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SOME NOTES ON THE RADICAL OF A FINITE GROUP RING

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1. Introduction

Let p be a prime number and G a finite group with a Sylow p -subgroup P of order p^a . Let \mathfrak{N} be the radical of the group ring kG of G taken over a field k of characteristic p . If \mathfrak{Z} is the radical of the center of kG , then we see easily that $kG \cdot \mathfrak{Z} \subset \mathfrak{N}$. We shall show that $\mathfrak{N} = kG \cdot \mathfrak{Z}$ holds if and only if G is p -nilpotent and P is abelian.

The nilpotency index of \mathfrak{N} , which is denoted by $t(G)$, is the smallest integer t such that $\mathfrak{N}^t = 0$. Suppose G is p -solvable, then it is known that $a(p-1)+1 \leq t(G) \leq p^a$ (Passman [11], Tsushima [12], Wallace [16]). Furthermore if G has the p -length one, it holds that $t(G) = t(P)$ (Clarke [2]). We see easily from this that the first equality holds in the above if P is elementary, while the second holds if P is cyclic. However the equality $t(G) = a(p-1)+1$ does not necessarily imply that P is elementary, as is remarked by Motose (e.g. $G = S_4$, $p=2$, see Ninomiya [10]). In contrast with this, we shall show that if $t(G) = p^a$, then P is cyclic.

NOTATION: p is a fixed prime number. G is always a finite group and P a Sylow p -subgroup of order p^a . As usual, $|X|$ denotes the cardinality of a set X . Let K be an algebraic number field containing the $|G|$ -th roots of unity and \mathfrak{o} the ring of integers in K . We fix a prime divisor \mathfrak{p} of p in \mathfrak{o} and we let $k = \mathfrak{o}/\mathfrak{p}$. We denote by $\{\varphi_1, \dots, \varphi_r\}$ and $\{\eta_1, \dots, \eta_r\}$ the set of irreducible Brauer characters and principal indecomposable Brauer characters of G respectively, in which the arrangement is such that $(\eta_i, \varphi_j) = \delta_{ij}$ and φ_1 is the trivial character. We put $s(G) = \sum_{i=1}^r \varphi_i(1)^2$.

For a block B of kG , we denote by δ_B and ψ_B its block idempotent and the associated linear character respectively. $\mathfrak{N}(G)$ (or \mathfrak{N} for brevity) denotes the radical of the group ring kG and \mathfrak{Z} the radical of the center of kG . The nilpotency index of $\mathfrak{N}(G)$, which will be denoted by $t(G)$, is defined to be the smallest integer t such that $\mathfrak{N}(G)^t = 0$. If $G \triangleright H$, then $kG \cdot \mathfrak{N}(H) = \mathfrak{N}(H) \cdot kG$ is a two sided ideal of kG contained in \mathfrak{N} , which will be denoted by \mathfrak{N}_H (or \mathfrak{N} for brevity). Other notations are standard.

We shall several times refer to the following Theorem of Green (Green [7], Dornhoff [4] § 52).

Theorem. *Let $G \triangleright H$ and G/H is a p -group.*

If V is a finitely generated absolutely indecomposable kH -module, then V^G is also absolutely indecomposable.

2. Square sum of the degrees of irreducible characters

In this section, we mention some remarks about the dimension of $\mathfrak{N} = \mathfrak{N}(G)$, most of which are direct consequences of our results [14].

Let S be the set of the p -elements of G and $c = \sum_{x \in S} x \in kG$. In [14], we have shown that $\mathfrak{N} \subset (0: c)$ and we have the equality provided G is p -solvable. For $\lambda = \sum_{x \in G} a_x x \in kG$, $a_x \in k$, we put $\sigma_p(\lambda) = \sum_{x \in S} a_x$. Note that $\sigma_p(\lambda)$ is the coefficient of the identity in $c\lambda$. Hence $c\lambda = 0$ if and only if $\sigma_p(x\lambda) = 0$ for any $x \in G$, or

$$(0: c) = \{ \lambda \in kG \mid \sigma_p(x\lambda) = 0 \text{ for any } x \in G \} \quad \dots\dots\dots (1)$$

Therefore, our result quoted above is written as

Proposition 1. *If $\lambda \in \mathfrak{N}$, then $\sigma_p(x\lambda) = 0$ for any x of G .*

We next discuss the dimension of $(0: c)$. Let $M = M_G = (a_{g,h})$ be the $(|G|, |G|)$ -matrix over k defined as

$$a_{g,h} = \begin{cases} 1, & \text{if } gh \text{ is a } p\text{-element} \\ 0, & \text{otherwise} \end{cases}$$

Then, we have

$$\dim_k(0: c) = |G| - r(M), \text{ where } r(M) \text{ denotes the rank of } M \text{ over } k. \quad \dots\dots\dots (2)$$

Indeed, for $\lambda = \sum a_x x \in kG$, we have $\sigma_p(x\lambda) = \sum_{y \in S} a_y$, that is $M \begin{pmatrix} \vdots \\ a_x \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \sigma_p(x\lambda) \\ \vdots \end{pmatrix}$ for $x \in G$. From this and (1), we get easily (2).

Furthermore from that $\mathfrak{N} \subset (0: c)$ and (2), we have

$$s(G) = |G| - \dim_k \mathfrak{N} \geq r(M) \quad \dots\dots\dots (3)$$

If H is a subgroup of G , then M_H appears in M_G as a submatrix. In particular $r(M_G) \geq r(M_H)$. Now, recall that we have $\mathfrak{N} = (0: c)$ and hence $s(G) = r(M)$ provided G is p -solvable. Summarizing the aboves, we have

Proposition 2. *If G is p -solvable, then we have $s(G) \geq s(H)$ for any subgroup H of G .*

REMARK 1. If H is a p' -subgroup, then $r(M_H) = |H|$. Hence we have from (3) that $s(G) \geq |H|$ for any p' -subgroup H of G , which has been shown in Brauer and Nesbitt [1] by the inequalities $s(G) \geq \frac{|G|}{u} \geq |H|$, where $u = \eta_1(1)$.

In connection with the above remark, we give the following, which is essentially due to Wallace [15].

Proposition 3. *We have $s(G) = |H|$ for some p' -subgroup H of G if and only if $G \triangleright P$, in which case H is necessary a complement of P in G .*

Proof. "if part" is well known and easily shown (Curtis and Reiner [3] § 64 Exercise 1).

Suppose $s(G) = |H|$ for some p' -subgroup H of G . Then we have $s(G) = \frac{|G|}{u}$, which forces that $\eta_i = \varphi_i \eta_1$ for any i ($1 \leq i \leq r$) (see [1] pp. 580). We claim that $u = p^a$. If this would be shown, then H is necessary a complement of P and $\eta_1(x)$ is rational for any $x \in G$. Then the argument of Wallace [15] is valid, concluding $G \triangleright P$ (see also M.R. 22 # 12146 No. 12 (1966)).

Let

$$\theta(x) = \begin{cases} p^a & \text{if } x \text{ is } p\text{-regular} \\ 0 & \text{otherwise} \end{cases}$$

As is well known, θ is an integral linear combination of η_i 's: $\theta = \sum m_i \eta_i = \eta_1 \sum m_i \varphi_i$, where each m_i is a rational integer. Comparing the degrees of both sides, we get $u = p^a$ as claimed. This completes the proof.

3. LC type

For convenience, we call a (finite dimensional) algebra A over a field to be LC if its (Jacobson) radical is generated over A by the radical of its center.

The objective of this section is to prove

Theorem 4. *The followings are equivalent to each other.*

- (1) kG is LC
- (2) the principal block B_0 of kG is LC
- (3) G is p -nilpotent and P is abelian

"(1) \Rightarrow (2)" is trivial. On the other hand, we have already shown "(3) \Rightarrow (1)" in [13] assuming P is cyclic. The same argument, being simplified by virtue of the Green's Theorem quoted in the introduction, will be made below to yield the present assertion.

We begin with

Lemma 5. *Let $G \triangleright H$ and b a block of kH . Let B_1, \dots, B_s be the blocks*

of kG which cover b . If a defect group of each B_i is contained in H , then we have $\mathfrak{N}B_i = \mathfrak{B}B_i$ for each i ($1 \leq i \leq s$).

Proof. Let b_1, \dots, b_t be the blocks of kH which are conjugate to b under G and ε_i the block idempotent of b_i .

From the choice of B_i 's we have

$$\varepsilon = \varepsilon_1 + \dots + \varepsilon_t = \delta_1 + \dots + \delta_s, \quad \text{where } \delta_i = \delta_{B_i}$$

Let $\Lambda = kG\varepsilon/\mathfrak{L}\varepsilon \supset \Gamma = kH\varepsilon/\mathfrak{N}(H)\varepsilon$. We show that Λ is semisimple. Let M be a Λ -module and N any submodule of M . The inclusion map $N \rightarrow M$ splits as Γ -modules, since Γ is semisimple and then it does as Λ -modules, since M is (G, H) projective by the assumption. Therefore Λ is semisimple and our assertion is clear.

The following remark is useful.

REMARK 2.

(1) (well known) If G/H is a p' -group, then the assumption of Lemma 5 is always satisfied and hence we have $\mathfrak{N} = \mathfrak{B}_H$.

(2) (Feit [5] pp. 268) If G/H is a p -group, then there is a unique block which covers b .

The following Lemma is not so essential here, but we write down it for its own interest. The result is noticed by Y. Nobusato.

Lemma 6. Suppose $G \triangleright H$ and G/H is a p -group. Then for any simple kH -module N , N^G has the composition length $[I: H]$, where I is the inertia group of N in G .

Proof. Clear from the Green's Theorem and the orthogonality relations $(\eta_i, \varphi_j) = \delta_{ij}$.

The following result has been shown in our previous paper [13].

Lemma 7. Let $G \triangleright H$ and $[G: H] = p$. Let B be a block of kG . Suppose there is a conjugate class C of G such that $C \not\subseteq H$ and $\psi_B(\tilde{C}) \neq 0$, where $\tilde{C} = \sum_{x \in C} x$.

Then, we have $\mathfrak{N}B = \mathfrak{B}B + kG(\tilde{C} - \psi_B(\tilde{C}))\delta_B$.

Proof. We put $\delta = \delta_B$ and $\psi = \psi_B$ for brevity. Let $\delta = \sum e$ be a decomposition into the sum of primitive idempotents. We may assume each e is contained in kH by the Green's Theorem. It suffices to show that $\mathfrak{N}e = \mathfrak{B}e + kG(\tilde{C} - \psi(\tilde{C}))e$. Let $a \in G$ be any element not contained in H . We have

$$(\tilde{C} - \psi(\tilde{C}))^{p-1}e = a^{p-1}\lambda_1 + \dots + a\lambda_{p-1} - \psi(\tilde{C})^{p-1}e, \quad \text{where } \lambda_i \in kH.$$

Since $\psi(\tilde{C}) \neq 0$, this implies that $(\tilde{C} - \psi(\tilde{C}))^{p-1}e$ is not contained in $\mathfrak{B}e = a^{p-1}\mathfrak{N}(H)e \oplus \dots \oplus \mathfrak{N}(H)e$. Therefore we have a sequence (note that $(\tilde{C} - \psi(\tilde{C}))\delta \in \mathfrak{N}$)

$kG\bar{e} \equiv (\bar{C} - \psi(\bar{C}))kG\bar{e} \equiv \dots \equiv (\bar{C} - \psi(\bar{C}))^{p-1}kG\bar{e} \equiv 0$, where $kG\bar{e} = kGe/\mathfrak{N}e \cong kG \otimes_{kH} kHe/\mathfrak{N}(H)e$.

However, since $kG\bar{e}$ has at most p composition factors by Lemma 6, we have $(\bar{C} - \psi(\bar{C}))kG\bar{e} = \mathfrak{N}\bar{e}$, that is $\mathfrak{N}e = \mathfrak{N}e + kG(\bar{C} - \psi(\bar{C}))e$ as required. This completes the proof.

Before proceeding, we mention a remark. If B is a block of kG of full defect, then there is an ordinary irreducible character χ belonging to B whose degree is not divisible by p . If x is a p -element, then $\chi(x) \equiv \chi(1) \pmod{p}$. Hence it follows that if C is a conjugate class of a p -element, then $\psi_B(\bar{C}) = |C|$.

The following proposition proves “(3) \Rightarrow (1)” of Theorem 4.

Proposition 8. *Suppose G is p -nilpotent and P is abelian. Let $\{C_1, \dots, C_s\}$ be the set of the conjugate classes of p -elements of G . For a (normal) subgroup H of G containing $O_{p'}(G)$, let Δ_H be the sum of the block idempotents of kH of full defect and for any C_i such that $C_i \subset H$, let $\Delta(C_i, H) = (\bar{C}_i - |C_i|)\Delta_H$.*

Then we have $\mathfrak{N} = \sum_{i,H} kG\Delta(C_i, H)$, where H is taken over the subgroups of G containing $O_{p'}(G)$. In particular, kG is LC.

Proof. Let B be any block of kG . If B has the defect smaller than a , then there is a normal subgroup H of index p which contains a defect group of B . Then by Lemma 5 and Remark 2, we have $\mathfrak{N}B = \mathfrak{N}_H B$. On the other hand, assume B has full defect. Let H be any normal subgroup of G of index p . There is some C_i such that $C_i \not\subset H$ and $\psi_B(\bar{C}_i) = |C_i| \neq 0$, since P is abelian. Hence by Lemma 7, we have $\mathfrak{N}B = \mathfrak{N}_H B + kG(\bar{C}_i - |C_i|)\delta_B$. From the aboves, we have $\mathfrak{N} = \sum_H \mathfrak{N}_H + \sum_{i=1}^s kG\Delta(C_i, G)$, where H is taken over the normal subgroups of G of index p and thus the result will follow by the induction on the order of G (note that if $H \supset C_i$, where $H \supset O_{p'}(G)$, then C_i is also a conjugate class of H).

We next go into the proof of “(2) \Rightarrow (3)”.

Lemma 9. *Let I be the augmentation ideal of kG and δ_0 the block idempotent of the principal block B_0 of kG . If $I\mathfrak{N}\delta_0 = \mathfrak{N}I\delta_0$, then G is p -nilpotent.*

Proof. Let e be a primitive idempotent of kG such that $kGe/\mathfrak{N}e$ is the trivial G -module. It is easy to see that $Ie = \mathfrak{N}e$. Hence we have $I\mathfrak{N}e = I\mathfrak{N}\delta_0 e = \mathfrak{N}I\delta_0 e = \mathfrak{N}Ie = \mathfrak{N}^2 e$. Recurring this, we get $I\mathfrak{N}^s e = \mathfrak{N}^{s+1} e$ for any $s \geq 0$. This implies that G acts trivially on each factor of the series,

$kGe \supset \mathfrak{N}e \supset \dots \supset \mathfrak{N}^s e = 0$, in other words, kGe has the only (non isomorphic) simple constituent, the trivial one. Hence G is p -nilpotent.

Lemma 10. *Suppose G is a p -group. If kG is LC, then G is abelian.*

Proof. We prove by the induction on the order of G . It is clear that if kG is LC, then $k(G/H)$ is also LC for any normal subgroup H of G .

Let Z be the center of G and let z be an element of Z of order p . We may assume $G/\langle z \rangle$ is abelian by the induction hypothesis. Assume G is not abelian. Then we have $G' = [G, G] = \langle z \rangle$. Since $|gG'| = p$, gG' is the conjugate class of g unless g is central. Therefore, \mathfrak{Z} is spanned over k by the set $\{u-1, x\sigma \mid u \in Z, x \in G-Z\}$, where $\sigma = \sum_{x \in G'} x$. Let $t = t(Z)$ be the nilpotency index of $\mathfrak{N}(Z)$. We show that $\mathfrak{Z}^t = 0$. This will be deduced from the following observations.

$$(1) \quad x\sigma \cdot y\sigma = xy\sigma^2 = 0.$$

$$(2) \quad (x\sigma) \prod_{i=1}^{t-1} (z_i - 1) \in (x\sigma)\mathfrak{N}(Z)^{t-1} = (x\sigma)k\tau = 0, \text{ where } \tau = \sum_{z \in Z} z. \text{ In fact, } \mathfrak{N}(Z)^{t-1} = k\tau, \text{ as is easily shown (for any } p\text{-group } Z) \text{ and } \sigma\tau = p\tau = 0, \text{ since } G' \subset Z.$$

$$(3) \quad \prod_{i=1}^t (z_i - 1) = 0, \text{ since } t = t(Z), \text{ where } z_1, \dots, z_t \text{ are arbitrary elements of } Z.$$

Now, from the assumption, we conclude that $\mathfrak{N}^t = 0$. Take $y \in G - Z$. Then $(y-1)\tau$ is not zero and is contained in $(y-1)\mathfrak{N}(Z)^{t-1} \subset \mathfrak{N}^t = 0$, a contradiction. This completes the proof.

Proof of “(2) \Rightarrow (3)”. Let $\delta_0 = \delta_{B_0}$. Since by the assumption $\mathfrak{N}\delta_0$ is generated by central elements over kG , we have $\mathfrak{N}I\delta_0 = \mathfrak{N}I\delta_0$ and hence G is p -nilpotent by Lemma 9. In particular, B_0 is isomorphic to $k(G/O_{p'}(G)) \cong kP$. Hence kP is also LC , implying P is abelian by Lemma 10. This completes the proof of Theorem 4.

4. Application of a result of Clarke

In this section we shall show,

Theorem 11. *Suppose G is p -solvable. If $t(G) = p^a$, then P is cyclic.*

To prove this, the following Theorem is essential.

Theorem (Clarke [2]). *If G is a p -solvable group of p -length one, then $t(G) = t(P)$.*

Proof (of Theorem 11). We prove by the induction on the order of G . If G is a p -group, then our result follows from the Theorem 3.7 of Jennings [9]. If G has a proper normal subgroup H of index prime to p , then we have $\mathfrak{N} = \mathfrak{Z}_H$ and the result follows from the induction hypothesis on H . Hence we may assume G has no proper normal subgroup of index prime to p . Furthermore, by the Theorem of Clarke, it suffices to show that G is p -nilpotent.

Let H be a normal subgroup of index p . Since $\mathfrak{N}^p \subset \mathfrak{Z}_H$ ([11] or [12]), we find $t(H) = p^{a-1}$. Hence a Sylow p -subgroup Q of H is cyclic by the induction hypothesis. In particular H has the p -length one. Let $K = O_{p'}(G) = O_{p'}(H)$. Then $G/K \triangleright QK/K = O_p(H/K)$. Now, assume $G \not\cong PK$. Then we have $O_p(G/K) = QK/K$ and $C_{G/K}(QK/K) = QK/K$, as is well known (Hall and Higman [8]).

Therefore, G/QK is isomorphic to a subgroup of $\text{Aut}(QK/K)$, whence G/QK is abelian, since the automorphism group of a cyclic group is abelian. Since we have assumed that G has no normal subgroup of index prime to p , G/QK must be a p -group, contradicting that $G \neq PK$. This completes the proof.

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Added in proof.

Lemma 5 has been obtained in

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