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AN APPLICATION OF THE THEORY OF DESCENT TO THE $S\otimes_R S$-MODULE STRUCTURE OF S/R-AZUMAYA ALGEBRAS

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Introduction. Let R be a commutative ring and S a commutative R-algebra which is a finitely generated faithful projective R-module. An R-Azumaya algebra A is called an S/R-Azumaya algebra if A contains S as a maximal commutative subalgebra and is left S-projective. S-S-bimodule structure (for which we shall call $S\otimes_R S$-module structure) of S/R-Azumaya algebras is determined in [5] when S/R is a separable Galois extension and in [8] when S/R is a Hopf Galois extension, both are connected with one which is so called seven terms exact sequence due to Chase, Harrison and Rosenberg [3].

In this paper we shall investigate the $S\otimes_R S$-module structure of S/R-Azumaya algebras assuming only that S is a finitely generated faithful projective R-module. So S/R-Azumaya algebras are not necessarily $S\otimes_R S$-projective (c.f. [8] Th. 2.1). But in §1 we shall show for any S/R-Azumaya algebra A, there exists a unique finitely generated projective $S\otimes_R S$-module P of rank one with certain cohomological properties such that A is $S\otimes_R S$-isomorphic to $P\otimes_{S\otimes_R S}\mathrm{End}_R(S)$. In §2, we shall investigate S/R-Azumaya algebras resulting from Amitsur's 2-cocycles. Finally we shall deal with the seven terms exact sequence in §3.

Throughout R will be a fixed commutative ring with unit, a commutative R-algebra S is a finitely generated faithful projective as R-module, each $\otimes$, End, etc. is taken over R unless otherwise stated. Repeated tensor products of S are denoted by exponents, $S^q=S\otimes\cdots\otimes S$ with q-factors. We shall consider S^q as an S-algebra on first term. To indicate module structure, we write if necessary, $S_1\otimes S_2$ instead of $S^2=S\otimes S$, ${}_{S_1}M_{S_2}$ instead of $S^2=S_1\otimes S_2$-module M etc.. $H^q(S/R, U)$ and $H^q(S/R, Pic)$ denote the q-th Amitsur's cohomology groups of the extension S/R with respect to the unit functor U and Picard group functor Pic respectively.

1. S/R-Azumaya algebras and $H^1(S/R, Pic)$

First we prove the following, which clarify the S^2-module structure of split S/R-Azumaya algebras.

Lemma 1.1. *Let M be a finitely generated projective S-module of rank one, then* End (M) *is isomorphic to* $(M\otimes S)\otimes_{S^2}(S\otimes M^*)\otimes_{S^2}\mathrm{End}(S)$ *as S^2-modules, where* $M^*=\mathrm{Hom}_S(M, S)$.

Proof. We define ψ: $(M\otimes S)\otimes_{S^2}(S\otimes M^*)\otimes_{S^2}\mathrm{End}(S)\rightarrow\mathrm{End}(M)$ as follows;

$$\psi((m\otimes s)\otimes(t\otimes f)\otimes g)(n)=tg(f(sn))m$$

$m,n\in M, s,t\in S, f\in M^*, g\in\mathrm{End}(S)$. Then ψ is a well-defined S^2-homomorphism and by localization we get ψ is an isomorphism.

REMARK. By ψ, the multiplication of $(M\otimes S)\otimes_{S^2}(S\otimes M^*)\otimes_{S^2}\mathrm{End}(S)$ is given by the formula

$$\begin{aligned}&((m\otimes s)\otimes(t\otimes f)\otimes g)\cdot((n\otimes u)\otimes(v\otimes p)\otimes q)\\&=(m\otimes u)\otimes(t\otimes p)\otimes g\cdot f(n)\cdot s\cdot vq\,.\end{aligned}$$

Now let A be an S/R-Azumaya algebra then A is split by S. Hence there exists a finitely generated faithful projective S-module M such that $S\otimes A$ is isomorphic to $\mathrm{End}_S(M)$ as S-algebras. As is well known, M inherits the S^2-module structure and is S^2-projective of rank one. By Lemma 1.1, $S\otimes A\cong\mathrm{End}_S(M)\cong(M\otimes_S S^2)\otimes_{S^3}(S^2\otimes_S M^*)\otimes_{S^3}\mathrm{End}_S(S^2)=({}_{S_1}M_{S_2}\otimes S_3)\otimes_{S^3}({}_{S_1}M^*{}_{S_3}\otimes S_2)\otimes_{S^3}\mathrm{End}_S(S^2)$, $M^*=\mathrm{Hom}_{S^2}(M, S^2)$. If we put $P=((M\otimes_S S^2)\otimes_{S^3}(S^2\otimes_S M^*))\otimes_{S^3}S^2=(({}_S M_{S_2}\otimes S_3)\otimes_{S^3}({}_{S_1}M^*{}_{S_3}\otimes S_2))\otimes_{S^3}S^2=((M\otimes_{S^2}S_1)\otimes S_2)\otimes_{S^2}{}_{S_1}M^*{}_{S_2}$, where we ragard S^2 (resp. S) as an S^3(resp. S^2)-module by $\mu\otimes 1: S^3\rightarrow S^2$ (resp. $\mu: S^2\rightarrow S$), μ is the multiplication of S, then $A\cong P\otimes_{S^2}\mathrm{End}(S)$ as S^2-modules. Define the S^2-algebra isomorphism $\Phi:\mathrm{End}_{S^2}(M\otimes S)=\mathrm{End}_{S_1\otimes S_2}({}_{S_1}M_{S_4}\otimes S_2)\rightarrow\mathrm{End}_{S^2}(S\otimes M)=\mathrm{End}_{S_1\otimes S_2}(S_1\otimes{}_{S_2}M_{S_3})$ by the composite of the isomorphisms $\mathrm{End}_{S_1\otimes S_2}({}_{S_1}M_{S_3}\otimes S_2)\cong S_1\otimes A\otimes S_2\cong S_1\otimes S_2\otimes A\cong\mathrm{End}_{S_1\otimes S_2}(S_1\otimes{}_{S_2}M_{S_3})$, where the middle isomorphism is the one induced from the twisting homomorphism $A\otimes S_2\rightarrow S_2\otimes A\,(a\otimes s\mapsto s\otimes a)$ and the others are induced from $S\otimes A\cong\mathrm{End}_S(M)$. Then from Morita theory there exists a finitely generated projective S^2-module Q of rank one such that $({}_{S_1}M_{S_3}\otimes S_2)\otimes_{S_1\otimes S_2}{}_{S_1}Q_{S_2}\cong S_1\otimes{}_{S_2}M_{S_3}$ as $\mathrm{End}_{S^2}(S_1\otimes{}_{S_2}M_{S_3})$-modules, hence as S^3-modules. Tensoring with S^2 over S^3 (regarding S^2 as an S^3-module by $1\otimes\mu: S^3\rightarrow S^2$), we get an S^2-isomorphism ${}_{S_1}M_{S_2}\otimes_{S_1\otimes S_2}{}_{S_1}Q_{S_2}\cong S_1\otimes(M\otimes_{S^2}S_2)$. Therefore,

$$\begin{aligned}S\otimes P&=(S\otimes(M\otimes_{S^2}S)\otimes S)\otimes_{S^3}(S\otimes M^*)\\&\cong((M\otimes_{S^2}Q)\otimes S)\otimes_{S^3}(S\otimes M^*)\\&=(M\otimes S)\otimes_{S^3}(Q\otimes S)\otimes_{S^3}(S\otimes M^*)\\&\cong(M\otimes S)\otimes_{S^3}(S^2\otimes_S M^*)\end{aligned}$$

$$\cong (M\otimes S)\otimes_{S^3}((M^*\otimes_{S^2}S)\otimes S^2)\otimes_{S^3}(S^2\otimes_S M^*)\otimes_{S^3}((M\otimes_{S^2}S)\otimes S^2)$$
$$= (M\otimes S)\otimes_{S^3}((M^*\otimes_{S^2}S)\otimes S^2)\otimes_{S^3}(S^2\otimes_S M^*)\otimes_{S^3}(S^2\otimes_S(M\otimes_{S^2}S)\otimes S)$$
$$= (P^*\otimes S)\otimes_{S^3}(S^2\otimes_S P),\ P^* = \mathrm{Hom}_{S^2}(P, S^2)\,.$$

This means P is a 1-cocycle of the extension S/R with respect to the functor *Pic* (we call simply 1-cocycle). Since $P^*=((M^*\otimes_{S^2}S)\otimes S)\otimes_{S^2}M$, $\mathrm{End}_S(P^*)\cong\mathrm{End}_S(M)$ as S-algebras.
If $S\otimes A\cong\mathrm{End}_S(N)$ for another N, then $\mathrm{End}_S(M)\cong\mathrm{End}_S(N)$ as S-algebras. So there exists a finitely generated projective S-module Q' of rank one such that ${}_{S_1}M_{S_2}\otimes_{S_1}Q'\cong N$ as S^2-modules. Easy calculation shows that the 1-cocycles obtained from M and N are S^2-isomorphic.

To prove the uniqueness of 1-cocycle P, we prepare the following

Lemma 1.2. *Let T be a commutative R-algebra, which is a finitely genreated faithful projective R-module. And let P, Q be finitely generated projective T-modules of rank one. Then*

$$\mathrm{Hom}_{T\otimes T}(P\otimes Q, Q\otimes P)\cong\mathrm{Hom}_{T\otimes T}(\mathrm{End}(P), \mathrm{End}(Q))$$

Especially, $\mathrm{Iso}_{T\otimes T}(P\otimes Q, Q\otimes P)$ *corresponds to* $\mathrm{Iso}_{T\otimes T}(\mathrm{End}(P, \mathrm{Ecd}(Q))$.

Proof. For any T-module $M_i, N_i (i=1, 2)$, we have the following isomorphism ρ: $\mathrm{Hom}_{T\otimes T}(M_1\otimes M_2, \mathrm{Hom}(N_2, N_1))\cong\mathrm{Hom}_{T\otimes T}(M_1\otimes N_2, \mathrm{Hom}(M_2, N_1))$ given by $(\rho(\varphi)(m_1\otimes n_2))(m_2)=(\varphi(m_1\otimes m_2))(n_2)$, $m_i\in M_i$, $n_i\in N_i$, $\varphi\in\mathrm{Hom}_{T\otimes T}(M_1\otimes M_2, \mathrm{Hom}(N_2,N_1))$, ([6] I.4.2). Put $M_1=P, M_2=N_1=Q$, $N_2=\mathrm{Hom}(P,R)$, then we get easily. Further assertion follows easily by localization.

Let P, P' be 1-cocycles such that $P\otimes_{S^2}\mathrm{End}(S)\cong P'\otimes_{S^2}\mathrm{End}(S)\cong A$ as S^2-modules. Then $\mathrm{End}_S(P^*)\cong\mathrm{End}_S(P'^*)$ as S^3-modules by Lemma 1.1 and the cocycle condition of P, P'. From Lemma 1.2 we get an S^3-isomorphism $P^*\otimes_S P'^*=({}_{S_1}P^*{}_{S_2}\otimes S_3)\otimes_{S^3}({}_{S_1}P'^*{}_{S_3}\otimes S_2)\cong P'^*\otimes_S P^*=({}_{S_1}P'^*{}_{S_2}\otimes S_3)\otimes_{S^3}({}_{S_1}P^*{}_{S_3}\otimes S_2)$. Thus $({}_{S_1}P^*{}_{S_2}\otimes S_3)\otimes_{S^3}({}_{S_1}P_{S_3}\otimes S_2)\cong({}_{S_1}P'^*{}_{S_2}\otimes S_3)\otimes_{S^3}({}_{S_1}P'_{S_3}\otimes S_2)$, the left side is isomorphic to $S_1\otimes_{S_2}P_{S_3}$ and the right side is isomorphic to $S_1\otimes_{S_2}P'_{S_3}$ by the cocycle condition of P, P'. Tensoring with S^2 over S^3 (regarding S^2 as an S^3-module by $\mu\otimes 1\colon S^3\to S^2$), we get $P\cong P'$.
Summing up we get

Theorem 1.3. *Let A be an S/R-Azumaya algebra, then there exists a unique* 1*-cocycle P such that A is isomorphic to $P\otimes_{S^2}\mathrm{End}(S)$ as S^2-modules and $S\otimes A$ is isomorphic to* $\mathrm{End}_S(P^*)$ *as S-algebras, where* $P^*=\mathrm{Hom}_{S^2}(P, S^2)$.

REMARK. In proving the above theorem, we used the S-algebra isomorphism

$S\otimes A\simeq \mathrm{End}_S(M)$. If we assume this isomorphism is only an S^3-module isomorphism, then by using Lemma 1.2 in suitable situations we shall get Theorem 1.3 only replaceing "*S-algebras*" to "*S^3-modules*" in the last statement. So Theorem 1.3 does not fully characterize S/R-Azumaya algebras.

Proposition 1.4. *Let A, B be S/R-Azumaya algebras, P, Q be 1-cocycles obtained from A, B respectively. Then the 1-cocycle obtained from $A\cdot B=\mathrm{End}_{A\otimes B}(S\otimes_{S^2}(A\otimes B))$ is $P\otimes_{S^2} Q$.*

Proof. $S\otimes A\simeq \mathrm{End}_S(P^*)$ and $S\otimes B\simeq \mathrm{End}_S(Q^*)$, so $S\otimes(A\cdot B)=(S\otimes A)\cdot(S\otimes B)\simeq \mathrm{End}_S(P^*\otimes_{S^2}Q^*)$, (c.f. [3] 2.13.). Thus the 1-cocycle obtained from $A\cdot B$ equals to $P\otimes_{S^2}Q$.

Next we shall start from a 1-cocycle P and an S^3-isomorphism ϕ: $S^2\otimes_S P^*={}_{S_1}P^*{}_{S_3}\otimes S_2\simeq(S_1\otimes{}_{S_2}P^*{}_{S_3})\otimes_{S^3}({}_{S_1}P^*{}_{S_2}\otimes S_3)=(S\otimes P^*)\otimes_{S^3}(P^*\otimes S)$. Define the S^4-isomorphisms ϕ_1, ϕ_2, ϕ_3 as follows;

$\phi_1=1\otimes\phi$: $S_1\otimes{}_{S_2}P^*{}_{S_4}\otimes S_3\simeq(S_1\otimes S_2\otimes{}_{S_3}P^*{}_{S_4})\otimes_{S^4}(S_1\otimes{}_{S_2}P^*{}_{S_3}\otimes S_4)$, identity on S_1

ϕ_2 : ${}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3\simeq(S_1\otimes S_2\otimes{}_{S_3}P^*{}_{S_4})\otimes_{S^4}({}_{S_1}P^*{}_{S_3}\otimes S_2\otimes S_4)$, identity on S_2

ϕ_3 : ${}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3\simeq(S_1\otimes{}_{S_2}P^*{}_{S_4}\otimes S_3)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)$, identity on S_3.

Further we define $u(\phi)\in\mathrm{End}_{S^4}({}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3)$ by the composite

$$\begin{aligned}&{}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3\xrightarrow{\phi_2}(S_1\otimes S_2\otimes{}_{S_3}P^*{}_{S_4})\otimes_{S^4}({}_{S_1}P^*{}_{S_3}\otimes S_2\otimes S_4)\\ &\xrightarrow{1\otimes_{S^4}(\phi\otimes 1)}(S_1\otimes S_2\otimes{}_{S_3}P^*{}_{S_4})\otimes_{S^4}(S_1\otimes{}_{S_2}P^*{}_{S_3}\otimes S_4)\otimes_{S^4}\\ &({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)\xrightarrow{\phi_1^{-1}\otimes_{S^4}1}(S_1\otimes{}_{S_2}P^*{}_{S_4}\otimes S_3)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)\\ &\xrightarrow{\phi_3^{-1}}{}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3.\end{aligned}$$

Then we may think $u(\phi)$ is a unit of S^4 by homothety. As easily checked, $u(\alpha\phi)=\delta(\alpha^{-1})u(\phi)$ for a unit $\alpha\in S^3$, where δ is the coboundary operator in Amitsur's complex with respect to the unit functor U.

Lemma 1.5. *$u(\phi)$ is a 3-cocycle.*

Proof. By localization it follows readily.

Theorem 1.6. *Let P be a 1-cocycle with a S^3-isomorphism ϕ: ${}_{S_1}P^*{}_{S_3}\otimes S_2\simeq(S_1\otimes{}_{S_2}P^*{}_{S_3})\otimes_{S^3}({}_{S_1}P^*{}_{S_2}\otimes S_3)$. Then $A=P\otimes_{S^2}\mathrm{End}(S)$ has an S/R-Azumaya algebra structure, if and only if, $u(\phi)$ is a coboundary. If $u(\phi)=\delta(\beta)$ where β is a* *unit of* S^3, *then* $(\beta\phi)^*$ *induces a* S*-algebra isomorphism* $S\otimes A\cong \mathrm{End}_S(P^*)$, *where* $(\beta\phi)^*$ *is the isomorphism* $S\otimes P\cong (P^*\otimes S)\otimes_{S^3}({}_{S_1}P_{S_3}\otimes S_2)$ *induced from* $\beta\phi$.

Proof. First we assume $A=P\otimes_{S^2}\mathrm{End}(S)$ is an S/R-Azumaya algebra, then $S\otimes A\cong \mathrm{End}_S(P^*)$ as S-algebras from the uniqueness of 1-cocycle. Define the S^2-algebra isomorphism

$$\Phi\colon \mathrm{End}_{S_1\otimes S_2}({}_{S_1}P^*{}_{S_3}\otimes S_2) = S_1\otimes A\otimes S_2 \to S_1\otimes S_2\otimes A = \mathrm{End}_{S_1\otimes S_2}(S_1\otimes_{S_2}P^*{}_{S_3})$$

by the twisting homomorphism $A\otimes S_2\to S_2\otimes A$. Φ is a descent homomorphism, that is if we put $\Phi_1=1\otimes\Phi\colon S_1\otimes \mathrm{End}_S(P^*)\otimes S\to S_1\otimes S\otimes \mathrm{End}_S(P^*)$ identity on S_1, $\Phi_2\colon \mathrm{End}_S(P^*)\otimes S_2\otimes S\to S\otimes S_2\otimes \mathrm{End}_S(P^*)$ identity on S_2, $\Phi_3=\Phi\otimes 1\colon \mathrm{End}_S(P^*)\otimes S\otimes S_3\to S\otimes \mathrm{End}_S(P^*)\otimes S_3$ identity on S_3, then $\Phi_2=\Phi_1\cdot\Phi_3$. Since Φ is an S^2-algebra isomorphism, there exists a finitely generated projective S^2-module Q of rank one such that ${}_{S_1}P^*{}_{S_3}\otimes S_2$ is isomorphic to $(S_1\otimes_{S_2}P^*{}_{S_3})\otimes_{S_1\otimes S_2}{}_{S_1}Q_{S_2}=(S_1\otimes_{S_2}P^*{}_{S_3})\otimes_{S^3}({}_{S_1}Q_{S_2}\otimes S_3)$ as S^3-modules and Φ is induced by this isomorphism ϕ'. From the cocycle condition of P, Q is isomorphic to P^*. From the definition of Φ_1, Φ_2, Φ_3, the following diagram is commutative for any $f\in \mathrm{End}_{S_1\otimes S_2\otimes S_3}({}_S P^*{}_{S_4}\otimes S_2\otimes S_3)$.

$$(S_1\otimes S_2\otimes_{S_3}P^*{}_{S_4})\otimes_{S^4}(S_1\otimes_{S_2}P^*{}_{S_3}\otimes S_4)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)\xrightarrow{\Phi_2(f)\otimes_{S^4}1\otimes_{S^4}1}(S_1\otimes S_2\otimes_{S_3}P^*{}_{S_4})\otimes_{S_4}(S_1\otimes_{S_2}P^*{}_{S_3}\otimes S_4)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)$$

$$\Big\Uparrow 1\otimes_{S^4}(\phi'\otimes 1) \qquad\qquad \Big\Uparrow 1\otimes_{S^4}(\phi'\otimes 1)$$

$$(S_1\otimes S_2\otimes_{S_3}P^*{}_{S_4})\otimes_{S^4}({}_{S_1}P^*{}_{S_3}\otimes S_2\otimes S_4)\xrightarrow{\Phi_2(f)\otimes_{S^4}1}(S_1\otimes S_2\otimes_{S_3}P^*{}_{S_4})\otimes_{S^4}({}_{S_1}P^*{}_{S_3}\otimes S_2\otimes S_4)$$

$$\Big\Uparrow \phi_2' \qquad\qquad \Big\Uparrow \phi_2'$$

$${}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3 \xrightarrow{\quad f\quad} {}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3$$

$$\Big\Downarrow \phi_3' \qquad\qquad \Big\Downarrow \phi_3'$$

$$(S_1\otimes_{S_2}P^*{}_{S_4}\otimes S_3)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)\xrightarrow{\Phi_3(f)\otimes_{S^4}1}(S_1\otimes_{S_2}P^*{}_{S_4}\otimes S_3)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)$$

$$\phi_1'\Big\Downarrow \otimes_{S^4}1 \qquad\qquad \phi_1'\Big\Downarrow \otimes_{S^4}1$$

$$(S_1\otimes S_2\otimes_{S_3}P^*{}_{S_4})\otimes_{S^4}(S_1\otimes_{S_2}P^*{}_{S_3}\otimes S_4)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)\xrightarrow{\Phi_1\cdot\Phi_3(f)\otimes_{S_4}1\otimes_{S^4}1}(S_1\otimes S_2\otimes_{S_3}P^*{}_{S_4})\otimes_{S^4}(S_1\otimes_{S_2}P^*{}_{S_3}\otimes S_4)\otimes_{S^4}({}_{S_1}P^*{}_{S_2}\otimes S_3\otimes S_4)$$

Thus $(1\otimes_{S^4}(\phi'\otimes 1))\cdot\phi_2'\cdot f\cdot\phi_2'^{-1}\cdot(1\otimes_{S^4}(\phi'^{-1}\otimes 1))=(\phi_1'\otimes_{S^4}1)\cdot\phi_3'\cdot f\cdot\phi_3'^{-1}\cdot(\phi_1'^{-1}\otimes_{S^4}1)$. Hence $f\cdot u(\phi')=u(\phi')\cdot f$ for any $f\in \mathrm{End}_{S_1\otimes S_2\otimes S_3}({}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3)$. Therefore 3-cocycle $u(\phi')$ is contained in the center of $\mathrm{End}_{S_1\otimes S_2\otimes S_3}({}_{S_1}P^*{}_{S_4}\otimes S_2\otimes S_3)$, which is $S_1\otimes S_2\otimes S_3$. Easily we get $u(\phi')$ is a coboundary. Thus $u(\phi)=$

$u(\alpha^{-1}\phi')=\delta(\alpha)u(\phi')$ is a coboundary.
Conversely let $u(\phi)$ be a coboundary then we may assume $u(\phi)=1\otimes 1\otimes 1\otimes 1$. Let ϕ^* be the isomorphism $S\otimes P\cong(P^*\otimes S)\otimes_{S^3}({}_{S_1}P_{S_3}\otimes S_2)$ induced from ϕ by duality pairing. We consider $S\otimes A=(S\otimes P)\otimes_{S_3}\mathrm{End}_S(S^2)$ equals $\mathrm{End}_S(P^*)$ $=(P^*\otimes S)\otimes_{S^3}({}_{S_1}P_{S_3}\otimes S_2)\otimes_{S^3}\mathrm{End}_S(S^2)$ by $\phi^*\otimes_{S^3}1$. Thus $S\otimes A$ has an S-algebra structure. Define Φ: $S\otimes A\otimes S\cong S\otimes S\otimes A$ by the twisting homomorphism $A\otimes S\to S\otimes A$. Clearly $\Phi_2=\Phi_1\cdot\Phi_3$. From the theory of faithfully flat descent, if Φ is an S^2-algebra isomorphism, then the descented module A has an R-algebra structure (necessarily an S/R-Azumaya algebra structure) such that the induced S-algebra structure of $S\otimes A$ coincides the original one of $S\otimes A$. Therefore all is settled if we show Φ is an S^2-algebra homomorphism. So we may assume R is a local ring. Thus $P=S^2$, $A=\mathrm{End}\,(S)$ and ϕ^* is the homothety by $\sum_i x_i\otimes_i y_i\otimes z_i$. Since $u(\phi)=1\otimes 1\otimes 1\otimes 1$, $\sum_i x_i\otimes y_i\otimes z_i$ is a 2-cocycle. The multiplication in $S\otimes\mathrm{End}(S)\otimes S$ is given by $(s\otimes f\otimes t)\cdot(u\otimes g\otimes v)$ $=(\sum_i x_i\otimes y_i\otimes z_i\otimes 1)^{-1}\cdot(\sum_{i,j} x_ix_jsu\otimes y_ifz_iy_jgz_j\otimes tv)$, $s\otimes f\otimes t$, $u\otimes g\otimes v\in S\otimes\mathrm{End}$ $(S)\otimes S$, which is equal to $\sum_i su\otimes x_ify_igz_i\otimes tv$ since $\sum_i x_i\otimes y_i\otimes z_i$ is a 2-cocycle. The multiplication in $S\otimes S\otimes\mathrm{End}\,(S)$ is given similarly. As easily checked, Φ is an S^2-algebra homomorphism. This completes the proof.

Proposition 1.7. *If P is a 1-coboundary then $u(\phi)$ is a 3-coboundary.*

Proof. Since $P=(Q\otimes S)\otimes_{S^2}(S\otimes Q^*)$ for some finitely generated projective S-module Q of rank one, $Q^*=\mathrm{Hom}_S(Q,S)$, $A=P\otimes\mathrm{End}(S)\cong\mathrm{End}(Q)$ has an algebra structure. Hence $u(\phi)$ is a coboundary by Theorem 1.6.

Let $Br(S/R)$ denotes the Brauer group of R-Azumaya algebras split by S. For an element of $Br(S/R)$, we can choose an S/R-Azumaya algebra as its representative, and this representative is uniquely determined modulo $\{\mathrm{End}\,(Q)\,|\,Q$ is a finitely generated projective S-module of rank one$\}$ (c.f. [3] 2.13).
Thus summing up the results of this section, we get

Corollary 1.8. *The following sequence is exact*

$$Br(S/R)\xrightarrow{\theta_5}H^1(S/R,\,Pic)\xrightarrow{\theta_6}H^3(S/R,\,U)$$

where θ_5 is the homomorphism induced from the one which carries S/R-Azumaya algebras to 1-cocycles determined by Theorem 1.3, *θ_6 is the one induced by Lemma* 1.5.

2. S/R-Azumaya algebras and $H^2(S/R,\,U)$

Let $\sigma=\sum_i x_i\otimes y_i\otimes z_i$ be an Amitsur's 2-cocycle (of the extension S/R with respect to the unit functor U). We shall define a new multiplication "$*$"

on End(S) by setting

$$(f*g)(s)=\sum_i x_i f(y_i g(z_i s))$$

for all $f, g\in \mathrm{End}(S)$, $s\in S$. Then Sweedler [7] proved this algebra $A(\sigma)$ is isomorphic to the Rosenberg Zelinsky central separable algebra coming from the 2-cocycle σ^{-1}.

We shall call that a 2-cocycle σ is normal if $\sum_i x_i y_i\otimes z_i=\sum_i x_i\otimes y_i z_i=1\otimes 1$. As can be easily proved, every 2-cocycle σ is cohomologeous to a normal 2-cocycle σ' and $A(\sigma)\cong A(\sigma')$. For a normal 2-cocycle σ', the S/R-Azumaya algebra $A(\sigma')$ is isomorphic to End(S) as S^2-modules. The following asserts the converse is true.

Proposition 2.1. *An S/R-Azumaya algebra A is obtained from a normal 2-cocycle, if and only if, A is isomorphic to* End(S) *as S^2-modules.*

Proof. If A is isomorphic to End(S), then the 1-cocycle P obtained from A is isomorphic to S^2. The method of the proof of the well-known fact that "$H^2(S/R, U)\cong Br(S/R)$ *if* $Pic(S\otimes S)=0$" can be applied in this case (c.f. [6] V.2.1).

Corollary 2.2. *The sequence* $H^2(S/R, U)\xrightarrow{\theta_4} Br(S/R)\xrightarrow{\theta_5} H^1(S/R, Pic)$, *where θ_4 is induced from the homomorphism which carries a 2-cocycle σ to $A(\sigma)$, is exact.*

Lemma 2.3. *The homomorphisms* ρ: $S\otimes \mathrm{End}(S)\to \mathrm{End}_S(\mathrm{End}(S))$, ρ': $S\otimes S\otimes \mathrm{End}(S)\to \mathrm{Hom}_S(\mathrm{End}(S)\otimes_S \mathrm{End}(S), \mathrm{End}(S))$ *defined by setting* $(\rho(s\otimes f))(g))=sg\cdot f$, $(\rho'(s\otimes t\otimes f))(g\otimes h)=sg\cdot th\cdot f$, $f, g, h\in \mathrm{End}(S)$, $s, t\in S$, are isomorphisms.

Proof. σ is nothing else the well-known isomorphism $S\otimes \mathrm{End}(S)^0\cong \mathrm{End}_S(\mathrm{End}(S))$. The composite of the isomorphisms $S\otimes S\otimes \mathrm{End}(S)\cong S\otimes \mathrm{End}_S(\mathrm{End}(S))\cong \mathrm{Hom}_S(\mathrm{End}(S), S\otimes \mathrm{End}(S))\cong \mathrm{Hom}_S(\mathrm{End}(S), \mathrm{End}_S(\mathrm{End}(S))\cong \mathrm{Hom}_S(\mathrm{End}(S)\otimes_S \mathrm{End}(S), \mathrm{End}(S))$ is ρ'.

Poroposition 2.4. *Let* $\sigma=\sum_i x_i\otimes y_i\otimes z_i$, $\tau=\sum_i x_i'\otimes y_i'\otimes z_i'$ *be normal 2-cocycles, then $A(\sigma)\cong A(\tau)$ as S/R-Azumaya algebras (that is isomorphic as R-algebras and compatible with the maximal commutative imbeddings of S), if and only only if, σ is cohomologeous to τ.*

Proof. "If part" is trivial. Let Ψ: $A(\sigma)\cong A(\tau)$ be the given isomorphism, then by Lemma 1.2 with $T=P=Q=S$, Ψ corresponds to the homothety by the unit $\sum_i u_i\otimes v_i\in S^2$.

$$\Psi(f)(s)=\sum_i u_i f(v_i s),\ f\in\mathrm{End}(S)=A(\sigma),\ s\in S.$$

Since Ψ is an algebra isomorphism,

$$\Psi(f*g)(s)=\sum_i u_i(f*g)(v_i s)=\sum_{i,j} u_i x_j f(y_j g(z_j v_i s))$$
$$=(\Psi(f)*\Psi(g))(s)=\sum_{i,j,k} u_i x'_k f(v_i y'_k u_j g(v_j z'_k s))$$

for all $f, g\in\mathrm{End}(S)=A(\sigma)$, $s\in S$. Hence by Lemma 2.3

$$\sum_{i,j} u_i x_j\otimes y_j\otimes z_j v_i=\sum_{i,j,k} u_i x'_k\otimes v_i u_j y'_k\otimes v_j z'_k.$$

Thus σ is cohomologeous to τ.

Now let P be a finitely generated projective S-module of rank one with the S^2-isomorphism ζ: $S\otimes P\cong P\otimes S$, (this means that P is a 0-cocycle with respect to the functor *Pic*). Define S^3-isomorphisms $\zeta_1, \zeta_2, \zeta_3$ as follows;

$$\begin{array}{lll}\zeta_1=1\otimes\zeta: & S_1\otimes S\otimes P\cong S_1\otimes P\otimes S & \text{identity on } S_1\\ \zeta_2 \quad : & S\otimes S_2\otimes P\cong P\otimes S_2\otimes S & \text{identity on } S_2\\ \zeta_3=\zeta\otimes 1: & S\otimes P\otimes S_3\cong P\otimes S\otimes S_3 & \text{identity on } S_3.\end{array}$$

Define the S^3-automorphism of $S\otimes S\otimes P$ by $\zeta_2^{-1}\cdot\zeta_3\cdot\zeta_1$ then $\zeta_2^{-1}\cdot\zeta_3\cdot\zeta_1$ is the homothety by the unit $v(\zeta)\in S^3$. By localization we can easily check that $v(\zeta)$ is a 2-cocycle.

Proposition 2.5. *Let σ be a normal 2-cocycle and assume that $A(\sigma)=0$ in $Br(S/R)$. Then there exists a finitely generated projective S-module P such that $S\otimes P\overset{\zeta}{\cong}P\otimes S$, and σ is cohomologeous to $v(\zeta)$ or equivalently $A(\sigma)\cong A(v(\zeta))$.*

Proof. Since $A(\sigma)=0$ in $Br(S/R)$, $A(\sigma)\cong\mathrm{End}(P)$ for some finitely generated faithful projective R-module P. P inherits the S-module structure and S-projective of rank one. $\mathrm{End}(P)\cong(P\otimes S)\otimes_{S^2}(S\otimes P^*)\otimes_{S^2}\mathrm{End}(S)$ as S^2-modules and $(P\otimes S)\otimes_{S^2}(S\otimes P^*)$ is a 1-cocycle. From the uniqueness of 1-cocycle (Theorem 1.3), there exists an S^2-isomorphism ζ: $S\otimes P\cong P\otimes S$. We may assume $v(\zeta)$ is a normal 2-cocycle. Therefore by Proposition 2.4, all is settled if we prove $A(v(\zeta))\cong\mathrm{End}(P)$. Define Ψ: $A(v(\zeta))=\mathrm{End}(S)\to\mathrm{End}(P)$ by the following commutative diagram

$$\begin{array}{ccc}P & \longrightarrow & S\otimes P\overset{\zeta}{\cong}P\otimes S\\ \downarrow\Psi(f) & & \downarrow 1\otimes f\\ P & \overset{cont.}{\longleftarrow} & S\otimes P\overset{\zeta}{\cong}P\otimes S\end{array}$$

where "*cont.*" is the contraction homomorphism, $f\in A(v(\zeta))=\mathrm{End}(S)$. By localization technique, we get that Ψ is an S/R-algebra isomorphism.

Corollary 2.6. *The sequence* $H^0(S/R, Pic) \overset{\theta_3}{\to} H^2(S/R, U) \overset{\theta_4}{\to} Br(S/R)$, *where* θ_3 *is induced from the homomorphism which carries a* 0-*cocycle* P, ζ: $S\otimes P \cong P\otimes S$, *to* $v(\zeta)$ *is exact.*

Proof. The only thing that we must show is that θ_3 is a homomorphism. But it follows readily.

3. The seven terms exact sequence

Let $\rho=\sum_i x_i\otimes y_i \in S^2$ be a 1-cocycle of the extension S/R with respect to the unit functor U. From the cocycle condition of ρ, $\sum_i x_i y_i=1$. We make a new End (S)-module ${}_\rho S$ as follows;

${}_\rho S=S$ as S-modules, $f\cdot s=\sum_i x_i f(y_i s)$, $f\in \mathrm{End}(S)$, $s\in S$. By the cocycle condition of ρ, ${}_\rho S$ is in fact an End(S)-module. From Morita theory

$$\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)\otimes S \cong {}_\rho S\,.$$

And $\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)$ is a finitely generated projective R-module of rank one. If ρ is a coboundary (that is $\rho=x\otimes x^{-1}$, $x\in S$), then the homomorphism $\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)\to \mathrm{Hom}_{\mathrm{End}(S)}(S, S)\ (\cong R)$ which carries $g\in \mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)$ to $x^{-1}g\in \mathrm{Hom}_{\mathrm{End}(S)}(S, S)$ is an isomorphism. For another 1-cocycle ρ', we have a canonical isomorphism $\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)\otimes \mathrm{Hom}_{\mathrm{End}(S)}(S, {}_{\rho'} S)\cong \mathrm{Hom}_{\mathrm{End}(S)}(S, {}_{\rho\rho'} S)$. Hence the homomorphism which carries the 1-cocycle ρ to $\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)$ induces the homomorphism θ_1: $H^1(S/R, U)\to Pic(R)$.

Lemma 3.1. θ_1 *is a monomorphism.*

Proof. Let $\rho=\sum_i x_i\otimes y_i$ be a 1-cocycle and assume that $\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)$ is a free R-module of rank one with a free base g. If we put $g(1_S)=x$ then x is a unit of S since $\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)\otimes S\cong {}_\rho S=S$ as S-modules. The condition $g\in \mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S)$ claims

$$g(f(s)) = f(s)x = f\cdot(g(s)) = \sum_i x_i f(y_i s x)$$

for all $f\in \mathrm{End}(S)$, $s\in S$. By Lemma 2.3, we get $\rho=\sum_i x_i\otimes y_i=x\otimes x^{-1}$. Thus ρ is a coboundary.

Next we define θ_2: $Pic(R)\to H^0(S/R, Pic)$ as the homomorphism induced by tensoring with S over R.

Lemma 3.2. *The sequence*

$$H^1(S/R, U)\xrightarrow{\theta_1} Pic(R)\xrightarrow{\theta_2} H^0(S/R, Pic)$$

is exact.

Proof. $\theta_2 \cdot \theta_1 = 0$ since $\mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\rho S) \otimes S \cong {}_\rho S$ for a 1-cocycle ρ. Conversely, let P be a finitely generated projective R-module of rank one and assume that $S \otimes P$ is isomorphic to S as S-modules. From the theory of faithfully flat descent, there exists an S^2-isomorphism η: $S \otimes S \cong S \otimes S$ with property $\eta_2 = \eta_3 \eta_1$ and P is characterized as $\{s \in S \mid s \otimes 1 = \eta(1 \otimes s)$ *in* $S \otimes S\}$, where η_i, $i = 1, 2, 3$, is defined similarly as ζ_i in §2. Since η is a homothety, we may put $\eta = \sum_i x_i \otimes y_i$, $x_i, y_i \in S$. Then η is a 1-cocycle by the relation $\eta_2 = \eta_3 \eta_1$. Define the homomorphisms Ψ, Ψ', $P \underset{\Psi'}{\overset{\Psi}{\rightleftarrows}} \mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\eta S)$, by setting $\Psi(p)(s) = sp$, $\Psi'(g) = g(1_S)$, $p \in P$, $s \in S$, $g \in \mathrm{Hom}_{\mathrm{End}(S)}(S, {}_\eta S)$. By Lemma 2.3 and the characterization of $P = \{s \in S \mid s \otimes 1 = \eta(1 \otimes s)\}$, Ψ and Ψ' are well-defined homomorphisms and are inverse to each other. This completes the proof.

Lemma 3.3. *The sequence*

$$Pic(R) \xrightarrow{\theta_2} H^0(S/R, Pic) \xrightarrow{\theta_3} H^2(S/R, U)$$

is exact, where θ_3 is the homomorphism induced by the one which carries a 0-cocycle P, ζ: $S \otimes P \cong P \otimes S$ to $v(\zeta)$.

Proof. $\theta_3 \cdot \theta_2 = 0$ as easily proved. Let P be a finitely generated projective S-module of rank one such that $S \otimes P \overset{\zeta}{\cong} P \otimes S$. Further assume that $v(\zeta) = \zeta_2^{-1} \zeta_3 \zeta_1$ is a 2-coboundary. Then we may assume $v(\zeta) = 1 \otimes 1 \otimes 1$. Thus ζ is a descent homomorphism. Hence there exists a finitely generated projective R-module P' of rank one such that $P \cong P' \otimes S$. This completes the proof.

Summing up Corollary 1.8, 2.2, 2.6, Lemma 3.1, 3.2, 3.3 we get

Theorem 3.4. *The sequence*

$$\begin{aligned} 0 \to H^1(S/R, U) \xrightarrow{\theta_1} Pic(R) \xrightarrow{\theta_2} H^0(S/R, Pic \xrightarrow{\theta_3} H^2(S/R, U) \\ \xrightarrow{\theta_4} Br(S/R) \xrightarrow{\theta_5} H^1(S/R, Pic) \xrightarrow{\theta_6} H^3(S/R, U) \end{aligned}$$

is an exact sequence of abelian groups.

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