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SOME NOTES ON THE RADICAL OF A FINITE GROUP RING

YUKIO TSUSHIMA

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1. Introduction

Let $p$ be a prime number and $G$ a finite group with a Sylow $p$-subgroup $P$ of order $p^a$. Let be $\mathfrak{R}$ the radical of the group ring $kG$ of $G$ taken over a field $k$ of characteristic $p$. If $\mathfrak{Z}$ is the radical of the center of $kG$, then we see easily that $kG \cdot \mathfrak{Z} \subset \mathfrak{R}$. We shall show that $\mathfrak{R} = kG \cdot \mathfrak{Z}$ holds if and only if $G$ is $p$-nilpotent and $P$ is abelian.

The nilpotency index of $\mathfrak{R}$, which is denoted by $t(G)$, is the smallest integer $t$ such that $\mathfrak{R}^t = 0$. Suppose $G$ is $p$-solvable, then it is known that $a(p-1)+1 \leq t(G) \leq p^a$ (Passman [11], Tsushima [12], Wallace [16]). Furthermore if $G$ has the $p$-length one, it holds that $t(G) = t(P)$ (Clarke [2]). We see easily from this that the first equality holds in the above if $P$ is elementary, while the second holds if $P$ is cyclic. However the equality $t(G) = a(p-1)+1$ does not necessarily imply that $P$ is elementary, as is remarked by Motose (e.g. $G = S_4$, $p = 2$, see Ninomiya [10]). In contrast with this, we shall show that if $t(G) = p^a$, then $P$ is cyclic.

**Notation:** $p$ is a fixed prime number. $G$ is always a finite group and $P$ a Sylow $p$-subgroup of order $p^a$. As usual, $|X|$ denotes the cardinality of a set $X$. Let $K$ be an algebraic number field containing the $|G|$-th roots of unity and $\mathfrak{o}$ the ring of integers in $K$. We fix a prime divisor $\mathfrak{p}$ of $p$ in $\mathfrak{o}$ and we let $k = \mathfrak{o}/\mathfrak{p}$. We denote by $\{\varphi_1, \ldots, \varphi_s\}$ and $\{\eta_1, \ldots, \eta_r\}$ the set of irreducible Brauer characters and principal indecomposable Brauer characters of $G$ respectively, in which the arrangement is such that $(\eta_i, \varphi_j) = \delta_{ij}$ and $\varphi_1$ is the trivial character. We put $s(G) = \sum_{i=1}^{s} \varphi_i(1)^2$.

For a block $B$ of $kG$, we denote by $\delta_B$ and $\psi_B$ its block idempotent and the associated linear character respectively. $\mathfrak{R}(G)$ (or $\mathfrak{R}$ for brevity) denotes the radical of the group ring $kG$ and $\mathfrak{Z}$ the radical of the center of $kG$. The nilpotency index of $\mathfrak{R}(G)$, which will be denoted by $t(G)$, is defined to be the smallest integer $t$ such that $\mathfrak{R}(G)^t = 0$. If $G > H$, then $kG \cdot \mathfrak{R}(H) = \mathfrak{R}(H) \cdot kG$ is a two sided ideal of $kG$ contained in $\mathfrak{R}$, which will be denoted by $2_H$ (or $\mathfrak{S}$ for brevity). Other notations are standard.
We shall several times refer to the following Theorem of Green (Green [7], Dornhoff [4] § 52).

**Theorem.** Let $G > H$ and $G/H$ is a $p$-group.
If $V$ is a finitely generated absolutely indecomposable $kH$-module, then $V^G$ is also absolutely indecomposable.

2. Square sum of the degrees of irreducible characters

In this section, we mention some remarks about the dimension of $\mathcal{R} = \mathcal{R}(G)$, most of which are direct consequences of our results [14].

Let $S$ be the set of the $p$-elements of $G$ and $c = \sum_{x \in S} x \in kG$. In [14], we have shown that $\mathcal{R} \subseteq (0 : c)$ and we have the equality provided $G$ is $p$-solvable. For $\lambda = \sum_{x \in G} a_x x \in kG$, $a_x \in k$, we put $\sigma_p(\lambda) = \sum_{x \in S} a_x$. Note that $\sigma_p(\lambda)$ is the coefficient of the identity in $c\lambda$. Hence $c\lambda = 0$ if and only if $\sigma_p(\lambda x) = 0$ for any $x \in G$, or

$$0 : c = \{ \lambda \in kG \mid \sigma_p(x \lambda) = 0 \text{ for any } x \in G \} \quad \cdots \cdots \cdots \cdots (1)$$

Therefore, our result quoted above is written as

**Proposition 1.** If $\lambda \in \mathcal{R}$, then $\sigma_p(x \lambda) = 0$ for any $x$ of $G$.

We next discuss the dimension of $(0 : c)$. Let $M = M_G = (a_{g, h})$ be the $(|G|, |G|)$-matrix over $k$ defined as

$$a_{g, h} = \begin{cases} 1, & \text{if } gh \text{ is a } p\text{-element} \\ 0, & \text{otherwise} \end{cases}$$

Then, we have

$$\dim_k (0 : c) = |G| - r(M), \text{ where } r(M) \text{ denotes the rank of } M \text{ over } k. \quad \cdots \cdots (2)$$

Indeed, for $\lambda = \sum_{x \in G} a_x x \in kG$, we have $\sigma_p(x \lambda) = \sum_{x \in S} a_x$, that is

$$M \left( \begin{array}{c} a_x \\ \vdots \end{array} \right) = \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) \sigma_p(x \lambda) \text{ for } x \in G. \text{ From this and (1), we get easily (2).}$$

Furthermore from that $\mathcal{R} \subseteq (0 : c)$ and (2), we have

$$s(G) = |G| - \dim_N \geq r(M) \quad \cdots \cdots \cdots \cdots (3)$$

If $H$ is a subgroup of $G$, then $M_H$ appears in $M_G$ as a submatrix. In particular $r(M_G) \geq r(M_H)$. Now, recall that we have $\mathcal{R} = (0 : c)$ and hence $s(G) = r(M)$ provided $G$ is $p$-solvable. Summarizing the aboves, we have

**Proposition 2.** If $G$ is $p$-solvable, then we have $s(G) \geq s(H)$ for any subgroup $H$ of $G$. 
Remark 1. If $H$ is a $p'$-subgroup, then $r(M_H) = |H|$. Hence we have from (3) that $s(G) \geq |H|$ for any $p'$-subgroup $H$ of $G$, which has been shown in Brauer and Nesbitt [1] by the inequalities $s(G) \geq \frac{|G|}{u} \geq |H|$, where $u = \eta_i(1)$.

In connection with the above remark, we give the following, which is essentially due to Wallace [15].

**Proposition 3.** We have $s(G) = |H|$ for some $p'$-subgroup $H$ of $G$ if and only if $G \triangleright P$, in which case $H$ is necessary a complement of $P$ in $G$.

Proof. "if part" is well known and easily shown (Curtis and Reiner [3] § 64 Exercise 1).

Suppose $s(G) = |H|$ for some $p'$-subgroup $H$ of $G$. Then we have $s(G) = \frac{|G|}{u}$, which forces that $\eta_i = \varphi_i \eta_i$ for any $i$ ($1 \leq i \leq r$) (see [1] pp. 580). We claim that $u = p^a$. If this would be shown, then $H$ is necessary a complement of $P$ and $\eta_i(x)$ is rational for any $x \in G$. Then the argument of Wallace [15] is valid, concluding $G \triangleright P$ (see also M.R. 22 # 12146 No. 12 (1966)).

Let

\[ \theta(x) = \begin{cases} 
\ p^a & \text{if } x \text{ is } p \text{-regular} \\
\ 0 & \text{otherwise}
\end{cases} \]

As is well known, $\theta$ is an integral linear combination of $\eta_i$: $\theta = \sum m_i \eta_i = \eta_i \sum m_i \varphi_i$, where each $m_i$ is a rational integer. Comparing the degrees of both sides, we get $u = p^a$ as claimed. This completes the proof.

3. LC type

For convenience, we call a (finite dimensional) algebra $A$ over a field to be LC if its (Jacobson) radical is generated over $A$ by the radical of its center.

The objective of this section is to prove

**Theorem 4.** The followings are equivalent to each other.

(1) $kG$ is LC

(2) the principal block $B_0$ of $kG$ is LC

(3) $G$ is $p$-nilpotent and $P$ is abelian

“(1) $\Rightarrow$ (2)” is trivial. On the other hand, we have already shown “(3) $\Rightarrow$ (1)” in [13] assuming $P$ is cyclic. The same argument, being simplified by virtue of the Green's Theorem quoted in the introduction, will be made below to yeild the present assertion.

We begin with

**Lemma 5.** Let $G \triangleright H$ and $b$ a block of $kH$. Let $B_1, \cdots, B_s$ be the blocks
of \( kG \) which cover \( b \). If a defect group of each \( B_i \) is contained in \( H \), then we have \( \mathcal{R}B_i = \mathcal{S}B_i \) for each \( i \) (1 \( \leq i \leq s \)).

Proof. Let \( b_1, \ldots, b_t \) be the blocks of \( kH \) which are conjugate to \( b \) under \( G \) and \( \varepsilon_i \) the block idempotent of \( b_i \).

From the choice of \( B_j \)'s we have

\[
\mathcal{E} = \varepsilon_1 + \cdots + \varepsilon_t = \delta_1 + \cdots + \delta_s, \quad \text{where } \delta_i = \delta_{B_i}.
\]

Let \( \Lambda = \mathcal{S}G \varepsilon \mathcal{S}G \supset \Gamma = \mathcal{S}H \varepsilon \mathcal{S}H \). We show that \( \Lambda \) is semisimple. Let \( M \) be a \( \Lambda \)-module and \( N \) any submodule of \( M \). The inclusion map \( N \rightarrow M \) splits as \( \Gamma \)-modules, since \( \Gamma \) is semisimple and then it does as \( \Lambda \)-modules, since \( M \) is \((G, H)\) projective by the assumption. Therefore \( \Lambda \) is semisimple and our assertion is clear.

The following remark is useful.

Remark 2.

(1) (well known) If \( G \lhd H \) is a \( p' \)-group, then the assumption of Lemma 5 is always satisfied and hence we have \( \mathcal{R} = \mathcal{S}H \).

(2) (Feit [5] pp. 268) If \( G \lhd H \) is a \( p \)-group, then there is a unique block which covers \( b \).

The following Lemma is not so essential here, but we write down it for its own interest. The result is noticed by Y. Nobusato.

**Lemma 6.** Suppose \( G \lhd H \) and \( G \lhd H \) is a \( p \)-group. Then for any simple \( kH \)-module \( N \), \( N^G \) has the composition length \([I : H]\), where \( I \) is the inertia group of \( N \) in \( G \).

Proof. Clear from the Green’s Theorem and the orthogonality relations

\[
(\eta_i, \varphi_j) = \delta_{ij}.
\]

The following result has been shown in our previous paper [13].

**Lemma 7.** Let \( G \lhd H \) and \([G : H] = p\). Let \( B \) be a block of \( kG \). Suppose there is a conjugate class \( C \) of \( G \) such that \( C \lhd H \) and \( \varphi_B(C) \neq 0 \), where \( C = \sum \lambda \). Then, we have \( \mathcal{R}B = \mathcal{S}B + \mathcal{S}G(C - \varphi_B(C))\delta_B \).

Proof. We put \( \delta = \delta_B \) and \( \varphi = \varphi_B \) for brevity. Let \( \delta = \sum e \) be a decomposition into the sum of primitive idempotents. We may assume each \( e \) is contained in \( kH \) by the Green’s Theorem. It suffices to show that \( \mathcal{R}e = \mathcal{S}e + \mathcal{S}G(C - \varphi_B(C))e \). Let \( a \in G \) be any element not contained in \( H \). We have

\[
(C - \varphi(C))^{p-1}e = a^{p-1}\lambda_1 + \cdots + a^{p-1}\lambda_{p-1} - \varphi_B(C)^{p-1}e,
\]

where \( \lambda_i \in kH \).

Since \( \varphi(C) \neq 0 \), this implies that \((C - \varphi(C))^{p-1}e\) is not contained in \( \mathcal{S}e = a^{p-1}\mathcal{S}G(H)e + \cdots + \mathcal{S}G(H)e \). Therefore we have a sequence (note that \((C - \varphi(C))\delta \in \mathcal{R}\))
where $kG\varepsilon \cong (C - \psi(C))kG\varepsilon \cong \cdots \cong (C - \psi(C))^{p-1}kG\varepsilon \cong 0$, where $kG\varepsilon = kG/\mathfrak{e} \cong kG \otimes_{kH} kH e/\mathcal{R}(H)e$.

However, since $kG\varepsilon$ has at most $p$ composition factors by Lemma 6, we have $(C - \psi(C))kG\varepsilon = \mathfrak{e}$, that is $\mathfrak{e} = \mathfrak{e} + kG(C - \psi(C))\varepsilon$ as required. This completes the proof.

Before proceeding, we mention a remark. If $B$ is a block of $kG$ of full defect, then there is an ordinary irreducible character $\chi$ belonging to $B$ whose degree is not divisible by $p$. If $x$ is a $p$-element, then $\chi(x) \equiv \chi(1) \mod p$. Hence it follows that if $C$ is a conjugate class of a $p$-element, then $\psi_B(C) = |C|$.

The following proposition proves "(3) $\Rightarrow$ (1)" of Theorem 4.

**Proposition 8.** Suppose $G$ is $p$-nilpotent and $P$ is abelian. Let $\{C_1, \cdots, C_r\}$ be the set of the conjugate classes of $p$-elements of $G$. For a (normal) subgroup $H$ of $G$ containing $O_p(G)$, let $\Delta_H$ be the sum of the block idempotents of $kH$ of full defect and for any $C_i$ such that $C_i \subset H$, let $\Delta(C_i, H) = (C_i - |C_i|)\Delta_H$.

Then we have $\mathcal{R} = \sum_i kG\Delta(C_i, H)$, where $H$ is taken over the subgroups of $G$ containing $O_p(G)$. In particular, $kG$ is LC.

Proof. Let $B$ be any block of $kG$. If $B$ has the defect smaller than $a$, then there is a normal subgroup $H$ of index $p$ which contains a defect group of $B$. Then by Lemma 5 and Remark 2, we have $\mathcal{R}B = \mathfrak{R}H B$. On the other hand, assume $B$ has full defect. Let $H$ be any normal subgroup of $G$ of index $p$. There is some $C_i$ such that $C_i \subset H$ and $\psi_B(C_i) = |C_i| \equiv 0$, since $P$ is abelian. Hence by Lemma 7, we have $\mathcal{R}B = \mathfrak{R}H B + kG(C_i - |C_i|)\Delta_H$. From the above, we have $\mathcal{R} = \sum_H \mathfrak{R}H + \sum_i kG\Delta(C_i, G)$, where $H$ is taken over the normal subgroups of $G$ of index $p$ and thus the result will follow by the induction on the order of $G$ (note that if $H \supset C_i$, where $H \supset O_p(G)$, then $C_i$ is also a conjugate class of $H$).

We next go into the proof of "(2) $\Rightarrow$ (3)".

**Lemma 9.** Let $I$ be the augmentation ideal of $kG$ and $\delta_0$ the block idempotent of the principal block $B_0$ of $kG$. If $I\mathfrak{R}\delta_0 = \mathfrak{R}\delta_0$, then $G$ is $p$-nilpotent.

Proof. Let $e$ be a primitive idempotent of $kG$ such that $kGe/\mathfrak{e}e$ is the trivial $G$-module. It is easy to see that $Ie = \mathfrak{e}e$. Hence we have $I\mathfrak{R}e = I\mathfrak{R}\delta_0 e = \mathfrak{R}I\delta_0 e = \mathfrak{R}e = \mathfrak{e}e$. Recurring this, we get $I^{s+1}\mathfrak{e} = \mathfrak{e}$ for any $s \geq 0$. This implies that $G$ acts trivially on each factor of the series, $kGe \supset \mathfrak{e}e \supset \cdots \supset \mathfrak{e}e = 0$, in other words, $kGe$ has the only (non isomorphic) simple constituent, the trivial one. Hence $G$ is $p$-nilpotent.

**Lemma 10.** Suppose $G$ is a $p$-group. If $kG$ is LC, then $G$ is abelian.

Proof. We prove by the induction on the order of $G$. It is clear that if $kG$ is LC, then $k(G/H)$ is also LC for any normal subgroup $H$ of $G$. 

Let \( Z \) be the center of \( G \) and let \( z \) be an element of \( Z \) of order \( p \). We may assume \( G/\langle z \rangle \) is abelian by the induction hypothesis. Assume \( G \) is not abelian. Then we have \( G' = [G, G] = \langle z \rangle \). Since \( |gG'| = p \), \( gg' \) is the conjugate class of \( g \) unless \( g \) is central. Therefore, \( \mathcal{B} \) is spanned over \( k \) by the set \( \{ u - 1, x\sigma | u \in Z, x \in G - Z \} \), where \( \sigma = \sum x \). Let \( t = t(Z) \) be the nilpotency index of \( \mathcal{R}(Z) \). We show that \( \mathcal{B} = 0 \). This will be deduced from the following observations.

1. \( x\sigma \cdot y\sigma = xy\sigma^2 = 0 \).

2. \( (x\sigma) \prod_{i=1}^{t-1} (x_i - 1) \in (x\sigma)\mathcal{R}(Z)^{t-1} = (x\sigma)k\tau = 0 \), where \( \tau = \sum z \). In fact, \( \mathcal{R}(Z)^{t-1} = k\tau \), as is easily shown (for any \( p \)-group \( Z \)) and \( \sigma\tau = pt = 0 \), since \( G' \subseteq Z \).

3. \( \prod_{i=1}^{t} (x_i - 1) = 0 \), since \( t = t(Z) \), where \( z_1, \ldots, z_t \) are arbitrary elements of \( Z \).

Now, from the assumption, we conclude that \( \mathcal{B} = 0 \). Take \( y \in G - Z \). Then \( (y - 1)\tau \) is not zero and is contained in \( (y - 1)\mathcal{R}(Z)^{t-1} \subseteq \mathcal{B} = 0 \), a contradiction. This completes the proof.

Proof of "(2) \( \Rightarrow \) (3)". Let \( \delta_0 = \delta_{z_0} \). Since by the assumption \( \mathcal{R}\delta_0 \) is generated by central elements over \( kG \), we have \( \mathcal{R}\delta_0 \subseteq \mathcal{R}\delta_0 \) and hence \( G \) is \( p \)-nilpotent by Lemma 9. In particular, \( B_0 \) is isomorphic to \( k(G/O_{p'}(G)) \cong kP \). Hence \( kP \) is also \( LC \), implying \( P \) is abelian by Lemma 10. This completes the proof of Theorem 4.

4. Application of a result of Clarke

In this section we shall show,

**Theorem 11.** Suppose \( G \) is \( p \)-solvable. If \( t(G) = p^s \), then \( P \) is cyclic.

To prove this, the following Theorem is essential.

**Theorem** (Clarke [2]). If \( G \) is a \( p \)-solvable group of \( p \)-length one, then \( t(G) = t(P) \).

Proof (of Theorem 11). We prove by the induction on the order of \( G \). If \( G \) is a \( p \)-group, then our result follows from the Theorem 3.7 of Jennings [9]. If \( G \) has a proper normal subgroup \( H \) of index prime to \( p \), then we have \( \mathcal{R} = \mathcal{R}_H \) and the result follows from the induction hypothesis on \( H \). Hence we may assume \( G \) has no proper normal subgroup of index prime to \( p \). Furthermore, by the Theorem of Clarke, it suffices to show that \( G \) is \( p \)-nilpotent.

Let \( H \) be a normal subgroup of index \( p \). Since \( \mathcal{R}^p \subseteq \mathcal{R}_H \) ([11] or [12]), we find \( t(H) = p^{s-1} \). Hence a Sylow \( p \)-subgroup \( Q \) of \( H \) is cyclic by the induction hypothesis. In particular \( H \) has the \( p \)-length one. Let \( K = O_{p'}(G) = O_{p'}(H) \). Then \( G/K \supseteq QK/K = O_{p'}(H/K) \). Now, assume \( G \neq PK \). Then we have \( O_{p'}(G/K) = QK/K \) and \( C_{gdk}(QK/K) = QK/K \), as is well known (Hall and Higman [8]).
Therefore, $G/QK$ is isomorphic to a subgroup of $\text{Aut}(QK/K)$, whence $G/QK$ is abelian, since the automorphism group of a cyclic group is abelian. Since we have assumed that $G$ has no normal subgroup of index prime to $p$, $G/QK$ must be a $p$-group, contradicting that $G \neq PK$. This completes the proof.

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References


Added in proof.

Lemma 5 has been obtained in
