$. 

Since $\Psi$ is an algebra isomorphism,

$$
\Psi(f * g)(s) = \sum u_i f(y, v, s) = \sum u_i x_j f(y_j g(z_j v, s)) = (\Psi(f) * \Psi(g))(s) = \sum u_i x_j f(v_i y_i u_i g(v, z v))
$$

for all $f, g \in \text{End}(S) = A(\sigma)$, $s \in S$. Hence by Lemma 2.3

$$
\sum u_i x_j y_j z_i v_i = \sum u_i x_k y_k v_i.
$$

Thus $\sigma$ is cohomologous to $\tau$.

Now let $P$ be a finitely generated projective $S$-module of rank one with the $S^2$-isomorphism $\zeta: S \otimes P \cong P \otimes S$, (this means that $P$ is a 0-cocycle with respect to the functor $P \otimes \cdot$). Define $S^3$-isomorphisms $\zeta_1, \zeta_2, \zeta_3$ as follows;

$$
\zeta_1 = 1 \otimes \zeta: S_1 \otimes S \otimes P \cong S_1 \otimes P \otimes S \text{ identity on } S_1
$$

$$
\zeta_2 : S \otimes S_2 \otimes P \cong P \otimes S_2 \otimes S \text{ identity on } S_2
$$

$$
\zeta_3 = \zeta \otimes 1 : S \otimes P \otimes S_3 \cong P \otimes S \otimes S_3 \text{ identity on } S_3.
$$

Define the $S^3$-automorphism of $S \otimes S \otimes P$ by $\zeta_2^{-1} \cdot \zeta_3 \cdot \zeta_1$ then $\zeta_2^{-1} \cdot \zeta_3 \cdot \zeta_1$ is the homothety by the unit $v(\zeta) \in S^3$. By localization we can easily check that $v(\zeta)$ is a 2-cocycle.

**Proposition 2.5.** Let $\sigma$ be a normal 2-cocycle and assume that $A(\sigma) = 0$ in $Br(S/R)$. Then there exists a finitely generated projective $S$-module $P$ such that $S \otimes P \cong P \otimes S$, and $\sigma$ is cohomologous to $v(\zeta)$ or equivalently $A(\sigma) = A(v(\zeta))$.

Proof. Since $A(\sigma) = 0$ in $Br(S/R)$, $A(\sigma) = \text{End}(P)$ for some finitely generated faithful projective $R$-module $P$. $P$ inherits the $S$-module structure and $S$-projective of rank one. $\text{End}(P) \cong (P \otimes S) \otimes_{S^2}(S \otimes \text{End}(S))$ as $S^2$-modules and $(P \otimes S) \otimes_{S^2}(S \otimes \text{End}(S))$ is a 1-cocycle. From the uniqueness of 1-cocycle (Theorem 1.3), there exists an $S^2$-isomorphism $\zeta: S \otimes P \cong P \otimes S$. We may assume $v(\zeta)$ is a normal 2-cocycle. Therefore by Proposition 2.4, all is settled if we prove $A(v(\zeta)) = \text{End}(P)$. Define $\Psi: A(v(\zeta)) \rightarrow \text{End}(P)$ by the following commutative diagram

$$
P \xrightarrow{\Psi(f)} S \otimes P \xrightarrow{\zeta} P \otimes S
$$

where "cont." is the contraction homomorphism, $f \in A(v(\zeta)) = \text{End}(S)$. By localization technique, we get that $\Psi$ is an $S/R$-algebra isomorphism.
Corollary 2.6. The sequence $H^n(S/R, Pic) \overset{\theta_3}{\rightarrow} H^n(S/R, U) \overset{\theta_4}{\rightarrow} Br(S/R)$, where $\theta_3$ is induced from the homomorphism which carries a 0-cocycle $P, \zeta: S \otimes P \simeq P \otimes S$, to $v(\zeta)$ is exact.

Proof. The only thing that we must show is that $\theta_3$ is a homomorphism. But it follows readily.

3. The seven terms exact sequence

Let $\rho=\sum_i x_i \otimes y_i \in S^2$ be a 1-cocycle of the extension $S/R$ with respect to the unit functor $U$. From the cocycle condition of $\rho$, $\sum_i x_i y_i = 1$. We make a new End $(S)$-module, $\rho S$ as follows;

$\rho S = S$ as $S$-modules, $f \cdot s = \sum_i x_i f(y_i s)$, $f \in \text{End}(S)$, $s \in S$. By the cocycle condition of $\rho$, $\rho S$ is in fact an End $(S)$-module. From Morita theory $\text{Hom}_{\text{End}(S)}(S, \rho S) \otimes S \simeq \rho S$.

And $\text{Hom}_{\text{End}(S)}(S, \rho S)$ is a finitely generated projective $R$-module of rank one. If $\rho$ is a coboundary (that is $\rho = x \otimes x^{-1}$, $x \in S$), then the homomorphism $\text{Hom}_{\text{End}(S)}(S, \rho S) \rightarrow \text{Hom}_{\text{End}(S)}(S, S)$ ($\simeq R$) which carries $g \in \text{Hom}_{\text{End}(S)}(S, \rho S)$ to $x^{-1} g \in \text{Hom}_{\text{End}(S)}(S, S)$ is an isomorphism. For another 1-cocycle $\rho'$, we have a canonical isomorphism $\text{Hom}_{\text{End}(S)}(S, \rho S) \otimes \text{Hom}_{\text{End}(S)}(S, \rho' S) \simeq \text{Hom}_{\text{End}(S)}(S, \rho' S)$. Hence the homomorphism which carries the 1-cocycle $\rho$ to $\text{Hom}_{\text{End}(S)}(S, \rho S)$ induces the homomorphism $\theta_2: H^0(S/R, U) \rightarrow \text{Pic}(R)$.

Lemma 3.1. $\theta_1$ is a monomorphism.

Proof. Let $\rho = \sum_i x_i \otimes y_i$ be a 1-cocycle and assume that $\text{Hom}_{\text{End}(S)}(S, \rho S)$ is a free $R$-module of rank one with a free base $g$. If we put $g(1_S) = x$ then $x$ is a unit of $S$ since $\text{Hom}_{\text{End}(S)}(S, \rho S) \otimes S \simeq \rho S = S$ as $S$-modules. The condition $g \in \text{Hom}_{\text{End}(S)}(S, \rho S)$ claims

$g(f(s)) = f(s)x = f \cdot g(s) = \sum_i x_i f(y_i s)$

for all $f \in \text{End}(S), s \in S$. By Lemma 2.3, we get $\rho = \sum_i x_i \otimes y_i = x \otimes x^{-1}$. Thus $\rho$ is a coboundary.

Next we define $\theta_2: \text{Pic}(R) \rightarrow H^0(S/R, Pic)$ as the homomorphism induced by tensoring with $S$ over $R$.

Lemma 3.2. The sequence

$$H^0(S/R, U) \overset{\theta_1}{\rightarrow} \text{Pic}(R) \overset{\theta_2}{\rightarrow} H^0(S/R, \text{Pic})$$
Proof. $\theta_2 \cdot \theta_i = 0$ since $\text{Hom}_{\text{End}(S)}(S \otimes S) \otimes S \cong \pi S$ for a 1-cocycle $\rho$. Conversely, let $P$ be a finitely generated projective $R$-module of rank one and assume that $S \otimes P$ is isomorphic to $S$ as $S$-modules. From the theory of faithfully flat descent, there exists an $S'$-isomorphism $\eta: S \otimes S \cong S \otimes S$ with property $\eta_2 = \eta_3 \eta_1$ and $P$ is characterized as $\{s \in S | s \otimes 1 = \eta(1 \otimes s) \text{ in } S \otimes S\}$, where $\eta_i, i=1, 2, 3$, is defined similarly as $\xi_i$ in §2. Since $\eta$ is a homothety, we may put $\eta = \sum_i x_i \otimes y_i, x_i, y_i \in S$. Then $\eta$ is a 1-cocycle by the relation $\eta_2 = \eta_3 \eta_1$. Define the homomorphisms $\Psi, \Psi', P \xrightarrow{\Psi} \text{Hom}_{\text{End}(S)}(S \otimes S)$, by setting $\Psi(p)(s) = sp, \Psi'(g) = g(1_s), p \in P, s \in S, g \in \text{Hom}_{\text{End}(S)}(S \otimes S)$. By Lemma 2.3 and the characterization of $P = \{s \in S | s \otimes 1 = \eta(1 \otimes s)\}$, $\Psi$ and $\Psi'$ are well-defined homomorphisms and are inverse to each other. This completes the proof.

**Lemma 3.3.** The sequence

$$\text{Pic}(R) \xrightarrow{\theta_2} H^0(S|R, \text{Pic}) \xrightarrow{\theta_3} H^0(S|R, U)$$

is exact, where $\theta_3$ is the homomorphism induced by the one which carries a 0-cocycle $P, \zeta: S \otimes P \cong P \otimes S$ to $v(\zeta)$.

Proof. $\theta_3 \cdot \theta_2 = 0$ as easily proved. Let $P$ be a finitely generated projective $S$-module of rank one such that $S \otimes P \cong P \otimes S$. Further assume that $v(\zeta) = \xi_2 \xi_3 \xi_1$ is a 2-coboundary. Then we may assume $v(\zeta) = 1 \otimes 1 \otimes 1$. Thus $\zeta$ is a descent homomorphism. Hence there exists a finitely generated projective $R$-module $P'$ of rank one such that $P \cong P' \otimes S$. This completes the proof.

Summing up Corollary 1.8, 2.2, 2.6, Lemma 3.1, 3.2, 3.3 we get

**Theorem 3.4.** The sequence

$$0 \to H^1(S|R, U) \xrightarrow{\theta_1} \text{Pic}(R) \xrightarrow{\theta_2} H^0(S|R, \text{Pic}) \xrightarrow{\theta_3} H^0(S|R, U) \xrightarrow{\theta_5} Br(S|R) \xrightarrow{\theta_6} H^1(S|R, \text{Pic}) \xrightarrow{\theta_6} H^2(S|R, U)$$

is an exact sequence of abelian groups.

**References**
