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Osaka J. Math.
5 (1968), 299-312

# REPRESENTATION OF GAUSSIAN PROCESSES EQUIVALENT TO WIENER PROCESS[1]

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(Received June 24, 1968)

## 1. Introduction

The purpose of this paper is to get a canonical representation of Gaussian processes which are equivalent (or mutually absolutely continuous) to Wiener process. The main result is this. Suppose we are given a Gaussian process $Y_t$ on a probability space $(\Omega, \mathfrak{B}, P)$, which is equivalent to Wiener process. Then a Wiener process $X_t$ is constructed on $(\Omega, \mathfrak{B}, P)$ as a functional of $\{Y_s; s \leqq t\}$ and, conversely, $Y_t$ is represented as a measurable functional of $\{X_s; s \leqq t\}$ for each $t \in [0, T]$. In case of $E(Y_t)=0$, $t \in [0, T]$, $Y_t$ is represented by the formula

$$(1.1) \qquad Y_t = X_t - \int_0^t \left( \int_0^s l(s, u) dX_u \right) ds ,$$

where $l(s, t)$ is a Volterra kernel belonging to $L^2([0, T]^2)$[2], and the representation of $Y_t$ is unique (Theorem 1). Conversely, if a Wiener process $X_t$ is given and if $Y_t$ is represented by (1.1), $Y_t$ is equivalent to Wiener process (Theorem 2). The density of the transformed measure with respect to which $Y_t$ is a Wiener process is easily evaluated in the proof of Theorem 2. The proof of these facts is based heavily upon martingale theory due to Meyer [8] and Kunita-S. Watanabe [6].

The conditions for a Gaussian process to be equivalent to Wiener process have been obtained in terms of the mean and the covariance by Shepp [11] and Golosov [3]. Moreover, by the method of linear transformations of Wiener space, Shepp [11] and H. Sato [10] have obtained the following representation

$$Y_t = X_t - \int_0^t \left( \int_0^T m(s, u) dX_u \right) ds ,$$

where $m(s, u)$ is a kernel of $L^2([0, T]^2)$ with some additional conditions. This representation involves the stochastic integral on the fixed time interval $[0, T]$, so that it does not assert even the fact that if the Gaussian process is equivalent

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1) This work was partially supported by the Yukawa Foundation.
2) The Volterra kernel $l(s, t) \in L^2([0, T]^2)$ means $l(s, u)=0$ for $s<u$.

to Wiener process in the time interval $[0, T]$, it is so in any subinterval $[0, t_0]$ for each $t_0 \in [0, T]$. Such a fact is clarified in the representation (1.1).

The author wishes to thank Professor H. Kunita for his helpful comments; especially the proof of Lemma 3 is due to his idea. The author also enjoyed many advices from the members of "Seminar on Probability" in Osaka-Kyoto and Nagoya.

## 2. Preliminaries

Let $(\Omega, \mathfrak{B}, P)$ be a complete probability space, $\{\mathfrak{F}_t; t \in [0, T]\}$ a system of $\sigma$-subalgebras of $\mathfrak{B}$ which are increasing in $t$, and $\{X_t; t \in [0, T]\}$ a stochastic process on $(\Omega, \mathfrak{B}, P)$. In the following discussion, time interval $[0, T]$ will be fixed.

DEFINITION 1. When $(X_t, \mathfrak{F}_t, P)$ satisfies the following conditions 1), 2) and 3), it is called a *Wiener process*:

1) The sample paths of $X_t$ are continuous in $t$, and $X_0 = 0$.

2) For $t \geqq s$, $t, s \in [0, T]$, $E(X_t | \mathfrak{F}_s) = X_s$ with P-measure 1, where $E(\cdot | \cdot)$ denotes the conditional expectation with respect to the measure $P$.

3) $E((X_t - X_s)^2 | \mathfrak{F}_s) = t - s$ with P-measure 1, for $t \geqq s$, $t, s \in [0, T]$.

This definition of Wiener process is due to Doob [1].

DEFINITION 2. A stochastic process $Y_t$, definied on $(\Omega, \mathfrak{B}, P)$ (or simply, $(Y_t, P)$) is called a *Gaussian process*, when the distribution of $(Y_{t_1}, Y_{t_2}, \cdots, Y_{t_N})$ with respect to $P$ is subject to an $N$-dimensional Gaussian distribution.

Let $(Y_t, P)$ be a stochastic process. Let $\tilde{P}$ be a probability measure on $(\Omega, \mathfrak{B})$ such that $P$ and $\tilde{P}$ are mutually absolutely continous, that is, $\tilde{P}(d\omega) = \varphi(\omega) P(d\omega)$ with a strictly positive $\varphi$. Let $\mathfrak{Y}_t$ be the $\sigma$-subalgebra of $\mathfrak{B}$, generated by $\{Y_s, s \leqq t\}$, adjoined with all $P$-negligible sets. Note that the notion of negligible sets is identical for both $P$ and $\tilde{P}$.

DEFNITION 3. A stochastic process $(Y_t, P)$ is said to be *equivalent to Wiener process* when there is a probability measure

$$\tilde{P}(d\omega) = \varphi(\omega) P(d\omega),$$

such that $P$ and $\tilde{P}$ are mutually absolutely continuous and such that $(Y_t, \mathfrak{Y}_t, \tilde{P})$ is a Wiener process.

REMARK 1. Suppose $\mathfrak{B} = \mathfrak{Y}_T$. Let $\tilde{P}$ be absolutely continous relative to $P$, that is, $\tilde{P}(d\omega) = \varphi(\omega) P(d\omega)$ with non-negative $\varphi$. If $Y_t$ is Gaussian with respect to both $P$ and $\tilde{P}$, then $P$ and $\tilde{P}$ are mutually absolutely continous by Hajék and Feldman's result (see Rozanov [9]).

REMARK 2. When $(Y_t, P)$ is equivalent to Wiener process, we can assume that the sample paths of $Y_t$ are continous by choosing a suitable modification,

## 3. Necessary condition and uniqueness

The purpose of this section is to prove the following theorem.

**Theorem 1.** *Suppose that a Gaussian process* $(Y_t, P)$, $t\in[0, T]$, *with mean* 0, *is equivalent to Wiener process. Then there exists a Wiener process* $(X_t, \mathfrak{Y}_t, P)$ *and a Volterra kernel* $l(s, u)\in L^2([0, T]^2)$ *such that* $Y_t$ *is represented, with P-measure* 1, *by the formula*

$$Y_t = X_t - \int_0^t \left(\int_0^s l(s, u)\, dX_u\right) ds \qquad \text{for every } t\in[0, T]\,. \tag{3.1}$$

*Moreover, such* $X_t$ *and* $l(s, u)\in L^2([0, T]^2)$ *are unique.*

For the proof of Theorem 1, we need some lemmas. Let $\varphi(\omega)$ be the density $d\tilde{P}(\omega)/dP(\omega)$ in Definition 3.

**Lemma 1.** *Let* $M_t$ *be a right continuous modification of the martingle* $\tilde{E}\left(\frac{1}{\varphi}\,|\,\mathfrak{Y}_t\right)$ *with respect to* $(\mathfrak{Y}_t, P)$. *Then,*

1) $$P(M_t>0 \quad \text{for} \quad t\in[0, T]) = 1\,.$$

2) $M_t$ *is represented by*

$$M_t = \exp\left\{\int_0^t f(s, \omega)\, dY_s - \frac{1}{2}\int_0^t f^2(s, \omega)\, ds\right\} \qquad \text{for any} \quad t\in[0, T]\,,$$

*where* $f(s, \omega)$ *is a function which is* (i) $(s, \omega)$*-measurable and* (ii) $\mathfrak{Y}_s$*-measurable for each* $s\in[0, T]$, *and which satisfies*

(iii) $$P\left(\int_0^T f^2(s, \omega)\, ds<\infty\right) = 1\,.$$

Proof. Let us put

$$\tau_0 = \begin{cases} \inf\{t;\, M_t = 0\} & \\ T\,, & \text{if} \quad \{t;\, M_t = 0\} = \phi\,, \end{cases}$$

then $\tau_0$ is a stopping time relative to $\{\mathfrak{Y}_t\}$. By the optional sampling theorem

$$\tilde{E}(M_T\,|\,\mathfrak{Y}_{\tau_0}) = M_{\tau_0} \qquad \text{with } P\text{-measure } 1.$$

So we get

$$\int_{\{\tau_0<T\}} M_T \tilde{P}(d\omega) = \int_{\{\tau_0<T\}} M_{\tau_0} \tilde{P}(d\omega) = 0.$$

On the other hand, since $M_T(\omega)=\tilde{E}(\varphi^{-1}\,|\,\mathfrak{Y}_T)>0$, we get $P(\tau_0<T)=0$ from the above equality. The proof of 1) is finished,

Since $(Y_t, \mathfrak{Y}_t, \tilde{P})$ is a Wiener process, the martingle $M_t$ is represented by

$$M_t = \int_0^t g(s, \omega) dY_s + 1 \qquad \text{for any} \quad t \in [0, T],^{3)}$$

according to Kunita-S. Watanabe [6], where $g(s, \omega)$ satisfies (i), (ii) and (iii) of this lemma by replacing $f(s, \omega)$ with $g(s, \omega)$. We can now apply Itô's formula [5] and we get

$$\begin{aligned} \log M_t &= \log(1 + \int_0^t g(s, \omega) dY_s) \\ &= \int_0^t \frac{1}{M_s} g(s, \omega) dY_s - \frac{1}{2} \int_0^t \frac{1}{M_s^2} g(s, \omega)^2 ds . \end{aligned}$$

Put

$$f(s, \omega) = \frac{1}{M_s} g(s, \omega),$$

then $f(s, \omega)$ satisfies (i), (ii) and (iii). The proof of this lemma is finished.

**Lemma 2.** (Girsanov [2]). *Let $f(s, \omega)$ be the function of Lemma* 1 *and let*

$$X_t = Y_t - \int_0^t f(s, \omega) ds . \tag{3.2}$$

*Then $(X_t, \mathfrak{Y}_t, P(d\omega))$ is a Wiener process.*

The next lemma plays an important role.

**Lemma 3.** *Under the same assumption as in Lemma* 1, *if $(Y_t, P)$ is a Gaussian process, then it follows that*

$$E\left(\int_0^T f(s, \omega)^2 ds\right) < \infty \quad \textit{and} \quad \tilde{E}\left(\int_0^T f(s, \omega)^2 ds\right) < \infty .$$

Proof. First let us note the fact that

$$\tilde{E}(M_T \log M_T) = K < \infty ,$$

which is due to Hajék-Feldman (see Rozanov [9] for a simple proof), because $Y_t$ is a Gaussian process with respect to two measures $P$ and $\tilde{P}$. Since $x \log x$ is a convex function, $\{M_t \log M_t\}_{t \in [0,T]}$ is a submartingale with respect to $\{\mathfrak{Y}_t\}$ in the probability space $\{\Omega, \mathfrak{B}, \tilde{P}\}$. Therefore, by the optional sampling theorem,

$$K \geqq \tilde{E}(M_{T_n \wedge T} \log M_{T_n \wedge T}),$$

where $\{T_n\}_{n=1,2,\cdots}$ is an arbitrary sequence of stopping times increasing to $T$. Moreover,

3) See Supplement.

$$\begin{aligned}\tilde{E}(M_{T_n\wedge T}\log M_{T_n\wedge T}) &= \tilde{E}(M_T \log M_{T_n\wedge T})\\ &= E\Big(\int_0^{T_n\wedge T} f\,dY_s - \frac{1}{2}\int_0^{T_n\wedge T} f^2 ds\Big)\\ &= E\Big(\int_0^{T_n\wedge T} f\,dX_s + \frac{1}{2}\int_0^{T_n\wedge T} f^2 ds\Big).\end{aligned}$$

The last equality holds by Girsanov [2]. The process $\left\{\int_0^t f(s,\omega)\,dX_s\right\}_{t\in[0,T]}$ is a local martingale[4] with respect to $\{\mathfrak{Y}_t\}$ in $(\Omega, \mathfrak{B}, P)$, so we can choose a sequence $\{T_n\}$ such that $\left\{\int_0^{t\wedge T_n} f(s,\omega)dX_s\right\}_{t\in[0,T]}$ is a martingale for every $n$. Then, the last expectation above is $\frac{1}{2}E\Big(\int_0^{T_n\wedge T} f^2 ds\Big)$ for every $n$, and we get the first part of this lemma as $n$ tends to $\infty$. On the other hand,

$$M_t^{-1} = \exp\Big(-\int_0^t f dX_s - \frac{1}{2}\int_0^t f^2 ds\Big)$$

is a martingale with respect to $\{\mathfrak{Y}_t\}$ in $(\Omega, \mathfrak{B}, P)$. Therefore, we can similarly get the second part.

**Lemma 4.** *Under the same assumption as in Lemma* 3, *let* $\mathfrak{M}_t$ *be the linear manifold spanned by* $\{Y_s; s\leqq t\}$ *and let* $\overline{\mathfrak{M}}_t^{(P)}$ *and* $\overline{\mathfrak{M}}_t^{(\tilde{P})}$ *be the closure of* $\mathfrak{M}_t$ *by* $L^2$-*norm relative to the measure* $P$ *and* $\tilde{P}$, *respectively. Then* $\overline{\mathfrak{M}}_t^{(P)} = \overline{\mathfrak{M}}_t^{(\tilde{P})}$. *Moreover,*

$$F_t(\omega) = \int_0^t f(s,\omega)ds$$

*belongs to* $\overline{\mathfrak{M}}_t^{(P)}$, *where* $f(s,\omega)$ *is the function of Lemma* 1.

Proof. To prove the first half, let $Z\in\overline{\mathfrak{M}}_t^{(P)}$ and $\{Z_n\}$ be a sequence of $\mathfrak{M}_t$ converging to $Z$ in $L^2(P)$ sense. Then, there is a subsequence $\{Z_{n_k}\}$ of $\{Z_n\}$ converging to $Z$ with $P$-measure 1. Since $\{Z_{n_k}\}$ is a Gaussian system relative to the measure $\tilde{P}$, so is $\{z_{n_k}\}\cup\{Z\}$, and the convergence $\{Z_{n_k}\}\to\{Z\}$ takes place in $L^2(\tilde{P})$-sense. Therefore, $Z\in\overline{\mathfrak{M}}_t^{(\tilde{P})}$ or equivalently $\overline{\mathfrak{M}}_t^{(P)}\subseteq\overline{\mathfrak{M}}_t^{(P)}$. The converse relation is carried out by the same way.

Since $(X_t, \mathfrak{Y}_t, P)$ is a martingale, the relation (3.2) implies that, for each $h>0$, the equality

$$\int_0^t \frac{E(F_{s+h}\,|\,\mathfrak{Y}_s)-F_s}{h}ds = \int_0^t \frac{E(Y_{s+h}\,|\,\mathfrak{Y}_s)-Y_s}{h}ds \quad \text{for any } t^{5)} \tag{3.3}$$

4) We say a stochastic process $L_t$, $t\in[0,T]$, is a local martingale with respect to $\{\mathfrak{Y}_t\}$ in $(\Omega, \mathfrak{B}, P)$ when there exists an increasing sequence of stopping times $\{T_n\}_{n=1,2,\dots}$ with respect to $\{\mathfrak{Y}_t\}$ such that $T_n\to T$ with $P$-measure 1 and $L_{t\wedge T_n}$ is a martingale with respect to $\{\mathfrak{Y}_t\}$ for each $n$ in $(\Omega, \mathfrak{B}, P)$ (see [6]).

5) We define $F_t=F_T$ and $Y_t=Y_T$ for $t>T$, for convenient.

holds with $P$-measure 1. On the other hand, it is known that every $E[Y_{s+h}|\mathfrak{Y}_s]$, $s\leqq t$, belongs to $\mathfrak{M}_{h+t}^{(P)}$ because $(Y_t, P)$ is Gaussian. Hence it is sufficient to show that the left hand of (3.3) converges to $F_t$ in probability. Put

$$F_t^+ = \int_0^t (f\vee 0)ds, \quad F_t^- = -\int_0^t (f\wedge 0)ds$$

and denote by $F_{h,t}^{\pm}$ the left hand of (3.3) replacing $F$ by $F^{\pm}$ there, respectively. Then $\{F_t^{\pm}\}$ is a continous and increasing process adapted to $\{\mathfrak{Y}_t\}$ in the sense of Meyer [8]. Moreover, $\{F_t^{\pm}\}$ is integrable by Lemma 3. Hence, $F_t^{\pm}$ converges to $F_t^{\pm}$ as $h\to 0$ in $L^1(P)$-sense ([8] p. 126), respectively. Similarly $F_{h,t}$ converges to $F_t$ as $h\to 0$. Thus the proof is complete.

Proof of Theorem 1. 1°. Under the assumption of this theorem, we shall first prove that, with $P$-measure 1 the function $f(s, \omega)$ of Lemma 1 can be represented by

$$f(s, \omega) = \int_0^s k(s, u)dY_u \qquad \text{for almost all} \quad s\in[0, T],$$

where $k(s, u)$ is a Volterra kernel in $L^2([0, T]^2)$. It is well known that each element of $\overline{\mathfrak{M}}_t^{(P)}$ can be represented by a stochastic integral of the form $\int_0^t K(u)dY_u$. Hence $F_t$ of Lemma 4 is represented by $\int_0^t K(t, u)\, dY_u$. Noting that $F_t$ is continuous, we can choose $K(t, u)$ to be $(t, u)$-measurable by means of Slutsky's method [12]. Now, let $\Lambda$ denote the $(s, \omega)$-set

$$\{(s, \omega);\ \lim_{n\to\infty} n(F_s - F_{s-1/n}) \quad \text{does not exist or} \quad \lim_{n\to\infty} n(F_s - F_{s-1/n}) \neq f(s, \omega)\},$$

where we define $F_s=0$, for $s<0$. Then $\Lambda$ is $(s, \omega)$-measurable and $\mu(\Lambda)=0$, where $\mu$ is the product measure of $\tilde{P}(d\omega)$ and Lebesgue measure $m(ds)$ on $[0, T]$. In fact, $m(\Lambda_\omega)=0$ with $P$-measure 1, where $\Lambda_\omega=\{s;\ (s, \omega)\in\Lambda\}$. By Fubini's theorem it follows that $\tilde{P}(\tilde{\Lambda}_s)=0$ for almost all $s$, where $\Lambda_s=\{\omega;\ (s, \omega)\in\Lambda\}$. Therefore, for almost all $s$,

$$\lim_{n\to\infty} n(F_s - F_{s-1/n}) = f(s, \omega) \qquad \text{for} \quad \omega\notin\Lambda_s,$$

and $f(s, \omega)\in\overline{\mathfrak{M}}_s^{(P)}$ for such $s$. Hence for almost all $s\in[0, T]$,

$$f(s, \omega) = \int_0^s k(s, u)dY_u \qquad \text{with } P\text{-measure } 1,$$

where $k(s, u)$ belongs to $L^2(du)$ for such $s$. Moreover, we can choose $k(s, u)$ to be $(s, u)$-measurable. Put

$$\Lambda'_s = \left\{\omega\,; f(s, \omega) \neq \int_0^s k(s, u) dY_u\right\} \cup \Lambda_s\,,$$

then $P(\Lambda'_s)=0$ for almost all $s\in[0, T]$. Since

$$\Lambda' = \left\{(s, \omega)\,; f(s, \omega) \neq \int_0^s k(s, u) dY_u\right\} \cup \Lambda$$

is $(s, \omega)$-measurable,

$$f(s, \omega) = \int_0^s k(s, u) dY_u$$

holds for $s \in \Lambda_\omega' = \{s;\, (s, \omega) \in \Lambda'\}$ with $P$-measure 1. By this fact, we can get

$$X_t = Y_t - \int_0^t \Big(\int_0^s k(s, u) dY_s\Big) ds\,.$$

By Lemma 2,

$$\begin{aligned}\tilde{E}\Big(\int_0^T \Big(\int_0^s k(s, u) dY_u\Big)^2 ds\Big) &= \int_0^T \tilde{E}\Big(\int_0^s k(s, u) dY_u\Big)^2 ds \\ &= \int_0^T \int_0^s k(s, u)^2 ds du < \infty,\end{aligned}$$

therefore we can see that $k(s, u)$ is a Volterra kernel of $L^2([0, T]^2)$.

2°. Next, we want to represent $Y_t$ in the form of (3.1), by constructing the kernel $l(s, t)$. For the Volterra kernel $k(s, t)$, there is a resolvent kernel $l(s, t)$ such that

$$\begin{aligned}&l(s, t) + k(s, t) - \int_t^s l(s, u) k(u, t) du = 0 \qquad \text{in} \quad L^2([0, T]^2) \\ &l(s, t) + k(s, t) - \int_t^s k(s, u) l(u, t) du = 0 \qquad \text{in} \quad L^2([0, T]^2)\,.\end{aligned} \tag{3.4}$$

For, the Neumann series for the Volterra kernel $k(s, t)$ converges in the sense of $L^2([0, T]^2)$, and the limit is the kernel $l(s, t)$ (see Smithesis [13]). Thus the equations

$$\begin{aligned}&X_t - \int_0^t \Big(\int_0^s l(s, u) dX_u\Big) ds \\ &= Y_t - \int_0^t \Big(\int_0^s k(s, u) dY_u\Big) ds - \int_0^t \Big(\int_0^s l(s, u) dY_u\Big) ds \\ &\qquad + \int_0^t \Big(\int_0^s l(s, u) \int_0^u k(u, v) dY_v du\Big) ds \\ &= Y_t - \int_0^t \Big(\int_0^s k(s, u) dY_u\Big) ds - \int_0^t \Big(\int_0^s l(s, u) dY_u\Big) ds \\ &\qquad + \int_0^t \Big(\int_0^s \Big(\int_v^s l(s, u) k(u, v) du\Big) dY_v\Big) ds\end{aligned} \tag{3.5}$$

hold with $P$-measure 1 for each $t\in[0, T]$. The last equality follows by using the formula

$$\int_0^s\left(\int_0^s m(u, v)dY_u\right)dv=\int_0^s\left(\int_0^s m(u, v)dv\right)dY_u$$

where $m(u, v)\in L^2[0, T]^2)$. By the equation (3.4), the stochastic process

$$\int_0^t\left\{\int_0^s\left(-k(s, u)-l(s, u)+\int_u^s l(s, v)k(v, u)dv\right)dY_u\right\}ds$$

is identically 0 with $P$-measure 1. Therefore the right side of (3.5) is equal to $Y_t$ with $P$-measure 1 for each $t\in[0, T]$.

This proves (3.1), for both side of (3.1) are continuous with $P$-measure 1.

3°. Finally we will discuss the uniqueness of the representation (3.1). In fact, we will prove a slightly stronger result than the uniqueness statement in the theorem. Suppose $Y_t$ has two representations such as

$$\begin{aligned} Y_t &= X_t^1-\int_0^t\left(\int_0^s l(s, u)dX_u^1\right)ds \\ &= X_t^2-\int_0^t h(s, \omega)ds, \end{aligned}$$

where $(X_t^1, \mathfrak{Y}_t, P)$ and $(X_t^2, \mathfrak{Y}_t, P)$ are Wiener processes and $h(s, \omega)$ is a function satisfying (i), (ii) and (iii) in Lemma 1. Then,

$$X_t^1-X_t^2=\int_0^t\left(h(s, \omega)-\int_0^s l(s, u)dX_u^1\right)ds$$

is a martingale with respect to $\{\mathfrak{Y}_t\}$. By the uniqueness of Meyer's decomposition ([7] p. 113), it follows easily that

$$\begin{gathered} X_t^1-X_t^2=X_0^1-X_0^2=0, \\ \int_0^t h(s, \omega)ds=\int_0^t\left(\int_0^s l(s, u)dX_u^1\right)ds \end{gathered}$$

hold for any $t\in[0, T]$ with $P$-measure 1. By Fubini's theorem

$$h(s, \omega)=\int_0^s l(s, u)dX_u^1, \qquad \text{for almost all} \quad (s, \omega)\in[0, T]\times\Omega.$$

Thus, the proof of this theorem is completed.

REMARK 3. Theorem 1 shows that $\mathfrak{X}_t=\mathfrak{Y}_t$ for each $t\in[0, T]$, where $\mathfrak{X}_t$ is the $\sigma$-algebra generated by $\{X_s; s\leqq t\}$ and $P$-negligible sets. Therefore $(Y_t, P)$ has the proper canonical representation (3, 1) with respect to the Wiener process $(X_t, \mathfrak{X}_t, P)$ in the sense of Hida [4].

In case of $E(Y_t)\neq 0$, we get the following theorem by the same method as in Theorem 1.

**Theorem 1′.** *Suppose that a Gaussian process* $(Y_t, P)$, $t\in[0, T]$, *is equivalent to Wiener process. Then there exists a Wiener process* $(X_t, \mathfrak{Y}_t, P)$ *such that* $Y_t$ *is represented, with P-measure* 1, *by the formula*

$$Y_t = X_t - \int_0^t \left(\int_0^s l(s, u)dX_u\right) ds - \int_0^t a(s)ds, \quad \textit{for any } t\in[0, T]$$

*where* $l(s, u)$ *is a Volterra kernel belonging to* $L^2([0, T]^2)$ *and* $a(s)\in L^2([0, T])$. *Moreover such* $X_t$, $l(s, u)$ *and* $a(s)$ *are unique.*

## 4. Sufficient condition

In this section we will prove the converse of Theorem 1. This result is implied in Shepp [11] or H. Sato [10], if we note that the Volterra kernel has only zero as its eigenvalues. But, our proof is simpler than theirs and the density is explicitly evaluated.

**Theorem 2.** *If* $(X_t, \mathfrak{F}_t, P)$ *is a Wiener process, then the Gaussian process with respect to the measure P*

$$Y_t = X_t - \int_0^t \left(\int_0^s l(s, u)dX_u\right) ds$$

*is equivalent to Wiener process, where* $l(s, u)\in L^2([0, T]^2)$ *is a Volterra kernel. In this case the density* $\varphi(\omega)$ *in Definition* 3 *can be taken as follows*[6];

$$(4.1) \qquad \varphi(\omega) = \exp\left\{\int_0^T \left(\int_0^s l(s, u)dX_u\right)dX_s - \frac{1}{2}\int_0^T \left(\int_0^s l(s, u)dX_u\right)^2 ds\right\}.$$

Proof. This theorem is established if we can show that

$$E\left(\exp\left\{\int_0^T \left(\int_0^s l(s, u)dX_u\right)dX_s - \frac{1}{2}\int_0^T \left(\int_0^s l(s, u)dX_u\right)^2 ds\right\}\right) = 1,$$

for then $(Y_t, \mathfrak{F}_t, N_T(\omega)P(d\omega))$ is a Wiener process by virtue of Girsanov [2]. Let

$$N_t = \exp\left\{\int_0^t \left(\int_0^s l(s, u)dX_u\right)dX_s - \frac{1}{2}\int_0^t \left(\left(\int_0^s l(s, u)dX_u\right)^2 ds\right)\right\}.$$

Then the process $(N_t, \mathfrak{F}_t, P)$ is a local martingale (see [6]) and we may consider $N_t$ has continous paths. Therefore, there is an increasing sequence of stopping times $\{T_n\}$ which tends to $T$ with $P$-measure 1 such that $\{N_t^n = N_{t\wedge T_n}, \mathfrak{F}_t, P\}$ is a martingale for each $n$. Hence, it is enough to prove that $\{N_T^n\}$ is uniformly integrable, because $E(N_T)=\lim_{n\to\infty} E(N_{T_n})=1$. Observing that

6) The density $\varphi(\omega)$ may not be unique in general. But, the function of (4.1) is the only density which is measurable with respect to $\{X_t;\ 0\leqq t\leqq T\}$.

$$N_t^n = \exp\left\{\int_0^t \left(\chi_{[0,T_n]}(s)\int_0^s l(s,u)dX_u\right)dX_s - \frac{1}{2}\int_0^t \left(\chi_{[0,T_n]}(s)\int_0^s l(s,u)dX_u\right)^2 ds\right\},$$

where

$$\chi_{[0,T_n]}(s) = \begin{cases} 1 & \text{if } s \leqq T_n \\ 0 & \text{if } s > T_n, \end{cases}$$

define

$$\begin{aligned} Y_t^n &= X_t - \int_0^t \chi_{[0,T_n]}(s)\left(\int_0^s l(s,u)dX_u\right)ds \\ &= X_t - \int_0^{T_n\wedge t}\left(\int_0^s l(s,u)dX_u\right)ds. \end{aligned}$$

Then Girsanov's theorem, applied to $N_T^n$, tells us that $(Y_t^n, \mathfrak{F}_t, N_T^n P(d\omega))$ is a Wiener process for each $n$. On the other hand, a similar calculation as in (3.5) shows that

$$X_t = Y_t - \int_0^t \left(\int_0^s k(s,u)dY_u\right)ds$$

holds for any $t\in[0, T]$ with $P$-measure 1, where $k(s, u)$ is the resolvent kernel of $l(s, u)$ which satisfies (3.4). Therefore, it follows that

$$\int_0^t\left(\int_0^s k(s,u)dY_u\right)ds = \int_0^t\left(\int_0^s l(s,u)dX_u\right)ds, \qquad t\in[0, T],$$

or

$$\int_0^s k(s,u)dY_u = \int_0^s l(s,u)dX_u \qquad \text{for almost all } \ s\in[0, T].$$

Then

$$\begin{aligned} \log N_t^n &= \int_0^{t\wedge T_n}\left(\int_0^s l(s,u)dX_u\right)dX_s - \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s l(s,u)dX_u\right)^2 ds \\ &= \int_0^{t\wedge T_n}\left(\int_0^s l(s,u)dX_u\right)dY_s + \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s l(s,u)dX_u\right)^2 ds \\ &= \int_0^{t\wedge T_n}\left(\int_0^s k(s,u)dY_u\right)dY_s + \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s k(s,u)dY_u\right)^2 ds \\ &= \int_0^{t\wedge T_n}\left(\int_0^s k(s,u)dY_u^n\right)dY_s^n + \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s k(s,u)dY_u^n\right)^2 ds, \end{aligned}$$

for any $t\in[0, T]$, with $P$-measure 1. The last equality follows from the fact that

$$\int_0^{t\wedge T_n} f(s,\omega)dY_s = \int_0^{t\wedge T_n} f(s,\omega)dY_s^n$$

holds for any $f(s, \omega)$ satisfying (i), (ii) and (iii) in Lemma 1. As $(Y_t^n, \mathfrak{F}_t, \tilde{P}^n)$, $n=1, 2, \cdots$, are Wiener processes, where $\tilde{P}^n(d\omega)=N_T^n(\omega)P(d\omega)$, we have

$$E((\log N_T^n)\,N_T^n) = \tilde{E}^n(\log N_T^n)$$
$$= \tilde{E}^n\Big(\int_0^{T\wedge T_n}\Big(\int_0^s k(s,u)dY_u^n\Big)dY_n^s + \frac{1}{2}\int_0^{T\wedge T_n}\Big(\int_0^s k(s,u)dY_u^n\Big)^2 ds\Big)$$
$$= \frac{1}{2}\tilde{E}^n\Big(\int_0^{T\wedge T_n}\Big(\int_0^s k(s,u)dY_u^n\Big)^2 ds\Big)$$
$$\leqq \frac{1}{2}\int_0^T\Big(\int_0^s k(s,u)^2 du\Big)ds = K.$$

Hence, the family $\{N_T^n\}_{n=1,2,\cdots}$ is uniformly integrable. This is our desired result.

More generally, the following theorem can be proved analogously.

**Theorem 2′.** *Let $(X_t, \mathfrak{F}_t, P)$, $l(s,t)$ be as in Theorem 2 and $a(s)$ be of $L^2([0,T])$. Then the Gaussian process with respect to the measure $P$*

$$Y_t = X_t - \int_0^t\Big(\int_0^s l(s,u)dX_u\Big)ds - \int_0^t a(s)ds$$

*is equivalent to Wiener process. In this case, the function*

$$\varphi(\omega) = \exp\Big\{\int_0^T\Big(\int_0^s l(s,u)dX_u + a(s)\Big)dX_s - \frac{1}{2}\int_0^T\Big(\int_0^s l(s,u)dX_u + a(s)\Big)^2 ds\Big\}$$

*defines a density in Definition 3 such that $(X_t, \mathfrak{F}_t, \varphi P(d\omega))$ is a Wiener process.*

## 5. Related topics

1. A decomposition of a positive definite operator $(I-H)$ on $L^2([0,T])$.

**Proposition 1.** *A Gaussain process $(Y_t, P)$, $t\in[0,T]$, with mean $0$ is equivalent to the Wiener process if and only if $(Y_t, P)$ has the covariance*

$$E(Y_{t_1}\,Y_{t_2}) = (t_1\wedge t_2) - \int_0^{t_1\wedge t_2}\Big(\int_u^{t_1} l(s,u)ds\Big)du - \int_0^{t_1\wedge t_2}\Big(\int_u^{t_2} l(s,u)ds\Big)du + \int_0^{t_1}\int_0^{t_2}\Big(\int_0^{s_1\wedge s_2} l(s_1,u)\,l(s_2,u)du\Big)ds_1\,ds_2,$$

*with a Volterra kernel $l(s,u)$ in $L^2([0,T]^2)$. Moreover such $l(s,u)$ is unique.*

This proposition follows immediately from Theorem 1 and Theorem 2. As an application to $L^2$-theory, we can get

**Proposition 2.** *Let $H$ be a symmetric integral operator on $L^2([0,T])$.*

*Then $I-H$ is strictly positive definite if and only if there is an integral operator $L$ of Volterra type such that*

$$I\text{-}H=(I-L)(I-L^*)\,, \tag{5.1}$$

*where $L^*$ is the adjoint of $L$. Furthermore, such a decomposition is unique.*

Proof. "If" part: Since $L$ is of Volterra type, the integral equation

$$(I-L)f=0$$

has the unique solution $f=0$ in $L^2([0, T])$ (Simthesis [13]). Therefore

$$(\mathrm{I}-L^*)g=0$$

has the unique solution $g=0$ in $L^2([0, T])$. Hence, for $g\neq 0$,

$$\begin{aligned}((I-H)g, g) &= ((I-L)(I-L^*)g, g)\\ &= ((I-L^*)g, (I-L^*)g)>0\,.\end{aligned}$$

"Only if" part: Let $h(u, v)$ be the kernel which defines the operator $H$. Then, by a result of Shepp [11], there is a Gaussain process $(Y_t, P)$, equivalent to Wiener process, with covariance

$$E(Y_{t_1}\ Y_{t_2})=(t_1\wedge t_2)-\int_0^{t_1}\int_0^{t_2}h(u, v)\,du\,dv\,.$$

Hence, by Proposition 5.1, there is a unique Volterra kernel $l(u, v)$ such that

$$h(u, v)=l(u, v)+l(v, u)-\int_0^T l(u, w)\ l(u, w)\ dw\,.$$

If we define the operator $(I-L)$ by

$$(I-L)f(u)=f(u)-\int_0^T l(u, v)f(v)dv=f(u)-\int_0^u l(u, v)f(v)dv\,,$$

then

$$(I-L)\ (I-L^*)\,f(u)=(I-H)f(u).$$

2. Pinned Wiener process (Lévy [7] p. 318).
If $(X_t, \mathfrak{F}_t, P)$, $t\in[0, 1]$, is a Wiener process, then

$$\begin{aligned}Y_t &= (1-t)\int_0^t\frac{dX_u}{1-u}=X_t-\int_0^t\left(\int_0^s\frac{-1}{1-u}dX_u\right)ds & 0\leqq t<1\\ &=0 & t=1\end{aligned}$$

is the so-called pinned Wiener process with mean 0 and covariance

$E(Y_t Y_s)=(1-t)s$, for $t<s$. In this case,

$$l(s, u)=\begin{cases}\dfrac{-1}{1-u} & u\leqq s \\ 0 & u>s\end{cases}, \quad k(s, u)=\begin{cases}\dfrac{1}{1-s} & u\leqq s \\ 0 & u>s\,.\end{cases}$$

Evidently, the Gaussian process $(Y_t, P)$ is equivalent to Wiener process in $[0, t_0]$, $t_0<1$, by Theorem 2, while $(Y_t, P)$ is not equivalent to Wiener process in $[0, 1]$, because $Y_1=0$, with $P$-measure 1. This phenomenon can be explained from that the kernel $l(s, u)$ does not belong to $L^2([0, 1]^2)$. The process $Y_t$ is the unique solution of the stochastic integral equation

$$Y_t = X_t+\int_0^t \frac{1}{1-s}\int_0^s dY_u \qquad t<1\,,$$

with the initial condition $Y_0=0$.

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*Supplement* for the proof of Lemma 1.

Professor T. Watanabe pointed out that, for the representation $M_t=\int_0^u g(s, \omega)dY_s+1$, it is necessary to prove that there is an increasing sequence $\{T_n\}_{n=1,2,\dots}$ of stopping times wh'ch converges to $T$ and $\{M_{t\wedge T_n}\}_{t\in[0,T]}$ $(n=1, 2, \cdots)$ are square integrable martingales with respect to $\{\mathfrak{Y}_t\}$ in $(\Omega, \mathfrak{B}, P)$ (see Kunita-S. Watanabe [6]).

For the proof, we will first show that $M_t$ has cotinuous paths. Set

$$M_t^N = \tilde{E}\left(\frac{1}{\varphi}\wedge N \mid \mathfrak{Y}_t\right),$$

then $M_t-M_t^N$ is a positive martingale and $M_T^N$ converges to $M_T$ in $L^1(\tilde{P})$ sense. Using Doob's inequality ([1] p. 353),

$$\tilde{P}(\sup_{0\leqq t\leqq T}(M_t-M_t^N)\geqq\lambda)\leqq\frac{\tilde{E}(M_T-M_T^N)}{\lambda}.$$

This shows $M_t^N$ converges to $M_t$ uniformly in probability $\tilde{P}$. On the other hand, $\{M_t^N\}_{t\in[0,T]}$ are square integrable martingale, and so they have continuous paths. Hense $M_t$ has continuous paths.

Next, if we choose the sequence of stopping times $\{T_n\}_{n=1,2\cdots}$ such that

$$T_n = \begin{cases} \min\{t;\ M_t\geqq n\} \\ T \quad \text{if} \quad \{t;\ M_t\geqq n\}\neq\phi, \end{cases}$$

then $T_n$ converges to $T$ and $\{M_{t\wedge T_n}\}_{t\in[0,T]}$ $(n=1, 2\cdots)$ are square integrable martingales, because of the cotinuity of paths of $M_t$.