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Osaka University
SOME NOTES ON THE RADICAL OF A FINITE GROUP RING

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(Received September 6, 1977)

1. Introduction

Let \( p \) be a prime number and \( G \) a finite group with a Sylow \( p \)-subgroup \( P \) of order \( p^a \). Let be \( \mathfrak{R} \) the radical of the group ring \( kG \) of \( G \) taken over a field \( k \) of characteristic \( p \). If \( \mathfrak{Z} \) is the radical of the center of \( kG \), then we see easily that \( kG \cdot \mathfrak{Z} \subset \mathfrak{R} \). We shall show that \( \mathfrak{R} = kG \cdot \mathfrak{Z} \) holds if and only if \( G \) is \( p \)-nilpotent and \( P \) is abelian.

The nilpotency index of \( \mathfrak{R} \), which is denoted by \( t(G) \), is the smallest integer \( t \) such that \( \mathfrak{R}^t = 0 \). Suppose \( G \) is \( p \)-solvable, then it is known that \( a(p-1)+1 \leq t(G) \leq p^a \) (Passman [11], Tsushima [12], Wallace [16]). Furthermore if \( G \) has the \( p \)-length one, it holds that \( t(G) = t(P) \) (Clarke [2]). We see easily from this that the first equality holds in the above if \( P \) is elementary, while the second holds if \( P \) is cyclic. However the equality \( t(G) = a(p-1)+1 \) does not necessarily imply that \( P \) is elementary, as is remarked by Motose (e.g. \( G = S_4, p = 2 \), see Ninomiya [10]). In contrast with this, we shall show that if \( t(G) = p^a \), then \( P \) is cyclic.

**Notation:** \( p \) is a fixed prime number. \( G \) is always a finite group and \( P \) a Sylow \( p \)-subgroup of order \( p^a \). As usual, \( |X| \) denotes the cardinality of a set \( X \). Let \( K \) be an algebraic number field containing the \( |G| \)-th roots of unity and \( o \) the ring of integers in \( K \). We fix a prime divisor \( \mathfrak{p} \) of \( p \) in \( o \) and we let \( k = o/\mathfrak{p} \). We denote by \( \{ \varphi_1, \ldots, \varphi_r \} \) and \( \{ \eta_1, \ldots, \eta_r \} \) the set of irreducible Brauer characters and principal indecomposable Brauer characters of \( G \) respectively, in which the arrangement is such that \( (\eta_i, \varphi_j) = \delta_{ij} \) and \( \varphi_1 \) is the trivial character. We put \( s(G) = \sum_{i=1}^{r} \varphi_i(1)^2 \).

For a block \( B \) of \( kG \), we denote by \( \delta_B \) and \( \psi_B \) its block idempotent and the associated linear character respectively. \( \mathfrak{R}(G) \) (or \( \mathfrak{R} \) for brevity) denotes the radical of the group ring \( kG \) and \( \mathfrak{Z} \) the radical of the center of \( kG \). The nilpotency index of \( \mathfrak{R}(G) \), which will be denoted by \( t(G) \), is defined to be the smallest integer \( t \) such that \( \mathfrak{R}(G)^t = 0 \). If \( G > H \), then \( kG \cdot \mathfrak{R}(H) = \mathfrak{R}(H) \cdot kG \) is a two sided ideal of \( kG \) contained in \( \mathfrak{R} \), which will be denoted by \( \mathcal{Z} \) (or \( \mathcal{Z} \) for brevity). Other notations are standard.
We shall several times refer to the following Theorem of Green (Green [7], Dornhoff [4] § 52).

**Theorem.** Let $G > H$ and $G/H$ is a $p$-group.

If $V$ is a finitely generated absolutely indecomposable $kH$-module, then $V^G$ is also absolutely indecomposable.

2. **Square sum of the degrees of irreducible characters**

In this section, we mention some remarks about the dimension of $\mathfrak{R} = \mathfrak{R}(G)$, most of which are direct consequences of our results [14].

Let $S$ be the set of the $p$-elements of $G$ and $c = \sum_{x \in S} x kG$. In [14], we have shown that $\mathfrak{R} \subset (0 : c)$ and we have the equality provided $G$ is $p$-solvable. For $\lambda = \sum_{x \in G} a_x \in kG$, $a_x \in k$, we put $\sigma_p(\lambda) = \sum_{x \in S} a_x$. Note that $\sigma_p(\lambda)$ is the coefficient of the identity in $c \lambda$. Hence $c \lambda = 0$ if and only if $\sigma_p(\lambda x) = 0$ for any $x \in G$, or

$$(0 : c) = \{ \lambda \in kG \mid \sigma_p(\lambda x) = 0 \text{ for any } x \in G \} \quad \text{........................(1)}$$

Therefore, our result quoted above is written as

**Proposition 1.** If $\lambda \in \mathfrak{R}$, then $\sigma_p(\lambda x) = 0$ for any $x$ of $G$.

We next discuss the dimension of $(0 : c)$. Let $M = M_G = (a_{x,h})$ be the $(|G|, |G|)$-matrix over $k$ defined as

$$a_{x,h} = \begin{cases} 1, & \text{if } gh \text{ is a } p \text{-element} \\ 0, & \text{otherwise} \end{cases}$$

Then, we have

$$\dim_k (0 : c) = |G| - r(M), \text{ where } r(M) \text{ denotes the rank of } M \text{ over } k. \quad \text{......(2)}$$

Indeed, for $\lambda = \sum_{x \in G} a_x \in kG$, we have $\sigma_p(\lambda x) = \sum_{x \in S} a_x$, that is $M \left( \begin{array}{c} a_x \\ \vdots \end{array} \right) = \left( \begin{array}{c} \sigma_p(\lambda x) \\ \vdots \end{array} \right)$ for $x \in G$. From this and (1), we get easily (2).

Furthermore from that $\mathfrak{R} \subset (0 : c)$ and (2), we have

$$s(G) = |G| - \dim N \geq r(M) \quad \text{........................(3)}$$

If $H$ is a subgroup of $G$, then $M_H$ appears in $M_G$ as a submatrix. In particular $r(M_G) \geq r(M_H)$. Now, recall that we have $\mathfrak{R} = (0 : c)$ and hence $s(G) = r(M)$ provided $G$ is $p$-solvable. Summarizing the aboves, we have

**Proposition 2.** If $G$ is $p$-solvable, then we have $s(G) \geq s(H)$ for any subgroup $H$ of $G$. 
Remark 1. If $H$ is a $p'$-subgroup, then $r(M_H) = |H|$. Hence we have from (3) that $s(G) \geq |H|$ for any $p'$-subgroup $H$ of $G$, which has been shown in Brauer and Nesbitt [1] by the inequalities $s(G) \geq \frac{|G|}{u} \geq |H|$, where $u = \eta_i(1)$.

In connection with the above remark, we give the following, which is essentially due to Wallace [15].

**Proposition 3.** We have $s(G) = |H|$ for some $p'$-subgroup $H$ of $G$ if and only if $G \triangleright P$, in which case $H$ is necessary a complement of $P$ in $G$.

Proof. “if part” is well known and easily shown (Curtis and Reiner [3] § 64 Exercise 1).

Suppose $s(G) = |H|$ for some $p'$-subgroup $H$ of $G$. Then we have $s(G) = \frac{|G|}{u}$, which forces that $\eta_i = \varphi_i \eta_i$ for any $i (1 \leq i \leq r)$ (see [1] pp. 580). We claim that $u = p^a$. If this would be shown, then $H$ is necessary a complement of $P$ and $\eta_i(x)$ is rational for any $x \in G$. Then the argument of Wallace [15] is valid, concluding $G \triangleright P$ (see also M.R. 22 # 12146 No. 12 (1966)).

Let

$$\theta(x) = \begin{cases} p^a & \text{if } x \text{ is } p\text{-regular} \\ 0 & \text{otherwise} \end{cases}$$

As is well known, $\theta$ is an integral linear combination of $\eta_i$'s: $\theta = \sum m_i \eta_i = \eta_1 \sum m_i \varphi_i$, where each $m_i$ is a rational integer. Comparing the degrees of both sides, we get $u = p^a$ as claimed. This completes the proof.

3. LC type

For convenience, we call a (finite dimensional) algebra $A$ over a field to be LC if its (Jacobson) radical is generated over $A$ by the radical of its center.

The objective of this section is to prove

**Theorem 4.** The followings are equivalent to each other.

1. $kG$ is LC
2. the principal block $B_0$ of $kG$ is LC
3. $G$ is $p$-nilpotent and $P$ is abelian

“(1)⇒(2)” is trivial. On the other hand, we have already shown “(3)⇒(1)” in [13] assuming $P$ is cyclic. The same argument, being simplified by virtue of the Green's Theorem quoted in the introduction, will be made below to yeild the present assertion.

We begin with

**Lemma 5.** Let $G \triangleright H$ and $b$ a block of $kH$. Let $B_1, \ldots, B_s$ be the blocks
of \( kG \) which cover \( b \). If a defect group of each \( B_i \) is contained in \( H \), then we have \( \mathcal{R}B_i = 2B_i \) for each \( i \) \((1 \leq i \leq \epsilon)\).

Proof. Let \( b_1, \ldots, b_t \) be the blocks of \( kH \) which are conjugate to \( b \) under \( G \) and \( \varepsilon_i \) the block idempotent of \( b_i \).

From the choice of \( B_i \)'s we have

\[
\varepsilon = \varepsilon_1 + \cdots + \varepsilon_i = \delta_1 + \cdots + \delta_i, \quad \text{where } \delta_i = \delta_{B_i}.
\]

Let \( \Lambda = kG\varepsilon \mathcal{R} \ni \Gamma = kH\varepsilon \mathcal{R}(H)\varepsilon \). We show that \( \Lambda \) is semisimple. Let \( M \) be a \( \Lambda \)-module and \( N \) any submodule of \( M \). The inclusion map \( N \rightarrow M \) splits as \( \Gamma \)-modules, since \( \Gamma \) is semisimple and then it does as \( \Lambda \)-modules, since \( M \) is \((G, H)\) projective by the assumption. Therefore \( \Lambda \) is semisimple and our assertion is clear.

The following remark is useful.

Remark 2.

(1) (well known) If \( G/H \) is a \( p'\)-group, then the assumption of Lemma 5 is always satisfied and hence we have \( \mathcal{R} = 2_H \).

(2) (Feit [5] pp. 268) If \( G/H \) is a \( p \)-group, then there is a unique block which covers \( b \).

The following Lemma is not so essential here, but we write down it for its own interest. The result is noticed by Y. Nobusato.

Lemma 6. Suppose \( G \triangleright H \) and \( G/H \) is a \( p \)-group. Then for any simple \( kH \)-module \( N \), \( N^G \) has the composition length \([I: H]\), where \( I \) is the inertia group of \( N \) in \( G \).

Proof. Clear from the Green’s Theorem and the orthogonality relations \((\eta_i, \varphi_j) = \delta_{i,j}\).

The following result has been shown in our previous paper [13].

Lemma 7. Let \( G \triangleright H \) and \([G: H] = p\). Let \( B \) be a block of \( kG \). Suppose there is a conjugate class \( C \) of \( G \) such that \( C \triangleleft H \) and \( \psi_B(C) = 0 \), where \( C = \sum_{\lambda \in H} x \).

Then, we have \( \mathcal{R}B = 2B + kG(C - \psi(B))(\psi_B) \delta_B \).

Proof. We put \( \delta = \delta_B \) and \( \psi = \psi_B \) for brevity. Let \( \delta = \sum e \) be a decomposition into the sum of primitive idempotents. We may assume each \( e \) is contained in \( kH \) by the Green’s Theorem. It suffices to show that \( \mathcal{R}e = 2e + kG(C - \psi(B))e \). Let \( a \in G \) be any element not contained in \( H \). We have

\[
(C - \psi(B))e = a^{p-1}e = a^{p-1}e + \cdots + a^{p-1}e,
\]

where \( \lambda_i \in kH \).

Since \( \psi(C) = 0 \), this implies that \( (C - \psi(C))e \) is not contained in \( 2e + kG(C - \psi(B))e \). Therefore we have a sequence (note that \( (C - \psi(C)) \delta \in \mathcal{R} \))
$kG\bar{e} = (\bar{C} - \psi(\bar{C}))kG\bar{e} \supseteq \cdots \supseteq (\bar{C} - \psi(\bar{C}))^{p-1}kG\bar{e} \supseteq 0$, where $kG\bar{e} = kGe/\mathcal{L}e \cong kG \otimes_{kH} kHe/\mathcal{R}(H)e$.

However, since $kG\bar{e}$ has at most $p$ composition factors by Lemma 6, we have $(\bar{C} - \psi(\bar{C}))kG\bar{e} = \mathcal{R}e$, that is $\mathcal{R}e = \mathcal{L}e + kG(\bar{C} - \psi(\bar{C}))e$ as required. This completes the proof.

Before proceeding, we mention a remark. If $B$ is a block of $kG$ of full defect, then there is an ordinary irreducible character $\chi$ belonging to $B$ whose degree is not divisible by $p$. If $x$ is a $p$-element, then $\chi(x) \equiv \chi(1) \mod p$. Hence it follows that if $C$ is a conjugate class of a $p$-element, then $\psi_B(C) = |C|$.

The following proposition proves "(3) $\Rightarrow$ (1)" of Theorem 4.

**Proposition 8.** Suppose $G$ is $p$-nilpotent and $P$ is abelian. Let $\{C_1, \cdots, C_n\}$ be the set of the conjugate classes of $p$-elements of $G$. For a (normal) subgroup $H$ of $G$ containing $O_p'(G)$, let $\Delta_H$ be the sum of the block idempotents of $kH$ of full defect and for any $C_i$ such that $C_i \subseteq H$, let $\Delta(C_i, H) = (C_i - |C_i|)\Delta_H$.

Then we have $\mathcal{R} = \sum_{H} kG\Delta(C_i, H)$, where $H$ is taken over the subgroups of $G$ containing $O_p'(G)$. In particular, $kG$ is LC.

Proof. Let $B$ be any block of $kG$. If $B$ has the defect smaller than $a$, then there is a normal subgroup $H$ of index $p$ which contains a defect group of $B$. Then by Lemma 5 and Remark 2, we have $\mathcal{R}B = \mathcal{L}eB$. On the other hand, assume $B$ has full defect. Let $H$ be any normal subgroup of $G$ of index $p$. There is some $C_i$ such that $C_i \subseteq H$ and $\psi_B(C_i) = |C_i| \pm 0$, since $P$ is abelian. Hence by Lemma 7, we have $\mathcal{R}B = \mathcal{L}eB + kG(C_i - |C_i|)\Delta_B$. From the above, we have $\mathcal{R} = \sum_{H} \mathcal{L}eH + \sum_{i} kG\Delta(C_i, G)$, where $H$ is taken over the normal subgroups of $G$ of index $p$ and thus the result will follow by the induction on the order of $G$ (note that if $H \supseteq C_i$, where $H \supseteq O_p'(G)$, then $C_i$ is also a conjugate class of $H$).

We next go into the proof of "(2) $\Rightarrow$ (3)".

**Lemma 9.** Let $I$ be the augmentation ideal of $kG$ and $\delta_0$ the block idempotent of the principal block $B_0$ of $kG$. If $I\mathcal{R}\delta_0 = \mathcal{R}I\delta_0$, then $G$ is $p$-nilpotent.

Proof. Let $e$ be a primitive idempotent of $kG$ such that $kGe/\mathcal{R}e$ is the trivial $G$-module. It is easy to see that $Ie = \mathcal{R}e$. Hence we have $I\mathcal{R}e = I\mathcal{R}\delta_0 e = \mathcal{R}I\delta_0 e = \mathcal{R}e/\mathcal{R}e$. Recurring this, we get $I\mathcal{R}^s e = \mathcal{R}^{s+1} e$ for any $s \geq 0$. This implies that $G$ acts trivially on each factor of the series, $kGe \supseteq \mathcal{R}e \supseteq \cdots \supseteq \mathcal{R}^s e = 0$, in other words, $kG\bar{e}$ has the only (non isomorphic) simple constituent, the trivial one. Hence $G$ is $p$-nilpotent.

**Lemma 10.** Suppose $G$ is a $p$-group. If $kG$ is LC, then $G$ is abelian.

Proof. We prove by the induction on the order of $G$. It is clear that if $kG$ is LC, then $(G/H)$ is also LC for any normal subgroup $H$ of $G$. 


Let \( Z \) be the center of \( G \) and let \( z \) be an element of \( Z \) of order \( p \). We may assume \( G/\langle z \rangle \) is abelian by the induction hypothesis. Assume \( G \) is not abelian. Then we have \( G'=[G, G]=\langle z \rangle \). Since \( |gG'|=p, gG' \) is the conjugate class of \( g \) unless \( g \) is central. Therefore, \( \mathcal{B} \) is spanned over \( k \) by the set \( \{u-1, x\sigma \mid u \in Z, x \in G-Z \} \), where \( \sigma = \sum_{X \in G} x \). Let \( t=t(Z) \) be the nilpotency index of \( \mathfrak{N}(Z) \). We show that \( \mathfrak{B}'=0 \). This will be deduced from the following observations.

1. \( x\sigma \cdot y\sigma = xy\sigma^2 = 0 \).
2. \( (x\sigma)^{t-1} \prod_{i=1}^{t} (z_i-1) \in (x\sigma)\mathfrak{N}(Z)^{t-1} = (x\sigma)k\tau = 0 \), where \( \tau = \sum_{\sigma \in G} \sigma \). In fact, \( \mathfrak{N}(Z)^{t-1} = k\tau \), as is easily shown (for any \( p \)-group \( Z \)) and \( \sigma\tau = pt = 0 \), since \( G' \subset Z \).
3. \( \prod_{i=1}^{t} (z_i-1) = 0 \), since \( t=t(Z) \), where \( z_1, \ldots, z_t \) are arbitrary elements of \( Z \).

Now, from the assumption, we conclude that \( \mathfrak{B}'=0 \). Take \( y \in G-Z \). Then \( (y-1)\tau \) is not zero and is contained in \( (y-1)\mathfrak{N}(Z)^{t-1} \subset \mathfrak{B}'=0 \), a contradiction. This completes the proof.

Proof of "(2) \( \Rightarrow \) (3)". Let \( \delta_0 = \delta_{B_0} \). Since by the assumption \( \mathfrak{N}\delta_0 \) is generated by central elements over \( kG \), we have \( \mathfrak{N}\delta_0 = \mathfrak{N}\delta_0 \) and hence \( G \) is \( p \)-nilpotent by Lemma 9. In particular, \( B_0 \) is isomorphic to \( k(G/O_{p'}(G)) \approx kP \). Hence \( kP \) is also \( LC \), implying \( P \) is abelian by Lemma 10. This completes the proof of Theorem 4.

4. Application of a result of Clarke

In this section we shall show,

**Theorem 11.** Suppose \( G \) is \( p \)-solvable. If \( t(G)=p^a \), then \( P \) is cyclic.

To prove this, the following Theorem is essential.

**Theorem** (Clarke [2]). If \( G \) is a \( p \)-solvable group of \( p \)-length one, then \( t(G)=t(P) \).

Proof (of Theorem 11). We prove by the induction on the order of \( G \). If \( G \) is a \( p \)-group, then our result follows from the Theorem 3.7 of Jennings [9]. If \( G \) has a proper normal subgroup \( H \) of index prime to \( p \), then we have \( \mathfrak{N} = \mathfrak{N}_H \) and the result follows from the induction hypothesis on \( H \). Hence we may assume \( G \) has no proper normal subgroup of index prime to \( p \). Furthermore, by the Theorem of Clarke, it suffices to show that \( G \) is \( p \)-nilpotent.

Let \( H \) be a normal subgroup of index \( p \). Since \( \mathfrak{N}^p \subset \mathfrak{N}_H \) ([11] or [12]), we find \( t(H)=p^{a-1} \). Hence a Sylow \( p \)-subgroup \( Q \) of \( H \) is cyclic by the induction hypothesis. In particular \( H \) has the \( p \)-length one. Let \( K=O_{p'}(G)=O_{p'}(H) \). Then \( G/K \supset Q/K = O_{p'}(H/K) \). Now, assume \( G \neq PK \). Then we have \( O_{p'}(G/K) = Q/K \) and \( C_{GL}(Q/K/K)=Q/K, \) as is well known (Hall and Higman [8]).
Therefore, $G/QK$ is isomorphic to a subgroup of $\text{Aut}(QK/K)$, whence $G/QK$ is abelian, since the automorphism group of a cyclic group is abelian. Since we have assumed that $G$ has no normal subgroup of index prime to $p$, $G/QK$ must be a $p$-group, contradicting that $G \neq PK$. This completes the proof.

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References


Added in proof.
Lemma 5 has been obtained in