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REPRESENTATION OF GAUSSIAN PROCESSES
EQUIVALENT TO WIENER PROCESS

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1. Introduction

The purpose of this paper is to get a canonical representation of Gaussian processes which are equivalent (or mutually absolutely continuous) to Wiener process. The main result is this. Suppose we are given a Gaussian process $Y_t$ on a probability space $(\Omega, \mathcal{F}, P)$, which is equivalent to Wiener process. Then a Wiener process $X_t$ is constructed on $(\Omega, \mathcal{F}, P)$ as a functional of $\{Y_s; s \leq t\}$ and, conversely, $Y_t$ is represented as a measurable functional of $\{X_s; s \leq t\}$ for each $t \in [0, T]$. In case of $E(Y_t)=0$, $t \in [0, T]$, $Y_t$ is represented by the formula

$$(1.1) \quad Y_t = X_t - \int_0^t \left( \int_0^s l(s, u) dX_u \right) ds,$$

where $l(s, t)$ is a Volterra kernel belonging to $L^2([0, T]^2)$, and the representation of $Y_t$ is unique (Theorem 1). Conversely, if a Wiener process $X_t$ is given and if $Y_t$ is represented by (1.1), $Y_t$ is equivalent to Wiener process (Theorem 2). The density of the transformed measure with respect to which $Y_t$ is a Wiener process is easily evaluated in the proof of Theorem 2. The proof of these facts is based heavily upon martingale theory due to Meyer [8] and Kunita-S. Watanabe [6].

The conditions for a Gaussian process to be equivalent to Wiener process have been obtained in terms of the mean and the covariance by Shepp [11] and Golosov [3]. Moreover, by the method of linear transformations of Wiener space, Shepp [11] and H. Sato [10] have obtained the following representation

$$(1.2) \quad Y_t = X_t - \int_0^t \left( \int_0^T m(s, u) dX_u \right) ds,$$

where $m(s, u)$ is a kernel of $L^2([0, T]^2)$ with some additional conditions. This representation involves the stochastic integral on the fixed time interval $[0, T]$, so that it does not assert even the fact that if the Gaussian process is equivalent

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1) This work was partially supported by the Yukawa Foundation.
2) The Volterra kernel $l(s, t) \in L^2([0, T]^2)$ means $l(s, u)=0$ for $s<u$. 

to Wiener process in the time interval \([0, T]\), it is so in any subinterval \([0, t_0]\) for each \(t_0 \in [0, T]\). Such a fact is clarified in the representation (1.1).

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2. Preliminaries

Let \((\Omega, \mathcal{B}, P)\) be a complete probability space, \(\{\mathcal{F}_t; t \in [0, T]\}\) a system of \(\sigma\)-subalgebras of \(\mathcal{B}\) which are increasing in \(t\), and \(\{X_t; t \in [0, T]\}\) a stochastic process on \((\Omega, \mathcal{B}, P)\). In the following discussion, time interval \([0, T]\) will be fixed.

**Definition 1.** When \((X_t, \mathcal{F}_t, P)\) satisfies the following conditions 1), 2) and 3), it is called a Wiener process:

1) The sample paths of \(X_t\) are continuous in \(t\), and \(X_0 = 0\).
2) For \(t \geq s, t, s \in [0, T]\), \(E(X_t | \mathcal{F}_s) = X_s\) with P-measure 1, where \(E(\cdot | \cdot)\) denotes the conditional expectation with respect to the measure \(P\).
3) \(E((X_t - X_s)^2 | \mathcal{F}_s) = t - s\) with P-measure 1, for \(t \geq s, t, s \in [0, T]\). This definition of Wiener process is due to Doob [1].

**Definition 2.** A stochastic process \(Y_t\), defined on \((\Omega, \mathcal{B}, P)\) (or simply, \((Y_t, P)\)) is called a Gaussian process, when the distribution of \((Y_t, Y_{t_2}, \ldots, Y_{t_n})\) with respect to \(P\) is subject to an \(N\)-dimensional Gaussian distribution.

Let \((Y_t, P)\) be a stochastic process. Let \(\bar{P}\) be a probability measure on \((\Omega, \mathcal{B})\) such that \(P\) and \(\bar{P}\) are mutually absolutely continuous, that is, \(\bar{P}(d\omega) = \varphi(\omega)P(d\omega)\) with a strictly positive \(\varphi\). Let \(\mathcal{F}_t\) be the \(\sigma\)-subalgebra of \(\mathcal{B}\), generated by \(\{Y_s, s \leq t\}\), adjoined with all \(P\)-negligible sets. Note that the notion of negligible sets is identical for both \(P\) and \(\bar{P}\).

**Definition 3.** A stochastic process \((Y_t, P)\) is said to be equivalent to Wiener process when there is a probability measure \(\bar{P}(d\omega) = \varphi(\omega)P(d\omega)\), such that \(P\) and \(\bar{P}\) are mutually absolutely continuous and such that \((Y_t, \mathcal{F}_t, \bar{P})\) is a Wiener process.

**Remark 1.** Suppose \(\mathcal{B} = \mathcal{F}_T\). Let \(\bar{P}\) be absolutely continuous relative to \(P\), that is, \(\bar{P}(d\omega) = \varphi(\omega)P(d\omega)\) with non-negative \(\varphi\). If \(Y_t\) is Gaussian with respect to both \(P\) and \(\bar{P}\), then \(P\) and \(\bar{P}\) are mutually absolutely continuous by Hajek and Feldman's result (see Rozanov [9]).

**Remark 2.** When \((Y_t, P)\) is equivalent to Wiener process, we can assume that the sample paths of \(Y_t\) are continuous by choosing a suitable modification,
3. Necessary condition and uniqueness

The purpose of this section is to prove the following theorem.

**Theorem 1.** Suppose that a Gaussian process \((Y_t, P), t \in [0, T]\), with mean 0, is equivalent to Wiener process. Then there exists a Wiener process \((X_t, \mathbb{Y}_t, P)\) and a Volterra kernel \(l(s, u) \in L^2([0, T]^2)\) such that \(Y_t\) is represented, with \(P\)-measure 1, by the formula

\[
Y_t = X_t - \int_0^t \left( \int_0^s l(s, u) \, dX_u \right) \, ds \quad \text{for every } t \in [0, T].
\]

Moreover, such \(X_t\) and \(l(s, u) \in L^2([0, T]^2)\) are unique.

For the proof of Theorem 1, we need some lemmas. Let \(\varphi(\omega)\) be the density \(dP(\omega)/dP(\omega)\) in Definition 3.

**Lemma 1.** Let \(M_t\) be a right continuous modification of the martingale \(E\left( \frac{1}{\varphi} | \mathbb{Y}_t \right)\) with respect to \((\mathbb{Y}_t, P)\). Then,

1) \(P(M_t > 0 \text{ for } t \in [0, T]) = 1\).

2) \(M_t\) is represented by

\[
M_t = \exp \left\{ \int_0^t f(s, \omega) \, dY_s - \frac{1}{2} \int_0^t f^2(s, \omega) \, ds \right\} \quad \text{for any } t \in [0, T],
\]

where \(f(s, \omega)\) is a function which is (i) \((s, \omega)\)-measurable and (ii) \(\mathbb{Y}_s\)-measurable for each \(s \in [0, T]\), and which satisfies

\[
(iii) \quad P\left( \int_0^T f^2(s, \omega) \, ds < \infty \right) = 1.
\]

**Proof.** Let us put

\[
\tau_0 = \begin{cases} \inf \{t; M_t = 0\} \\ T, \quad \text{if } \{t; M_t = 0\} = \emptyset, \end{cases}
\]

then \(\tau_0\) is a stopping time relative to \((\mathbb{Y}_t)\). By the optional sampling theorem

\[
E(M_T | \mathbb{Y}_{\tau_0}) = M_{\tau_0} \quad \text{with } P\text{-measure 1.}
\]

So we get

\[
\int_{\{\tau_0 < T\}} M_T P(d\omega) = \int_{\{\tau_0 < T\}} M_{\tau_0} P(d\omega) = 0.
\]

On the other hand, since \(M_T(\omega) = E(\varphi^{-1}| \mathbb{Y}_T) > 0\), we get \(P(\tau_0 < T) = 0\) from the above equality. The proof of 1) is finished.
Since \((Y_t, \mathcal{F}_t, \tilde{P})\) is a Wiener process, the martingle \(M_t\) is represented by

\[
M_t = \int_0^t g(s, \omega) dY_s + 1 \quad \text{for any } t \in [0, T],
\]

according to Kunita-S. Watanabe [6], where \(g(s, \omega)\) satisfies (i), (ii) and (iii) of this lemma by replacing \(f(s, \omega)\) with \(g(s, \omega)\). We can now apply Itô's formula [5] and we get

\[
\log M_t = \log(1 + \int_0^t g(s, \omega) dY_s)
\]

\[
= \int_0^t \frac{1}{M_s} g(s, \omega) dY_s - \frac{1}{2} \int_0^t \frac{1}{M_s^2} g(s, \omega)^2 ds.
\]

Put

\[
f(s, \omega) = \frac{1}{M_s} g(s, \omega),
\]

then \(f(s, \omega)\) satisfies (i), (ii) and (iii). The proof of this lemma is finished.

**Lemma 2.** (Girsanov [2]). Let \(f(s, \omega)\) be the function of Lemma 1 and let

\[
X_t = Y_t - \int_0^t f(s, \omega) ds.
\]

Then \((X_t, \mathcal{F}_t, P(d\omega))\) is a Wiener process.

The next lemma plays an important role.

**Lemma 3.** Under the same assumption as in Lemma 1, if \((Y_t, P)\) is a Gaussian process, then it follows that

\[
E\left(\int_0^T f(s, \omega) ds\right) < \infty \quad \text{and} \quad \tilde{E}\left(\int_0^T f(s, \omega) ds\right) < \infty.
\]

Proof. First let us note the fact that

\[
\tilde{E}(M_T \log M_T) = K < \infty,
\]

which is due to Hajek-Feldman (see Rozanov [9] for a simple proof), because \(Y_t\) is a Gaussian process with respect to two measures \(P\) and \(\tilde{P}\). Since \(x \log x\) is a convex function, \(\{M_t \log M_t\}_{t \in [0, T]}\) is a submartingale with respect to \(\{\mathcal{F}_t\}\) in the probability space \(\{\Omega, \mathcal{B}, \tilde{P}\}\). Therefore, by the optional sampling theorem,

\[
K \geq \tilde{E}(M_{T_{n \wedge T}} \log M_{T_{n \wedge T}}),
\]

where \(\{T_n\}_{n=1, 2, \ldots}\) is an arbitrary sequence of stopping times increasing to \(T\). Moreover,

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3) See Supplement.
\[ \tilde{E}(M_{T_n \wedge T} \log M_{T_n \wedge T}) = \tilde{E}(M_T \log M_{T_n \wedge T}) \]
\[ = E\left( \int_0^{T_n \wedge T} f dY_s - \frac{1}{2} \int_0^{T_n \wedge T} f^2 ds \right) \]
\[ = E\left( \int_0^{T_n \wedge T} f dX_s + \frac{1}{2} \int_0^{T_n \wedge T} f^2 ds \right). \]

The last equality holds by Girsanov [2]. The process \( \left\{ \int_0^t f(s, \omega) dX_s \right\}_{t \in [0, T]} \) is a local martingale with respect to \( \mathcal{F}_n \) in \((\Omega, \mathcal{F}, P)\), so we can choose a sequence \( \{T_n\} \) such that \( \left\{ \int_0^{T_n \wedge T} f(s, \omega) dX_s \right\}_{t \in [0, T]} \) is a martingale for every \( n \). Then, the last expectation above is \( \frac{1}{2} E\left( \int_0^{T_n \wedge T} f^2 ds \right) \) for every \( n \), and we get the first part of this lemma as \( n \) tends to \( \infty \). On the other hand,
\[ M_t^{-1} = \exp \left( -\int_0^t f dX_s - \frac{1}{2} \int_0^t f^2 ds \right) \]
is a martingale with respect to \( \{\mathcal{F}_n\} \) in \((\Omega, \mathcal{F}, P)\). Therefore, we can similarly get the second part.

**Lemma 4.** Under the same assumption as in Lemma 3, let \( \mathcal{K}_t \) be the linear manifold spanned by \( \{Y^s; s \leq t\} \) and let \( \mathcal{K}_t^{(P)} \) and \( \mathcal{K}_t^{(\tilde{P})} \) be the closure of \( \mathcal{K}_t \) by \( L^2 \)-norm relative to the measure \( P \) and \( \tilde{P} \), respectively. Then \( \mathcal{K}_t^{(P)} = \mathcal{K}_t^{(\tilde{P})} \). Moreover,
\[ F_t(\omega) = \int_0^t f(s, \omega) ds \]

belongs to \( \mathcal{K}_t^{(P)} \), where \( f(s, \omega) \) is the function of Lemma 1.

**Proof.** To prove the first half, let \( Z \in \mathcal{K}_t^{(P)} \) and \( \{Z_n\} \) be a sequence of \( \mathcal{K}_t \) converging to \( Z \) in \( L^2(P) \) sense. Then, there is a subsequence \( \{Z_{n_k}\} \) of \( \{Z_n\} \) converging to \( Z \) with \( P \)-measure 1. Since \( \{Z_{n_k}\} \) is a Gaussian system relative to the measure \( \tilde{P} \), so is \( \{Z_{n_k}\} \cup \{Z\} \), and the convergence \( \{Z_{n_k}\} \to \{Z\} \) takes place in \( L^2(\tilde{P}) \)-sense. Therefore, \( Z \in \mathcal{K}_t^{(\tilde{P})} \) or equivalently \( \mathcal{K}_t^{(P)} \subseteq \mathcal{K}_t^{(\tilde{P})} \). The converse relation is carried out by the same way.

Since \( (X_t, \mathcal{H}_t, P) \) is a martingale, the relation (3.2) implies that, for each \( h > 0 \), the equality
\begin{equation}
\int_0^t E(F_{s+h} | \mathcal{H}_s) - F_s ds = \int_0^t E(Y_{s+h} | \mathcal{H}_s) - Y_s ds \quad \text{for any } t > 0.
\end{equation}

4) We say a stochastic process \( L_t, t \in [0, T] \), is a local martingale with respect to \( \{\mathcal{F}_n\} \) in \((\Omega, \mathcal{F}, P)\) when there exists an increasing sequence of stopping times \( \{T_n\}_{n=1,2,...} \) with respect to \( \{\mathcal{F}_n\} \) such that \( T_n \to T \) with \( P \)-measure 1 and \( L_{T_n \wedge T} \) is a martingale with respect to \( \{\mathcal{F}_n\} \) for each \( n \) in \((\Omega, \mathcal{F}, P)\) (see [6]).

5) We define \( F_\cdot \equiv F_T \) and \( Y_\cdot = Y_T \) for \( t > T \), for convenient.
holds with $P$-measure 1. On the other hand, it is known that every $E[Y_{s+h} | \mathcal{F}_s]$, $s \leq t$, belongs to $\mathcal{M}_s^{(P)}$ because $(Y_t, P)$ is Gaussian. Hence it is sufficient to show that the left hand of (3.3) converges to $F_t$ in probability. Put

$$F^+_t = \int_0^t (f \vee 0) ds, \quad F^-_t = -\int_0^t (f \wedge 0) ds$$

and denote by $F^\pm_{h,t}$ the left hand of (3.3) replacing $F$ by $F^\pm$ there, respectively. Then $\{F^\pm_t\}$ is a continuous and increasing process adapted to $\mathcal{F}_t$ in the sense of Meyer [8]. Moreover, $\{F^\pm_t\}$ is integrable by Lemma 3. Hence, $F^\pm_t$ converges to $F_t$ as $h \to 0$ in $L^1(P)$-sense ([8] p. 126), respectively. Similarly $F^\pm_{h,t}$ converges to $F_t$ as $h \to 0$. Thus the proof is complete.

Proof of Theorem 1. 1°. Under the assumption of this theorem, we shall first prove that, with $P$-measure 1 the function $f(s, \omega)$ of Lemma 1 can be represented by

$$f(s, \omega) = \int_0^s k(s, u) dY_u \quad \text{for almost all } s \in [0, T],$$

where $k(s, u)$ is a Volterra kernel in $L^2([0, T]^2)$. It is well known that each element of $\mathcal{M}_s^{(P)}$ can be represented by a stochastic integral of the form $\int_0^t K(u) dY_u$. Hence $F_t$ of Lemma 4 is represented by $\int_0^t K(t, u) dY_u$. Noting that $F_t$ is continuous, we can choose $K(t, u)$ to be $(t, u)$-measurable by means of Slutsky's method [12]. Now, let $\Lambda$ denote the $(s, \omega)$-set

$$\{(s, \omega); \lim_{n \to \infty} n(F_s - F_{s-1/n}) \text{ does not exist or } \lim_{n \to \infty} n(F_s - F_{s-1/n}) = f(s, \omega)\},$$

where we define $F_s = 0$, for $s < 0$. Then $\Lambda$ is $(s, \omega)$-measurable and $\mu(\Lambda) = 0$, where $\mu$ is the product measure of $\bar{P}(d\omega)$ and Lebesgue measure $m(ds)$ on $[0, T]$. In fact, $m(\Lambda_{\omega}) = 0$ with $P$-measure 1, where $\Lambda_{\omega} = \{s; (s, \omega) \in \Lambda\}$. By Fubini's theorem it follows that $\bar{P}(\Lambda_s) = 0$ for almost all $s$, where $\Lambda_s = \{\omega; (s, \omega) \in \Lambda\}$. Therefore, for almost all $s$,

$$\lim_{n \to \infty} n(F_s - F_{s-1/n}) = f(s, \omega) \quad \text{for } \omega \in \Lambda_s,$$

and $f(s, \omega) \in \mathcal{M}_s^{(P)}$ for such $s$. Hence for almost all $s \in [0, T]$,

$$f(s, \omega) = \int_0^s k(s, u) dY_u \quad \text{with } P\text{-measure 1},$$

where $k(s, u)$ belongs to $L^2(du)$ for such $s$. Moreover, we can choose $k(s, u)$ to be $(s, u)$-measurable. Put
\[
\Lambda' = \{ \omega; f(s, \omega) + \int_0^s k(s, u) dY_u \} \cup \Lambda,
\]
then \( P(\Lambda'_s) = 0 \) for almost all \( s \in [0, T] \). Since

\[
\Lambda' = \{ (s, \omega); f(s, \omega) + \int_0^s k(s, u) dY_u \} \cup \Lambda
\]
is \((s, \omega)\)-measurable,

\[
f(s, \omega) = \int_0^s k(s, u) dY_u
\]
holds for \( s \in \Lambda'_s = \{ s; (s, \omega) \in \Lambda' \} \) with \( P\)-measure 1. By this fact, we can get

\[
X_t = Y_t - \int_0^t \left( \int_0^s k(s, u) dY_u \right) ds.
\]

By Lemma 2,

\[
E\left( \int_0^T \left( \int_0^s k(s, u) dY_u \right)^2 ds \right) = \int_0^T \mathbb{E}\left( \int_0^s k(s, u) dY_u \right)^2 ds
\]

\[
= \int_0^T \int_0^s k(s, u)^2 ds du < \infty,
\]
therefore we can see that \( k(s, u) \) is a Volterra kernel of \( L^2([0, T]^2) \).

2°. Next, we want to represent \( Y_t \) in the form of (3.1), by constructing the kernel \( l(s, t) \). For the Volterra kernel \( k(s, t) \), there is a resolvent kernel \( l(s, t) \) such that

\[
l(s, t) + k(s, t) - \int_t^s l(s, u) k(u, t) du = 0 \quad \text{in} \quad L^2([0, T]^2)
\]

(3.4)

\[
l(s, t) + k(s, t) - \int_t^s k(s, u) l(u, t) du = 0 \quad \text{in} \quad L^2([0, T]^2).
\]

For, the Neumann series for the Volterra kernel \( k(s, t) \) converges in the sense of \( L^2([0, T]^2) \), and the limit is the kernel \( l(s, t) \) (see Smith thesis [13]). Thus the equations

\[
(3.5)
\]

\[
X_t = \int_0^t \left( \int_0^s l(s, u) dX_u \right) ds
\]

\[
= Y_t - \int_0^t \left( \int_0^s k(s, u) dY_u \right) ds - \int_0^t \left( \int_0^s l(s, u) dY_u \right) ds
\]

\[
+ \int_0^t \left( \int_0^s l(s, u) \int_0^u k(u, v) dY_v du \right) ds
\]

\[
= Y_t - \int_0^t \left( \int_0^s k(s, u) dY_u \right) ds - \int_0^t \left( \int_0^s l(s, u) dY_u \right) ds
\]

\[
+ \int_0^t \left( \int_0^s l(s, u) k(u, v) dY_v \right) ds.
\]
hold with P-measure 1 for each $t \in [0, T]$. The last equality follows by using the formula
\[
\int_0^s \left( \int_0^v m(u, v) dY_u \right) dv = \int_0^s \left( \int_0^v m(u, v) dv \right) dY_u
\]
where $m(u, v) \in L^2([0, T]^2)$. By the equation (3.4), the stochastic process
\[
\int_0^t \left\{ -k(s, u) - l(s, u) + \int_u^t l(s, v) k(v, u) dv \right\} dY_u \bigg| ds
\]
is identically 0 with P-measure 1. Therefore the right side of (3.5) is equal to $Y_t$ with P-measure 1 for each $t \in [0, T]$.

This proves (3.1), for both side of (3.1) are continuous with P-measure 1.

3°. Finally we will discuss the uniqueness of the representation (3.1). In fact, we will prove a slightly stronger result than the uniqueness statement in the theorem. Suppose $Y_t$ has two representations such as
\[
Y_t = X_t^1 - \int_0^t \left( \int_0^v l(s, u) dX_u^1 \right) ds
\]
\[
= X_t^2 - \int_0^t h(s, \omega) ds,
\]
where $(X_t^1, \mathcal{F}, P)$ and $(X_t^2, \mathcal{F}, P)$ are Wiener processes and $h(s, \omega)$ is a function satisfying (i), (ii) and (iii) in Lemma 1. Then,
\[
X_t^1 - X_t^2 = \int_0^t \left[ h(s, \omega) - \int_0^v l(s, u) dX_u^1 \right] ds
\]
is a martingale with respect to $\{\mathcal{F}_t\}$. By the uniqueness of Meyer's decomposition ([7] p. 113), it follows easily that
\[
X_t^1 - X_t^2 = X_0^1 - X_0^2 = 0,
\]
\[
\int_0^t h(s, \omega) ds = \int_0^t \left( \int_0^v l(s, u) dX_u^1 \right) ds
\]
hold for any $t \in [0, T]$ with P-measure 1. By Fubini's theorem
\[
h(s, \omega) = \int_0^v l(s, u) dX_u^1, \quad \text{for almost all } (s, \omega) \in [0, T] \times \Omega.
\]
Thus, the proof of this theorem is completed.

Remark 3. Theorem 1 shows that $\mathfrak{F}_t = \mathcal{F}_t$ for each $t \in [0, T]$, where $\mathfrak{F}_t$ is the $\sigma$-algebra generated by $\{X_s; s \leq t\}$ and $P$-negligible sets. Therefore $(Y_t, P)$ has the proper canonical representation (3.1) with respect to the Wiener process $(X_t, \mathfrak{F}, P)$ in the sense of Hida [4].

In case of $\mathbb{E}(Y_t) \equiv 0$, we get the following theorem by the same method as in Theorem 1.
Theorem 1'. Suppose that a Gaussian process \((Y_t, P), t \in [0, T]\), is equivalent to Wiener process. Then there exists a Wiener process \((X_t, \mathcal{F}_t, P)\) such that \(Y_t\) is represented, with \(P\)-measure 1, by the formula

\[ Y_t = X_t - \int_0^t \left( \int_0^s l(s, u)dX_u \right) ds - \int_0^t a(s) ds, \quad \text{for any } t \in [0, T] \]

where \(l(s, u)\) is a Volterra kernel belonging to \(L^2([0, T])\) and \(a(s) \in L^2([0, T])\). Moreover such \(X_t, l(s, u)\) and \(a(s)\) are unique.

4. Sufficient condition

In this section we will prove the converse of Theorem 1. This result is implied in Shepp [11] or H. Sato [10], if we note that the Volterra kernel has only zero as its eigenvalues. But, our proof is simpler than theirs and the density is explicitly evaluated.

Theorem 2. If \((X_t, \mathcal{F}_t, P)\) is a Wiener process, then the Gaussian process with respect to the measure \(P\)

\[ Y_t = X_t - \int_0^t \left( \int_0^s l(s, u)dX_u \right) ds \]

is equivalent to Wiener process, where \(l(s, u) \in L^2([0, T])\) is a Volterra kernel. In this case the density \(\varphi(\omega)\) in Definition 3 can be taken as follows:

\[ \varphi(\omega) = \exp \left\{ \int_0^T \left( \int_0^s l(s, u)dX_u \right) dX_s - \frac{1}{2} \int_0^T \left( \int_0^s l(s, u)dX_u \right)^2 ds \right\} . \]

Proof. This theorem is established if we can show that

\[ E\left( \exp \left\{ \int_0^T \left( \int_0^s l(s, u)dX_u \right) dX_s - \frac{1}{2} \int_0^T \left( \int_0^s l(s, u)dX_u \right)^2 ds \right\} \right) = 1, \]

for then \((Y_t, \mathcal{F}_t, N_T(\omega)P(d\omega))\) is a Wiener process by virtue of Girsanov [2]. Let

\[ N_t = \exp \left\{ \int_0^t \left( \int_0^s l(s, u)dX_u \right) dX_s - \frac{1}{2} \int_0^t \left( \int_0^s l(s, u)dX_u \right)^2 ds \right\} . \]

Then the process \((N_t, \mathcal{F}_t, P)\) is a local martingale (see [6]) and we may consider \(N_t\) has continuous paths. Therefore, there is an increasing sequence of stopping times \(\{T_n\}\) which tends to \(T\) with \(P\)-measure 1 such that \(\{N_t^\infty = N_t_{t \leq T_n}, \mathcal{F}_t, P\}\) is a martingale for each \(n\). Hence, it is enough to prove that \(\{N_T^\infty\}\) is uniformly integrable, because \(E(N_T) = \lim_{n \to \infty} E(N_{T_n}) = 1\). Observing that

6) The density \(\varphi(\omega)\) may not be unique in general. But, the function of (4.1) is the only density which is measurable with respect to \(\{X_t, 0 \leq t \leq T\}\).
\[ N_t^n = \exp \left\{ \int_0^t \left( X_{t_0, T_n}(s) \int_0^s l(s, u) dX_u \right) ds - \frac{1}{2} \int_0^t \left( X_{t_0, T_n}(s) \int_0^s l(s, u) dX_u \right)^2 ds \right\}, \]

where

\[ X_{t_0, T_n}(s) = \begin{cases} 1 & \text{if } s \leq T_n, \\ 0 & \text{if } s > T_n, \end{cases} \]

define

\[ Y_t^n = X_t - \int_0^t X_{t_0, T_n}(s) \left( \int_0^s l(s, u) dX_u \right) ds \]

\[ = X_t - \int_0^{T_n} \left( \int_0^s l(s, u) dX_u \right) ds. \]

Then Girsanov's theorem, applied to \( N_T^n \), tells us that \( (Y_t^n, \mathcal{F}_t, N_T^n P(d\omega)) \) is a Wiener process for each \( n \). On the other hand, a similar calculation as in (3.5) shows that

\[ X_t = Y_t - \int_0^t \left( \int_0^s k(s, u) dY_u \right) ds \]

holds for any \( t \in [0, T] \) with \( P \)-measure 1, where \( k(s, u) \) is the resolvent kernel of \( l(s, u) \) which satisfies (3.4). Therefore, it follows that

\[ \int_0^t \left( \int_0^s k(s, u) dY_u \right) ds = \int_0^t \left( \int_0^s l(s, u) dX_u \right) ds, \quad t \in [0, T], \]

or

\[ \int_0^t k(s, u) dY_u = \int_0^t l(s, u) dX_u \quad \text{for almost all } s \in [0, T]. \]

Then

\[ \log N_t^n = \int_0^{t \wedge T_n} \left( \int_0^s l(s, u) dX_u \right) dX_s - \frac{1}{2} \int_0^{t \wedge T_n} \left( \int_0^s l(s, u) dX_u \right)^2 ds \]

\[ = \int_0^{t \wedge T_n} \left( \int_0^s l(s, u) dX_u \right) dY_s + \frac{1}{2} \int_0^{t \wedge T_n} \left( \int_0^s l(s, u) dX_u \right)^2 ds \]

\[ = \int_0^{t \wedge T_n} \left( \int_0^s k(s, u) dY_u \right) dY_s + \frac{1}{2} \int_0^{t \wedge T_n} \left( \int_0^s k(s, u) dY_u \right)^2 ds \]

\[ = \int_0^{t \wedge T_n} \left( \int_0^s k(s, u) dY_u \right) dY_s + \frac{1}{2} \int_0^{t \wedge T_n} \left( \int_0^s k(s, u) dY_u \right)^2 ds, \]

for any \( t \in [0, T] \), with \( P \)-measure 1. The last equality follows from the fact that

\[ \int_0^{t \wedge T_n} f(s, \omega) dY_s = \int_0^{t \wedge T_n} f(s, \omega) dY_s \]

holds for any \( f(s, \omega) \) satisfying (i), (ii) and (iii) in Lemma 1. As \( (Y_t^n, \mathcal{F}_t, \tilde{P}^n, n=1, 2, \ldots) \), are Wiener processes, where \( \tilde{P}^n(d\omega) = N_T^n(\omega) P(d\omega) \), we have
$E((\log N_T^n) N_T^n) = E^n(\log N_T^n)$

$= E^n\left(\int_0^{T/2} \left(\int_0^s k(s, u)dY_u^2\right)ds + \frac{1}{2} \int_0^{T/2} \left(\int_0^s k(s, u)dY_u^2\right)^2 ds\right)$

$= \frac{1}{2} E^n\left(\int_0^{T/2} \left(\int_0^s k(s, u)dY_u^2\right)ds\right)$

$\leq \frac{1}{2} \int_0^T \left(\int_0^s k(s, u)^2 du\right)ds = K.$

Hence, the family $\{N_T^n\}_{n=1,2,...}$ is uniformly integrable. This is our desired result.

More generally, the following theorem can be proved analogously.

**Theorem 2'.** Let $(X_t, \mathcal{F}_t, P)$, $l(s, t)$ be as in Theorem 2 and $a(s)$ be of $L^2([0, T])$. Then the Gaussian process with respect to the measure $P$

$$Y_t = X_t - \int_0^t \left(\int_0^s l(s, u)dX_u + a(s)\right)ds$$

is equivalent to Wiener process. In this case, the function

$$\varphi(\omega) = \exp\left\{T \left(\int_0^T \left(\int_0^s l(s, u)dX_u + a(s)\right)dX_s\right)\right. - \frac{1}{2} \int_0^T \left(\int_0^s l(s, u)dX_u + a(s)\right)^2 ds\right\}$$

defines a density in Definition 3 such that $(X_t, \mathcal{F}_t, \varphi P(\omega))$ is a Wiener process.

5. **Related topics**

1. A decomposition of a positive definite operator $(I - H)$ on $L^2([0, T])$.

**Proposition 1.** A Gaussian process $(Y_t, P)$, $t \in [0, T]$, with mean 0 is equivalent to the Wiener process if and only if $(Y_t, P)$ has the covariance

$$E(Y_t, Y_s) = (t \wedge s) - \int_0^{t \wedge s} \left(\int_u^{t \wedge s} l(s, u)du\right)ds$$

$$- \int_0^{t \wedge s} \left(\int_u^{t \wedge s} l(s, u)du\right)ds$$

$$+ \int_0^{t \wedge s} \left(\int_u^{t \wedge s} l(s, u)du\right)ds,$$

with a Volterra kernel $l(s, u)$ in $L^2([0, T])$. Moreover such $l(s, u)$ is unique.

This proposition follows immediately from Theorem 1 and Theorem 2. As an application to $L^2$-theory, we can get

**Proposition 2.** Let $H$ be a symmetric integral operator on $L^2([0, T])$. 
Then $I-H$ is strictly positive definite if and only if there is an integral operator $L$ of Volterra type such that

\[(5.1) \quad I-H = (I-L)(I-L^*),\]

where $L^*$ is the adjoint of $L$. Furthermore, such a decomposition is unique.

Proof. "If" part: Since $L$ is of Volterra type, the integral equation

\[(I-L)f = 0\]

has the unique solution $f=0$ in $L^2([0, T])$ (Simthesis [13]). Therefore

\[(I-L^*)g = 0\]

has the unique solution $g=0$ in $L^2([0, T])$. Hence, for $g \neq 0$,

\[
((I-H)g, g) = ((I-L)(I-L^*)g, g) = ((I-L^*)g, (I-L^*)g)>0.
\]

"Only if" part: Let $h(u, v)$ be the kernel which defines the operator $H$. Then, by a result of Shepp [11], there is a Gaussian process $(Y_t, P)$, equivalent to Wiener process, with covariance

\[
E(Y_{t_1}Y_{t_2}) = (t_1 \wedge t_2) - \int_0^{t_1} \int_0^{t_2} h(u, v) \, du \, dv.
\]

Hence, by Proposition 5.1, there is a unique Volterra kernel $l(u, v)$ such that

\[
h(u, v) = l(u, v) + l(v, u) - \int_0^T l(u, w) l(u, w) \, dw.
\]

If we define the operator $(I-L)$ by

\[(I-L)f(u) = f(u) - \int_0^T l(u, v) f(v) \, dv = f(u) - \int_0^u l(u, v) f(v) \, dv,
\]

then

\[(I-L)(I-L^*) f(u) = (I-H)f(u).
\]


If $(X_t, \mathcal{F}_t, P)$, $t \in [0, 1]$, is a Wiener process, then

\[
Y_t = (1-t) \int_0^t \frac{dX_u}{1-u} = X_t - \int_0^t \left( \int_0^s \frac{-1}{1-u} \, dX_u \right) ds = 0 \quad 0 \leq t < 1
\]

is the so-called pinned Wiener process with mean 0 and covariance
$E(Y_t, Y_s) = (1-t)s$, for $t<s$. In this case,

$$l(s, u) = \begin{cases} -1/u & u \leq s \\ 1-u & u > s \end{cases}, \quad k(s, u) = \begin{cases} 1 & u \leq s \\ 1-s & u > s \end{cases}.$$  

Evidently, the Gaussian process $(Y_t, P)$ is equivalent to Wiener process in $[0, t_s]$, $t_s < 1$, by Theorem 2, while $(Y_t, P)$ is not equivalent to Wiener process in $[0, 1]$, because $Y_1 = 0$, with $P$-measure 1. This phenomenon can be explained from that the kernel $l(s, u)$ does not belong to $L^2([0, 1]^2)$. The process $Y_t$ is the unique solution of the stochastic integral equation

$$Y_t = X_t + \int_0^t \frac{1}{1-s} \int_0^s dY_u \quad t < 1,$$

with the initial condition $Y_0 = 0$.

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**References**


Supplement for the proof of Lemma 1.

Professor T. Watanabe pointed out that, for the representation
\[ M_t = \int_0^t g(s, \omega) dY_s + 1, \]
it is necessary to prove that there is an increasing sequence \( \{T_n\}_{n=1,2,...} \) of stopping times which converges to \( T \) and \( \{M_{t \wedge T_n}\}_{t \in [0,T]} \) \((n=1,2,\ldots)\) are square integrable martingales with respect to \( \mathcal{F}_t \) in \((\Omega, \mathcal{F}, P)\) (see Kunita-S. Watanabe [6]).

For the proof, we will first show that \( M_t \) has continuous paths. Set
\[ M_t^N = \mathbb{E}\left( \frac{1}{\varphi} \wedge N \mid \mathcal{F}_t \right), \]
then \( M_t - M_t^N \) is a positive martingale and \( M_t^N \) converges to \( M_T \) in \( L^1(\hat{P}) \) sense. Using Doob’s inequality ([1] p. 353),
\[ \hat{P}(\sup_{0 \leq s \leq T} (M_t - M_t^N) \geq \lambda) \leq \frac{\mathbb{E}(M_T - M_T^N)}{\lambda}. \]
This shows \( M_t^N \) converges to \( M_t \) uniformly in probability \( \hat{P} \). On the other hand, \( \{M_t^N\}_{t \in [0,T]} \) are square integrable martingale, and so they have continuous paths. Hence \( M_t \) has continuous paths.

Next, if we choose the sequence of stopping times \( \{T_n\}_{n=1,2,...} \) such that
\[ T_n = \begin{cases} \min \{t; M_t \geq n\} & \text{if } \{t; M_t \geq n\} \neq \emptyset, \\ T & \text{if } \{t; M_t \geq n\} = \emptyset, \end{cases} \]
then \( T_n \) converges to \( T \) and \( \{M_{t \wedge T_n}\}_{t \in [0,T]} \) \((n=1,2,\ldots)\) are square integrable martingales, because of the continuity of paths of \( M_t \).