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On the Fundamental Solution of the Parabolic Equation in a Riemannian Space

By Kôsaku YOSIDA

1. Introduction. Let $\boldsymbol{R}$ be a connected domain of an infinitely differentiable, m-dimensional Riemannian space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$. We consider the general parabolic equation

$$L_{tx}f = \frac{\partial f(t, x)}{\partial t} - A_{tx}f(t, x), \tag{1.1}$$

where

$$\begin{aligned} A_{tx}f(t, x) &= g(x)^{-1/2} \frac{\partial^2}{\partial x^i \partial x^j} (g(x)^{1/2} a^{ij}(t, x) f(t, x)) \\ &\quad - g(x)^{-1/2} \frac{\partial}{\partial x^i} (g(x)^{1/2} b^i(t, x) f(t, x)) + c(t, x) f(t, x), \end{aligned} \tag{1.2}$$

$$g(x) = \det (g_{ij}(x)).$$

The operator A_{tx} is assumed to be elliptic in x in the sense that

$$a^{ij}(t, x)\xi_i\xi_j > 0 \quad \text{for} \quad \sum_i (\xi_i)^2 > 0. \tag{1.3}$$

Since the value of $A_{tx}f(t, x)$ must be independent of the local coordinates $(x^1, \ldots, x^m)$, we must have, by the coordinates change $x \rightarrow \bar{x}$, the transformation rule

$$\bar{a}^{ij}(t, \bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^s} a^{ks}(t, x), \tag{1.4}$$

$$\bar{b}^i(t, \bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} b^k(t, x) + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^s} a^{ks}(t, x). \tag{1.4}$$

For the sake of simplicity, we assume that the coefficients $a^{ij}(t, x)$, $b^i(t, x)$, $c(t, x)$ and $g_{ij}(x)$ are infinitely differentiable function of the local coordinates $(x^1, \ldots, x^m)$. The purpose of the present note is to construct, under a certain HYPOTHESIS, which is surely satisfied for compact Riemannian space $\boldsymbol{R}$, the fundamental solution

$$P(s, t, y, x), \qquad (s < t \text{ and } y, x \in \boldsymbol{R}) \tag{1.6}$$

of (1.1) with the following four properties:

i) For $s < t$,

$$L'_{sy}P = -\frac{\partial P}{\partial s} - A'_{sy}P = 0, \quad L_{tx}P = \frac{\partial P}{\partial t} - A_{tx}P = 0. \tag{1.7}$$

where

$$A'_{sy}h(s, y) = a^{ij}(s, y)\frac{\partial^2 h(s, y)}{\partial y^i \partial y^j} + b^i(s, y)\frac{\partial h(s, y)}{\partial y^i} + c(s, y)h(s, y). \tag{1.8}$$

ii) When $t \downarrow s_0$, $s \uparrow s_0$ and distance $(x, x_0) \to 0$, distance $(y, x_0) \to 0$, the function $P(s, t, y, x)$ exhibits the principal singularity

$$\pi^{-m/2}(a(s_0, x_0)/g(x_0))^{1/2}(t-s)^{-m/2}\exp(-a_{ij}(s_0, x_0)(x^i-y^i)(x^j-y^j) \times 4^{-1}(t-s)^{-1}), \quad \text{where} \tag{1.9}$$

$$a(s, x) = \det(a_{ij}(s, x)), \ (a_{ij}(s, x)) = (a^{ij}(s, x))^{-1}.$$

iii) We have

(1.10) $P(s, t, y, x)$ is, for any $\varepsilon > 0$, bounded in $y(x)$ for fixed $x(y)$ when s and $t(> s+\varepsilon)$ are bounded.

(1.11) $\int_R |P(s, t, y, x)| dx$, where $dx = g(x)^{1/2}dx^1 \cdots dx^m$, is bounded in y when s and $t(> s)$ are bounded.

$$\int_R P(s, t, y, x)dx = 1 \quad \text{when} \quad c(t, x) \equiv 0. \tag{1.11$'$}$$

iv) The Chapman-Kolmogoroff's equation holds, viz.

$$P(s, t, y, x) = \int_R P(s, u, y, z)P(u, t, z, x)dz, \quad s < u < t. \tag{1.12}$$

An Application to the Stochastic Processes. Let $c(t, x) \equiv 0$. When $\boldsymbol{R}$ is a compact Riemannian space, we have, besides i)-iv), the condition

(1.13) $P(s, t, y, x)$ is everywhere non-negative.

Thus, in such a case, $P(s, t, y, x)$ may be considered as the transition probability governed by the corresponding pair of Kolmogoroff's equations.

The following construction of P is based upon a construction[1] of a "fairly regular" parametrix for the adjoint equaticn of (1.1). Mr. Seizô Itô kindly discussed the manuscript and remarked that, when $\boldsymbol{R}$ is an Euclidean space, the fundamental solution for (1.1) was constructed by F. G. Dressel[2] starting with an entirely different parametrix.

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1) Cf. K. Yosida: On the integration of diffusion equations in Riemannian spaces, the Proc. Amer. Math. Soc. 3, 1952, 864-873.

2) The fundamental solution of the parabolic equations, Duke Math. J., 7 (1940), 186-203.

His method is an extension of W. Feller's paper[3] for the case $m=1$. Mr. Itô also has succeeded in constructing the fundamental solution for the differentiable manifold $\boldsymbol{R}$ by extending Feller-Dressel's method. See the immediately following paper by Mr. Itô.

2. The Parametrix for the Adjoint Equation of (1. 1). Let, according to the new metric $dr(\tau)^2 = a_{ij}(\tau, x)dx^i dx^j$,

$$\Gamma = \Gamma(\tau, y, x) = r(\tau, y, x)^2 \tag{2.1}$$

be the square of the smallest distance of y and x of $\boldsymbol{R}$. Then we have the

Lemma. *Let the positive integer k be $>(2+m/2)$. We may construct a parametrix for the adjoint equation of* (1.1)

$$H_1(\tau, t, y, x) = (t-\tau)^{-m/2} \exp\left(-\frac{\Gamma(\tau, y, x)}{4(t-\tau)}\right)\sum_{i=0}^{k} u_i(\tau, y, x)(t-\tau)^i, \ t>\tau, \tag{2.2}$$

such that

(2.3) $u_i(\tau, y, x)$ *are infinitely differentiable in the vicinity of* $y=x$ *and* $u_0(\tau, x, x)=1$,

(2.4) $L'_{\tau y}H_1(\tau, t, y, x) = (t-\tau)^{k-m/2} \exp\left(-\dfrac{\Gamma(\tau, y, x)}{4(t-\tau)}\right) c_k(\tau, y, x)$, *where* $c_k(\tau, y, x)$ *is infinitely differentiable in the vicinity* of $y=x$.

Proof. We regard the point z on the geodesic (according to the new metric $dr(\tau)^2 = a_{ij}(\tau, x)dx^i dx^j$) joining x and y as a function of $r = r(\tau, x, z)$. We have then the well-known identities[4)]

$$L(\tau, z, \dot{z}) = a_{ij}(\tau, z)\dot{z}^i\dot{z}^j = 1, \ \dot{z}^i = \frac{dz^i}{dr}, \tag{2.5}$$

$$\frac{\partial\Gamma(\tau, y, x)}{\partial y^i} = r(\tau, y, x)\frac{\partial L(\tau, y, \dot{y})}{\partial \dot{y}^i} = 2r(\tau, y, x)a_{ij}(\tau, y)\dot{y}^j.$$

Hence we have the important identity

$$\begin{aligned} \Gamma(\tau, y, x) &= r(\tau, y, x)^2 2^{-1}\dot{y}^k \frac{\partial L(\tau, y, \dot{y})}{\partial \dot{y}^k} \\ &= r(\tau, y, x)^2 4^{-1} a^{jk}(\tau, y)\frac{\partial L(\tau, y, \dot{y})}{\partial \dot{y}^j}\frac{\partial L(\tau, y, \dot{y})}{\partial \dot{y}^k} \\ &= 4^{-1}a^{jk}(\tau, y)\frac{\partial\Gamma(\tau, y, x)}{\partial y^i}\frac{\partial\Gamma(\tau, y, x)}{\partial y^k}. \end{aligned} \tag{2.6}$$

3) W. Feller: Zur Theorie der stochastischen Prozesse, Math. Ann. **113** (1936), 113-160.

4) See, for example, M. Riesz: L'intégrale de Riemann-Liouville et le problème de Cauchy, Acta Math, **81** (1948), p. 171.

Thus the operator $A'_{\tau y}$, when applied to a function $F(\Gamma, y)$, where Γ being looked as a function of y, may be written as

$$A'_{\tau y}F = 4\Gamma \frac{\partial^2 F}{\partial\Gamma^2} + M\frac{\partial F}{\partial\Gamma} + 2\left(a^{\sigma j}\frac{\partial\Gamma}{\partial y^j}\right)\frac{\partial^2 F}{\partial\Gamma\partial y^\sigma} + N(F), \quad \text{where} \tag{2.7}$$

$$M = a^{ij}\frac{\partial^2\Gamma}{\partial y^i\partial y^j} + b^i\frac{\partial\Gamma}{\partial y^i} = 2m + \sum_i 0(x^i - y^i),$$

$$N(F) = a^{ij}\frac{\partial^2 F}{\partial y^i\partial y^j} + b^i\frac{\partial F}{\partial y^i} + cF.$$

Here the differentiation must be performed as if Γ and y are independent variables. Hence we have

$$-A'_{\tau y}H_1(\tau, t, y, x) = \sum_{i=0}^{k} -\frac{\tau}{4}(t-\tau)^{i-2-m/2}\exp\left(-\frac{\Gamma}{4(t-\tau)}\right)u_i$$
$$+\sum_{i=0}^{k}(t-\tau)^{i-1-m/2}\exp\left(-\frac{\Gamma}{4(t-\tau)}\right)\left\{\frac{1}{2}\left(a^{\sigma j}\frac{\partial\Gamma}{\partial y^j}\right)\frac{\partial u_i}{\partial y^\sigma} + \frac{M}{2}u_i - N(u_{i-1})\right.$$
$$-(t-\tau)^{k-m/2}\exp\left(-\frac{\Gamma}{4(t-\tau)}\right)N(u_k),$$

where $u_{-1}\equiv 0$ and hence $N(u_{-1})\equiv 0$. Therefore, by

$$-\frac{\partial}{\partial\tau}H_1 = \sum_{i=0}^{k}(t-\tau)^{i-1-m/2}\exp\left(-\frac{\Gamma}{4(t-\tau)}\right)u_i\left(-\frac{m}{2}+i+\frac{1}{4}\frac{\partial\Gamma}{\partial\tau}+\frac{\Gamma}{4(t-\tau)}\right)$$
$$-\sum_{i=0}^{k}(t-\tau)^{i-m/2}\exp\left(-\frac{\Gamma}{4(t-\tau)}\right)\frac{\partial u_i}{\partial\tau},$$

we obtain the lemma if u_i are successively so determined that

$$\frac{1}{2}\left(a^{\sigma j}\frac{\partial\Gamma}{\partial y^j}\right)\frac{\partial u_i}{\partial y^\sigma} + \left(-\frac{m}{2}+i+\frac{M}{4}+\frac{1}{4}\frac{\partial\Gamma}{\partial\tau}\right)u_i = N(u_{i-1}) + \frac{\partial u_{i-1}}{\partial\tau}, \tag{2.8}$$

where u_i are infinitely differentiable in the vicinity of $y = x$ and $u_{-1}\equiv 0$, $u_0(\tau, x, x) = 1$.

To this purpose, we introduce the normal coordinates of y around x according to the new metric $dr(\tau)^2 = a_{ij}(\tau, y)dy^i dy^j$

$$\eta^i = r(\tau, y, x)\left(\frac{dy^i}{dr}\right)_{r=0} = r\xi^i. \tag{2.9}$$

Then we have, by (2.5),

$$\frac{1}{2}a^{\sigma j}\frac{\partial\Gamma}{\partial y^j}\frac{\partial u_i}{\partial y^\sigma} = r\xi^\sigma\frac{\partial u_i}{\partial\eta^\sigma}.$$

We have also the order relations

$$M = 2m + 0(r),$$

$$\frac{\partial \Gamma}{\partial \tau} = 0(r).$$

Hence the equations (2.8) are transformed into ordinary differential equations in r containing the parameters ξ

$$(2.10) \qquad r\frac{du_i}{dr} + \left(-\frac{m}{2} + i + \frac{M}{4} + \frac{1}{4}\frac{\partial \Gamma}{\partial \tau}\right)u_i = N(u_{i-1}) + \frac{\partial u_{i-1}}{\partial \tau}.$$

By $u_{-1} \equiv 0$ and $u_0(\tau, y, x) = 1$, these equations may be integrated as

$$(2.11) \qquad u_0(\tau, y, x) = \exp\left(-\int_0^r \rho^{-1}\left(-\frac{m}{2} + \frac{M}{4} + \frac{1}{4}\frac{\partial \Gamma}{\partial \tau}\right)\right)d\rho,$$

$$u_i(\tau, y, x) = u_0 r^{-i}\int_0^r \rho^{i-1}u_0^{-1}\left(N(u_{i-1}) + \frac{\partial u_{i-1}}{\partial \tau}\right)d\rho, \quad (i=1, 2, 3, \dots, k).$$

2. The Fundamental Solution of the Adjoint Equation of (1.1). We assume the following

Hypothesis. There exists a positive constant η with the properties: Let $\delta(S)$ be infinitely differentiable and $\geqq 0$ for $S \geqq 0$ such that

$$(3.1) \qquad \delta(S) = 1 \quad \text{for} \quad 0 \leqq S \leqq \eta \quad \text{and} \quad \delta(S) = 0 \quad \text{for} \quad S \geqq 2\eta.$$

Let $S(x, y)$ denote the distance of x and y according to the original metric $ds^2 = g_{ij}(x)dx^i dx^j$. Then

i) the function

$$(3.2) \qquad H(s, t, y, x) = \pi^{-m/2}(a(t, x)/g(x))^{1/2}H_1(s, t, y, x)\delta(S(y, x))$$

is defined everywhere and the integral

(3.3) $\int_{\boldsymbol{R}} |H(s, t, y, x)|dx$ is bounded in y when s and $t(>s)$ are bounded.

ii) The function

(3.4) $K(s, t, y, x) = L'_{sy}H(s, t, y, x)$ is bounded everywhere when s and $t(>s)$ are bounded.

iii) The integral

(3.5) $\displaystyle\int_{S(x, y) \leqq 2\eta} dy$ is bounded in x.

The above HYPOTHESIS is surely satisfied when $\boldsymbol{R}$ is a compact Riemannian space. In the general case, the HYPOTHESIS will impose conditions upon the coefficients $g_{ij}(x)$, $a^{ij}(t, x)$, $b^i(t, x)$ and $c(t, x)$.

Theorem 1. *Let the Hypothesis*[5]. *be satisfied. Then the function*

$$P(s, t, y, x) = H(s, t, y, x) - \int_s^t d\tau \int_{\boldsymbol{R}} H(s, \tau, y, z) Q(\tau, t, z, x) dz, \tag{3.6}$$

where

$$Q(s, t, y, x) = \sum_{n=1}^{\infty} (-1)^{n+1} K_n(s, t, y, x), \tag{3.7}$$

$$K_1 = K, \ K_n(s, t, y, x) = \int_s^t d\tau \int_{\boldsymbol{R}} K(s, \tau, y, z) K_{n-1}(\tau, t, z, x) dz,$$

satisfies $L'_{sy}P(s, t, y, x) = 0$, (1.10) *and* (1.11).

Proof. We obtain, by the integral formula due to Dirichlet

$$\int_s^t d\tau \int_s^\tau M(\sigma, \tau) N(\tau, t) d\sigma = \int_s^t d\sigma \int_\sigma^t M(\sigma, \tau) N(\tau, t) d\tau,$$

the associative law

$$(K \otimes L) \otimes M = K \otimes (L \otimes M) \tag{3.8}$$

for the "convolution"

$$(L \otimes M)(s, t, y, x) = \int_s^t d\tau \int_{\boldsymbol{R}} L(s, \tau, y, z) M(\tau, t, z, x) dz. \tag{3.9}$$

Let us, by (3.4)-(3.5), put

$$\sup_{\substack{s_0 \leq s < t \leq t_0 \\ x, y \in \boldsymbol{R}}} |K(s, t, y, x)| = N, \quad \sup_x \int_{S(x, y) \leq 2\eta} dy = A.$$

Then since $K(s, t, y, x)$ vanishes for $S(y, x) \geq 2\eta$ independently of s and t, we have, for $s_0 \leq s < t \leq t_0$,

$$\sup_{x, y \in \boldsymbol{R}} |K_n(s, t, y, x)| \leq N^n A^{n-1} (t-s)^{n-1}/(n-1)!, \tag{3.10}$$

$$\sup_y \int_{\boldsymbol{R}} |K_n(s, t, y, x)| dx \leq N^n A^n (t-s)^{n-1}/(n-1)!.$$

This proves the convergence of (3.7). Thus we have, by (3.3)-(3.8),

$$P = H - H \otimes Q = H - P \otimes K. \tag{3.11}$$

We have also (1.9)-(1.11) by applying Fubini's theorem.

The proof of $L'_{sy}P(s, t, y, x) = 0$ may be obtained as follows. We first prove the fundamental limit theorem

5) It is to be noted that our HYPOTHESIS is independent of the choice of the local coordinates, whereas Mr. Itô's conditions are dependent upon the local coordinates since his conditions are referred to the "canonical coordinates system".

$$(3.12)\quad f(x)=\lim_{s\uparrow t_0,\ t\downarrow t_0}\int_{\boldsymbol{R}} f(y)H(s,t,y,x)dy=\lim_{s\uparrow t_0,\ t\downarrow t_0}\int_{\boldsymbol{R}} f(y)H(s,t,x,y)dy$$

for any continuous function $f(y)$.

This may be proved as in the note referred to 1). Thus, if we know

$$(3.13)\qquad \lim_{\varepsilon\downarrow 0} L'_{sy}\int_s^{s+\varepsilon} d\tau\int_{\boldsymbol{R}} H(s,\tau,y,z)Q(\tau,t,z,x)dz=0\,,$$

we have

$$\begin{aligned}L'_{sy}P &= L'_{sy}H+\lim_{\varepsilon\downarrow 0}\frac{\partial}{\partial s}\int_{s+\varepsilon}^{t} d\tau\int_{\boldsymbol{R}} H(s,\tau,y,z)Q(\tau,t,z,x)dz\\ &\qquad -\lim_{\varepsilon\downarrow 0}\int_{s+\varepsilon}^{t} d\tau\, L'_{sy}\int_{\boldsymbol{R}} H(s,\tau,y,z)Q(\tau,t,z,x)dz\\ &= L'_{sy}H-\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}} H(s,s+\varepsilon,y,z)Q(s+\varepsilon,t,z,x)dz\\ &\qquad -\lim_{\varepsilon\downarrow 0}\int_{s+\varepsilon}^{t} d\tau\int_{\boldsymbol{R}} Q(\tau,t,z,x)L'_{sy}H(s,\tau,y,z)dz\\ &= K-Q-K\otimes Q=0\,.\end{aligned}$$

The proof of (3.13) may be obtained by changing z into the normal coordinates ζ around y according to the metric $dr(s)^2=a_{ij}(s,z)dz^i dz^j$ and then changing the coordinates ζ and τ into ξ and κ:

$$\zeta^i=(\tau-s)^{1/2}\xi^i,\ (\tau-s)^{1/2}=\kappa\,.$$

4. The Fundamental Formula and the Identity of the Fundamental Solution of (1.1) with the that of the Adjoint Equation of (1.1). Starting with the parametrix $H^*(s,t,y,x)$ for L_{tx}, we may construct the fundamental solution $P^*(s,t,y,x)$ for L_{tx} with the same properties as those given in (1.9)-(1.11). Of course we must impose the HYPOTHESIS for H^* similar to that for H. We may prove the identity

$$(4.1)\qquad P^*(s,t,y,x)=P(s,t,y,x)\,.$$

Proof. We will make use of the FUNDAMENTAL FORMULA

$$\begin{aligned}(4.2)\qquad &\int_{\boldsymbol{R}} h(t,y)f(t,y)dy-\int_{\boldsymbol{R}} h(s,y)f(s,y)dy\\ &=\int_s^t d\tau\int_{\boldsymbol{R}}\{h(\tau,y)L_{\tau y}f(\tau,y)-f(\tau,y)L'_{\tau y}h(\tau,y)\}dy\,,\end{aligned}$$

if $h(\tau,y)$ and $f(\tau,y)$ are continuously differentiable once in τ and twice in y and if, moreover, $h(\tau,y)$ vanishes outside a compact set of y which is independent of τ. This may be proved by

$$\left[\int_{\boldsymbol{R}} h(\tau, y)f(\tau, y)dy\right]_s^t = \int_s^t d\tau \frac{d}{d\tau}\int_{\boldsymbol{R}} h(\tau, y)f(\tau, y)dy$$

and

$$\int_{\boldsymbol{R}} \{f(\tau, y)A'_{\tau y}h(\tau, y) - h(\tau, y)A_{\tau y}f(\tau, y)\}dy = 0,$$

the latter being proved by Green's integral theorem and the vanishing of $h(\tau, y)$ outside a compact set of y independently of τ.

Now let $t_2 < s < t < t_1$, and apply (4.2) to

$$h(\tau, y) = H(\tau, t_1, y, z),\ f(\tau, y) = P^*(t_2, \tau, x, y).$$

Thus

$$\int_{\boldsymbol{R}} H(t, t_1, y, z)P^*(t_2, t, x, y)dy - \int_{\boldsymbol{R}} H(s, t_1, y, z)P^*(t_2, s, x, y)dy$$
$$= \int_s^t d\tau \int_{\boldsymbol{R}} H(\tau, t, y, z)L_{\tau y}P^*(t_2, \tau, x, y)dy$$
$$- \int_s^t d\tau \int_{\boldsymbol{R}} P^*(t_2, \tau, x, y)L'_{\tau y}H(\tau, t_1, y, z)dy$$
$$= - \int_s^t d\tau \int_{\boldsymbol{R}} P^*(t_2, \tau, x, y)K(\tau, t_1, y, z)dy.$$

By letting $t_1 \downarrow t$ and remembering (3.12), we obtain

$$P^*(t_2, t, x, z) - \int_{\boldsymbol{R}} H(s, t, y, z)P^*(t_2, s, x, y)dy$$
$$= - \int_s^t d\tau \int_{\boldsymbol{R}} P^*(t_2, \tau, x, y)K(\tau, t, y, z)dy.$$

Next, by letting $t_2 \uparrow s$ and remembering the limit theorem

(3.12)′ $f(x)\left(= \lim_{s\uparrow t_0, t\downarrow t_0} \int_{\boldsymbol{R}} f(y)P^*(s, t, y, x)dy\right) = \lim_{s\uparrow t_0, t\downarrow t_0} \int_{\boldsymbol{R}} f(y)P^*(s, t, x, y)dy$

for any (integrable and) bounded continuous function $f(y)$, which may be proved as (3.12), we obtain

(4.3) $$P^*(s, t, x, z) - H(s, t, x, z) = - \int_s^t d\tau \int_{\boldsymbol{R}} P^*(s, \tau, x, y)K(\tau, t, y, z)dy,$$

viz.

(4.3)′ $$P^* = H - P^* \otimes K.$$

Therefore the continuous kernel

$$S(s, t, y, x) = P(s, t, y, x) - P^*(s, t, y, x)$$

satisfies the conditions

$$S = -S \otimes K,$$

$\sup_y \int_{\boldsymbol{R}} |S(s, t, y, x)| dx$ is bounded if s and $t(>s)$ are bounded.

Hence

$$S = -S \otimes K_n \quad (n = 1, 2, \ldots),$$

and thus, by (3.10), we must have $S(s, t, y, x) \equiv 0$.

5. The Uniqueness Lemmas and their Application to the Proof of (1.11)′, (1.12) and (1.13).

The Uniqueness Lemma 1. *Let $f(t, x)$ be a continuous (for $t \geqq s$) solution of $L_{tx} f = 0$, $t > s$, such that*

$$f(s, x) = 0\,, \quad x \in \boldsymbol{R}\,,$$

$$\int_{\boldsymbol{R}} |f(t, x)| dx \textit{ is bounded for bounded } t\,(>s)\,.$$

Then we must have $f(t, x) \equiv 0$.

Proof. By applying the same argument as was used in the proof of (4.3), we obtain, for $\varepsilon > 0$,

$$\int_{\boldsymbol{R}} f(t, y) H(t, t+\varepsilon, y, x) dy = -\int_s^t d\tau \int_{\boldsymbol{R}} f(\tau, y) K(\tau, t+\varepsilon, y, x) dy\,.$$

Hence, by letting $\varepsilon \downarrow 0$ and remembering (3.12), we have

$$f(t, x) = -\int_s^t d\tau \int_{\boldsymbol{R}} f(\tau, y) K(\tau, t, y, x) dy\,.$$

Thus we obtain $f(t, x) \equiv 0$ by the same argument as was used in the proof of $P = P^*$.

Similarly we obtain the

Uniqueness Lemma 2. *Let $h(s, y)$ be a continuous (for $s \geqq t$) solution of $L'_{sy} h = 0$, $s < t$, such that*

$$h(t, y) = 0, \quad y \in \boldsymbol{R}$$

$$\sup_y |h(s, y)| \quad \textit{is bounded if } s(<t) \textit{ is bounded.}$$

Then $h(s, y) \equiv 0$.

We are now able to prove (1.11)′, (1.12) and (1.13).

The proof of (1.12). Let $s < u < t$, and consider

$$T(s, t, y, x) = \int_{\boldsymbol{R}} P(s, u, y, z) P(u, t, z, x) dz\,.$$

It is easy to see from (3.6), (3.10) and (1.11), that

$$L_{tx}T = \int_{R} P(s, u, y, z) L_{tx} P(u, t, z, x) dz = 0\,, \quad t > u\,.$$

Moreover $T(s, u, y, x) = P(s, u, y, x)$, by

(3.12)'' $f(x)\left(= \lim_{s\uparrow t_0,\, t\downarrow t_0} \int_{R} f(y) P(s, t, y, x) dy\right) = \lim_{s\uparrow t_0,\, t\downarrow t_0} \int_{R} f(y) P(s, t, x, y) dy$

for any (integrable and) bounded continuous function $f(y)$, which may be proved as (3.12). Thus we obtain

$$T(s, t, y, x) = P(s, t, y, x) \quad \text{for} \quad t > u$$

by the uniqueness lemma 1. Similarly we may prove

$$T(s, t, y, x) = P(s, t, y, x) \quad \text{for} \quad s < u\,.$$

The proof of (1.11)'. The function

$$p(s, t, y) = \int_{R} P(s, t, y, x) dx$$

is bounded when s and $t(> s)$ are bounded and satisfies

$$-\frac{\partial p}{\partial s} - a^{ij}(s, y)\frac{\partial^2 p}{\partial y^i \partial y^j} - b^i(s, y)\frac{\partial p}{\partial y^i} = 0\,,$$

$$\lim_{s\ t} p(s, t, y) = 1 \quad (\text{by } (3.12)'')\,.$$

Hence, by the uniqueness lemma 2, we have

$$p(s, t, y) \equiv 1\,.$$

The proof of (1.13). Let $f(x)$ be non-negative and continuous. It is sufficient to prove the non-negativity of

$$F(\varepsilon, s, t, y) = \exp(\varepsilon s) \int_{R} P(s, t, y, x) f(x) dx$$

for any $\varepsilon < 0$ and for any such $f(x)$. We have, by $L'_{sy}P = 0$,

(5.1) $$-\frac{\partial F}{\partial s} - a^{ij}(s, y)\frac{\partial^2 F}{\partial y^i \partial y^j} - b^i(s, y)\frac{\partial F}{\partial y^i} - \varepsilon F = 0\,, \quad s < t\,.$$

$F(\varepsilon, t, t, y)$ is non-negative by (3.12)''. Let $F(\varepsilon, s, t, y)$ be, for fixed ε and t, negative somewhere and let $F(\varepsilon, s_0, t, y_0) < 0$. Then $\hat{F}(s, y) = F(\varepsilon, s, t, y)$ must, in the product space

$$\{s\,;\ s_0 \leqq s \leqq t\} \times \boldsymbol{R}\,,$$

reach its negative minimum at a certain point (s_1, y_1), $s_0 \leqq s_1 < t$. We have, at (s_1, y_1),

$$\frac{\partial F}{\partial s} \geqq 0\,, \quad a^{ij}\frac{\partial^2 F}{\partial y^i \partial y^j} \geqq 0\,, \quad b^i\frac{\partial F}{\partial y^i} = 0\,, \quad \varepsilon F > 0\,,$$

contrary to (5.1).

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