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ON THE NORMAL BUNDLES OF S^2 MINIMALLY IMMERSED INTO THE UNIT SPHERES

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Introduction

Let $F: S^2 \rightarrow S^m(1)$ be a full minimal isometric immersion of the 2-dimensional sphere S^2 into the m -dimensional unit sphere $S^m(1)$. Let $N(S^2)$ be the normal bundle of S^2 and $\Gamma(N(S^2))$ the space of all C^∞ cross-sections of $N(S^2)$. We denote by \tilde{J} the Jacobi operator acting on $\Gamma(N(S^2))$. The operator \tilde{J} is diagonalisable (Simons [6]).

The 2-dimensional sphere S^2 may be considered as the homogeneous space $SU(2)/S(U(1) \times U(1))$. Then the isometric immersion F is $SU(2)$ -equivariant (Calabi [1], Do Carmo & Wallach [2]). Let V_λ be the complexification of the λ -eigenspace of \tilde{J} . Then V_λ is a $SU(2)$ -module and the multiplicities of any complex irreducible $SU(2)$ -modules contained in V_λ are all equal to 2 (Nagura [4]).

In this paper we show that the normal bundle $N(S^2)$ has a holomorphic vector bundle structure (Proposition 2). Therefore $\Gamma(N(S^2))$ is a complex vector space. Secondly we show that the Jacobi operator \tilde{J} is complex linear (Proposition 3). Hence every eigenspace of \tilde{J} is a complex linear subspace of $\Gamma(N(S^2))$. Thirdly we show that if we decompose an eigenspace of \tilde{J} into a direct sum of complex irreducible $SU(2)$ -modules, then any pairs of the components are not $SU(2)$ -isomorphic (Proposition 4). This result explains the above fact on the multiplicities.

1. Preliminaries

We denote by G (resp. by K) the special unitary group $SU(2)$ of degree 2 (resp. the subgroup $S(U(1) \times U(1))$ of $SU(2)$), i.e.

$$K = \left\{ \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}; b \in \mathbf{C}, |b| = 1 \right\},$$

where \bar{b} is the complex conjugate of b . Let \mathfrak{g} be the Lie algebra of G and \mathfrak{k} the Lie subalgebra of \mathfrak{g} corresponding to the subgroup K of G , i.e.

$$\mathfrak{g} = \left\{ \begin{pmatrix} \sqrt{-1}a & b \\ -\bar{b} & -\sqrt{-1}a \end{pmatrix}; a \in \mathbf{R}, b \in \mathbf{C} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} \sqrt{-1}a & 0 \\ 0 & -\sqrt{-1}a \end{pmatrix}; a \in \mathbf{R} \right\}.$$

Then \mathfrak{k} is a Cartan subalgebra of \mathfrak{g} . We define an $Ad(G)$ -invariant inner product $(,)$ on \mathfrak{g} by

$$(X, Y) = -\frac{1}{2} Tr(XY) \quad \text{for } X, Y \in \mathfrak{g},$$

where $Tr(XY)$ denotes the trace of the matrix XY . Let \mathfrak{p} be the orthogonal complement of \mathfrak{k} . Then

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}; b \in \mathbf{C} \right\}.$$

We may consider \mathfrak{p} as the tangent space $T_o(S^2)$ of S^2 at $o = \pi(e)$, where π is the natural projection of G onto $S^2 = G/K$. The inner product $(,)$ defines a G -invariant Riemannian metric on S^2 which coincides with the inner product $(,)$ on $\mathfrak{p} = T_o(S^2)$. We also denote by $(,)$ this G -invariant Riemannian metric. Then the Riemannian manifold $(S^2, (,))$ is of constant sectional curvature 4.

We choose an orthonormal basis $\{h, x, y\}$ of \mathfrak{g} as follows:

$$h = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad x = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An irreducible orthogonal representation $\rho: G \rightarrow GL(V)$ is said to be a *real spherical representation* of the pair (G, K) , if there exists a unit vector $v \in V$ such that $\rho(k)v = v$ for any $k \in K$. We have

Lemma 1 (cf. Serre [5]). *Let $\rho: G \rightarrow GL(V)$ be a real spherical representation of (G, K) . Then*

(1) *The complexification $\rho: G \rightarrow GL(V^c)$ of ρ is a complex irreducible representation with highest weight $2nh$, where V^c is the complexification of the vector space V and n is a non-negative integer.*

(2) *We can choose an orthogonal basis $\{u, v_i, w_i; i=1, 2, \dots, n\}$ of V with the following properties:*

$$d\rho(h)u = 0, \quad d\rho(h)v_i = 2iw_i, \quad d\rho(h)w_i = -2iv_i,$$

$$i = 1, 2, \dots, n.$$

$$d\rho(x)u = 2nv_1, \quad d\rho(y)u = -2nw_1.$$

If i is even,

$$\begin{aligned} d\rho(x)v_i &= (n+i)v_{i-1}+(n-i)v_{i+1}, \\ d\rho(x)w_i &= (n+i)w_{i-1}+(n-i)w_{i+1}, \\ d\rho(y)v_i &= (n+i)v_{i-1}-(n-i)v_{i+1}, \\ d\rho(y)w_i &= -(n+i)w_{i-1}+(n-i)w_{i+1}. \end{aligned}$$

If i is odd,

$$\begin{aligned} d\rho(x)v_i &= \begin{cases} -(n+1)u-(n-1)v_2 & i=1, \\ -(n+i)v_{i-1}-(n-i)v_{i+1} & i>1, \end{cases} \\ d\rho(x)w_i &= \begin{cases} -(n-1)w_2 & i=1, \\ -(n+i)w_{i-1}-(n-i)w_{i+1} & i>1, \end{cases} \\ d\rho(y)v_i &= \begin{cases} (n-1)w_2 & i=1, \\ -(n+i)w_{i-1}+(n-i)w_{i+1} & i>1, \end{cases} \\ d\rho(y)w_i &= \begin{cases} (n+1)u-(n-1)v_2 & i=1, \\ (n+i)v_{i-1}-(n-i)v_{i+1} & i<1. \end{cases} \end{aligned}$$

Here $d\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is the differential of the representation ρ .

For

$$k = \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} \in K, \quad \theta \in \mathbf{R},$$

we have by the above lemma

$$(1.1) \quad \begin{cases} \rho(k)v_i = \cos(2i\theta)v_i + \sin(2i\theta)w_i, \\ \rho(k)w_i = -\sin(2i\theta)v_i + \cos(2i\theta)w_i. \end{cases}$$

Let (M, g) (resp. (\bar{M}, \bar{g})) be a Riemannian manifold of dimension k (resp. of dimension m). Let $F: M \rightarrow \bar{M}$ be an isometric immersion of M into \bar{M} . We identify the tangent space $T_p(M)$ of M at $p \in M$ with a linear subspace of the tangent space $T_{F(p)}(\bar{M})$ of \bar{M} at $F(p) \in \bar{M}$. We denote by $N_p(M)$ the orthogonal complement of $T_p(M)$ in $T_{F(p)}(\bar{M})$. Let $T(M)$ (resp. $N(M)$) be the tangent bundle (resp. the normal bundle) of M . We denote by $\mathfrak{X}(M)$ (resp. by $\Gamma(N(M))$) the space of all C^∞ cross-sections of $T(M)$ (resp. of $N(M)$). Let ∇ (resp. $\bar{\nabla}$) be the Riemannian connection of M (resp. of \bar{M}). Let D be the normal connection of F . Let $B: T_p(M) \times T_p(M) \rightarrow N_p(M)$ be the second fundamental form of F , and $A: N_p(M) \times T_p(M) \rightarrow T_p(M)$ the Weingarten form of F . For any vector fields $X, Y \in \mathfrak{X}(M)$ and for any normal vector field $\xi \in \Gamma(N(M))$, we have the followings (cf. Kobayashi & Nomizu [3] Vol. II Chap. 7 section 3):

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

$$\begin{aligned} \nabla_X \xi &= -A_\xi X + D_X \xi, \\ \bar{g}(\xi, B(X, Y)) &= g(A_\xi X, Y). \end{aligned}$$

We denote by H the mean curvature of F . Let $\{e_1, e_2, \dots, e_k\}$ be an orthonormal basis of $T_p(M)$. Then we have

$$H_p = \sum_{i=1}^k B(e_i, e_i).$$

The isometric immersion $F: M \rightarrow \bar{M}$ is said to be *minimal*, if the mean curvature H of F vanishes identically.

Let \bar{R} be the curvature tensor of \bar{M} . We define linear mappings \bar{A}, \bar{R} , of $N_p(M)$ as follows:

$$(1.2) \quad \bar{A}(v) = \sum_{i,j=1}^k \bar{g}(v, B(e_i, e_j)) B(e_i, e_j),$$

$$(1.3) \quad \bar{R}(v) = \sum_{i=1}^k (\bar{R}(e_i, v)e_i)^N \quad \text{for } v \in N_p(M),$$

where $\{e_1, e_2, \dots, e_k\}$ is an orthonormal basis of $T_p(M)$ and $(\bar{R}(e_i, v)e_i)^N$ denotes the normal component of $\bar{R}(e_i, v)e_i$. The linear mappings \bar{A} and \bar{R} are independent of the choice of an orthonormal basis. We denote by Δ the Laplace operator on $N(M)$. Let $\{E_1, E_2, \dots, E_k\}$ be an orthonormal local basis of $T(M)$ on a neighborhood of $p \in M$. Then we have

$$\Delta f(p) = \sum_{i=1}^k (D_{E_i} D_{E_i} f)(p) - \sum_{i=1}^k (D_{\nabla_{E_i} E_i} f)(p) \quad \text{for } f \in \Gamma(N(M)).$$

The Jacobi operator \bar{J} is the operator on $N(M)$ defined by

$$(1.4) \quad \bar{J} = -\Delta - \bar{A} + \bar{R}.$$

2. A complex structure on the normal bundle $N(S^2)$

In the rest of this paper we assume that $F: S^2 \rightarrow S^m(1)$ is a full minimal isometric immersion of $(S^2, c(\cdot, \cdot))$, $c > 0$, into the m -dimensional unit sphere $S^m(1)$. We may consider $S^m(1)$ as the unit sphere of an $(m+1)$ -dimensional Euclidean vector space V with the center 0. Then the following results are known (Calabi [1] p. 123, Do Carmo & Wallach [2] p. 103): The minimal immersion F is rigid, and there exist a real spherical representation $\rho: G \rightarrow GL(V)$ of (G, K) and a unit vector $u_0 \in V$ such that

$$F(gK) = \rho(g)u_0 \quad \text{for any } g \in G.$$

Let $\{u, v_i, w_i; i=1, 2, \dots, n\}$ ($m=2n$) be the orthogonal basis of V in Lemma 1. We identify the tangent space of V with V itself in a canonical way. Then we have

$$T_0(S^2) = \{v_1, w_1\}_{\mathbf{R}},$$

$$N_0(S^2) = \{v_i, w_i; i = 2, 3, \dots, n\}_{\mathbf{R}},$$

where $N_0(S^2)$ is the normal space of S^2 at o in the unit sphere $S^m(1)$. Put

$$V^0 = \mathbf{R}u_0 = \mathbf{R}u, \quad V^T = T_0(S^2), \quad V^N = N_0(S^2).$$

Then we have the following orthogonal decomposition:

$$(2.1) \quad T_{u_0}(V) = V^0 + V^T + V^N.$$

REMARK. The number c can be explicitly computed (cf. Nagura [4] I p. 128). We have

$$c = 2n(n+1).$$

Let $\phi: K \rightarrow GL(V^N)$ be the representation of K defined by

$$\phi(k) = \rho(k)|_{V^N} \quad \text{for } k \in K.$$

Let $G \times_K V^N$ be the vector bundle associated to G by ϕ . The vector bundle homomorphism $\iota: G \times_K V^N \rightarrow N(S^2)$ defined by

$$\iota(g \circ v) = \rho(g)v \quad \text{for } g \in G \text{ and } v \in V$$

is isomorphic (Nagura [4] I p. 123). Here $x \circ v$ is the image of $(x, v) \in G \times V^N$ by the natural projection of $G \times V^N$ onto $G \times_K V^N$. Put

$$C^\infty(G; V^N)_K = \left\{ f: G \rightarrow V^N \text{ } C^\infty \text{ mapping; } f(gk) = \phi(k)^{-1}f(g) \right\} \\ \left. \text{for } g \in G \text{ and } k \in K \right\}.$$

The isomorphism $\iota: G \times_K V^N \rightarrow N(S^2)$ induces an isomorphism of $C^\infty(G; V^N)_K$ onto $\Gamma(N(S^2))$. We also denote by ι this isomorphism

We denote by \tilde{G} the complex special linear group $SL(2, \mathbf{C})$ of degree 2. Let \tilde{U} be the subgroup of \tilde{G} defined by

$$\tilde{U} = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}; a, b \in \mathbf{C}, a \neq 0 \right\}.$$

The 2-dimensional sphere S^2 may be considered as the 1-dimensional complex projective space. In fact, the mapping $i: S^2 = G/K \rightarrow P^1(\mathbf{C}) = \tilde{G}/\tilde{U}, i(gK) = g\tilde{U}$ for $g \in G$, gives this identification. We define a complex structure I on V^N by

$$Iv_i = w_i, \quad Iw_i = -v_i \quad i = 2, 3, \dots, n.$$

We denote by \tilde{V}^N this complex vector space (V^N, I) . We have by

$$(1.1) \quad \phi(k) \circ I = I \circ \phi(k) \quad \text{for } k \in K.$$

Therefore the bundle $G \times_K V^N$ has a complex vector bundle structure. In addition the following proposition asserts that $G \times_K V^N$ is a holomorphic vector bundle.

Proposition 2. *Let $F: (S^2, c(\cdot, \cdot)) \rightarrow S^m(1)$, $c > 0$, be a full minimal isometric immersion. Then the normal bundle $N(S^2)$ has a holomorphic vector bundle structure.*

Proof. We shall show that $G \times_K V^N$ has a holomorphic vector bundle structure. We define a mapping $\psi: \tilde{U} \rightarrow GL(\tilde{V}^N)$ by

$$\begin{aligned} \psi(\tilde{u})v_i &= (Re a^{2i})v_i + (Im a^{2i})w_i, \\ \psi(\tilde{u})w_i &= -(Im a^{2i})v_i + (Re a^{2i})w_i \end{aligned}$$

$$\text{for } \tilde{u} = \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \in \tilde{U},$$

where $Re a^{2i}$ (resp. $Im a^{2i}$) is the real part (resp. the imaginary part) of $a^{2i} \in \mathbb{C}$. Since $\psi(\tilde{u}_1\tilde{u}_2) = \psi(\tilde{u}_1)\psi(\tilde{u}_2)$ for $\tilde{u}_1, \tilde{u}_2 \in \tilde{U}$, ψ is a holomorphic representation of \tilde{U} . Let $\tilde{G} \times_{\tilde{v}} \tilde{V}^N$ be the vector bundle associated to \tilde{G} by ψ . This vector bundle $\tilde{G} \times_{\tilde{v}} \tilde{V}^N$ is a holomorphic vector bundle. Since the restriction $\psi|_K$ of ψ to K coincides with ϕ , the bundle homomorphism $i: G \times_K V^N \rightarrow \tilde{G} \times_{\tilde{v}} \tilde{V}^N$, $i(x \circ v) = x \circ v$, is an isomorphism as C^∞ vector bundle. Hence $G \times_K V^N$ has a holomorphic vector bundle structure. Q.E.D.

3. On the Jacobi operator \tilde{J}

We also denote by I the complex structure on $C^\infty(G; V^N)_K$ induced from the complex structure I on V^N . Let \tilde{I} be the complex structure on $\Gamma(N(S^2))$ corresponding to this complex structure I on $C^\infty(G; V^N)_K$ under the isomorphism $\iota: C^\infty(G; V^N)_K \rightarrow \Gamma(N(S^2))$. We define an action L (resp. an action σ) of G on $C^\infty(G; V^N)_K$ (resp. on $\Gamma(N(S^2))$) as follows:

$$\begin{aligned} (L_g f)(h) &= f(g^{-1}h) \quad \text{for } g, h \in G \text{ and } f \in C^\infty(G; V^N)_K, \\ (\sigma_g \tilde{f})(hK) &= d(\rho(g))\tilde{f}(hK) \quad \text{for } g, h \in G \text{ and } \tilde{f} \in \Gamma(N(S^2)), \end{aligned}$$

where $d(\rho(g))$ is the differential of the isometry $\rho(g)$ of $S^m(1)$. Then we have (Nagura [4] I p. 124)

$$\iota \circ L_g = \sigma_g \circ \iota \quad \text{for } g \in G.$$

We have easily

$$(3.1) \quad I \circ L_g = L_g \circ I, \quad \tilde{I} \circ \sigma_g = \sigma_g \circ \tilde{I} \quad \text{for } g \in G.$$

The above result shows that the action L (resp. σ) is a complex representation of G with representation space $(C^\infty(G; V^N)_K, I)$ (resp. $(\Gamma(N(S^2)), \tilde{I})$).

Let J be the operator on $C^\infty(G; V^N)_K$ corresponding to the Jacobi operator \tilde{J} . We have (Nagura [4] I p. 131)

$$(3.2) \quad Jf = -\frac{1}{c} \left[\sum_{i=1}^3 E_i E_i f - 2c_\rho f + 2 \sum_{i=1}^3 \{d\rho(E_i)(E_i f)\}^N + 2 \sum_{i=1}^3 \{d\rho(E_i)(d\rho(E_i)f)^N\}^N \right]$$

for $f \in C^\infty(G; V^N)_K$,

where $E_1=h, E_2=x, E_3=y, c_\rho=-2c$ and $(v)^N$ denotes the V^N -component of $v \in V$ with respect to the decomposition (2.1). In (3.2) we consider \mathfrak{g} as the Lie algebra of left invariant vector fields on G .

Proposition 3. *The Jacobi operator \tilde{J} is complex linear on $(\Gamma(N(S^2)), \tilde{I})$.*

Proof. We shall show that $J \circ I = I \circ J$. Since $Z \circ I = I \circ Z$ for $Z \in \mathfrak{g}$, it is sufficient to show that

$$(3.3) \quad \{d\rho(Z)(I(v))\}^N = I(d\rho(Z)v)^N \quad \text{for } Z \in \mathfrak{g} \text{ and } v \in V^N.$$

Applying Lemma 1, we have

$$\{d\rho(Z)(I(v))\}^N = I(d\rho(Z)v)^N$$

for $Z = h, x, y$ and $v = v_i, w_i, i = 2, 3, \dots, n$.

This proves (3.3). Q.E.D.

Let U_λ be the λ -eigenspace of \tilde{J} in $\Gamma(N(S^2))$. Since the space U_λ is G -invariant (Nagura [4] I p. 119), U_λ is a complex G -invariant subspace of $(\Gamma(N(S^2)), I)$ by Proposition 3. Therefore we have the following proposition by Nagura [4] (III (2) of Theorem 12.3.3).

Proposition 4. *If we decompose an eigenspace of \tilde{J} into a direct sum of complex irreducible G -modules, then any pairs of the irreducible components are not G -isomorphic.*

Let \tilde{L} be the space of Killing vector fields on the unit sphere $S^m(1)$. Put

$$\tilde{W} = \{(\tilde{k}_{1,S^2})^N; \tilde{k} \in \tilde{L}\},$$

where $(\tilde{k}_{1,S^2})^N$ is an element of $\Gamma(N(S^2))$ obtained by the normal projection of $\tilde{k} \in \tilde{L}$. This space \tilde{W} is a G -module. A cross-section $\tilde{f} \in \Gamma(N(S^2))$ is called a *Jacobi field*, if it satisfied the equation $\tilde{J}\tilde{f}=0$. An element of \tilde{W} is a Jacobi field (Simons [6] p. 74). Let \tilde{W}^c be the complexification of the space \tilde{W} . Then the multiplicities of any complex irreducible G -modules contained in \tilde{W}^c are

equal to 1 (Nagura [4] III Lemma 12.4.2). Hence we have

$$(3.4) \quad \tilde{W} \cap W = \{0\} .$$

Let U_0 be the space of all Jacobi fields. Then we have by (3.4) and Nagura [4] (III Theorem 12.4.1 and Lemma 12.4.2)

$$U_0 = W + \tilde{W} \text{ (direct sum) .}$$

Thus we could obtain the space U_0 . However the author does not know the geometric meaning of this decomposition.

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