Sumioka, T.
Osaka J. Math.
**21** (1984), 629–648

# TACHIKAWA'S THEOREM ON ALGEBRAS OF LEFT COLOCAL TYPE

Dedicated to Professor Hirosi Nagao on his 60th birthday

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(Received December 14, 1982)

## Introduction

Let $A$ be an artinian ring. Then $A$ is said to be of right local type if any finitely generated indecomposable right $A$-module $M$ is local (i.e. $M$ has a unique maximal submodule) and a ring of left colocal type is defined as the dual notion. We say $A$ is left serial if a left $A$-module $A$ is a direct sum of uniserial submodules. Tachikawa [4, 5] gave characterizations of algebras of right local (or equivalently of left colocal) type.

**Theorem** (Tachikawa). *For a finite dimensional algebra $A$ with the Jacobson radical $N$, the following conditions* (a)–(d) *are equivalent.*

(a) *$A$ is of right local type.*

(b) *$A$ is of left colocal type.*

(c) ($c_1$) *$A$ is left serial.*

($c_2$) *For any uniserial left $A$-modules $L_1$ and $L_2$ with $|L_1|\leq|L_2|$, any isomorphism $\theta$: $S_1(L_1)\rightarrow S_1(L_2)$ is $(L_1, L_2)$-maximal or $(L_1, L_2)$-extendible (see Section 1 for the definitions), where $|L_i|$ is the composition length of $L_i$ and $S_1(L_i)$ is the socle of $L_i$ for $i=1, 2$.*

($c_3$) $|eN/eN^2|\leq 2$ *for any primitive idempotent $e$ of $A$.*

(d) ($d_1$) *$A$ is left serial.*

($d_2$) *$eN=M_1\oplus M_2$ for any primitive idempotent $e$ of $A$, where $M_i$ is either zero or a uniserial submodule of the right $A$-module $eN$ for each $i=1, 2$.*

More precisely Tachikawa [4] gave a proof of the equivalence of (b) and (c) for *any* artinian ring. But in the proof of the implication from (c) to (b), there were two gaps. He himself pointed out one of them, namely [4, Lemma 4.9], and informed Fuller of it and that the lemma holds for any artinian ring under a suitable assumption (D) which is satisfied for any finite dimensional algebra over a field (cf. Section 3 for the definition of (D). See also Fuller [3, Note p. 165].). Now the other one (which is related to [4, Corollary 4, 6]) can be filled with an elementary lemma (i.e. Lemma 1.1 below, which is essentially used in [5, Proposition 4, 2]) under the additional assumption (D).

In Section 3 we shall give a self-contained proof for the above stated implication from (c) to (b). On the other hand we shall point out in Section 4 that the equivalence of (c) and (d) holds for any artinian rings. Unfortunately it remains open whether any ring of colocal type satisfies (D), however in the last Section we shall give an example of an artinian ring which satisfies (c) but not (b) and remark that some simultaneous equations with 6-unknowns are closely related to this problem.

For the sake of completeness we shall also give a proof of the implication from (b) to (c) together with proofs for results which have been shown in [4] and [1].

Throughout this paper $A$ is a left and right artinian ring with unity, $N$ is the Jacobson radical of $A$ and all modules are finitely generated (unitary) left $A$-module unless otherwise stated. For a module $M$, we denote the top $M/NM$ of $M$ by $\bar{M}$, the composition length of $M$ by $|M|$. For any integer $i\geq 0$ we define a submodule $S_i(M)$ of a module $M$ inductively as following: $S_0(M)=0$ and $S_i(M)/S_{i-1}(M)$ is the socle of $M/S_{i-1}(M)$. We denote by $p(A)$ the set of primitive idempotents of $A$. Symbols (a), $\cdots$, (d) always mean the conditions in the theorem above.

The author wishes to express his thanks to Professor H. Tachikawa for his valuable advice.

## 1. Preliminaries

Let $M_1$ and $M_2$ be modules with submodules $T_1$ and $T_2$, respectively. If a homomorphism $\varphi$: $M_1\to M_2$ canonically induces a map $T_1\to T_2$, the map is also denoted by $\varphi$: $T_1\to T_2$. Let $\theta$: $T_1\to T_2$ be a homomorphism. We say $\theta$ is $(M_1, M_2)$-*extendible* if $\theta$ is induced from some homomorphism $\varphi$: $M_1\to M_2$, and in this case $\varphi$ is an extension of $\theta$. We say $\theta$ is $(M_1, M_2)$-*maximal* if there is no module $U$ such that $T_1\subsetneqq U\subset M_1$ and $\theta$ is $(U, M_2)$-extendible. In case $T=T_1=T_2$ and $\theta$ is $1_T$ the identity map of $T$, we simply say $T$ is $(M_1, M_2)$-*extendible* (resp.-*maximal*) if $1_T$ is $(M_1, M_2)$-extendible (resp.-maximal).

The following lemma is clear.

**Lemma 1.1.** *Let $M_1$, $M_2$ and $T$ be submodules of a module $M$ such that $M=M_1+M_2$ and $T=M_1\cap M_2$. If $T'$ is a submodule of $T$ and $\varphi$: $M_1\to M_2$ is an extension of $1_{T'}$, then for $M_1'=\{x-x\varphi \mid x\in M_1\}$ the following hold.*

(1) $M=M_1'+M_2$.

(2) $M_1'\cap M_2=\{x-x\varphi \mid x\in T\}$.

(3) *The epimorphism $M_1\to M_1'$ defined by $x\to(x-x\varphi)$; $x\in M_1$, induces epimorphisms $M_1/T'\to M_1'$ and $T/T'\to M_1'\cap M_2$, in particular $|M_1'\cap M_2|\leq|T|-|T'|$.*

The following lemmas 1.2 and 1.3 are due to Tachikawa [4, Lemma 1.3 and Lemma 4.4].

**Lemma 1.2.** *Let $M_1$, $M_2$ and $T$ be submodules of a module $M$ such that $M=M_1+M_2$ and $T=M_1\cap M_2$. Then*

*(1) $T$ is $(M_1, M_2)$-extendible if and only if $M=M_1'\oplus M_2$ for some submodule $M_1'$ of $M$.*

*(2) $T$ is $(M_1, M_2)$-maximal if and only if $S_1(M)=S_1(M_2)$.*

Proof. (1) 'Only if' part is immediate from Lemma 1.1. If $M=M_1'\oplus M_2$, then the restriction map $\pi_2\colon M_1\to M_2$ of the projection $\pi_2\colon M_1'\oplus M_2\to M_2$ is clearly an extension of $1_T$.

(2) 'Only if' part: Assume $S_1(M)=U'\oplus S_1(M_2)$ for a non-zero module $U'$. Since $U'\cap M_2=0$ and $U'\oplus M_2\supset M_2$, $U'\oplus M_2=(M_1+M_2)\cap(U'\oplus M_2)=(M_1\cap(U'\oplus M_2))+M_2$. Put $U=M_1\cap(U'+M_2)$. Then we have $U+M_2=U'\oplus M_2$, $T=U\cap M_2$ and $T\subsetneqq U\subset M_1$. Applying (1) to $U+M_2$, $T$ is $(U, M_2)$-extendible.

'If' part: Assume $\varphi\colon U\to M_2$ is an extension of $1_T$ with $T\subsetneqq U\subset M_1$. From (1) we have $U+M_2=U'\oplus M_2$ for some module $U'\neq 0$. Thus $S_1(M)\supset S_1(U')\oplus S_1(M_2)\supsetneqq S_1(M_2)$.

**Lemma 1.3.** *Let $M_i$ $(i=1, 2, 3)$ and $T$ be submodules of a module $M$ such that $M=M_1+(M_2\oplus M_3)$ and $T=M_1\cap(M_2\oplus M_3)$, and $\pi_3\colon T\to M_3$ the restriction map of the projection $M_2\oplus M_3\to M_3$. Then $\pi_3$ is $(M_1, M_3)$-extendible if and only if $M=(M_1'+M_2)\oplus M_3$ for some submodule $M_1'$ of $M$.*

Proof. This is shown by the method similar to the proof of (1) in Lemma 1.2.

Let $M$ and $P_i$ $(i=1, \cdots, n)$ be modules. Then a map $\varphi\colon M\to\bigoplus_{i=1}^{n}P_i$ has a matrix representation $\varphi=(\varphi_1, \cdots, \varphi_n)$ by the composition maps $\varphi_j\colon M\to P_j$ of $\varphi\colon M\to\bigoplus_{i=1}^{n}P_i$ and the projections $\bigoplus_{i=1}^{n}P_i\to P_j$. Similarly a map $\psi\colon\bigoplus_{i=1}^{n}P_i\to M$ has a matrix representation $\psi=(\psi_1, \cdots, \psi_n)^T$ (the transposed matrix of $(\psi_1, \cdots, \psi_n)$) by the maps $\psi_i\colon P_i\to M$. For idempotents $e$ and $f$ of $A$, we assume that $u\in t(eN^{r-1}f)$ means $eN^{r-1}f\supsetneqq eN^rf$, $u\in eN^{r-1}f$ and $u\notin eN^rf$.

Let $u_i\in t(eN^{r-1}f_i)$, where $e, f_i\in p(A)$ and $i=1, \cdots, n$. Denote a residue class of $x\in eA$ in $eA/eN^r$ by $\bar{x}$ and that of $y\in Af_i$ in $Af_i/N^rf_i$ by $[y]_i$ or simply by $[y]$.

**Lemma 1.4.** *Let $u_i\in t(eN^{r-1}f_i)$ and put $P_i=Af_i/N^rf_i$ for an integer $r\geq 1$, where $e$ is an idempotent of $A$, $f_i$ is a primitive idempotent and $i=1, \cdots, n$. Then under the above notation, the following conditions are equivalent.*

(1) $(\bar{u}_1A+\cdots+\bar{u}_{n-1}A)\cap\bar{u}_nA\neq 0$.

(2) $\bar{u}_n \in \bar{u}_1 A + \cdots + \bar{u}_{n-1} A$.

(3) *There is a homomorphism* $\psi\colon \bigoplus_{i=1}^{n-1} P_i \to P_n$ *such that* $(\sum_{i=1}^{n-1} [u_i]_i)\psi = [u_n]_n$.

(4) *There is a homomorphism* $\varphi\colon \bigoplus_{i=1}^{n} P_i \to P_n$ *such that* $(\sum_{i=1}^{n} [u_i]_i)\varphi = 0$ *and* $\varphi_n$ *is an identity map, where* $\varphi = (\varphi_1, \cdots, \varphi_n)^T$.

Proof. The equivalences (1)⇔(2) and (3)⇔(4) are clear since $\bar{u}_n A$ is a simple module.

(2)⇔(3). Note that any homomorphism $P_i \to P_n$ is induced from a right multiplication map $\tilde{a}_i\colon Af_i \to Af_n$ by $a_i \in A$ with $a_i = f_i a_i f_n$. The condition (2) is equivalent to one that there are elements $a_i = f_i a_i f_n$ of $A$, $i=1, \cdots, n-1$, with $\bar{u}_n = \bar{u}_1 a_1 + \cdots + \bar{u}_{n-1} a_{n-1}$ which is equivalent to $[u_n] = [u_1 a_1] + \cdots + [u_{n-1} a_{n-1}]$. This shows the equivalence of (2) and (3).

We say that a module $M$ is uniserial if $M$ has a unique composition series, and an artinian ring $A$ is left serial if a left $A$-module $A$ is a direct sum of uniserial submodules.

The following corollaries immediate from Lemma 1.4, noting $A[u_1] \simeq A[u_2] \simeq \overline{Ae}$ in Corollary 1.6.

**Corollary 1.5.** *Let $A$ be a left serial ring, $e$ a primitive idempotent of $A$ and $r$ and $n$ integers* $\geq 1$. *Then the following conditions are equivalent.*

(1) $|\overline{eN^{r-1}}| < n$.

(2) *If* $P_1, \cdots, P_n$ *are uniserial modules with* $|P_i| = r$ *and* $\alpha = (\alpha_1, \cdots, \alpha_n)\colon \overline{Ae} \to \bigoplus_{i=1}^{n} P_i$ *is a map with monomorphism* $\alpha_i$ *for each* $i=1, \cdots, n$, *then there exists a map* $\varphi = (\varphi_1, \cdots, \varphi_n)^T\colon \bigoplus_{i=1}^{n} P_i \to P_j$ *for some* $j$ $(1 \leq j \leq n)$ *such that* $\alpha\varphi = 0$ *and* $\varphi_j$ *is an identity map.*

**Corollary 1.6.** *An artinian ring $A$ is right serial if and only if for any* $u_i \in t(eN^{r-1}f_i)$ $(i=1, 2)$ *the isomorphism* $\theta\colon A[u_1] \to A[u_2]$ *with* $[u_1]\theta = [u_2]$ *is* $(P_1, P_2)$-*extendible, where* $e, f_i \in p(A)$, $P_i = Af_i/N^r f_i$ *and* $[u_i] = u_i + N^r f_i \in P_i$. *In particular, a left serial ring $A$ is (left and right) serial if and only if for any uniserial modules* $L_1$ *and* $L_2$ *with* $|L_1| \leq |L_2|$, *any isomorphism* $\theta\colon S_1(L_1) \to S_1(L_2)$ *is* $(L_1, L_2)$-*extendible.*

## 2. The implication from (b) to (c)

The results in this section were essentially delt with in [1] (see [1, Theorem 2.5 and Remark 4]).

Let (E): $0 \to T \xrightarrow{\alpha} \bigoplus_{i=1}^{n} P_i \xrightarrow{\beta} M \to 0$ be an exact sequence of modules with monomorphism $\alpha_i\colon T \to P_i$ for each $i=1, \cdots, n$, where $n \geq 2$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta =$

$(\beta_1, \cdots, \beta_n)^T$. Put $L_i = P_i\beta$. Let $\alpha'_j \colon T \to \bigoplus_{i\neq j} P_i$ and $\beta'_j \colon \bigoplus_{i\neq j} P_j \to M$ denote maps induced from $\alpha$ and $\beta$, respectively. Then as easily seen $\beta_j$ and $\beta'_j$ are monomorphisms for each $j$ and in particular $P_i \simeq L_i$ and $\sum_{i\neq j} L_i = \bigoplus_{i\neq j} L_i$. Moreover for any non-trivial partition $I = I_1 \cup I_2$ of $I = \{1, \cdots, n\}$ (i.e. $I_1$, $I_2 \subsetneq I$ and $I_1 \cap I_2 = \phi$) we have $\bigoplus_{I_1} L_i \cap \bigoplus_{I_2} L_i \simeq T$.

Conversely let $T$ be a module and $M = \sum_{i=1}^n L_i$ a sum of submodules $L_i$ of a module $M$ with the following property:

(A) For each $j = 1, \cdots, n$, $\sum_{i\neq j} L_i = \bigoplus_{i\neq j} L_i$ and for some non-trivial partition $\{1, \cdots, n\} = I_1 \cup I_2$, $\bigoplus_{I_1} L_i \cap \bigoplus_{I_2} L_i \simeq T$.

Put $P_i = L_i$ and let $\beta \colon \bigoplus_{i=1}^n P_i \to M = \sum_{i=1}^n L_i$ be a canonical map (i.e. $(x_1, \cdots, x_n)\beta = \sum_{i=1}^n x_i$; $x_i \in P_i$). Then it is easy to see that we have an exact sequence (E) with monomorphism $\alpha_i$ and $L_i = P_i\beta$ as above. We say a sum $M = \sum_{i=1}^n L_i$ of submodules $L_i$ with $n \geq 2$ is a *T-amalgamated sum* (by (E)) if it has the property (A) (and $L_i = P_i\beta$ in the exact sequence (E)).

REMARK 1. Consider the above exact sequence (E) and put $T_j = L_j \cap \bigoplus_{i\neq j} L_j$. Then we have commutative diagrams

$$\begin{array}{ccc} \bigoplus_{i\neq j} P_i & \xrightarrow{\beta'_j} & \bigoplus_{i\neq j} L_i \\ \uparrow & & \uparrow \\ T\alpha'_j & \xrightarrow{\beta'_j} & T_j \end{array} \qquad\qquad \begin{array}{ccc} P_j & \xrightarrow{\beta_j} & L_j \\ \uparrow & & \uparrow \\ T\alpha_j & \xrightarrow{\beta_j} & T_j \end{array}$$

with isomorphism rows and inclusion columns. Since $\alpha'_j \colon T \to \bigoplus_{i\neq j} P_i$ and $\alpha_j \colon T \to P_j$ are monomorphisms, a map $\theta \colon T\alpha'_j \to T\alpha_j$ defined by $t\alpha'_j\theta = t\alpha_j$ $(t \in T)$ is well-defined and an isomorphism. Moreover we have $(t\alpha'_j)(-\theta)\beta_j = -t\alpha_j\beta_j = (t\alpha'_j)\beta'_j 1_{T_j}$; $t \in T$. Therefore it follows from the above diagrams that $\theta$ orequivalently $-\theta$ is $(\bigoplus_{i\neq j} P_i, P_j)$-extendible (resp.-maximal) if and only if $T_j$ is $(\bigoplus_{i\neq j} L_i, L_j)$-extendible (resp.-maximal).

**Lemma 2.1.** *Let $S$ be a simple module and $L_1, \cdots, L_n$ local submodules of a module $M$ such that $M = \sum_{i=1}^n L_i$ is an $S$-amalgamated sum, where $n \geq 2$ and $|L_i| \geq 2$ for each $i = 1, \cdots, n$. Then $M$ is decomposable if and only if $S_j$ is $(\bigoplus_{i\neq j} L_i, L_j)$-extendible for some $j$, $1 \leq j \leq n$, where $S_j = (\bigoplus_{i\neq j} L_i) \cap L_j$.*

Proof. Assume $M$ has a non-trivial decomposition $M = M_1 \oplus M_2$. If $\sigma \colon M \to \bar{M} = M/NM$ is a canonical epimorphism, $L_i\sigma$ is simple and we have

$\bar{M}=L_1\sigma\oplus\cdots\oplus L_n\sigma=M_1\sigma\oplus M_2\sigma$ by the assumption. Then by [1, Lemma 1.1] there exists a non-trivial partition $\{1, \cdots, n\}=I_1\cup I_2$ such that $\bar{M}=M_1\sigma\oplus(\oplus_{I_2}L_i\sigma)=(\oplus_{I_1}L_i\sigma)\oplus M_2\sigma$. Hence we have $M=M_1+(\oplus_{I_2}L_i)=(\oplus_{I_1}L_i)+M_2$, for $NM$ is small in $M$. But it holds $2|M|=(\sum_{i=1}^{n}|L_i|-1)+(|M_1|+|M_2|)$ since $|S|=1$. This shows $M=M_1\oplus(\oplus_{I_2}L_i)$ or $M=(\oplus_{I_1}L_i)\oplus M_2$. Thus $L_j$ is a direct summand of $M$ for some $j$, which implies $S_j$ is $(\bigoplus_{i\neq j}L_i, L_j)$-extendible by Lemma 1.2. The converse is also immediate from Lemma 1.2.

REMARK 2. 'Only if part' of Lemma 2.1 is essentially used in Proposition 2.4 for $n=2$ or 3. In the case $n=2$ or 3, Lemma 2.1 is shown by applying the Krull-Schmidt Theorem instead of [1, Lemma 1.1].

**Corollary 2.2.** *Let $S$ be a simple module and $P_i$ a local module with $|P_i|\geq 2$ for each $i=1, \cdots, n$. Assume* (E): $0\to S\xrightarrow{\alpha}\bigoplus_{i=1}^{n}P_i\xrightarrow{\beta}M\to 0$ *is an exact sequence of modues with monomorphisms $\alpha_i$, where $\alpha=(\alpha_1, \cdots, \alpha_n)$. Then the following conditions are equivalent.*

(1) *$M$ is decomposable.*

(2) *There is a homomorphism $\psi$: $\bigoplus_{i\neq j}P_i\to P_j$ for some $j$ such that $\alpha'_j\psi=\alpha_j$, where $\alpha'_j$: $S\to\bigoplus_{i\neq j}P_i$ is a map induced from $\alpha$.*

(3) *There is a homomorphism $\varphi$: $\bigoplus_{i=1}^{n}P_i\to P_j$ for some $j$ such that $\alpha\varphi=0$ and $\varphi_j$ is an identity map, where $\varphi=(\varphi_1, \cdots, \varphi_n)^T$.*

Proof. Each condition of (1), (2) and (3) implies $n\geq 2$. Hence, considering the $S$-amalgamated sum by the exact sequence (E), the corollary is immediate from Lemma 2.1 (see Remark 1).

**Corollary 2.3.** *Let $u_i\in t(eN^{r-1}f_i)$ for $r\geq 2$ and put $S=\overline{Ae}$ and $P_i=Af_i/N^rf_i$, where $e, f_i\in p(A)$ and $i=1, \cdots, n$. Let $\alpha_i$: $S\to P_i$ denote the monomorphism defined by $[ae]\alpha_i=[aeu_i]$; $ae\in Ae$, where $[-]$ is a residue class in $S$ or $P_i$. If $0\to S\xrightarrow{\alpha}P_1\oplus\cdots\oplus P_n\xrightarrow{\beta}M\to 0$ is an exact sequence with $\alpha=(\alpha_1, \cdots, \alpha_n)$, then the following conditions are equivalent.*

(1) *$M$ is indecomposable.*

(2) $\bar{u}_1A\oplus\cdots\oplus\bar{u}_nA\subset\overline{eN^{r-1}}$, *where $\bar{u}_i\in\overline{eN^{r-1}}$ is a residue class of $u_i$.*

Proof. This is immediate from Corollary 2.2 and Lemma 1.4.

We say that an artinian ring $A$ is of *left colocal type* if any finitely generated indecomposable left $A$-module is colocal.

**Proposition 2.4.** *Let $A$ be an artinian ring of left colocal type. Then $A$*

*satisfies the condition* (c) (i.e. ($c_1$), ($c_2$) *and* ($c_3$)).

Proof. ($c_1$) If $\overline{N^{r-1}f}\neq 0$ for $f\in p(A)$ and an integer $r\geq 1$, then by the assumption an indecomposable module $Af/N^rf$ has a simple socle $S_1(Af/N^rf)$ which contains $\overline{N^{r-1}f}$. This shows $\overline{N^{r-1}f}$ is simple. Thus $A$ is left serial.

($c_2$) Let $\theta$: $S_1(L_1)\rightarrow S_1L_2)$ be an isomorphism, where $L_1$ and $L_2$ are uniserial modules with $|L_1|\leq|L_2|$. Then as easily seen we may assume $L_1$ and $L_2$ are submodules of a module $M$ such that $M=L_1+L_2$, $S=L_1\cap L_2$ is simple and $\theta$ is the identity map of $S$ (see Remark 1). If $\theta$ is not $(L_1, L_2)$-maximal, then by Lemma 1.2 $S_1(M)\supsetneqq S_1(L_2)=S$. Hence $M$ is not colocal, so $M$ is decomposable by the assumption. Thus $\theta$ is $(L_1, L_2)$-extendible by Lemma 2.1.

($c_3$) Suppose $|\overline{eN}|\geq 3$, where $e\in p(A)$. Then $\overline{eN}\supset\bar{u}_1A\oplus\bar{u}_2A\oplus\bar{u}_3A$ for some $u_i\in t(eNf_i)$; $f_i\in p(A)$. Then there exists an indecomposable module $M$ such that $|S_1(M)|\geq 2$ by Corollary 2.3. This is a contradiction. Thus it holds $|\overline{eN}|\leq 2$ for each $e\in p(A)$.

## 3. The implication from (c) to (b) under a condition ($D$)

Throughout this section, assume that $A$ is a left serial ring. In this case any local left $A$-module is quasi-projective. Let $L$ be a uniserial module with $|L|=n$ and put $L_i=S_i(L)$ and $D_i(L)=\mathrm{Hom}\,(\bar{L}_i, \bar{L}_i)$ for each $i=1, \cdots, n$. Then $D_i(L)$ is a division ring. If $n\geq i\geq j\geq 1$, any element $\bar{\varphi}_i$: $\bar{L}_i\rightarrow\bar{L}_i$ of $D_i(L)$ is induced from a map $\varphi_i$: $L_i\rightarrow L_i$, and moreover $\varphi_i$ induces an map $\bar{\varphi}_j$: $\bar{L}_j\rightarrow\bar{L}_j$. Now we define a map $\lambda_{ij}$: $D_i(L)\rightarrow D_j(L)$ by $(\bar{\varphi}_i)\lambda_{ij}=\bar{\varphi}_j$. Then as easily seen $\lambda_{ij}$ are well-defined and ring monomorphisms with equalities $\lambda_{ij}\lambda_{jk}=\lambda_{ik}$ for all $i$, $j$ and $k$ $(n\geq i\geq j\geq k\geq 1)$. Hence through the maps $\lambda_{ij}$, we can regard a sequence $D_1(L)$, $D_2(L)$, $\cdots$, $D_n(L)$ as a descending chain

$$D_1(L)\supset D_2(L)\supset\cdots\supset D_n(L)$$

of division rings (cf. [4, p. 211]).

**Lemma 3.1.** *Let $A$ be a left serial ring. For a uniserial module $L$ with $|L|=n$ and an integer $r$ with $1\leq r\leq n$, the following conditions are equivalent.*

(1) $D_r(L)=D_n(L)$.

(2) *Any isomorphism $\theta$: $S_1(L)\rightarrow S_1(L)$ is $(L, L)$-extendible whenever $\theta$ is $(S_r(L), S_r(L))$-extendible.*

Proof. Put $L_i=S_i(L)$, $i=1, \cdots, n$ and let $\bar{\varphi}_r$: $\bar{L}_r\rightarrow\bar{L}_r$ be a map induced from an isomorphism $\varphi_r$: $L_r\rightarrow L_r$. As easily seen (1) is equivalent to a condition that there is a map $\psi_n$: $L_n\rightarrow L_n$ with $(L_r)(\varphi_r-\psi_n)\subset L_{r-1}$. Since $L$ is uniserial, the last conditions is equivalent to $(L_1)(\varphi_r-\psi_n)=0$ which implies (2).

REMARK 3. For an integer $r\geq 2$, the condition (2) of Lemma 3.1 does not imply that any isomorphism $\varphi_r\colon S_r(L)\to S_r(L)$ is $(L, L)$-extendible (see Example 1).

It is called by *$S_r$-classes* isomorphism classes of uniserial modules with composition length $r$. Note that for $e$ and $f$ in $p(A)$ and an integer $r\geq 1$, $\overline{fA}$ is embedded in $\overline{eN^{r-1}}$ if and only if $\overline{Ae}$ is embedded in $\overline{N^{r-1}f}$, since these conditions are equivalent to $eN^{r-1}f/eN^rf\neq 0$.

**Lemma 3.2.** *Let $A$ be a left serial ring and $e$, $f_1, \cdots, f_s$ and $f$ be primitive idempotents with $f_iA\not\simeq f_jA$ for $i\neq j$. Then for any integer $r\geq 1$ the following hold.*

(1) *$\overline{f_1A}\oplus\cdots\oplus\overline{f_sA}$ is embedded in $\overline{eN^{r-1}}$ if and only if $L_i=Af_i/N^rf_i$ $(i=1, \cdots, s)$ satisfy $|L_i|=r$ and $S_1(L_i)\simeq\overline{Ae}$. (Thus in this case there are $s$ $S_r$-classes whose socles are isomorphic to $\overline{Ae}$.)*

(2) *$(\overline{fA})^t$ (i.e. a direct sum of t-copies of $\overline{fA}$) is embedded in $\overline{eN^{r-1}}$ if and only if $\dim D_1(L)/_{D_r(L)}\geq t$ and $S_1(L)=\overline{N^{r-1}f}\simeq\overline{Ae}$, where $L=Af/N^rf$.*

Proof. (1) This is clear by the note above.

(2) Put $\overline{eN^{r-1}f}=eN^{r-1}f/eN^rf$ and $D=fAf/fNf$. Then $(\overline{fA})^t$ is embedded in $\overline{eN^{r-1}}$ if and only if $\dim \overline{eN^{r-1}f}_D\geq t$. By the above note, $S_1(L)=\overline{N^{r-1}f}\simeq \overline{Ae}$ if $\overline{fA}$ i sembedded in $\overline{eN^{r-1}}$. Therefore $D_1(L)=\mathrm{Hom}_A(\overline{N^{r-1}f}, \overline{N^{r-1}f})\simeq \mathrm{Hom}_A(\overline{Ae}, \overline{N^{r-1}f})\simeq\overline{eN^{r-1}f}$ as right $D$-modules. The restriction maps $\varphi_1$: $S_1(L)\to S_1(L)$ of maps $\varphi_r\colon L\to L$ coincide with the right multiplication maps by elements of $D$. Therefore we can identify $D_r(L)$ with $D$, so the assertion is immediate from the above $D$-isomorphisms.

Let $S$ be a simple module and $L$ a uniserial module with $|L|\geq 2$. Denote by $c(S)$ the number of $S_2$-classes whose socles are isomorphic to $S$ and put $m(L)=\dim D_1(L)_{D_2(L)}$. The following lemma is easily seen by Lemma 3.2.

**Lemma 3.3.** *Let $A$ be a left serial ring and $e$ a primitive idempotent. Then $|\overline{eN}|\leq 2$ if and only if $c(S_1(L))+m(L)\leq 3$ for any uniserial module $L$ with the conditions $|L|\geq 2$ and $S_1(L)\simeq\overline{Ae}$.*

Let $S$ be a simple module. We call $S$ of *first kind* if $m(L)=1$ (i.e. $D_1(L)=D_2(L)$) for any uniserial module $L$ with $S\simeq S_1(L)\subsetneqq L$, and $S$ of *second kind* if $S$ is not if first kind. By Lemma 3.2 $\overline{Ae}$ is of first kind if and only if $\overline{eN}$ is (zero or) square free (i.e. a direct sum of pair-wise non-isomorphic simple modules).

**Lemma 3.4.** *Let $A$ be a ring satisfying* (c) *and let $L_1$ and $L_2$ be uniserial modules with $|L_1|\leq|L_2|$ and $S=S_1(L_1)\simeq S_1(L_2)$.*

(1) *If $S_2(L_1)\simeq S_2(L_2)$, then $L_1$ can be embedded in $L_2$.*

(2) *If $S$ is of first kind and $S_2(L_1)\simeq S_2(L_2)$, then any isomorphism $\theta\colon S_1(L_1)$*

$\to S_1(L_2)$ *is* $(L_1, L_2)$*-extendible.*

(3) *If* $S$ *is of second kind, then* $L_1$ *can be embedded in* $L_2$.

Proof. (1) is clear by ($c_2$), and (2) follows from Lemma 3.1 and ($c_2$). Moreover (3) is an immediate consequence of (1) since it holds $c(S)=1$ by Lemma 3.3.

**Lemma 3.5.** *Let* $A$ *be a ring satisfying* (c) *and let* $P_1, \cdots, P_n$ *be uniserial modules with* $|P_i|\geq 2$ *and* $\alpha_i\colon S\to P_i$ *a homomorphism for each* $i=1, \cdots, n$, *where* $S$ *is a simple module and* $n\geq 3$. *If* $0\to S\xrightarrow{\alpha} P_1\oplus\cdots\oplus P_n\xrightarrow{\beta} M\to 0$ *is an exact sequence with* $\alpha=(\alpha_1, \cdots, \alpha_n)$, *then* $M$ *is decomposable.*

Proof. We may assume that $|P_1|\leq|P_2|\leq\cdots\leq|P_n|$ and each $\alpha_i$ is a non-zero map. Put $S_i=S_1(P_i)$ and $P_i'=S_2(P_i)$ and consider an exact sequence $0\to S\xrightarrow{\alpha'} P_1'\oplus\cdots\oplus P_n'\xrightarrow{\beta'} M'\to 0$ induced from the above one. Then by ($c_3$), $n\geq 3$ and Corollary 1.5, there exists a map $\varphi'=(\varphi_1', \cdots, \varphi_n')^T\colon \bigoplus_{i=1}^{n} P_i'\to P_j'$ for some $j$ such that $\alpha'\varphi'=0$ and $\varphi_j'$ is an identity map. Put $I=\{i\,|\,\varphi_i'$ is an isomorphism (i.e. $(S_i)\varphi_i'\neq 0)\}$. Then we may assume $j=\max_{i\in I} i$ by considering a map $\varphi_i'\varphi_k'^{-1}$ instead of $\varphi_i'$ for each $i=1, \cdots, n$ if $k>j$ for some $k\in I$. By ($c_2$) for each $i\in I$, there exists a map $\varphi_i\colon P_i\to P_j$ such that $(S_i)(\varphi_i-\varphi_i')$, where we take an identity map as $\varphi_j$. For each $k\notin I$, let $\varphi_k\colon P_k\to P_j$ be a zero map. Then for $\varphi=(\varphi_1, \cdots, \varphi_n)^T$ we have $\alpha\varphi=0$, and therefore by Corollary 2.2 $M$ is decomposable.

We say that a module $M$ is of $I_1$*-type* (resp. $I_2$*-type*) if $M$ is indecomposable and $|\bar{M}|=1$ (resp. $|\bar{M}|=2$), and $M$ is of $I$*-type* if $M$ is of $I_1$- or $I_2$-type. Since $A$ is left serial, the modules of $I_1$-type coincide with the uniserial modules.

**Proposition 3.6.** *Let* $A$ *be a left serial ring satisfying* ($c_2$). *Then a module* $M$ *is of* $I_2$*-type if and only if there exist uniserial submodules* $L_1$ *and* $L_2$ *which satisfy the following conditions.*

(1) $M=L_1+L_2$ *and* $|L_1|, |L_2|\geq 2$.

(2) $S=L_1\cap L_2$ *is a simple module and* $S$ *is* $(L_1, L_2)$*-maximal. Moreover in this case* $S=S_1(M)$, *so* $M$ *is colocal.*

Proof. 'If' part and $S=S_1(M)$ are immediate from Lemma 1.2.

'Only if' part: Let $M$ be an indecomposable module with $|\bar{M}|=2$. Then we have clearly $M=L_1+L_2$ for some uniserial submodules $L_1$ and $L_2$ such that $L_1\cap L_2\neq 0$ and $2\leq|L_1|\leq|L_2|$. Assume $L_1\cap L_2$ is not simple. If $S'$ is a simple submodule $L_1\cap L_2$, then $S'$ is not $(L_1, L_2)$-maximal so $S'$ is $(L_1, L_2)$-extendible from ($c_2$). Thus by Lemma 1.1, $M=L_1'+L_2$ for some uniserial submodule $L_1'$ of $M$ such that $|L_1'\cap L_2|<|L_1\cap L_2|$. Iterating this argument, the assertion holds.

Let $M$, $L_1$ and $L_2$ be as the above proposition. If $|L_1|\leq|L_2|$, then $|L_2|$ is equal to the Loewy length $t$ of $M$ (i.e. $N^{t-1}M\neq 0$ and $N^tM=0$) and we have $|L_1|=|M|-|L_2|+1$. Thus we define an integer $s(M)$ as min $\{|L_1|, |L_2|\}$ determined by $M$. Moreover we define $s(L)$ as $|L|$ if $L$ is a uniserial module.

Now we consider the following condition (D) which is always satisfied for finite dimensional algebras over a field.

(D) $\dim_{D_2(L)} D_1(L)=\dim D_1(L)_{D_2(L)}$ for any uniserial left $A$-module $L$ with $|L|\geq 2$.

Note that the condition (D) is equivalent to the following: $\dim_D \mathrm{Hom}_A(\overline{Nf}, \overline{Nf})=\dim \mathrm{Hom}_A(\overline{Nf}, \overline{Nf})_D$ for any $f\in p(A)$, where $D$ denotes a division ring $fAf/fNf$ and $\mathrm{Hom}_A(\overline{Nf}, \overline{Nf})$ is canonically regarded as a $(D, D)$-bimodule.

**Lemma 3.7.** *Let $A$ be a ring satisfying the conditions* (c) *and* (D). *If $M$ is a module of $I_2$-type and $L$ is a uniserial module with $|L|\leq s(M)$, then any homomorphism $\theta$: $S_1(L)\to S_1(M)$ is $(L, M)$-extendible.*

Proof. Put $S=S_1(L)$ and $S'=S_1(M)$. From Proposition 3.6, there exist uniserial submodules $L_1$ and $L_2$ of $M$ such that $M=L_1+L_2$, $|L_i|\geq 2$ $(i=1, 2)$, $S'=L_1\cap L_2$ is simple and $(L_1, L_2)$-maximal. Then we have $|L|\leq|L_i|$; $i=1, 2$, from the definition of $s(M)$. We may assume $\theta$: $S\to S'$ is an isomorphism, since otherwise $\theta$ is a zero map.

(i) In case $S$ is of first kind. Since $S'$ $(\simeq S)$ is of first kind and $(L_1, L_2)$-maximal, we have $S_2(L_i)\not\simeq S_2(L_2)$ by Lemma 3.4. It follows from $c(S_1(L))\leq 2$ that $S_2(L)\simeq S_2(L_1)$ or $S_2(L)\simeq S_2(L_2)$. Thus by Lemma 3.4 $\theta$ is $(L, L_i)$-extendible for some $i=1, 2$, and consequently $(L, M)$-extendible.

(ii) In case $S$ is of second kind. Put $r=|L|$ and $M'=S_r(L_1)+S_r(L_2)\subset M$. It suffices to show that $\theta$: $S\to S_1(M')$ is $(L, M')$-extendible. Thus we may assume $M=M'$ and $r=|L|=|L_1|=|L_2|$. Since $S$ is of second kind and $S=S_1(L)\simeq S_1(L_1)\simeq S_1(L_2)$, we have isomorphisms $\beta_i$: $L\to L_i$ for $i=1, 2$ by Lemma 3.4. Let $s$ be an elements of $S$. Since the restriction maps $\beta_i$: $S\to S_1(L_i)=S'$ are isomorphisms, there is an isomorphism $\lambda$: $S\to S$ such that $s\lambda\beta_1=-s\beta_2$. Define $\alpha$: $S\to L\oplus L$ and $\beta$: $L\oplus L\to M$ as $s\alpha=(s\lambda, s)$ and $\beta=(\beta_1, \beta_2)^T$. Then we have an exact sequence $0\to S\xrightarrow{\alpha} L\oplus L\xrightarrow{\beta} M\to 0$. Since $S'$ is $(L_1, L_2)$-maximal, $\lambda$ is also $(L, L)$-maximal (see Remark 1). The maps $\beta_1$: $S\to S'$ and $\theta$: $S\to S'$ are isomorphisms, so we have an isomorphism $\mu$: $S\to S$ such that $s\theta=s\mu\beta_1$, i.e. $s\theta=s(\mu, 0)\beta$. By Lemma 3.1, Lemma 3.3 and the assumption, it holds that $D_2(L)=D_r(L)$ and $\dim_{D_r(L)} D_1(L)=\dim D_1(L)_{D_r(L)}=2$. On the other hand $\lambda$: $S\to S$ is $(L, L)$-maximal, so $\lambda\notin D_2L)=D_r(L)$. Consequently $D_1(L)=D_r(L)1_S+D_r(L)\lambda$ and there exist maps $\varphi_i$: $L\to L$ $(i=1, 2)$

such that $\mu=\varphi_1 1_S-\varphi_2\lambda_i$ in $D_1(L)$, i.e. $s\mu=s\varphi_1-s\varphi_2\lambda$. Put $\varphi=(\varphi_1, \varphi_2): L\to L\oplus L$. Since $(s\varphi_2\lambda, s\varphi_2)\beta=(s\varphi_2)\alpha\beta=0$, we have $s\varphi\beta=(s\varphi_1, s\varphi_2)\beta=(s\varphi_1-s\varphi_2\lambda, 0)\beta=s(\mu, 0)\beta=s\theta$. This shows $\varphi\beta: L\to M$ is an extension of $\theta: S\to S'$.

For any artinian ring $A$, the condition (b) implies (c) by Proposition 2.4. But its converse does not necessarily hold (see Example 3). The following proposition shows the converse holds under the condition (D).

**Proposition 3.8.** *Let $A$ be a ring satisfying conditions* (c) *and* (D). *Then $A$ is of left colocal type.*

Proof. Let $M$ be an $A$-module with $|\bar{M}|=n$. By induction on $n$, we show that $M$ has a decomposition $M=M_1\oplus\cdots\oplus M_r$ such that each $M_i$ is of $I$-type. If $n=1$ or 2, then the assertion holds by Proposition 3.6. Assume $n\geq 3$. Then it suffices to show that $M$ is decomposable, for any proper direct summands of $M$ has a decomposition as above by the inductional assumption. From $|\bar{M}|=n$ we have $M=L_1+\cdots+L_n$ for some uniserial modules $L_i$, $i=1, \cdots, n$, since $A$ is left serial. We may assume $|L_1|\leq|L_i|$ for each $i=1, \cdots, n$. By inductional assumption $L_2+\cdots+L_n=M_2\oplus\cdots\oplus M_r$ for some modules $M_i$ of $I$-type; $i=2, \cdots, r$. If there is a module $M_i$, $2\leq i\leq r$, such that $s(M_i)<|L_1|$, then we have $M=L_1'+\cdots+L_n'$ for some uniserial submodules $L_i'$ with $|L_1'|<|L_1|$. Iterating of this argument, we may assume that $M=L_1+(M_2\oplus\cdots\oplus M_r)$ and $|L_1|\leq s(M_i)$ for each $i$. Put $M'=M_2\oplus\cdots\oplus M_r$ and $T=L_1\cap M'$. If $T$ is a zero module, our assertion is clear. Assume $|T|\geq 2$. Let $S$ be the simple submodule of $T$ and denote by $\pi_i$: $T\to M_i$ the restriction map of a projection $M_2\oplus\cdots\oplus M_r\to M_i$ for each $i$. Then by ($c_2$) and Lemma 3.7, $\pi_i$: $S\to M_i$ is $(L_1, M_i)$-extendible, for this is clear in case $\pi_i$ is zero-map. Hence $S$ is $(L_1, M')$-extendible, so there exists a uniserial submodule $L_1'$ such that $M=L_1'+M'$, $|L_1'|<|L_1|$ and $|L_1'\cap M'|<|T|$ by Lemma 1.1. Iterating this argument, we may assume $M=L_1+(M_2\oplus\cdots\oplus M_r)$, $|L_1|\leq s(M_i)$ for each $i=2, \cdots, r$, and $T=L_1\cap(M_2\oplus\cdots\oplus M_r)$ is simple. If $M_j$ is of $I_2$-type for some $j\,(2\leq j\leq r)$, then $\pi_j$: $T\to M_j$ is $(L_1, M_j)$-extendible and therefore by Lemma 1.3 $M$ is decomposable. If $M_i$ is of $I_1$-type for any $i\,(2\leq i\leq r)$, then $M$ is decomposable by Lemma 3.5.

## 4. The equivalence of (c) and (d)

In this section we study the following condition (Er) (for any integer $r\geq 1$) which is a generalization of ($c_2$) (i.e. (E2) implies ($c_2$)).

(Er) For any uniserial modules $L_1$ and $L_2$ with $r\leq|L_1|\leq|L_2|$, any isomorphism $\theta$: $S_1(L_1)\to S_1(L_2)$ is $(L_1, L_2)$-extendible whenever $\theta$ is $(S_r(L_1), S_r(L_2))$-extendible, where $r$ is an integer $\geq 1$.

In particular the equivalence of (c) and (d) is shown as an immediate consequence of a necessary and sufficient condition for left serial rings to satisfy (Er) (c.f. Corollary 4.4).

For submodules $L_1, \cdots, L_n$ of a module $M$, we say that $L_1, \cdots, L_n$ are independent if the sum $\sum_{i=1}^{n} L_i$ is direct (i.e. $\sum_{i=1}^{n} L_i = \bigoplus_{i=1}^{n} L_i$).

**Lemma 4.1.** *Let $A$ be a left serial ring and $u_i \in t(eN^{r-1}f_i)$ for $i=1, \cdots, n$, where $r$ is an integer, $e$ is an idempotent and $f_i$ is a primitive idempotent. Then the following conditions are equivalent.*

(1) *$\bar{u}_1A, \cdots, \bar{u}_nA$ are independent, where $\bar{u}_i$ is a residue class $u_i+eN^r$ of $u_i$ in $eN^{r-1}/eN^r$.*

(2) *$u_1A, \cdots, u_nA$ are independent and $u_1A\oplus\cdots\oplus u_nA$ is a direct summand of $eN^{r-1}$.*

Proof. (1)$\Rightarrow$(2). Assume $u_1A, \cdots, u_nA$ are dependent. Then there are elements $a_i=f_ia_ig$ of $A$, $i=1, \cdots, n$ such that $u_1a_1+\cdots+u_na_n=0$ and $u_ka_k \neq 0$ for some $k$, $1\leq k\leq n$, where $g\in p(A)$. Since $Ag$ is uniserial by the assumption, there is an integer $j$, $1\leq j\leq n$, say $j=n$, with $Au_ia_i\subset Au_na_n\subset Ag$ for each $i=1,\cdots, n$. Clearly we have $Au_i=N^{r-1}f_i$ and $Au_na_n=N^{s-1}g$ for some integer $s$. Consider $\tilde{a}_i$: $Af_i\rightarrow Ag$ a right multiplication map by $a_i$. Then we have $(N^{r-1}f_n)\tilde{a}_n=Au_na_n=N^{s-1}g$, which shows $s\geq r$ and $\tilde{a}_n$ induces an isomorphism $\psi_n$: $Af_n/N^rf_n\rightarrow N^{s-r}g/N^sg$. Moreover $(N^rf_i)\tilde{a}_i=Nu_ia_i\subset Nu_no_n=N^sg$ and so $\tilde{a}_i$ induces a homomorphism $\psi_i$: $Af_i/N^rf_i\rightarrow N^{s-r}g/N^sg$. Put $\psi=(\psi_1, \cdots, \psi_n)^T$ and $\varphi=(\varphi_1, \cdots, \varphi_n)^T=\psi\psi_n^{-1}$. Then from $u_1a_1+\cdots+u_na_n=0$, we have $\sum_{i=1}^{n}[u_i]\varphi_i=(\sum_{i=1}^{n}[u_i]\psi_i)\psi_n^{-1}=[\sum_{i=1}^{n}u_ia_i]\psi_n^{-1}=0$, where $[u_i]$ and $[\sum_{i=1}^{n}u_ia_i]$ denote residue classes $u_i+N^rf_i$ and $\sum_{i=1}^{n}u^ia_i+N^sg$, respectively. Clearly $\varphi_n=\psi_n\psi_n^{-1}$ is an identity map. Hence by Lemma 1.4, $\bar{u}_aA, \cdots, \bar{u}_nA$ are dependent. Thus (1) implies that $u_1A, \cdots, u_nA$ are independent.

Next under the condition (1) we show $u_1A\oplus\cdots\oplus u_nA$ is a direct summand of $\overline{eN^{r-1}}$. Since $\bar{u}_1A, \cdots, \bar{u}_nA$ are independent, there are elements $v_i\in t(eN^{r-1}g_i)$; $g_i\in p(A)$, $i=1, \cdots, n$, such that $\overline{eN^{r-1}}=\bar{u}_1A\oplus\cdots\oplus\bar{u}_nA\oplus\bar{v}_1A\oplus\cdots\oplus\bar{v}_mA$. Therefore it holds $eN^{r-1}=u_1A\oplus\cdots\oplus u_nA\oplus v_1A\oplus\cdots\oplus v_mA$, for $\bar{u}_1A, \cdots, \bar{u}_nA, \bar{v}_1A, \cdots, \bar{v}_mA$ are independent and $eN^r$ is small in $eN^{r-1}$.

(2)$\Rightarrow$(1). This is clear.

**Corollary 4.2.** *Let $A$ be a left serial ring, $e$ an idempotent of $A$ and $r$ an integer$\geq 1$. Assume a right $A$-module $M$ is a direct summand of $eN^{r-1}$ with $|\bar{M}|=n$. Then we have $M=u_1A\oplus\cdots\oplus u_nA$ for some $u_i\in t(eN^{r-1}f_i)$, where $f_i\in p(A)$ and $i=1, \cdots, n$. Therefore $M$ is a direct sum of local right $A$-modules.*

Proof. If $\sigma$: $eN^{r-1} \to \overline{eN^{r-1}}$ is a canonical map, $\bar{M} \simeq M\sigma$. By the assumption $eN^{r-1} = M \oplus M'$ for some submodule $M'$ of $eN^{r-1}$. Hence $\overline{eN^{r-1}} = M\sigma \oplus M'\sigma = \bar{u}_1 A \oplus \cdots \oplus \bar{u}_n A \oplus M'\sigma$ for some $u_i \in t(eN^{r-1}f_i)$ with $u_i \in M$. Therefore by Lemma 4.1, $eN^{r-1} = u_1 A \oplus \cdots \oplus u_n A \oplus M'$. But $u_1 A \oplus \cdots \oplus u_n A \subset M$, and so $M = u_1 A \oplus \cdots \oplus u_n A$.

**Lemma 4.3.** *Let $A$ be a left serial ring and $e$ a primitive idempotent of $A$ and $r$ an integer $\geq 1$. Then the following conditions are equivalent.*

(1) *For any uniserial modules $L_1$ and $L_2$ such that $S_1(L_1) \simeq \overline{Ae}$ and $r \leq |L_1| \leq |L_2|$, any isomorphism $\theta$: $S_1(L_1) \to S_1(L_2)$ is $(L_1, L_2)$-extendible whenever $\theta$ is $(S_r(L_1), S_r(L_2))$-extendible.*

(2) *The right $A$-module $eN^{r-1}$ is a direct sum of uniserial submodules.*

Proof. Note by Lemma 4.1 and the Krull-Schmidt Theorem the condition (2) is equivalent to the following: For any $v \in t(eN^{r-1}g)$; $g \in p(A)$, $vA$ is a uniserial right $A$-module.

(1)$\Rightarrow$(2). Let $v \in t(eN^{r-1}g)$; $g \in p(A)$. By Lemma 4.1, $eN^{r-1} = vA \oplus M$ for some submodule $M$. Assume $vA$ is not uniserial. Then $\overline{vN^s}$ is not simple for some $s \geq 1$. Since $eN^{s+r-1} = vN^s \oplus MN^s$, by Lemma 4.2 $vN^s = u_1 A \oplus \cdots \oplus u_m A$ for some $u_i \in t(eN^{s+r-1}f_i)$; $f_i \in p(A)$, where $m \geq 2$ and $i = 1, \cdots, m$. Hence we have $u_i = va_i$ for an element $a_i$ of $A$ with $a_i = ga_i f_i$, $i = 1, 2$. By the assumption $Au_i = N^{s+r-1}f_i$ and $Av = N^{r-1}g$. Put $P = Ag/N^r g$ and $L_i = Af_i/N^{s+r}f_i$. Since $Ava_i = Au_i$ and $N^r ga_i = N^{s+r}f_i$, a right multiplication map $\tilde{a}_i$: $Ag \to Af_i$ induces an isomorphism $\psi_i$: $P \to S_r(L_i)$ with $[v]\psi_i = [u_i]$, where $[v] \in P$ and $[u_i] \in L_i$ are residue classes of $v$ and $u_i$, respectively. Put $\varphi' = \psi_1^{-1}\psi_2$. We have an isomorphism $\varphi'$: $S_r(L_1) \to S_r(L_2)$ with $[u_1]\varphi' = [u_2]$, and clearly $A[u_i] = S_1(L_i)$. Then by the condition (1), there is an isomorphism $\varphi$: $L_1 \to L_2$ with $[u_1]\varphi = [u_2]$. This is a contradiction by Lemma 1.4 and Lemma 4.1.

(2)$\Rightarrow$(1). Assume (2). Let $L_1$ and $L_2$ be uniserial modules as in (1) and $\theta$: $S_1(L_1) \to S_1(L_2)$ be an isomorphism which has an extension $\varphi_r$: $S_r(L_1) \to S_r(L_2)$. It suffices to show (1) in the case $|L_1| = |L_2| = s+r$ and $L_i = Af_i/N^{s+r}f_i$ where $s \geq 1$, $f_i \in p(A)$ and $i = 1, 2$. Since $L_i$ is uniserial, $P \simeq S_r(L_1) \simeq S_r L_2)$ for some $P = Ag/N^r g$; $g \in p(A)$. Hence we have isomorphism $\psi_i$: $P \to S_r(L_i)$; $i = 1, 2$, with $\psi_1\varphi_r = \psi_2$, which are induced from right multiplication maps $\tilde{a}_i$: $Ag \to Af_i$ by $a_i = ga_i f_i \in A$. Thus for some $v \in t(eN^{r-1}g)$ and $u_i \in t(eN^{s+r-1}f_i)$; $i = 1, 2$, it is satisfied $A[v] = S_1(P)$, $A[u_i] = S_1(L_i)$, $[va_i] = [u_i]$ and $[u_1]\theta = [u_2]$. By the assumption $vA$ is uniserial and hence it holds $va_1 A \supset va_2 A$ or $va_1 A \subset va_2 A$. If $va_1 A \supset va_2 A$, then we have $va_2 = va_1 c$ for some $c = f_1 c f_2 \in A$. Hence $\theta$ is extended to the right multiplication map $\tilde{c}$: $L_1 \to L_2$ since $[u_1]\tilde{c} = [va_1]\tilde{c} = [va_1 c] = [va_2] = [u_2]$. The assertion is similarly shown in the case $va_1 A \subset va_2 A$.

By Lemma 4.3 we have the following corollary. In case $r = 1$, the corollary implies the last assertion in Corollary 1.6.

**Corollary 4.4.** *A left serial ring $A$ satisfies the condition* (Er) *if and only if the right $A$-module $N^{r-1}$ is a direct sum of uniserial submodules.*

If $A$ is a finite dimensional algebra over a field, $A$ is of right local type if and only if $A$ is of left colocal type by the duality. Thus by Propositions 2.4 and 3.8 and Corollary 4.4, we have the following theorem which is shown by Tachikawa [4, 5] except for the equivalence (c) and (d) for artinian rings (see the introduction).

**Theorem 4.5** (Tachikawa). *Let $A$ be an artinian ring and consider the following conditions.*

(a) *$A$ is of right local type.*

(b) *$A$ is of left colocal type.*

(c) ($c_1$) *$A$ is left serial.*

($c_2$) *For any uniserial left $A$-modules $L_1$ and $L_2$ with $|L_1| \leq |L_2|$, any isomorphism $\theta$: $S_1(L_1) \to S_1(L_2)$ is $(L_1, L_2)$-maximal or $(L_1, L_2)$-extendible.*

($c_3$) *$|eN/eN^2| \leq 2$ for any primitive idempotent $e$ of $A$.*

(d) ($d_1$) *$A$ is left serial.*

($d_2$) *$eN = M_1 \oplus M_2$, for any primitive idempotent $e$ of $A$, where $M_i$ is either zero or a uniserial submodule of the right $A$-module $eN$ for each $i=1, 2$.*

*Then* (b) *implies* (c), *and* (c) *is equivalent to* (d). *If $A$ satisfies the condition* (D), *then* (c) *implies* (b). *In particular if $A$ is a finite dimensional algebra over a field, then the conditions* (a)–(d) *are equivalent.*

## 5. Examples

EXAMPLE 1. Let $K$ be a field and $A$ a subalgebra of a full matrix algebra $M_3(K)$ which is defined by the following:

$$A = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| \, a_{22} = a_{33}, \ a_{ij} \in K \right\}.$$

Then $A$ satisfies (d) (and so (c)). Let $e_{ij}$ be the $(i, j)$-matrix unit of $M_3(K)$ $(1 \leq i, j \leq 3)$ and put $e = e_{11}$. We have $Ne = Ke_{21} + Ke_{31}$, where $N = \operatorname{rad} A$. Define a map $\varphi$: $Ne \to Ne$ by $(be_{21} + ce_{31})\varphi = be_{21} + (b+c)e_{31}$; $b, c \in K$. It is easy to see that $\varphi$ is an automorphism of $Ne$. Since the restriction map $\varphi_1$: $S_1(Ne) \to S_1(Ne)$ of $\varphi$ is an identity map, $\varphi_1$ is $(Ae, Ae)$-extendible. (More generally $S_1(Ne)$ is of first kind, so any automorphism $S_1(Ne) \to S_1(Ne)$ is $(Ae, Ae)$-extendible.) But $\varphi$: $Ne \to Ne$ is not $(Ae, Ae)$-extendible, since any automorphism $Ae \to Ae$ is a right multiplication map $\tilde{a}$ by an element $a$ of $eAe\,(\simeq K)$. Thus the condition ($c_2$) does not necessarily imply the following: Any isomorphism $\theta'$: $S_2(L_1)\to S_2(L_2)$ is $(L_1, L_2)$-extendible for any uniserial module $L_1$ and $L_2$ with $2\leq |L_1|\leq |L_2|$. (This example shows the condition II in Introduction of [4] is not equivalent to the condition II in [4, Theorem 5.3].)

EXAMPLE 2. There exists an artinian ring of left colocal type which is not a finite dimensional algebra over a field and moreover is not serial: Let $K$ be a field and $F$ a field of quotients of the polynomial ring $K[x]$ in one indeterminate. Let $\tau$: $F\to F$ be a ring endomorphism extended from an endomorphism $K[x]\to K[x]$ which fixes $K$ and maps $x$ onto $x^2$. Put $M=F$ and consider an $(F, F)$-bimodule $M$ defined as $a\cdot m\cdot b=am(b)\tau$; $a, b\in F$, where the multiplication in right side of the equality are those in the field $M(=F)$. Let $A=F\ltimes M$ be a trivial extension of $F$ over $M$. Then $A$ is an artinian ring with Jacobson radical $N=M$ which satisfies the condition (d). Moreover $A$ satisfies the condition (D) since $\mathrm{Hom}_A(A/N, A/N)\simeq F$ is a field for a unique simple left $A$-module $A/N$ (up to isomorphism). Hence $A$ is of left colocal type by Theorem 4.5. On the other hand $A$ is not a finite dimensional algebra over a field since the center of $A$ is equal to $(K, 0)$. Clearly $A$ is not serial.

Next we give an example of an artinian ring which satisfies the condition (c) but is not of left colocal type. For modules $S$, $L'$ and a submodule $S'$ of $L'$ and a homomorphism $\theta$: $S\to S'$, we denote also by $\theta$: $S\to L'$ the composition map $S\to L'$ of the map $\theta$: $S\to S'$ and the inclusion map $S'\to L'$.

The following lemma is due to [1, Proposition 2.6].

**Lemma 5.1.** *Let* $0\to T\xrightarrow{\alpha}\bigoplus_{i=1}^{n} P_i\xrightarrow{\beta} M\to 0$ *be an exact sequence with colocal modules* $P_i$ *and* $\alpha=(\alpha_1, \cdots, \alpha_n)$. *Assume each map* $\alpha_i$: $T\to P_i$ *is a monomorphism with* $\mathrm{Im}\,\alpha_i=P_i$ *and* Coker $\alpha_1$ *is a simple module. Then* $M$ *is decomposable if and only if there exists a map* $\psi$: $\bigoplus_{i\neq j} P_i\to P_j$ *for some* $j\ (1\leq j\leq n)$ *such that* $\alpha'_j\psi=\alpha_j$, *where* $\alpha'_j$: $T\to\bigoplus_{i\neq j} P_i$ *is the map induced from* $\alpha$: $T\to\bigoplus_{i=1}^{n} P_i$.

Proof. Put $L_i=P_i\beta$ and $T_i=T\alpha_i\beta$. Then it follows from the assumption that $M=L_1+\cdots+L_n$, $T_j=L_j\cap(\bigoplus_{i\neq j} L_i)$ and $L_1/T_1$ is simple. If suffices to prove that $M$ is decomposable if and only if $T_j$ is $(\bigoplus_{i\neq j} L_i, L_j)$-extendible for some $j$ (see Remark 1). 'If' part is immediate from Lemma 1.2. Assume $M$ has a non-trivial decomposition $M=M_1\oplus M_2$. Since $L_2+\cdots+L_n=L_2\oplus\cdots\oplus L_n$ and $L_i$ is colocal for each $i=2, \cdots, n$, we have $S_1(L_2)\oplus\cdots\oplus S_1(L_n)\subset S_1(M_1)\oplus S_1(M_2)=S_1(M)$ and $S_1(L_i)$ is simple. Then by [1, Lemma 1.1] there exist a partition $\{2, \cdots, n\}=I_1\cup I_2$ and submodules $K_1$ and $K_2$ of $S_1(M)$ such that $S_1(M)=(\oplus_{I_1}S_1(L_i))\oplus S_1(M_2)\oplus K_1=S_1(M_1)\oplus(\oplus_{I_2}S_1(L_i))\oplus K_2$, which shows

$M\supset(\oplus_{I_1}L_i)\oplus M_2$ and $M\supset M_1\oplus(\oplus_{I_2}L_i)$. But $2|M|=\sum_{i=2}^{n}|L_i|+|M_1|+|M_2|+1$ since $L_1/T_1$ is simple and $T_1=L_1\cap(\bigoplus_{i=2}^{n}L_i)$. This shows that $M=(\oplus_I L_i)\oplus M_2$ (so $I_1\neq\phi$) or $M=M_1\oplus(\oplus_{I_2}L_i)$ (so $I_2\neq\phi$). Thus $L_j$ is a direct summand of $M$ for some $j$, so $T_j$ is $(\bigoplus_{i\neq j}L_i, L_j)$-extendible by Lemma 1.2.

**Lemma 5.2.** *Let $A$ be a left serial ring satisfying* (c$_2$) *and $L$ a uniserial module such that $|L|\geq 2$ and $D_1(L)\supsetneqq D_2(L)$. Then the following statements hold.*

(1) *$L$ is projective.*

(2) *Let $M$ be a module such that $M=L_1+\cdots+L_n$ is a sum of uniserial submodules $L_i$. If $|L|\leq|L_i|$ for each $i=1,\cdots,n$, then any homomorphism $\theta$: $S_1(L)\rightarrow S_1(M)$ is $(L, M)$-maximal or $(L, M)$-extendible.*

Proof. (1) Assume $L\simeq Af/N^rf$; $f\in p(A)$, and $N^rf\neq 0$. Then for $L'=Af/N^{r+1}f$ we have $D_1(L')\supset D_2(L')\supsetneqq D_3(L')$ since we may assume $D_2(L')=D_1(L)$ and $D_3(L')=D_2(L)$. This contradicts (c$_2$) by Lemma 3.1. Therefore $N^rf=0$, so $L$ is projective.

(2) We may clearly assume $\theta$ is a monomorphism. Put $P_i=L_i$ and let $P_1\oplus\cdots\oplus P_n$ be the outer direct sum of $P_1,\cdots,P_n$. We have an epimorphism $\beta$: $P_1\oplus\cdots\oplus P_n\rightarrow M$. Suppose $\theta$ is not $(L, M)$-maximal. Then $\theta$ is extended to a map $\theta'$: $S_2(L)\rightarrow M$. Since $S_2(L)$ is projective by (1), there is a map $\varphi'=(\varphi_1',\cdots,\varphi_n')$: $S_2(L)\rightarrow P_1\oplus\cdots\oplus P_n$ with $\varphi'\beta=\theta'$. Hence the restriction map $\varphi_i'$: $S_1(L)\rightarrow P_i$ of $\varphi_i'$: $S_2(L)\rightarrow P_i$ is not $(L, P_i)$-maximal and so is extended to a map $\varphi_i$: $L\rightarrow P_i$ for each $i=1,\cdots,n$ by (c$_2$). As easily seen $\varphi\beta$: $L\rightarrow M$ is an extension of $\theta$ for the map $\varphi=(\varphi_1,\cdots,\varphi_n)$: $L\rightarrow P_1\oplus\cdots\oplus P_n$.

Let $A$ be a ring satisfying the conditions (c). Let $S$ be a simple module of second kind and $L$, $L_1$ and $L_2$ uniserial modules such that $S\subsetneqq L\subset L_1\subset L_2$ (see Lemma 3.4 (3)). Consider an exact sequence $0\rightarrow S\xrightarrow{\alpha}L_1\oplus L_2\xrightarrow{\beta}M\rightarrow 0$ with $\alpha=(\lambda, 1_S)$; $\lambda$, $1_S\in D_1(L)$, where for a map $\gamma$: $S\rightarrow S$ we denote also by $\gamma$: $S\rightarrow L_i$ the composition map $S\rightarrow L_i$ of $\gamma$: $S\rightarrow S$ and the inclusion map $S\rightarrow L_i$. Then by Lemma 2.1 and (c$_2$), $M$ is indecomposable if and only if $\lambda$: $S\rightarrow S$ is $(L, L)$-maximal, i.e. $\lambda\notin D_2(L)$. Assume $M$ is indecomposable. In this case $M$ is of $I_2$-type and so colocal. Let $\theta$: $S_1(L)\rightarrow S_1(M)$ be an isomorphism. Then we have $\theta=\mu\beta_1=(\mu, 0)\beta$ for some $\mu\in D_1(L)$, where $\beta=(\beta_1, \beta_2)^T$ (see the proof of Lemma 3.7). Since by Lemma 5.2 $L$ is projective, $\theta$: $S_1(L)\rightarrow S_1(M)$ is $(L, M)$-extendible if and only if there exists a map $\varphi=(\varphi_1, \varphi_2)$: $L\rightarrow L_1\oplus L_2$ such that $\varphi\beta$: $L\rightarrow M$ is an extension of $\theta$. By the same argument in proof of Lemma 3.7 and the fact that $\beta_1$: $L_1\rightarrow M$ is a monomorphism, it is easily seen that $\varphi\beta$: $L\rightarrow M$ is an extension of $\theta$ if and only if an equality $\mu=\varphi_1 1_S-\varphi_2\lambda$ holds in $D_1(L)$, where we regard $\varphi_i$ as a map $\varphi_i$: $L\rightarrow L$ $(\subset L_i)$; $i=1, 2$. Thus by

Lemma 5.2 we have

**Lemma 5.3.** *Let $A$ be a ring satisfying* (c) *and $M$ be a module of $I_2$-type such that $S_1(M)$ is a simple module of second kind. Assume $L$ is a uniserial module with $|L|\leq s(M)$ and $\theta\colon S_1(L)\to S_1(M)$ is an isomorphism. Then under the above notation, $\theta$ is $(L, M)$-maximal if and only if $D_2(L)1_S\oplus D_2(L)\lambda\oplus D_2(L)\mu\subset D_1(L)$.*

EXAMPLE 3. There exists an artinian ring which satisfies the condition (c) (or equivalently (d)) but is not of left colocal type: Let $F$ and $G$ be division rings such that $G$ is a subring of $F$ and $\dim F_G=2$, $\dim_G F\geq 3$. There exist these rings by Cohn [2]. Let

$$A=\begin{pmatrix} G & & & \\ G & G & & \\ G & G & G & \\ F & F & F & F\end{pmatrix}$$

be a subring of a full matrix ring $M_4(F)$. Then as easily seen $A$ satisfies the condition (d) and consequently (c). But $A$ does not satisfy (b) (i.e. $A$ is not of left colocal type). In order to show it, we construct an indecomposable module which is not colocal. Put $L=Ae_{11}$, $L_i=N^{4-i}L$ $(1\leq i\leq 4)$ and $S=L_1$, where $e_{11}$ is a (1, 1)-matrix unit of $M_4(F)$, and $N=\operatorname{rad} A$. Then $L_i$ is uniserial with $|L_i|=i$. We can identify $D_1(L)$ and $D_2(L)$ with $F$ and $G$, respectively. Let $\lambda$ and $\mu$ be elements of $D_1(L)$ such that $D_2(L)1_S\oplus D_2(L)\lambda\oplus D_2(L)\mu\subset D_1(L)$ and let $\alpha'=(\lambda, 1_S)\colon S\to L_2\oplus L_4$ and $\alpha''=(\lambda, 1_S)\colon S\to L_3\oplus L_3$ be monomorphisms. Then by Lemma 1.2 we have the following exact sequences with colocal modules $P_2$ and $P_3$:

$$0\longrightarrow S\xrightarrow{\alpha'} L_2\oplus L_4\xrightarrow{\beta'} P_2\longrightarrow 0$$
$$0\longrightarrow S\xrightarrow{\alpha''} L_3\oplus L_3\xrightarrow{\beta''} P_3\longrightarrow 0\,.$$

Let $\alpha_2\colon S\to P_2$ be the composition map of $(\mu, 0)\colon S\to L_2\oplus L_4$ and $\beta'$, and let $\alpha_3\colon S\to P_3$ be the composition map of $(\mu, 0)\colon S\to L_3\oplus L_3$ and $\beta''$. Define a monomorphism $\alpha\colon S\to P_1\oplus P_2\oplus P_3$ by $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, where $P_1=L$ and $\alpha_1=1_S$, and consider an exact sequence $0\to S\xrightarrow{\alpha} P_1\oplus P_2\oplus P_3\xrightarrow{\beta} M\to 0$. Suppose $M$ is decomposable. Then by Lemma 5.1 there exists a map $\varphi_k=(\varphi_{ik}, \varphi_{jk})^T\colon P_i\oplus P_j\to P_k$ with $\alpha_i\varphi_{ik}+\alpha_j\varphi_{jk}=(\alpha_i, \alpha_j)\varphi_k=\alpha_k$, where $(i, j, k)$ is a permutation of $(1, 2, 3)$. Since the Loewy lengths of $P_2$ and $P_3$ are 4 and 3, respectively, $P_2$ is not isomorphic to $P_3$. But it holds $|P_1|<|P_2|=|P_3|$. This shows that there are no monomorphisms $P_i\to P_j$ for any $i$ and $j$ with $i\neq j$ and $i\neq 1$. Therefore $\alpha_i\varphi_{ij}=0$ if $i\neq j$ and $i\neq 1$. Thus for some $k$ $(k=2$ or $3)$, we have $s\varphi_{1k}=s\alpha_1\varphi_{1k}=s\alpha_k$ $(s\in S)$, which implies $\alpha_k\colon S\to P_k$ is $(P_1, P_k)$-extendible. This is a contradiction by Lemma 5.3. Hence $M$ is indecomposable. But the map $(\beta_2, \beta_3)\colon P_2\oplus P_3\to M$ induced from $\beta$ is a monomorphism. Therefore $M$ is not colocal.

From Theorem 4.5 and Example 3, the following question arises.

Question: *Whether does any ring of left colocal type satisfy the condition* (D)?

Though we can not answer the question, we study it in relation to simultaneous equations over a division ring. Let $A$ be a ring which satisfies (c) but not (D). Then there is a uniserial module $L$ with $|L|=2$ and $\dim_{D_2(L)}D_1(L)\geq 3$. Put $S=S_1(L)$ and let $\lambda_i$ and $\mu_i$ $(i=1, 2)$ be elements of $D_1(L)$ with $D_2(L)1_S\oplus D_2(L)\lambda_i\oplus D_2(L)\mu_i\subset D_1(L)$. Then as in Example 3, consider the following exact sequences:

$$0 \longrightarrow S \xrightarrow{(\lambda_i,\, 1_S)} L\oplus L \xrightarrow{\beta_i} M_i \longrightarrow 0$$
$$0 \longrightarrow S \xrightarrow{(1_S,\theta_1,\theta_2)} L\oplus M_1\oplus M_2 \longrightarrow M \longrightarrow 0\,,$$

where $\theta_i\colon S\to M_i$ is a map with $\theta_i=(\mu_i, 0)\beta_i$ for a map $(\mu_i, 0)\colon S\to L\oplus L$ $(i=1, 2)$. Then $M_i$ is a module of $I_2$-type and $\theta_i$ is $(L, M_i)$-maximal by Lemma 3.6 and 5.3. Moreover $M$ is not a colocal module. On the other hand by Lemma 5.1 $M$ is decomposable if and only if there exists a map $\psi\colon L\oplus M_i\to M_j$ with $(1_S, \theta_i)\psi=\theta_j$ for some permutation $(i, j)$ of $(1, 2)$ (see the proof of Example 3). Next we give a necessary and sufficient condition in order that there exists a map $\psi\colon L\oplus M_1\to M_2$ with $(1_S, \theta_1)\psi=\theta_2$. Consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r, "{(0,\, \lambda_1,\, 1_S)}"] & L\oplus(L\oplus L) \arrow[rr, "{(1_L,\, \beta_1)^T}"] \arrow[dd, dashed, "\varphi"'] & & L\oplus M_1 \arrow[r] \arrow[dd, dashed, "\psi"] & 0 \\
& & & S \arrow[ul, "{(1_S,\, \mu_1,\, 0)}"'] \arrow[ur, "{(1_S,\, \theta_1)}"] \arrow[dl, "{(\mu_2,\, 0)}"] \arrow[dr, "\theta_2"'] & & \\
0 \arrow[r] & S \arrow[r, "{(\lambda_2,\, 1_S)}"] & L\oplus L \arrow[rr, "\beta_2"] & & M_2 \arrow[r] & 0\,.
\end{tikzcd}$$

As easily seen there exists a map $\psi\colon L\oplus M_1\to M_2$ with $(1_S, \theta_1)\psi=\theta_2$ if and only if there exists a map $\varphi\colon L\oplus(L\oplus L)\to L\oplus L$ with $(\mathrm{Im}(0, \lambda_1, 1_S))\varphi(=(\mathrm{Ker}\,(1, \beta_1)^T)\varphi)\subset\mathrm{Ker}\,\beta_2$ and $(1, \mu_1, 0)\varphi\beta_2=(\mu_2, 0)\beta_2$, that is for any $s\in S$ it holds $(0, s\lambda_1, s)\varphi\beta_2=0$ and $(s, s\mu_1, 0)\varphi\beta_2=(s\mu_2, 0)\beta_2$. Put $\beta_2=(\beta_{12}, \beta_{22})^T\colon L\oplus L\to M_2$ and $\varphi=(\varphi_{ij})\colon L\oplus L\oplus L\to L\oplus L$, where $(\varphi_{ij})$ is a matrix of type $(3, 2)$ with coefficients $\varphi_{ij}$. Since $s'\lambda_2\beta_{12}+s'\beta_{22}=s'(\lambda_2, 1_S)\beta_2=0$, by using maps $\varphi_{ij}$, we can rewrite the above equalities as following:

$$((s\varphi_{31}+s\lambda_1\varphi_{21})-(s\varphi_{32}+s\lambda_1\varphi_{22})\lambda_2)\beta_{12}=0 \quad \text{and}$$
$$((s\varphi_{11}+s\mu_1\varphi_{21})-(s\varphi_{12}+s\mu_1\varphi_{22})\lambda_2)\beta_{12}=s\mu_2\beta_{12}\,.$$

But $\beta_{12}$: $L\to M_2$ is a monomorphism and $s$ is any element of $S$. Hence the equalities are equivalent to the system of equalities

$$(\varphi_{31}+\lambda_1\varphi_{21})-(\varphi_{32}+\lambda_1\varphi_2)_2\lambda_2=0 \quad \text{and}$$
$$(\varphi_{11}+\mu_1\varphi_{21})-(\varphi_{22}+\mu_1\varphi_{22})\lambda_2=\mu_{21} \quad \text{in} \quad D_1(L)\,.$$

Thus we have

**Proposition 5.4.** *Under the above notation, there exists a map* $\psi$: $L\oplus M_1\to M_2$ *with* $(1_S, \theta_1)\psi=\theta_2$ *if and only if the following simultaneous equations with* 6-*unknowns have a solution in* $D_2(L)$:

$$\text{(SE)} \quad \begin{cases} (x+\lambda_1 y)+(z+\lambda_1 u)\lambda_2=0 \\ (v+\mu_1 y)+(w+\mu_1 u)\lambda_2=\mu_2\,. \end{cases}$$

If $\lambda_1=\lambda_2$ and $\mu_1=\mu_2$, then the above simultaneous equations (SE) have a solution in $D_2(L)$ since $M_1=M_2$ and $\theta_1=\theta_2$. Moreover note that $D_2(L)\oplus\lambda_1 D_2(L)=D_2(L)\oplus\mu_1 D_2(L)=D_1(L)$ because $\lambda_1$, $\mu_1\notin D_2(L)$ and $\dim D_1(L)_{D_2(L)}$ $(=m(L))=2$ by Lemma 3.3.

Let (SE)$'$ denote the simultaneous equations obtained by exchanging 1 and 2 each other in the indices of (SE) above. Assume that for any division rings $F\supset G$ with $\dim_G F\geq 3$ and $\dim F_G=2$ there exist elements $\lambda_i$, $\mu_i$ of $F$ with $G1\oplus G\lambda_i\oplus G\mu_i\subset F\,(i=1,2)$ such that both (SE) and (SE)$'$ have no solution in $G$. Then the following conditions would be equivalent.

(1) $A$ is of left colocal type.

(2) $A$ is a ring satisfying (c) and (D).

EXAMPLE 4. An artinian ring which satisfies (c) but is not of right local type: Let $F\supset G$ be division rings as in Example 3 and put

$$A=\begin{pmatrix} G & 0 \\ F & F \end{pmatrix} \quad \text{and} \quad L=\begin{pmatrix} G \\ F \end{pmatrix}.$$

Then $A$ is a ring which satisfies (c) but does not (D) and $L$ is a unique non-simple projective module. Moreover we can regard division rings $D_1(L)\supset D_2(L)$ as $F\supset G$. It is an open problem whether $A$ is of left colocal type or not. If there exist elements $\lambda_i$, $\mu_i$ as above, then $A$ would be not of left colocal type. On the other hand we can show $A$ is of not right local type.

Let $\lambda_1$ and $\lambda_2$ be elements of $F$ with $G1\oplus G\lambda_1^{-1}\oplus G\lambda_2^{-1}\subset F$ and put $S=(G, 0)$, $S_i=(\lambda_i G, 0)$, $P=(F, F)$ and $L_i=P/S_i$. Denote by $\theta_i$: $S\to L_i$ the composition map of the inclusion $S\to P$ and the canonical epimorphism $\sigma_i$: $P\to L_i$. (We write homomorphisms between right modules on the left side.) Let $0\to S\xrightarrow{(\theta_1,\theta_2)^T} L_1\oplus L_2\to M\to 0$ be an exact sequence of right modules. We show $M$ is indecomposable. Suppose $M$ is decomposable. Then there exists an isomorphism $\psi$: $L_1\to L_2$ with $\psi\theta_1=\theta_2$ by Corollary 2.2. Since $P$ is a projective, the map $\psi$ can be lifted to a map $\tilde{\mu}$: $P\to P$ which is a left multiplication map by $\mu\in F$, so $\psi\sigma_1=\sigma_2\tilde{\mu}$. This shows $\mu(\lambda_1 G, 0)=\mu(\operatorname{Ker}\sigma_1)\subset\operatorname{Ker}\sigma_2=(\lambda_2 G, 0)$ and $\sigma_2(1,0)=\theta_2(1,0)=\psi\theta_1(1,0)=\psi\sigma_1(1,0)=\sigma_2\tilde{\mu}(1,0)=\sigma_2(\mu,0)$. Hence $\mu\lambda_1=\lambda_2 a$ and $\mu=1+\lambda_2 b$ for some $a$ and $b$ in $G$, so $b-a\lambda_1^{-1}+\lambda_2^{-1}=0$, which contradicts $G1\oplus G\lambda_1^{-1}\oplus G\lambda_2^{-1}\subset F$. Thus $M$ is indecomposable. But clearly $M$ is not local.

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