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Author(s)	Ito, T. Hiroshi; Tamura, Hideo
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Ito, H.T. and Tamura, H.
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SEMI-CLASSICAL ASYMPTOTICS FOR TOTAL SCATTERING CROSS SECTIONS OF N-BODY QUANTUM SYSTEMS

HIROSHI T. ITO and HIDEO TAMURA

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Introduction

One of the most fundamental quantities in scattering theory is the scattering cross section, which is directly related to experimental observations in laboratories. The total cross section is a quantity which measures how much the motion of particles scattered by potentials differs from the motion of free particles. In the present work, we study the semi-classical asymptotic behavior of total scattering cross sections with two-body initial states for N-body systems. The scattering process with two-body intial states is not only relatively easy to analyze from a theoretical viewpoint but also is practically important, because situations with many-body initial states are difficult to realize through actual experiments in laboratories. The problem of semi-classical bounds or asymptotics for total cross sections has been already studied by many works [5, 16, 19, 20, 21, 22, 23] in the case of two-body systems. On the other hand, some important properties of total cross sections in many-body systems have been also obtained by a series of works [2, 3, 4] (see also the recent work [9]). In these works, the following problems have been mainly considered: (1) finiteness of total cross section; (2) continuity as a function of energy; (3) behavior at high and low energies. The semi-classical asymptotics has not been discussed in detail in the works above. Many basic notations and definitions in many-body scattering theory are required to define precisely the total scttering cross section. We here mention the obtained result somewhat loosely. In section 2, the precise formulation of the main result is given as Theorem 2.1 together with the definition of total scattering cross section.

Throughout the entire discussion, the positive constant h, $0<h\ll 1$, denotes a small parameter corresponding to the Planck constant. We consider a system consisting of N, $N\geq 2$, particles which move in the three-dimensional space $\boldsymbol{R}^3$ and interact with each other through pair potentials V_{jk}, $1\leq j<k\leq N$. We denote by $r_j\in\boldsymbol{R}^3$, $1\leq j\leq N$, the position vector of the j-th particle. For notational brevity, we also assume that all the N particles take the identical masses $m_j=1$ for all j, $1\leq j\leq N$. For such a N-body system, the configuration space X is described as

$$X=\{r=(r_1, \cdots, r_N)\in \boldsymbol{R}^{3\times N}: \sum_{j=1}^{N} r_j=0\}$$

in the center-of-mass frame and the energy Hamiltonian (Schrödinger operator) takes the form

$$H(h)=-(h^2/2)\Delta+V \quad \text{on } L^2(X), \tag{0.1}$$

where Δ denotes the Laplacian over X and $V=V(r)$ is given as the sum of pair potentials

$$V(r)=\sum_{1\le j<k\le N} V_{jk}(r_j-r_k).$$

The pair potentials $V_{jk}(y)$, $y\in \boldsymbol{R}^3$, are assumed to fall off like $V_{jk}=O(|y|^{-\rho})$ at infinity for some $\rho>2$.

Let $a=\{1, (2, 3, \cdots, N)\}$ be a two-cluster decomposition. For one example of two-body initial states, we consider the situation in which at time $t=-\infty$, the $N-1$ particles labelled by 2, 3, ,$\cdots$,and N form a bound state at some energy $E_\alpha(h)<0$ and the remaning particle labelled by 1 comes into the scatterer from the long distance at relative energy $\lambda-E_\alpha(h)$, $\lambda>0$, and at incident direction $\omega\in S^2$, S^2 being the two-dimensional unit sphere. For such a two-body initial state α, the total scattering cross section $\sigma_\alpha(\lambda, \omega; h)$ can be defined for *a. e.* $(\lambda, \omega)\in(0, \infty)\times S^2$ ([2, 3]). The exceptional set $\{(\lambda, \omega): \sigma_\alpha(\lambda, \omega; h)=\infty\}$ is expected to be empty but it seems that this has not yet been established under the above decay assumption of pair potentials. The finiteness or smoothness in (λ, ω) of total scattering cross sections is one of the most important problems in many-body quantum scattering theory (see [9, 18] for the related problems). The exceptional set above may depend on the parameter h. Thus we here regard the quantity $\sigma_\alpha(\lambda, \omega; h)$ in the distributional sense $\mathcal{D}'((0, \infty)\times S^2)$ as a function of (λ, ω) and study its asymptotic behavior in the semi-classical limit $h\to 0$. The main result obtained here, somewhat loosely speaking, is that

$$\int_0^\infty \int_{S^2} F(\lambda, \omega)\sigma_\alpha(\lambda, \omega; h)d\omega d\lambda \sim h^{-2/(\rho-1)}, \ h\to 0,$$

for $F\in C_0^\infty((0, \infty)\times S^2)$. When $\rho>5/2$, the result above has been already proved by the authors [11] in the case of three-body systems and the method developed there extends to N-body systems without essential changes. A special emphasis in the present work is put on the case $2<\rho\le 5/2$.

The proof of the main theorem depends on the two basic results in spectral and scattering theory for many-body Schrödinger operators. One is the principle of limiting absorption proved by [13, 14] and the other is the asymptotic completeness of wave operators proved by [8, 17]. The principle of limiting absorption guarantees the existence of boundary values $R(\lambda\pm i0; H(h))$ to the positive real axis of resolvents $R(\lambda \pm ik\,;\, H(h))=(H(h)-\lambda \mp ik)^{-1}$ in an appropriate weighted L^2 space topology and makes it possible to represent scattering amplitudes with two-body initial states in terms of $R(\lambda+i0\,;\, H(h))$. On the other hand, the asymptotic completeness enables us to prove the optical relation through which total scattering cross sections are related to forward scattering amplitudes. The main theorem is proved by analyzing the resolvent $R(\lambda+i0\,;\, H(h))$ through the time-dependent representation formula. The proof also uses the microlocal resolvent estimate at high energies for the free Hamiltonian ([10, 12]), which makes it possible to improve the result obtained in the previous work [11].

1. N-body scattering systems

In this section, we fix several basic notations and definitions used in many-body scattering theory. We begin by making the assumption on pair potentials $V_{jk}(y)$, $y \in \boldsymbol{R}^3$. Let $\langle y\rangle=(1+|y|^2)^{1/2}$. The pair potentials V_{jk} are assumed to fulfill the following assumption:

$(V)_\rho$ $V_{jk}(y)$ is a real C^∞-smooth function and obeys

$$|\partial_y^\alpha V_{jk}| \le C_\alpha \langle y\rangle^{-\rho-|\alpha|} \quad \text{for some } \rho>2.$$

Throughout the entire discussion, we use the constant ρ with the meaning ascribed above. Under this assumption, the Hamiltonian $H(h)$ formally defined by (0.1) admits a unique self-adjoint realization in $L^2(X)$. We denote by the same notation $H(h)$ this self-adjoint realization.

The letter a or b is used to denote a partition of the set $\{1, 2, \cdots, N\}$ into non-empty disjoint subsets. Such a partition is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. We consider only a cluster decomposition a with $2 \le \#(a) \le N$. For pair (j, k), $1 \le j < k \le N$, we also use the notation $j\ a\ k$ if j and k are in the same cluster of a and $\sim j\ a\ k$ if they are in different clusters.

Let $\langle\ ,\ \rangle$ be the usual Euclidean scalar product in the configuration space X. For given cluster decomposition a, we define the two subspaces X^a and X_a of X as follows:

$$X^a=\{r=(r_1, \cdots, r_N) \in X : \sum_{j \in C} r_j=0 \text{ for all custers } C \text{ in } a\},$$

$$X_a=\{r=(r_1, \cdots, r_N) \in X : r_j=r_k \text{ for pairs } (j, k) \text{ with } j\ a\ k\}.$$

These two subspaces are mutually orthogonal with respect to the scalar product $\langle\ ,\ \rangle$ and span the total space X, $X=X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X)=L^2(X^a) \otimes L^2(X_a)$. We write x for a generic point in X and denote by x^a and x_a the projections of x onto X^a and X_a, respectively. Let

$$I_a(x)=\sum_{\sim jak} V_{jk}(r_j-r_k) \tag{1.1}$$

be the intercluster potential between clusters in a. The cluster Hamiltonian $H_a(h)$ is defined as

$$H_a(h)=H(h)-I_a=H^a(h)\otimes Id+Id\otimes T_a(h) \quad \text{on } L^2(X^a)\otimes L^2(X_a),$$

where $T_a(h)=-(h^2/2)\Delta$ acts on $L^2(X_a)$ and $H^a(h)$ is given by

$$H^a(h)=-(h^2/2)\Delta+\sum_{jak} V_{jk}(r_j-r_k) \quad \text{on } L^2(X^a).$$

Let $H^a(h)$ be as defined above. We denote by $d^a(h)$, $0\le d^a(h)\le\infty$, the number of eigenvalues of $H^a(h)$ with repetition according to their multiplicities. A pair $\alpha=(a, j)$, $1\le j\le d^a(h)$, is called a channel. The following notions are associated with channel α: (1) $E_\alpha(h)$ is the j-th eigenvalue of $H^a(h)$; (2) $\psi_\alpha=\psi_\alpha(x^a; h)\in L^2(X^a)$ is the normalized eigenstate corresponding to eigenvalue $E_\alpha(h)$; (3) $H_\alpha(h)$ is the channel Hamiltonian defined by

$$H_\alpha(h)=T_a(h)+E_\alpha(h) \quad \text{on } L^2(X_a); \tag{1.2}$$

(4) $J_\alpha(h)$: $L^2(X_a)\to L^2(X)$ is the channel identification operator defined by $J_\alpha u=\psi_\alpha\otimes u$; (5) $W_\alpha^\pm(h)$: $L^2(X_a)\to L^2(X)$ is the channel wave operator defined by

$$W_\alpha^\pm(h)=s-\lim_{t\to\pm\infty}\exp(ih^{-1}tH(h))J_\alpha(h)\exp(-ih^{-1}tH_\alpha(h)).$$

We know ([15]) that under assumption $(V)_\rho$, the channel wave operators really exist and that their ranges are mutually orthogonal

$$Range\ W_\alpha^\pm(h)\perp Range\ W_\beta^\pm(h),\ \alpha\neq\beta.$$

The channel wave operators are said to be asymptotically complete, if

$$\sum_\alpha\oplus Range\ W_\alpha^-(h)=\sum_\alpha\oplus Range\ W_\alpha^+(h),$$

where the summation is taken over all channels α. It is also known ([8, 17]) that under assumption $(V)_\rho$, the channel wave operators are asymptotically complete. Let α and β be two channels associated with cluster decompositions a and b, respectively. We define the scattering operator $S_{\alpha\to\beta}(h)$: $L^2(X_a)\to L^2(X_b)$ for scattering from the initial state to the final one as follows:

$$S_{\alpha\to\beta}(h)=W_\beta^+(h)^*W_\alpha^-(h).$$

By definition, it follows that $S_{\alpha\to\beta}(h)$ intertwines the channel Hamiltonians $H_\alpha(h)$ and $H_\beta(h)$ in the sense that

$$\exp(-ih^{-1}tH_\beta(h))S_{\alpha\to\beta}(h)=S_{\alpha\to\beta}(h)\exp(-ih^{-1}tH_\alpha(h)) \tag{1.3}$$

and also we obtain by the asymptotic completeness of channel wave operators that

$$\sum_\beta S_{\alpha\to\beta}(h)^*S_{\alpha\to\beta}(h)=Id \tag{1.4}$$

as an operator acting on $L^2(X_a)$. This relation plays a basic role in proving the optical theorem.

2. Total scattering cross sections

In this section, we give the precise definition of total scattering cross section and formulate the main theorem. We first construct the spectral representation for the operator $H_\alpha(h)$ defined by (1.2). Let S_a be the unit sphere in X_a and let $Y_\alpha = L^2(\Lambda_\alpha)\otimes L^2(S_a)$ with $\Lambda_\alpha=(E_\alpha(h), \infty)$. We define the generalized eigenfunction φ_α of $H_\alpha(h)$, $H_\alpha(h)\varphi_\alpha=\lambda\varphi_\alpha$, by

$$\varphi_\alpha(x_a\,;\,\lambda,\,\omega_a,\,h)=\exp(ih^{-1}\eta_\alpha(\lambda)\langle x_a,\,\omega_a\rangle) \tag{2.1}$$

for $(\lambda,\ \omega_a)\in\Lambda_\alpha\times S_a$, where $\eta_\alpha=\sqrt{2(\lambda-E_\alpha(h))}$. We also define the unitary mapping $F_\alpha(h)\colon L^2(X_a)\to Y_\alpha$ by

$$(F_\alpha(h)f)(\lambda,\,\omega_a)=c_\alpha(\lambda,\,h)\int\overline{\varphi}_\alpha(x_a\,;\,\lambda,\,\omega_a,\,h)f(x_a)dx_a$$

with $c_\alpha=(2\pi h)^{-\nu_a/2}\eta_\alpha^{(\nu_a-2)/2}$, $\nu_a=3(\#(a)-1)$, where the integration with no domain attached is taken over the whole space. This abbreviation is used throughout. The mapping $F_\alpha(h)$ yields the spectral representation for $H_\alpha(h)$ in the sense that $H_\alpha(h)$ is transformed into the multiplication by λ in the space Y_α

$$(F_\alpha(h)H_\alpha(h)f)(\lambda,\,\omega_a)=\lambda(F_\alpha(h)f)(\lambda,\,\omega_a).$$

From now on, we fix the two-cluster decomposition a, $\#(a)=2$, as $a=\{C_1,\,C_2\}$ and consider the two-body channel $\alpha=(a,\,j)$ as an initial state. Let $E_\alpha(h)<0$ be the binding energy of initial channel α. We assume, in addition to $(V)_\rho$, that for $c_0>0$ fixed arbitrarily,

$$E_\alpha(h)-\inf\sigma_{ess}(H^a(h))<-c_0<0 \tag{E}$$

uniformly in h, where $\sigma_{ess}(H^a(h))$ denotes the set of essential spectrum of $H^a(h)$. If (E) is fulfilled, then we can prove that for any $L\gg1$, the normalized eigenstate $\psi_\alpha\in L^2(X^a)$ associated with eigenvalue $E_\alpha(h)$ obeys the bound

$$\int\langle x^a\rangle^L|\psi_\alpha(x^a\,;\,h)|^2dx^a\le C_L \tag{2.2}$$

with C_L independent of h.

We proceed to defining the total scattering cross section $\sigma_\alpha(\lambda,\,\omega_a\,;\,h)$ with two-body initial channel α. Let β be a channel with b as a cluster decomposition. We define the operator $T_{\alpha\to\beta}(h)\colon L^2(X_a)\to L^2(X_b)$ as

$$T_{\alpha\to\beta}(h)=S_{\alpha\to\beta}(h)-\delta_{\alpha\beta}Id, \tag{2.3}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta notation. As is easily seen, this operator also has the same intertwining property as in (1.3). This enables us to represent $T_{\alpha\to\beta}(h)$ as a decomposable operator

$$T_{\alpha\to\beta}(h)=\int_{\Lambda_{\alpha\beta}}\oplus T_{\alpha\to\beta}(\lambda\,;\,h)d\lambda,\ \Lambda_{\alpha\beta}=(E_{\alpha\beta}(h),\infty),$$

where $E_{\alpha\beta}=\max(E_\alpha(h), E_\beta(h))$ and the fibers $T_{\alpha\to\beta}(\lambda\,;\,h): L^2(S_a)\to L^2(S_b)$ are defined for *a.e.* $\lambda\in\Lambda_{\alpha\beta}$. For example, $T_{\alpha\to\alpha}(\lambda\,;\,h)$ is defined through the relation

$$(F_\alpha(h)T_{\alpha\to\alpha}(h)f)(\lambda,\omega_a)=(T_{\alpha\to\alpha}(\lambda\,;\,h)(F_\alpha(h)f)(\lambda,\cdot))(\omega_a).$$

We can show (see Proposition 3.2) that under assumption $(V)_\rho$, $T_{\alpha\to\beta}(\lambda\,;\,h)$ is of Hilbert-Schmidt class for *a.e.* $\lambda>0$ and that its Hilbert-Schmidt norm is locally integrable as a function of energy $\lambda>0$. Denote by $T_{\alpha\to\beta}(\theta_b,\omega_a\,;\,\lambda,h)$, $(\lambda,\omega_a,\theta_b)\in(0,\infty)\times S_a\times S_b$, the integral kernel of $T_{\alpha\to\beta}(\lambda\,;\,h)$. Then the scattering amplitude $f_{\alpha\to\beta}(\omega_a\to\theta_b\,;\,\lambda,h)$ for scattering from the initial state α to the final one β at energy λ is defined by

$$f_{\alpha\to\beta}(\omega_a\to\theta_b\,;\,\lambda,h)=-2\pi i n_a^{-1/2}\eta_\alpha(\lambda)^{-1}hT_{\alpha\to\beta}(\theta_b,\omega_a\,;\,\lambda,h),$$

$\eta_\alpha=\sqrt{2(\lambda-E_\alpha(h))}$ being again as in (2.1), where n_a is the reduced mass for $a=\{C_1, C_2\}$ and is given as

$$n_a=N^{-1}\sum_{j\in C_1}m_j\cdot\sum_{k\in C_2}m_k$$

for the N-body system with the identical masses $m_j=1$, $1\le j\le N$. We refer to the book [1, p. 627] for the above definition of scattering amplitude. We now define the total scattering cross section $\sigma_\alpha(\lambda,\omega_a\,;\,h)$ for scattering initiated in the two-body channel α at energy $\lambda>0$ and at incident direction $\omega_a\in S_a$ as follows:

$$\sigma_\alpha(\lambda,\omega_a\,;\,h)=\sum_\beta\int_{S_b}|f_{\alpha\to\beta}(\omega_a\to\theta_b\,;\,\lambda,h)|^2d\theta_b.$$

As stated above, $\sigma_\alpha(\lambda,\omega_a\,;\,h)$ is defined only for *a.e.* $(\lambda,\omega_a)\in(0,\infty)\times S_a$. It should be noted that the exceptional set may depend on the parameter h.

We proceed to formulating the main theorem. Let $I_a(x)=I_a(x^a, x_a)$ be the intercluster potential defined by (1.1). We denote the intercluster coordinates for $a=\{C_1, C_2\}$ by

$$\zeta_a=(\sum_j m_jr_j)(\sum_j m_j)^{-1}-(\sum_k m_kr_k)(\sum_k m_k)^{-1}\in\boldsymbol{R}^3,$$

where the summations $\sum_j$ and $\sum_k$ are taken over $j\in C_1$ and $k\in C_2$, respectively. The coordinates x_a over X_a are represented only in terms of ζ_a and hence we can write

$$I_{a0}(x_a)=I_a(0,x_a)=\sum_{\sim jak}V_{jk}(\epsilon_{jk}\zeta_a),\tag{2.4}$$

where ϵ_{jk}, $1\le j<k\le N$, takes the value $+1$ or -1 according as $(j,k)\in C_1\times C_2$ or $(j,k)\in C_2\times C_1$. We identify S_a with the two-dimensional unit sphere S^2 and write

$\zeta_a \in \boldsymbol{R}^3$ as

$$\zeta_a = b + s\omega_a, \ b \in \Pi_\omega, \ s \in \boldsymbol{R}, \tag{2.5}$$

where Π_ω is the two-dimensional hyperplane (impact plane) orthogonal to ω_a. With these notations, we are now in a position to formulate the main theorem.

Theorem 2.1. *Let the notations be as above. Assume that the pair potential V_{jk} fulfills $(V)_\rho$ with $\rho > 2$ and that the binding energy $E_a(h)$ of two-body initial channel a satisfies (E). Then, as a function of $(\lambda, \omega_a) \in (0, \infty) \times S_a$, the total scattering cross section $\sigma_a(\lambda, \omega_a; h)$ behaves like*

$$\sigma_a(\lambda, \omega_a; h) = L_0(\lambda, \omega_a; h) + o(h^{-2/(\rho-1)}), \ h \to 0,$$

in the distributional sense $\mathscr{D}'((0, \infty) \times S_a)$, where L_0 is given as

$$L_0(\lambda, \omega_a; h) = 4\int_{\Pi_\omega} \sin^2\{(2h)^{-1} \sum_{\sim jak} \int V_{jk}(\epsilon_{jk} b + s\mu_a(\lambda)\omega_a)ds\} db$$

with $\mu_a = \sqrt{2(\lambda - E_a(h))/n_a}$, μ_a being the intercluster relative velocity along the incident direction ω_a.

The next result can be obtained as an immediate consequence of the above theorem, if the non-negativity of σ_a is taken into account.

Corollary 2.2. Suppose that the same assumptions as in Theorem 2.1 are fulfilled. Let

$$\sigma_a^{av}(\lambda; h) = (4\pi)^{-1} \int_{S_a} \sigma_a(\lambda, \omega_a; h) d\omega_a$$

be the averaged total scattering cross section. Then one has

$$\int_\Lambda \sigma_a^{av}(\lambda; h) d\lambda = (4\pi)^{-1} \int_\Lambda \int_{S_a} L_0(\lambda, \omega_a; h) d\omega_a d\lambda + o(h^{-2/(\rho-1)})$$

for any compact interval $\Lambda \subset (0, \infty)$.

We conclude the section by making some comments on the theorem above.

REMARK 1. The leading term L_0 is of order $O(h^{-2/(\rho-1)})$. If, in particular, V_{jk} behaves like $V_{jk}(y) = |y|^{-\rho}(c + o(1))$, $c \neq 0$, at infinity, then L_0 can be calculated as

$$L_0 = \sigma_0 \mu_a^{-2/(\rho-1)} h^{-2/(\rho-1)}(1 + o(1)), \ h \to 0,$$

for some $\sigma_0 > 0$ by making use of the spherical coordinates over R^3.

REMARK 2. The proof of the theorem makes only an essential use of the behavior at infinity of pair potentials V_{jk}. The theorem can be extended to a certain class of pair potentials with local singularities.

REMARK 3. Recently Isozaki [9] has proved that if $\rho>11/2$, $\sigma_\alpha(\lambda, \omega_a; h)$ is finite and continuous in $(\lambda, \omega_a)\in(0, \infty)\times S_a$ in the case of three-body systems. We can combine this with the semi-classical resolvent estimates obtained by [7] to derive the semi-classical asymptotic formula as in the theorem for (λ, ω_a) fixed, if λ is restricted to a non-trapping energy range in the sense of classical dynamics.

3. Optical theorem

We keep the same notations as in the previous sections and always assume the assumptions in the main theorem (Theorem 2.1). In particular, the two-cluster decompsiton a and the two-body initial channel α associated with a are fixed throughout the discussion below. The first step toward the proof of the main theorem is to represent $\sigma_\alpha(\lambda, \omega_a; h)$ in terms of the forward scattering amplitude $f_{\alpha\to\alpha}(\omega_a\to\omega_a; \lambda, h)$. This representation formula is called the optical theorem. The aim here is to prove this relation.

We begin by making a brief review on some important spectral properties of the N-body Schrödinger operator $H(h)$, which are required to formulate the optical theorem. The operator $H(h)$ is known to have the following spectral properties ([6, 14]): (1) $H(h)$ has no positive eigenvalues. (2) The boundary values $R(\lambda\pm i0; H(h))$ to the positive real axis exist as an operator from $L^2_\nu(X)$ into $L^2_{-\nu}(X)$ for any $\nu>1/2$ and have the local Hölder continuity as a function of $\lambda>0$ in the uniform topology, where $L^2_\nu(X)=L^2(X; \langle x\rangle^{2\nu}dx)$ denotes the weighted L^2 space with weight $\langle x\rangle^\nu$.

We denote by $(\cdot, \cdot)_0$ and $\|\cdot\|_0$ the L^2 scalar product and norm in $L^2(X)$. The proposition below is concerned with the representation formula for scattering amplitude $f_{\alpha\to\alpha}$ and it can be verified in almost the same way as in the two-body case.

Proposition 3.1. *Let φ_α be the generalized eigenfunction defined by* (2.1) *and let $\psi_\alpha\in L^2(X^a)$ be the normalized eigenstate associated with the binding energy $E_\alpha(h)$. Define $e_\alpha(\omega_a)$ as*

$$e_\alpha(\omega_a)=e_\alpha(x; \lambda, \omega_a, h)=\psi_\alpha(x^a; h)\otimes\varphi_\alpha(x_a; \lambda, \omega_a, h).$$

Then the operator $T_{\alpha\to\alpha}(\lambda; h): L^2(S_a)\to L^2(S_a)$ is of Hilbert-Schmidt class for all $\lambda>0$ and has the integral kernel

$$T_{\alpha\to\alpha}(\theta_a, \omega_a; \lambda, h)=c_{0\alpha}((-I_a+I_aR(\lambda+i0; H(h))I_a)e_\alpha(\omega_a), e_\alpha(\theta_a))_0$$

with $c_{0\alpha}=i(2\pi)^{-2}\eta_a h^{-3}$. In particular, the scattering amplitude $f_{\alpha\to\alpha}(\omega_a\to\theta_a; \lambda, h)$ is represented as

$$f_{\alpha\to\alpha}=(2\pi)^{-1}n_a{}^{-1/2}h^{-2}((-I_a+I_aR(\lambda+i0\,;\ H(h))I_a)e_a(\omega_a),\ e_a(\theta_a))_0.$$

The argument here based on the proposition above. By making use of this proposition, we first prove the following

Proposition 3.2. *The operator $T_{\alpha\to\beta}(\lambda\,;\ h):\ L^2(S_a)\to L^2(S_b)$ is of Hilbert-Schmidt class for a.e. $\lambda>0$.*

Proof. The proof uses the relation (1.4) follows from the asymptotic completeness of wave operators. If we use this relation, then we have by definition (2.3) that

$$\sum_\beta T_{\alpha\to\beta}(\lambda\,;\ h)^*T_{\alpha\to\beta}(\lambda\,;\ h)=-2\,\mathrm{Re}\ T_{\alpha\to\alpha}(\lambda\,;\ h) \tag{3.1}$$

for *a.e.* $\lambda>0$, where

$$\mathrm{Re}\ T_{\alpha\to\alpha}(\lambda\,;\ h)=(T_{\alpha\to\alpha}(\lambda\,;\ h)+T_{\alpha\to\alpha}(\lambda\,;\ h)^*)/2.$$

As is easily seen from Proposition 3.1, Re $T_{\alpha\to\alpha}(\lambda\,;\ h):\ L^2(S_a)\to L^2(S_a)$ is of trace class for all $\lambda>0$. This proves the proposition. □

The next result is called the optical theorem, which is obtained as a consequence of the asymptotic completeness of channel wave operators.

Theorem 3.3. *Assume that the same assumptions as in Theorem 2.1 are fulfilled. Then one has*

$$\sigma_\alpha(\lambda,\ \omega_a\,;\ h)=4\pi n_a^{-1/2}\eta_\alpha(\lambda)^{-1}h\,\mathrm{Im}\,f_{\alpha\to\alpha}(\omega_a\to\omega_a\,;\ \lambda,\ h)$$

in $\mathcal{D}'((0,\ \infty)\times S_a)$, where

$$\mathrm{Im}\,f_{\alpha\to\alpha}=(2\pi)^{-1}n_a^{-1/2}h^{-2}\,\mathrm{Im}(R(\lambda+i0\,;\ H(h))I_ae_\alpha(\omega_a),\ I_ae_\alpha(\omega_a))_0.$$

Proof. Let $F(\lambda,\ \omega_a)$ be a real smooth function with compact support in $(0,\ \infty)\times S_a$. We denote by F_λ the multiplication operator by $F(\lambda,\ \omega_a)$ acting on $L^2(S_a)$. Then we have by Proposition 3.1 and relation (3.1) that

$$\begin{aligned}&\iint F(\lambda,\ \omega_a)^2\sigma_\alpha(\lambda,\ \omega_a\,;\ h)d\omega_a d\lambda\\&\qquad=(2\pi)^2n_a^{-1}h^2\sum_\beta\int\eta_\alpha(\lambda)^{-2}\|F_\lambda T_{\alpha\to\beta}(\lambda\,;\ h)\|_{HS}^2d\lambda\\&\qquad=-2(2\pi)^2n_a^{-1}h^2\int\eta_\alpha(\lambda)^{-2}\ Trace(F_\lambda\,\mathrm{Re}\ T_{\alpha\to\alpha}(\lambda\,;\ h)F_\lambda)d\lambda\\&\qquad=4\pi n_a^{-1/2}h\iint F(\lambda,\ \omega_a)^2\eta_\alpha(\lambda)^{-1}\,\mathrm{Im}\,f_{\alpha\to\alpha}(\omega_a\to\omega_a\,;\ \lambda,\ h)d\omega_a d\lambda.\end{aligned}$$

This proves the theorem. □

4. Representation formula

By the optical theorem, the total scattering cross section in question is now represented as

$$\sigma_a(\lambda, \omega_a; h)=2n_a^{-1}\eta_a(\lambda)^{-1}h^{-1}Q(\lambda, h) \text{ in } \mathcal{D}'((0, \infty)\times S_a), \tag{4.1}$$

where

$$Q(\lambda, h)=\mathrm{Im}(R(\lambda+i0; H(h))I_a e_a(\omega_a), I_a e_a(\omega_a))_0.$$

The proof of the main theorem is reduced to analyzing the behavior as $h\to 0$ of $Q(\lambda, h)$. The aim here is to rewrite this quantity in a form adapted to this purpose. Throughout the discussion below, $\omega_a\in S_a$ is fixed and $\lambda>0$ is assumed to range over a compact interval $\Lambda\subset(0, \infty)$ fixed arbitrarily.

Several new notations are required. Let $I_{a0}(x_a)=I_a(0, x_a)$ be as in (2.4). We define $\theta_\pm=\theta_\pm(x_a; \lambda, h)$ by

$$\theta_\pm=\exp(ih^{-1}\int_0^{\pm\infty} I_{a0}(x_a+s\eta_a\omega_a)ds) \tag{4.2}$$

with $\eta_a=\sqrt{2(\lambda-E_a(h))}$ again. As is easily seen, $\theta_\pm$ solves the equation

$$\eta_a\langle\omega_a, \nabla_{x_a}\rangle\theta_\pm+ih^{-1}I_{a0}\theta_\pm=0. \tag{4.3}$$

Let $\varphi_0(s)\in C_0^\infty([0, \infty))$, $0\leq\varphi_0\leq 1$, be a basic cut-off function such that

$$\varphi_0=1 \text{ for } 0\leq s\leq 1, \quad \varphi_0=0 \text{ for } s>2$$

and let φ_∞ be defined by $\varphi_\infty=1-\varphi_0$. With these functions, we introduce a partition of unity over X_a as follows:

$$\begin{aligned}
&\chi_-(x_a; M, d)=\varphi_\infty(|x_a|/M)\varphi_0((\langle\hat{x}_a, \omega_a\rangle+1)/d),\\
&\chi_+(x_a; M, d)=\varphi_\infty(|x_a|/M)\varphi_\infty((\langle\hat{x}_a, \omega_a\rangle+1)/d),\\
&\chi(x_a; M)=\varphi_0(|x_a|/M)=1-\chi_+(x_a; M, d)-\chi_-(x_a; M, d)
\end{aligned} \tag{4.4}$$

for $M>1$ and $1>d>0$, where $\hat{x}_a=x_a/|x_a|$. We write ∂_a for $\partial/\partial x_a$. It follows from assumption $(V)_\rho$ that

$$\chi_\pm\partial_a^\alpha(\theta_\pm-1)=O(|x_a|^{-\rho+1-|\alpha|}), \quad |x_a|\to\infty. \tag{4.5}$$

Here the order relation depends on h but it does not matter to the argument below.

Let $I_{aR}(x_a)=\varphi_0(|x_a|/R)I_{a0}(x_a)$ for $R\gg 1$ and define $\theta_{-R}=\theta_{-R}(x_a; \lambda, h)$ by (4.2) with I_{a0} replaced by I_{aR}. We also define

$$w_{0R}=(1-\chi_-\theta_{-R}-\chi_+)e_a=(\chi+\chi_-(1-\theta_{-R}))e_a,$$

where we write $e_a=e_a(\omega_a)$ for notational brevity. As is easily seen, w_{0R} is of compact support as a function of x_a and converges to $w_0=(1-\chi_-\theta_--\chi_+)e_a$ as $R\to\infty$ in $L^2_{-\nu}(X)$ for any $\nu>1/2$. We now set

$$R(\lambda+i\varkappa\,;\ H(h))I_a e_\alpha = w_{0R} + w_R,\ \varkappa>0,$$

where the remainder term w_R solves the equation

$$(H(h)-\lambda-i\varkappa)w_R = I_a e_\alpha = i\varkappa w_{0R} + w_{+R}$$

with $w_{+R}=-(H(h)-\lambda)w_{0R}$. By the principle of limiting absorption, the resolvent $R(\lambda+i\varkappa\,;\ H(h))$ is bounded uniformly in $\varkappa$, $0<\varkappa\le 1$, as an operator from $L^2_\nu(X)$ into $L^2_{-\nu}(X)$ for any $\nu>1/2$, which implies

$$\varkappa\|R(\lambda+i\varkappa\,;\ H(h))w_{0R}\|_0\to 0,\ \varkappa\to 0.$$

Thus we have

$$R(\lambda+i0\,;\ H(h))I_a e_\alpha = w_{0R} + R(\lambda+i0\,;\ H(h))(I_a e_\alpha + w_{+R}).$$

We calculate w_{+R} on the right side and take the limit $R\to\infty$. Recall that the cluster Hamiltonian $H_a(h)$ is defined as

$$H_a(h)=H(h)-I_a=H^a(h)\otimes Id+Id\otimes T_a(h),$$

where $T_a(h)=-h^2\Delta/2$ acts on $L^2(X_a)$. Since e_α satisfies $(H_a(h)-\lambda)e_\alpha=0$, we have $(H(h)-\lambda)e_\alpha=I_a e_\alpha$. Similarly we have

$$(H(h)-\lambda)\chi_+e_\alpha=[T_a(h),\ \chi_+]e_\alpha+\chi_+I_{aR}e_\alpha+\chi_+(I_a-I_{aR})e_\alpha,$$

where the notation $[\cdot,\ \cdot]$ stands for the commutator relation. Since θ_{-R} satisfies the equation (4.3) with I_{a0} replaced by I_{aR}, it follows that

$$[T_a(h),\ \theta_{-R}]e_\alpha+I_{aR}\theta_{-R}e_\alpha=(T_a(h)\theta_{-R})e_\alpha$$

and hence we have

$$(H(h)-\lambda)\chi_-\theta_{-R}e_\alpha=[T_a(h),\ \chi_-]\theta_{-R}e_\alpha+\chi_-(T_a(h)\theta_{-R})e_\alpha+\chi_-\theta_{-R}(I_a-I_{aR})e_\alpha.$$

Let $\chi_0(x_a\,;\ M)=\varphi_0(|x_a|/2M)$. Then $\chi_0=1$ on the support of χ and hence

$$[T_a(h),\ \chi_++\chi_-]=\chi_0[T_a(h),\ \chi_++\chi_-].$$

The sum $[T_a(h),\ \chi_+]+[T_a(h),\ \chi_-]\theta_{-R}$ is written as

$$\chi_0([T_a(h),\ \chi_+]+[T_a(h),\ \chi_-]\theta_{-R})+(1-\chi_0)[T_a(h),\ \chi_-](\theta_{-R}-1).$$

Estimate (4.5) is still true for $\theta_{\pm R}$ uniformly in $R\gg 1$. Hence, w_{+R} converges to w_+ as $R\to\infty$ strongly in $L^2_\nu(X)$ for some $\nu>1/2$ and the limit w_+ is given as $w_+=-I_a e_\alpha+\sum_{1\le j\le 3}w_j$, where

$$\begin{aligned}
w_1&=\chi_0\{[T_a(h),\ \chi_-]\theta_-+[T_a(h),\ \chi_+]\}e_\alpha,\\
w_2&=\{(1-\chi_0)[T_a(h),\ \chi_-](\theta_--1)+\chi_+I_{a0}\}e_\alpha,\\
w_3&=\{\chi_-(T_a(h)\theta_-)+(\chi_-\theta_-+\chi_+)(I_a-I_{a0})\}e_\alpha.
\end{aligned}$$

Here the brackets $\{\cdots\}$ are regarded as operators acting on e_α. Thus we have

(4.6) $$R(\lambda+i0\,;\,H(h))I_a e_\alpha=w_0+\sum_{1\le j\le 3}R(\lambda+i0\,;\,H(h))w_j.$$

We fix the constant γ throughout as

(4.7) $$\gamma=1/(\rho-1)$$

and take $\beta>\gamma$ as

(4.8) $$\beta=(1+\delta)\gamma,\ 0<\delta\ll 1,$$

with δ to be determined later. According to notation (4.4), we also define

(4.9) $$p_0(x_a\,;\,h)=\varphi_0(h^\beta|x_a|/32),\ p_\pm(x_a\,;\,h)=\chi_\pm(x_a\,;\,16h^{-\beta},1/2)$$

with $M=16h^{-\beta}$ and $d=1/2$. Then we have

$$R(\lambda+i0\,;\,H(h))I_a e_\alpha=u_0+\sum_{j=1}^{3}R(\lambda+i0\,;\,H(h))u_j$$

by (4.6), where

(4.10) $$\begin{aligned}u_0(x\,;\,\lambda,h)&=(1-p_-\theta_--p_+)e_\alpha,\\ u_1(x\,;\,\lambda,h)&=p_0\{[T_a(h),p_-]\theta_-+[T_a(h),p_+]\}e_\alpha,\\ u_2(x\,;\,\lambda,h)&=\{(1-p_0)[T_a(h),p_-](\theta_--1)+p_+I_{a0}\}e_\alpha,\\ u_3(x\,;\,\lambda,h)&=\{p_-(T_a(h)\theta_-)+(p_-\theta_-+p_+)(I_a-I_{a0})\}e_\alpha.\end{aligned}$$

Hence $Q(\lambda,h)$ can be written as

(4.11) $$Q=\mathrm{Im}(u_0,I_a e_\alpha)_0+\sum_{j=1}^{3}\mathrm{Im}(u_j,R(\lambda-i0\,;\,H(h))I_a e_\alpha)_0.$$

The same argument as above applies to $R(\lambda-i0\,;\,H(h))I_a e_\alpha$. We again use the notation (4.4) to define

(4.12) $$q_0(x_a\,;\,h)=\varphi_0(h^\beta|x_a|/2),\ q_\pm(x_a\,;\,h)=\chi_\pm(x_a\,;\,h^{-\beta},1/16)$$

with $M=h^{-\beta}$ and $d=1/16$. Then we obtain

$$R(\lambda-i0\,;\,H(h))I_a e_\alpha=v_0+\sum_{j=1}^{3}R(\lambda-i0\,;\,H(h))v_j,$$

where

(4.13) $$\begin{aligned}v_0(x\,;\,\lambda,h)&=(1-q_--q_+\theta_+)e_\alpha,\\ v_1(x\,;\,\lambda,h)&=q_0\{[T_a(h),q_+]\theta_++[T_a(h),q_-]\}e_\alpha,\\ v_2(x\,;\,\lambda,h)&=\{(1-q_0)[T_a(h),q_+](\theta_+-1)+q_-I_{a0}\}e_\alpha,\\ v_3(x\,;\,\lambda,h)&=\{q_+(T_a(h)\theta_+)+(q_+\theta_++q_-)(I_a-I_{a0})\}e_\alpha.\end{aligned}$$

Hence it follows from (4.11) that

$$Q(\lambda, h)=Q_0(\lambda, h)+\sum_{1\le j,\, k\le 3} Q_{jk}(\lambda, h),$$

where Q_0 and Q_{jk} are given as

$$\begin{aligned} Q_0(\lambda, h)&=\operatorname{Im}(u_0, I_a e_a)_0+\sum_{j=1}^{3}\operatorname{Im}(u_j, v_0)_0, \\ Q_{jk}(\lambda, h)&=\operatorname{Im}(R(\lambda+i0\,;\, H(h))u_j, v_k)_0,\ 1\le j,\ k\le 3. \end{aligned} \tag{4.14}$$

This is a representation formula which plays a basic role in studying the asymptotic behavior as $h\to 0$ of quantity $Q(\lambda, h)$.

The remaining sections are devoted to evaluating each term defined in (4.14). Stating our conclusion first, only the term Q_{11} makes a contribution to the leading term of asymptotic formula in the main theorem and the other terms which are dealt with as remainder terms are shown to behave like $o(h^{1-2\gamma})$ as $h\to 0$ in $\mathcal{D}'((0, \infty))$ as a function of λ uniformly in $\omega_a\in S_a$, γ being as in (4.7).

By definitions (4.10) and (4.13), u_j and v_j take the form

$$u_j=f_j e_a,\ v_j=g_j e_a,\ 1\le j\le 3.$$

We end the section by formulating several simple properties of f_j and g_j as a series of lemmas, which are required in the proof of the main theorem. Below we again write ∂_a for $\partial/\partial x_a$ and use the following notations:

$$\begin{aligned} B(m, M)&=\{x_a\in X_a:\ m\le|x_a|<M\}, \\ \Gamma_\pm(M, d)&=\{x_a\in X_a:\ |x_a|>M,\ \langle \hat{x}_a, \omega_a\rangle \gtrless d\} \end{aligned}$$

for $M>m\ge 0$ and $1>d>-1$. If, in paricular, $m=0$, then $B(0, M)$ is simply denoted as $B(M)$.

Lemma 4.1. (i) *$\theta_\pm(x_a\,;\,\lambda, h)$ defined* (4.2) *satisfies the estimates*

$$\begin{aligned} &|p_-\operatorname{Re}(\theta_- -1)|\le Ch^{-2}\langle x_a\rangle^{-2(\rho-1)},\ |p_-\operatorname{Im}\theta_-|\le Ch^{-1}\langle x_a\rangle^{-(\rho-1)}, \\ &|q_+\operatorname{Re}(\theta_+ -1)|\le Ch^{-2}\langle x_a\rangle^{-2(\rho-1)},\ |q_+\operatorname{Im}\theta_+|\le Ch^{-1}\langle x_a\rangle^{-(\rho-1)}. \end{aligned}$$

(ii) *If $x_a\in\Gamma_-(Mh^{-\beta}, d)$, then $|\partial_\lambda^k\theta_-|\le C_k$ and*

$$|\partial_a^\alpha\partial_\lambda^k\theta_-|\le C_{\alpha k}h^{-1}\langle x_a\rangle^{-(\rho+|\alpha|-1)},\ |\alpha|\ge 1.$$

Similar estimates remain true for $\partial_\lambda^k\theta_+$ in $\Gamma_+(Mh^{-\beta}, d)$.

Lemma 4.2. (i) *$f_1=f_1(x_a\,;\,\lambda, h)$ is supported in $B(16h^{-\beta}, 64h^{-\beta})$ and satisfies the estimates*

$$|\operatorname{Re} f_1|\le C\langle x_a\rangle^{-\rho},\ |\operatorname{Im} f_1|\le Ch\langle x_a\rangle^{-1},$$

$$|\partial_a^\alpha \partial_\lambda^k f_1| \le C_{\alpha k} h \langle x_a \rangle^{-1-|\alpha|}.$$

(ii) *$g_1 = g_1(x_a\,;\ \lambda,\ h)$ is supported in $B(h^{-\beta}, 4h^{-\beta})$ and obeys the same bounds as above.*

Lemma 4.3. (i) *$f_2 = f_2(x_a\,;\ \lambda,\ h)$ is supported in $\Gamma_+(16h^{-\beta},\ -1/2)$ and satisfies the estimates*

$$|\mathrm{Re}\, f_2| \le C\langle x_a \rangle^{-\rho},\ |\mathrm{Im}\, f_2| \le C(h\langle x_a \rangle^{-\rho-1} + h^{-1}\langle x_a \rangle^{-2\rho+1}),$$
$$|\partial_a^\alpha \partial_\lambda^k f_2| \le C_{\alpha k} h \langle x_a \rangle^{-\rho-|\alpha|}.$$

(ii) $g_2 = g_2(x_a\,;\ \lambda,\ h)$ is supported in $\Gamma_-(h^{-\beta},\ -7/8)$ and obeys the same bounds as above.

Lemma 4.4. (i) *$f_3 = f_3(x_a\,;\ \lambda,\ h)$ has support in $|x_a| > 16h^{-\beta}$ and obeys the bound*

$$|\partial_\lambda^k f_3| \le C_{kL} \langle x_a \rangle^{-\rho-1} \langle x_a \rangle^{L},\ L \gg 1.$$

(ii) $g_3 = g_3(x_a\,;\ \lambda,\ h)$ has support in $|x_a| > h^{-\beta}$ and satisfies the same estimate as above.

These lemmas can be easily verified by a direct calculation. We give only a sketch for the proof.

Proof of Lemma 4.1. (i) follows from assumption $(V)_\rho$. Since $\beta(\rho-1) > \gamma(\rho-1) = 1$ by choice (4.8), we have $h^{-1} \le C\langle x_a \rangle^{\rho-1}$ for $|x_a| \ge mh^{-\beta}$. If this is taken into account, (ii) can be easily proved. $\square$

Proof of Lemma 4.2. We prove (i) only. By definition, it is clear that f_1 has support in $B(16h^{-\beta}, 64h^{-\beta})$. Denote by ∇_a the gradient over X_a. Then we have

$$u_1 = p_0(-h^2\langle \nabla_a p_+ + \theta_- \nabla_a p_-,\ \nabla_a e_a\rangle + O(h)\langle x_a \rangle^{-\rho-1} e_a + O(h^2)\langle x_a \rangle^{-2} e_a)$$

by Lemma 4.1. We can take δ in (4.8) so small that $h^2\langle x_a \rangle^{-2} \le C\langle x_a \rangle^{-\rho}$ on *supp* $p_0 \subset B(64h^{-\beta})$. The lemma follows from these facts. $\square$

Proof of Lemma 4.3. The lemma can be proved in the same way as Lemma 4. 2. Since $p_- = 1$ on $\Gamma_-(32h^{-\beta},\ -1/2)$, it is easy to see that f_2 has support in the outgoing region $\Gamma_+(16h^{-\beta},\ -1/2)$. By Lemma 4.1, we have

$$u_2 = p_+ I_{a0} e_a - h^2(1-p_0)(\theta_- - 1)\langle \nabla_a p_-,\ \nabla_a e_a\rangle + O(h)\langle x_a \rangle^{-\rho-1} e_a,$$

from which the lemma follows. $\square$

Proof of Lemma 4.4. We have $|p_- T_a(h)\theta_-| \le Ch\langle x_a \rangle^{-\rho-1}$ by Lemma 4.1 and also it follows from assumption $(V)_\rho$ that

(4.15) $$|I_a - I_{a0}| \le C_L \langle x_a \rangle^{-\rho-1} \langle x^a \rangle^L, \ L \gg 1.$$

This proves the lemma. □

5. Preparatory lemmas

In this section, we prepare three preparatory lemmas which are often used throughout the proof of the main theorem. These lemmas are stated without proofs and their proofs are given in section 8.

5.1. Let $u = u(x\,;\,\lambda,\,h)$ be a function such that u is C^2-smooth in λ and that for some $\sigma > 0$ and $\nu > 0$, u satisfies the estimates

(5.1) $$\begin{aligned} &\|u\|_0 = O(h^\sigma), \ \|\langle x_a \rangle^\nu u\|_0 = O(h^{\sigma-\beta\nu}), \\ &\|(h\partial_\lambda)u\|_0 = O(h^{\sigma-\beta}), \ \|\langle x_a \rangle^{\nu-2}(h\partial_\lambda)^2 u\|_0 = O(h^{\sigma-\beta\nu}) \end{aligned}$$

uniformly in $\lambda \in \Lambda$, $\Lambda \subset (0, \infty)$ being again a compact interval fixed arbitrarily.

Lemma 5.1. *Let $F(\lambda) \in C_0^\infty((0, \infty))$. Assume that $u = u(x\,;\,\lambda,\,h)$ and $v = v(x\,;\,\lambda,\,,h)$ belong to $L_\nu^2(X)$ and fulfill* (5.1) *with $\sigma = \sigma_1$ and $\sigma = \sigma_2$, respectively, for some $\nu > 1$. Define the integral J as*

$$J = \int_0^\infty F(\lambda)(R(\lambda + i0\,;\, H(h))u, \ v)_0 d\lambda.$$

Then one has $J = O(h^{\sigma_1+\sigma_2-\beta-1})$.

By Lemmas 4.2 and 4.4, u_1 and u_3 satisfy the condition of the lemma above. We apply this lemma to ovaluate the terms Q_{jk} with $(j, k) = (1, 3)$, $(3, 1)$ and $(3, 3)$. However the condition of Lemma 5.1 is not necessarily satisfied by u_2 behaving like $O(\langle x_a \rangle^{-\rho})$ under the assumption $\rho > 2$.

5.2. The second lemma is mainly used to evaluate the terms Q_{jk} with $j = 2$ or $k = 2$. Before formulating the lemma, we recall the notations. e_α is defined by $e_\alpha = \psi_\alpha \otimes \varphi_\alpha$. J_α : $L^2(x_a) \to L^2(X)$ is the channel identification operator defined by $J_\alpha u = \psi_\alpha(h) \otimes u$. $H_\alpha(h) = T_a(h) + E_\alpha(h)$ acting on $L^2(X_a)$ is the channel Hamiltonian associated with the two-body initial channel α.

Lemma 5.2. (i) *Assume that $f(x_a\,;\,\lambda,\,h)$ is supported in $\Gamma_+(Mh^{-\beta}, d)$ and obeys the bound $|\partial_a^\alpha \partial_\lambda^k f| \le C_{\alpha k} \langle x_a \rangle^{-\rho-|\alpha|}$ uniformly in $\lambda \in \Lambda$. Then one has*

$$R(\lambda + i0\,;\, H(h)) f e_\alpha = J_\alpha R(\lambda + i0\,;\, H_\alpha(h)) f \varphi_\alpha + R(\lambda + i0\,;\, H(h))w,$$

where the remainder term $w = w(x\,;\,\lambda,\,h) \in L_\nu^2(X)$ satisfies (5.1) *with $\sigma = \beta(2\rho - 5/2) - 1$ for any ν, $0 < \nu < \rho - 1$.*

(ii) *If* $g(x_a\,;\,\lambda,\,h)$ *is supported in* $\Gamma_-(Mh^{-\beta},\,d)$ *and obeys the same bound as above, then one has a similar representation for* $R(\lambda-i0\,;\,H(h))ge_a$.

5.3. The third lemma is employed in calculating the leading term which comes from the term Q_{11}. Let $\eta_a=\sqrt{2(\lambda-E_a(h))}$ be as before. We define the self- adjoint operator $A(\lambda,\,h)$ as

$$A(\lambda,\,h)=\eta_a\langle\omega_a,\,-ih\nabla_a\rangle+I_{a0}\quad\text{on } L^2(X_a) \tag{5.2}$$

and denote by $G_t(\lambda,\,h)=\exp(-ih^{-1}tA(\lambda,\,h))$ the unitary propagation group generated by $A(\lambda,\,h)$. This unitary group is explicitly expressed as

$$G_t(\lambda,\,h)f=f(x_a-t\eta_a\omega_a)\exp(-ih^{-1}\int_0^t I_{a0}(x_a-(t-s)\eta_a\omega_a)ds). \tag{5.3}$$

Lemma 5.3. *Let the notations be as above. Denote by* $(\cdot,\,\cdot)_a$ *the scalar product in* $L^2(X_a)$. *Assume that* $f(x_a\,;\,\lambda,\,h)$ *and* $g(x_a\,;\,\lambda,\,h)$ *are supported in* $B(mh^{-\beta},\,Mh^{-\beta})$ *and obey the bound*

$$|\partial_a^\alpha\partial_\lambda^k f|+|\partial_a^\alpha\partial_\lambda^k g|\le C_{\alpha k}h\langle x_a\rangle^{-1-|\alpha|}$$

uniformly in $\lambda\in\Lambda$. *Then one has in* $\mathcal{D}'((0,\,\infty))$

$$(R(\lambda+i0\,;\,H(h))fe_a,\,ge_a)_0=ih^{-1}\int_0^\infty(G_t(\lambda,\,h)f,\,g)_a dt+o(h^{1-2\gamma}).$$

5.4. Let $T_0=-\Delta/2$ be the free Hamiltonian acting on $L^2(X_a)$. Then we can write

$$R(\lambda+i0\,;\,H_a(h))=h^{-2}R(h^{-2}\zeta_a(\lambda)\pm i0\,;\,T_0)$$

with $\zeta_a=\lambda-E_a(h)$. The resolvent estimate at high energies for the free Hamiltonian T_0 plays an important role in the proof of Lemma 5.2 as well as in the proof of the main theorem. Such a result has been already established by [10, 12]. We here formulate this result in a form adapted to our purpose.

We require several new notations. Let $\xi_a\in X_a'$ be the coordinates dual to $x_a\in X_a$ and let $\hat{u}$ denote the Fourier transformation

$$\hat{u}(\xi_a)=(2\pi h)^{-3/2}\int\exp(-ih^{-1}\langle x_a,\,\xi_a\rangle)u(x_a)dx_a.$$

We denote by S_m the set of all $a(x_a,\,\xi_a)\in C^\infty(X_a\times X_a')$ such that

$$|(\partial/\partial x_a)^\alpha(\partial/\partial\xi_a)^\beta a|\le C_{\alpha\beta L}\langle x_a\rangle^{m-|\alpha|}\langle\xi_a\rangle^{-L}\quad\text{for any } L\gg 1.$$

A family of symbols $a(x_a,\,\xi_a\,;\,\epsilon)$ with parameter ϵ is said to belong to S_m uniformly in ϵ, if the constants $C_{\alpha\beta L}$ above can be taken uniformly in ϵ. Most of symbols which we consider in the later application have compact support in ξ_a and hence are of class S_m. For given symbol $a(x_a,\,\xi_a)\in S_m$, we define the pseudodifferential operator $a(x_a, hD_a)$ as

$$a(x_a, hD_a)u=(2\pi h)^{-3/2}\int \exp(ih^{-1}\langle x_a, \xi_a\rangle)a(x_a, \xi_a)\hat{u}(\xi_a)d\xi_a$$

and denote by OPS_m the class of such operators. We also use the notation

$$\Sigma_\pm(M, d, c)=\{(x_a, \xi_a): |x_a|>M, \xi_a\in\Omega(c), \langle\hat{x}_a, \hat{\xi}_a\rangle \gtrless d\},$$

where $\Omega(c)=\{\xi_a: |\xi_a-\eta_a\omega_a|<c\}$ for $0<c\ll 1$ small enough.

Proposition 5.4. *Write $R_{\pm a}(\lambda, h)$ for $R(\lambda\pm i0; H_a(h))$ and Q_a for the multiplication operator with $\langle x_a\rangle$. Denote by $\|\cdot\|$ the operator norm when considered as an operator from $L^2(X_a)$ into itself. Then one has following resolvent estimates uniformly in $\lambda\in\Lambda$.*

(i) *If $\mu>1/2$, then*

$$\|Q_a^{-\mu-k}\{(h\partial_\lambda)^k R_{\pm a}(\lambda, h)\}Q_a^{-\mu-k}\|=O(h^{-1}).$$

(ii) *Let $\mu<1/2$. If $b_\pm$ is of class OPS_0 with symbol supported in $\Sigma_\pm(M_\pm, d_\pm, c)$, then*

$$\|Q_a^{\mu-k-1}\{(h\partial_\lambda)^k R_{\pm a}(\lambda, h)\}b_\pm Q_a^{-\mu}\|=O(h^{-1}).$$

(iii) *Let $b_\pm\in OPS_0$ be as above. If $d_+>d_-$, then*

$$\|Q_a^{\mu}b_\mp\{(h\partial_\lambda)^k R_{\pm a}(\lambda, h)\}b_\pm Q_a^{\mu}\|=O(h^{L}).$$

for any $\mu\gg 1$ and $L\gg 1$.

(iv) *Let $b_\pm\in OPS_0$ be again as above and let $\psi_\pm=\varphi_0(|x_a|/m_\pm)$. If $m_\pm\ll M_\pm$, then*

$$\|\psi_\pm\{(h\partial_\lambda)^k R_{\pm a}(\lambda, h)\}b_\pm Q_a^{\mu}\|=O(h^{L})$$

for any $\mu\gg 1$ and $L\gg 1$.

REMARK. The proposition above is a special case of the results obtained in [10, 12]. Statement (iv) is not explicitly mentioned there but it can be verified in the same way as (iii), if we take account of the fact that classical free particles with initial satates in $\Sigma_\pm$ never pass over the support of $\psi_\pm$.

6. Remainder estimates

The present and next sections are devoted to proving the main theorem. Let $Q_0(\lambda, h)$ and $Q_{jk}(\lambda, h)$, $1\leq j, k\leq 3$, be defined in (4.14). The aim here is to prove that Q_0 and Q_{jk}, $(j, k)\neq(1, 1)$, behave like $o(h^{1-2\gamma})$ as $h\to 0$ in $\mathcal{D}'((0, \infty))$ as a function of λ.

(i) We first consider the term Q_0. This is split into $Q_0=Q_{00}+\sum_{j=1}^{3}Q_{0j}$, where

$Q_{0j}=\mathrm{Im}(f_je_a,\ v_0)_0$, $1\leq j\leq 3$, and

$$Q_{00}=\mathrm{Im}(u_0,\ I_ae_a)_0=\mathrm{Im}(-p_-\theta_-e_a,\ I_ae_a)_0.$$

We prove that each term obeys the bound

$$Q_{0j}(\lambda,\ h)=o(h^{1-2\gamma}),\ 0\leq j\leq 3. \tag{6.1}$$

The function p_- is supported in $|x_a|>16h^{-\beta}$ and also we have

$$|I_a|\leq C_L(\langle x_a\rangle^{-\rho}+\langle x_a\rangle^{-\rho-1}\langle x^a\rangle^L),\ L\gg 1,$$

by (4.15). Hence it follows from (2.2) and Lemma 4.1 that

$$Q_{00}=O(h^{-1})\int_{|x_a|>16h^{-\beta}}\langle x_a\rangle^{-2\rho+1}dx_a=O(h^{-1+2\beta(\rho-2)}). \tag{6.2}$$

Since $\beta>\gamma$ by choice (4.8), we obtain the bound (6.1) for Q_{00}.

Next we evaluate the terms Q_{0j}. By Lemmas 4.2~4.4, all the functions f_j, $1\leq j\leq 3$, vanish on $B(16h^{-\beta})$. If we write $v_0=(q+q_+(1-\theta_+))e_a$ with $q=1-q_+-q_-$ supported in $B(2h^{-\beta})$, then we have $Q_{0j}=\mathrm{Im}(f_je_a,\ q_+(1-\theta_+)e_a)_0$. By Lemmas 4.1 and 4.2, the term Q_{01} is estimated as in (6.2) and hence we have (6.1) for Q_{01}. By Lemma 4.3, f_2 also obeys the bounds $|\mathrm{Re}\,f_2|\leq C\langle x_a\rangle^{-\rho}$ and $|\mathrm{Im}\,f_2|\leq Ch\langle x_a\rangle^{-1}$, which implies (6.1) for Q_{02}. The bound for Q_{03} also follows from Lemma 4.4 at once. Thus we have proved that $Q_0=o(h^{1-2\gamma})$.

(ii) The aim here is to prove that

$$Q_{jk}(\lambda,\ h)=o(h^{1-2\gamma})\ \text{ in } \mathcal{D}'((0,\ \infty)) \tag{6.3}$$

for $(j,\ k)=(1,\ 3)$, $(3,\ 1)$ and $(3,\ 3)$. This is obtained as a simple application of Lemma 5.1. By Lemma 4.2, u_1 satisfies the estimates in (5.1) with $\sigma=\sigma_1=1-\beta/2$ for any $\nu>0$. On the other hand, u_3 satisfies these estimates with $\sigma=\sigma_3=\beta(\rho-1/2)>1+\beta/2$ for any ν, $0<\nu<\rho-1/2$, by Lemma 4.4. The functions v_1 and v_3 also satisfy the same estimates as u_1 and u_3, respectively. Since $\beta(\rho-2)>1-2\gamma$, (6.3) follows from Lemma 5.1 at once.

(iii) We deal with the term Q_{22} and prove that

$$Q_{22}(\lambda,\ h)=o(h^{1-2\gamma})\ \text{ in } \mathcal{D}'((0,\ \infty)). \tag{6.4}$$

We again write $(\cdot\ ,\cdot)_a$ for the scalar product in $L^2(X_a)$ and denote by $\|\cdot\|_a$ the norm in this space. By Lemma 4.3, f_2 and g_2 satisfy the assumption of Lemma 5.2. Hence we obtain

$$R(\lambda+i0\,;\ H(h))f_2e_a=J_aR(\lambda+i0\,;\ H_a(h))f_2\varphi_a+R(\lambda+i0\,;\ H(h))w_+,$$
$$R(\lambda-i0\,;\ H(h))g_2e_a=J_aR(\lambda-i0\,;\ H_a(h))g_2\varphi_a+R(\lambda+i0\,;\ H(h))w_-.$$

Here the remainder terms $w_\pm$ satisfy (5.1) with $\sigma=\beta(2\rho-5/2)-1$ for any ν, $0<\nu$

$<\rho-1$. Since

$$2\sigma-\beta-1=\beta(4\rho-6)-3>1-2\gamma,$$

it follows from Lemma 5.1 that

$$(R(\lambda+i0\,;\,H(h))w_+,\ w_-)_0=o(h^{1-2\gamma})\ \text{ in } \mathcal{D}'((0,\infty)).$$

Thus we have

$$Q_{22}=\mathrm{Im}(R(\lambda+i0\,;\,H_a(h))(f_2\varphi_a+J_a^*w_+),\ g_2\varphi_a)_a+o(h^{1-2\gamma})$$

in $\mathcal{D}'((0,\infty))$. We now use Proposition 5.4(i) to evaluate the first term on the right side. Take $\mu>1/2$ close enough to $1/2$. Then we have

$$\|\langle x_a\rangle^\mu f_2\|_a+\|\langle x_a\rangle^\mu g_2\|_a=O(h^{\beta(\rho-\mu-3/2)})$$

and $\|\langle x_a\rangle^\mu J_a^* w_+\|_a=O(h^{\sigma-\beta\mu})$, because w_+ satisfies (5.1) for any ν, $0<\nu<\rho-1$. We can choose μ so close to $1/2$ that $2\beta(\rho-\mu-3/2)-1>1-2\gamma$ and

$$\beta(\rho-\mu-3/2)+\sigma-\beta\mu-1=\beta(3\rho-2\mu-4)-2>1-2\gamma.$$

Hence (6.4) is obtained.

(iv) The terms Q_{23} and Q_{32} are estimated in the same way as Q_{22}. As stated previously, v_3 satisfies (5.1) with $\sigma=\sigma_3=\beta(\rho-1/2)$ for any ν, $0<\nu<\rho-1/2$, so that

$$\sigma_3+\sigma-\beta-1=\beta(3\rho-4)-2>1-2\gamma$$

for $\sigma=\beta(2\rho-5/2)-1$. This, together with Lemmas 5.1 and 5.2, implies that

$$Q_{23}(\lambda,\ h)=\mathrm{Im}(R(\lambda+i0\,;\,H_a(h))f_2\varphi_a,\ J_a^*v_3)_a+o(h^{1-2\gamma})\ \text{ in } \mathcal{D}'((0,\infty)).$$

By Lemma 4.4, $\|\langle x_a\rangle^\mu J_a^* v_3\|_a=O(h^{\beta(\rho-\mu-1/2)})$ for $\mu>1/2$, μ being close enough to 1/2. Hence we again use Proposition 5.4(i) to obtain that $Q_{23}=o(h^{1-2\gamma})$ in $\mathcal{D}'((0,\infty))$. A similar argument applies to Q_{32}.

(v) Finally we analyze the remaining two terms Q_{12} and Q_{21}. the function u_1 satisfies (5.1) with $\sigma=\sigma_1=1-\beta/2$ for any $\nu>0$ and hence

$$\sigma_1+\sigma-\beta-1=\beta(2\rho-4)-1>1-2\gamma$$

for $\sigma=\beta(2\rho-5/2)-1$. By Lemmas 5.1 and 5.2 again, we have

$$Q_{12}=\mathrm{Im}(f_1\varphi_a,\ R(\lambda-i0\,;\,H_a(h))g_2\varphi_a)_a+o(h^{1-2\gamma})\ \text{ in } \mathcal{D}'((0,\infty)).$$

We denote by T the first term on thr right side of the relation above. By Lemma 4.2, $f_1=f_1(x_a\,;\,\lambda,\ h)$ is supported in $B(16h^{-\beta},\ 64h^{-\beta})$ and obey the bound $|f_1|\leq Ch\langle x_a\rangle^{-1}$. However it does not necessarily satisfy $|f_1|\leq C\langle x_a\rangle^{-\rho}$ uniformly in h. Thus the first term T above cannot be controlled by a direct application of

Proposition 5.4(i) as in the previous steps (iii) and (iv). We employ a slightly different argument to evaluate this term.

By Lemma 4.3, $g_2 = g_2(x_a\,;\,\lambda,\,h)$ is supported in $\Gamma_-(h^{-\beta},\,-7/8)$ and satisfies the estimate $|\partial_a^\alpha g_2| \le C_\alpha \langle x_a\rangle^{-\rho-|\alpha|}$. We now take $M \gg 1$ large enough and decompose g_2 into

$$g_2 = g_{20} + g_{2\infty} = \varphi_0(h^\beta |x_a|/M) g_2 + \varphi_\infty(h^\beta |x_a|/M) g_2,$$

so that T is split into $T = T_0 + T_\infty$ according to the decomposition above. Let $a_-(x_a\,;\,\xi_a\,;\,\lambda,\,h) \in S_{-\rho}$ be the symbol defined by

$$a_-(x_a,\,\xi_a\,;\,\lambda,\,h) = g_{2\infty}(x_a\,;\,\lambda,\,h)\varphi_0(2|\xi_a - \eta_a\omega_a|/c),$$

where $c>0$ is chosen so small that it is supported in $\Sigma_-(Mh^{-\beta},\,-3/4,\,c)$. We write a_- for the pseudodifferential operator $a_-(x_a,\,hD_a\,;\,\lambda,\,h)$. Then we have $g_{2\infty} = a_-\varphi_a$ by definition and hence it follows from Proposition 5.4(iv) that $T_\infty = O(h^L)$ for any $L \gg 1$. On the other hand, the term T_0 can be written as

$$T_0 = \mathrm{Im}(R(\lambda + i0\,;\,H_a(h)) f_1 e_a,\; g_{20} e_a)_0.$$

Both the functions f_1 and g_{20} satisfy the assumption of Lemma 5.3. Hence we use this lemma with $I_a = 0$ to obtain that

$$T_0 = h^{-1}\int_0^\infty \mathrm{Re}(G_t^0(\lambda,\,h) f_1,\,g_{20})_a dt + o(h^{1-2\gamma}) \quad \text{in } \mathcal{D}'((0,\,\infty)),$$

where $G_t^0(\lambda,\,h) f_1 = f_1(x_a - t\eta_a\omega_a\,;\,\lambda,\,h)$. The integration on the right side is actually taken only over a finite interval $(0,\,Ch^{-\beta})$ for some $C \gg 1$, because $G_t^0 f_1$ vanishes on the support of g_{20} for $t > Ch^{-\beta}$. Since $\langle x_a\rangle^{-1} \sim h^{-\beta}$ is comparable on the support of f_1 and g_{20}, we have by Lemmas 4.2 and 4.3 that

$$\begin{aligned}&\|\mathrm{Re}\, f_1\|_a = O(h^{\beta(\rho-3/2)}),\ \|\mathrm{Im}\, f_1\|_a = O(h^{1-\beta/2}),\\ &\|\mathrm{Re}\, g_{20}\|_a = O(h^{\beta(\rho-3/2)}),\ \|\mathrm{Im}\, g_{20}\|_a = O(h^{-1+\beta(2\rho-5/2)}).\end{aligned}$$

Hence T_0 is estimated as $T_0 = o(h^{1-2\gamma})$ in $\mathcal{D}'((0,\,\infty))$. This yields the desired bound for Q_{12}. A similar argument applies to Q_{21} also.

Summing up the results obtained here, we can conclude that all the terms Q_0 and Q_{jk} except for Q_{11} in (4.14) obey the bound $o(h^{1-2\gamma})$ in $\mathcal{D}'((0,\,\infty))$ as a function of λ uniformly in $\omega_a \in S_a$ and hence it follows from relation (4.1) that

$$\sigma_a(\lambda,\,\omega\,;\,h) = 2n_a^{-1}\eta_a(\lambda)^{-1}h^{-1}Q_{11}(\lambda,\,h) + o(h^{-2\gamma}) \tag{6.5}$$

in $\mathcal{D}'((0,\,\infty)\times S_a)$.

7. Calculation of leading term

In this section, we complete the proof of the main theorem by calculating the leading term arising from Q_{11}.

(i) According to Lemma 4.2, f_1 and g_1 satisfy the assumption of Lemma 5.3. Hence

$$Q_{11}=h^{-1}\int_0^{\infty}\mathrm{Re}(G_t(\lambda,\ h)f_1,\ g_1)_a dt+o(h^{1-2\gamma})\quad \text{in } \mathcal{D}'((0,\ \infty)),$$

where the integration is actually taken only over a bounded interval $(0,\ Ch^{-\beta})$ for some $C\gg 1$. Recall definitions (4.10) and (4.13) of u_1 and v_1. We decompose $u_1=f_1e_a$ and $v_1=g_1e_a$ into $u_1=u_++u_-$ and $v_1=v_++v_-$, where

$$u_+=p_0[T_a(h),\ p_+]e_a,\ u_-=p_0[T_a(h),\ p_-]\theta_-e_a,$$
$$v_+=q_0[T_a(h),\ q_+]\theta_+e_a,\ v_-=q_0[T_a(h),\ q_-]e_a.$$

We further define $f_\pm$ and $g_\pm$ through the relations $u_\pm=f_\pm e_a$ and $v_\pm=g_\pm e_a$. Since f_+ is supported in $\Gamma_+(16h^{-\beta},\ -1/2)$, we can easily see from (5.3) that $G_t(\lambda,\ h)f_+$ vanishes on $\mathit{supp}\ g_1\subset B(4h^{-\beta})$ for $t>0$. Thus we obtain

$$Q_{11}=I_+(\lambda,\ h)+I_-(\lambda,\ h)+o(h^{1-2\gamma})\quad \text{in } \mathcal{D}'((0,\ \infty)),$$

where

$$I_\pm=h^{-1}\int_0^{\infty}\mathrm{Re}(G_t(\lambda,\ h)f_-,\ g_\pm)_a dt.$$

(ii) We analyze $I_\pm$ defined above. Write ∂_ω for $\langle\omega_a,\ \nabla_a\rangle$. Then f_- and g_- take the forms

$$f_-=-i\eta_a hp_0(\partial_\omega p_-)\theta_-+r,\ g_-=-i\eta_a hq_0(\partial_\omega q_-)+r_-,$$

where the remainder terms obey the bound $\|r\|_a+\|r_-\|_a=O(h^{2+\beta/2})$ by Lemma 4.1. We can choose $\delta>0$ so small that these remainder terms do not make any contribution to the leading term. If we neglect such a contribution from the remainder terms, then $G_t(\lambda,\ h)f_-$ takes the form

$$G_t(\lambda,\ h)f_-\sim -i\eta_a h(p_0(\partial_\omega p_-))(x_a-t\eta_a\omega_a)\theta_-(x_a\,;\ \lambda,\ h)$$

by (5.3). Since Re θ_- behaves like

$$\mathrm{Re}\ \theta_-(x_a\,;\ \lambda,\ h)=1+O(h^{-2})\langle x_a\rangle^{-2(\rho-1)}=1+O(h^{2\delta})$$

for $x_a\in \mathit{supp}\ g_-\subset\Gamma_-(h^{-\beta},\ -7/8)$, it follows that

$$I_-(\lambda,\ h)=L_-(\lambda,\ h)+o(h^{1-2\gamma}) \tag{7.1}$$

where

$$L_-=\eta_a^2 h\int_0^{\infty}\int(p_0\partial_\omega p_-)(x_a-t\eta_a\omega_a)(q_0\partial_\omega q_-)(x_a)dx_a dt.$$

A smilar argument applies to I_+ and we obtain

$$I_+(\lambda,\ h)=L_+(\lambda,\ h)+o(h^{1-2\gamma}), \tag{7.2}$$

where

$$L_+=\eta_a^2 h\int_0^\infty\int(p_0\partial_\omega p_-)(x_a-t\eta_a\omega_a)(q_0\partial_\omega q_+)(x_a)\Theta(x_a)dx_a dt$$

with

$$\Theta(x_a\,;\,\lambda,\,h)=\cos\left\{h^{-1}\int I_{a0}(x_a+s\eta_a\omega_a)ds\right\}.$$

We now write $x_a\in X_a$ as

$$x_a=y_a+z_a\eta_a\omega_a,\ y_a\in\Pi_\omega,\ z_a\in R, \tag{7.3}$$

where Π_ω again denotes the impact plane orthogonal to direction ω_a (see (2.5)). As is easily seen, Θ defined above depends only on $y_a\in\Pi_\omega$

$$\Theta=\Theta(y_a\,;\,\lambda,\,h)=\cos\left\{h^{-1}\int I_{a0}(y_a+z_a\eta_a\omega_a)dz_a\right\}$$

and behaves like $\Theta=1+h^{-2}O(|y_a|^{-2(\rho-1)})$ as $|y_a|\to\infty$ uniformly in h, so that we have

$$\int_{|y_a|>mh^{-\beta}}(1-\Theta(y_a))dy_a=O(h^{-2+2\beta(\rho-2)}). \tag{7.4}$$

(iii) The argument below uses the relation between the supports of cut-off functions p_0, p_-, q_0 and q_+ defined by (4.9) and (4.12). We here recall that:

(1) *supp* $p_0\subset B(64h^{-\beta})$ and $p_0=1$ on $B(32h^{-\beta})$.

(2) *supp* $p_-\subset\Gamma_-(16h^{-\beta},\,0)$ and $p_-=1$ on $\Gamma_-(32h^{-\beta},\,-1/2)$.

(3) *supp* $q_0\subset B(4h^{-\beta})$ and $q_0=1$ on $B(2h^{-\beta})$.

(4) *supp* $q_+\subset\Gamma_+(h^{-\beta},\,-15/16)$ and $q_+=1$ on $\Gamma_+(2h^{-\beta},\,-7/8)$.

We now assert that

$$\int_0^\infty\int(p_0\partial_\omega p_-)(x_a-t\eta_a\omega_a)(q_0\partial_\omega(q_-+q_+))(x_a)dx_a dt=0. \tag{7.5}$$

To see this, we first note the relation

$$q_0\partial_\omega(q_-+q_+)=-q_0\partial_\omega q=-\partial_\omega q, \tag{7.6}$$

where $q=1-q_+-q_-$ has support in $B(2h^{-\beta})$. If x_a represented by (7.3) satisfies $|y_a|\geq 2h^{-\beta}$, then $x_a\notin$ *supp* q and hence $q_0\partial_\omega(q_-+q_+)=0$. On the other hand, if x_a satisfies $|y_a|<2h^{-\beta}$, then $p_0\partial_\omega p_-=\partial_\omega p_-$ at such a point x_a and hence

$$\eta_a(p_0\partial_\omega p_-)(x_a-t\eta_a\omega_a)=-\partial_t p_-(x_a-t\eta_a\omega_a),$$

so that we have

$$\eta_a\int_0^\infty(p_0\partial_\omega p_-)(x_a-t\eta_a\omega_a)dt=-1 \tag{7.7}$$

for $x_a \notin supp\ p_-$ with $|y_a| < 2h^{-\beta}$ and hence, in particular, for $x_a \in supp\ q$. This, together with relation (7.6), proves (7.5). Therefore the sum of the two leading terms in (7.1) and (7.2) equals

$$L_- + L_+ = \eta_a^2 h \int_0^\infty \int (p_0 \partial_\omega p_-)(x_a - t\eta_a \omega_a)(q_0 \partial_\omega q_+)(x_a)(\Theta(y_a) - 1)\, dx_a dt.$$

If $x_a \in supp\ \partial_\omega q_+$ with $|y_a| < mh^{-\beta}$, $m > 0$ being small enough, then it follows that $x_a \notin supp\ p_-$ and $q_0 \partial_\omega q_+ = \partial_\omega q_+$ at such a point x_a. Hence we have

$$\eta_a \int (q_0 \partial_\omega q_+)(x_a)\, dz_a = 1$$

for $|y_a| < mh^{-\beta}$, $0 < m \ll 1$. By (7.3), we can write $dx_a = \eta_a dy_a dz_a$. If (7.4) and (7. 7) are further taken into account, then we see that the sum behaves like

$$L_- + L_+ = \eta_a h \int_{\Pi_\omega} (1 - \Theta(y_a\,;\ \lambda,\ \omega_a,\ h))\, dy_a + o(h^{1-2\gamma}).$$

This yields the leading term in the asymptotic behavior of Q_{11}

$$Q_{11} = 2^{-1} n_a \eta_a h L_0(\lambda,\ \omega_a\,;\ h) + o(h^{1-2\gamma}) \quad \text{in } \mathcal{D}'((0,\ \infty)),$$

where

$$L_0 = 4 n_a^{-1} \int_{\Pi_\omega} \sin^2 \left\{ (2h)^{-1} \int I_{a0}(y_a + z_a \eta_a \omega_a)\, dz_a \right\} dy_a.$$

By use of (2.4) and (2.5), L_0 can be put into the form as in the theorem. Thus we can obtain from (6.5) that

$$\sigma_a(\lambda,\ \omega_a\,;\ h) = L_0(\lambda,\ \omega_a\,;\ h) + o(h^{-2\gamma}) \quad \text{in } \mathcal{D}'((0,\ \infty) \times S_a)$$

and the proof of the main theorem is now completed.

8. Proof of Lemmas 5.1~5.3

In this section, we prove the three lemmas (Lemmas 5.1~5.3) which remain as unproved.

Proof of Lemma 5.1. The lemma can be easily proved. The proof uses the timedependent representation formula

$$R(\lambda + i0\,;\ H(h)) = ih^{-1} \int_0^\infty \exp(ih^{-1} t\lambda) \exp(-ih^{-1} tH(h))\, dt.$$

More precisely, we have to write

$$R(\lambda + i0\,;\ H(h)) = ih^{-1} \lim_{\kappa \downarrow 0} \int_0^\infty \exp(ih^{-1} t(\lambda + i\kappa)) \exp(-ih^{-1} tH(h))\, dt.$$

However we proceed with this formal representation for notational brevity. The rigorous justification can be easily done.

Let $F(\lambda)\in C_0^\infty((0, \infty))$. By the time-dependent representation formula above, the integral J can be written as $J=ih^{-1}I$, where

$$I=\int_0^\infty\int_0^\infty F(\lambda)\exp(ih^{-1}t\lambda)(\exp(-ih^{-1}tH(h))u,\ v)_0 dtd\lambda.$$

To prove the lemma, it suffices to show that $I=O(h^{\sigma_1+\sigma_2-\beta})$. Take τ as $\tau=h^{-\beta}$ and divide the intergal I into two parts

$$I=\int_0^\infty\left\{\int_0^\tau+\int_\tau^\infty\right\}\cdots dtd\lambda.$$

We denote by I_1 and I_2 the first and second integrals, respectively. By assumption, we can immediately obtain $I_1=O(h^{\sigma_1+\sigma_2-\beta})$.

Next we consider the integral I_2. We write $u=\varphi_0(|x_a|/t)u+\varphi_\infty(|x_a|/t)u$; similarly for v. Then I_2 is split into four integrals. Denote by I_2^2, I_2^3 and I_2^4 the integrals associated with decompositions $(\varphi_\infty-\varphi_0)$, $(\varphi_0-\varphi_\infty)$ and $(\varphi_\infty-\varphi_\infty)$, respectively. These integrals can be easily estimated by repeated use of the relation

$$\|\varphi_\infty(|x_a|/t)u\|_0=O(t^{-\nu})\|\langle x_a\rangle^\nu u\|_0=O(t^{-\nu})O(h^{\sigma_1-\beta\nu}),$$

which follows from assumption. Since $\nu>1$, we have $I_2^2=O(h^{\sigma_1+\sigma_2-\beta})$. Similarly I_2^3 and I_2^4 are shown to obey the same bound as above. The last integral I_2^1 associated with decomposition $(\varphi_0-\varphi_0)$ is estimated with aid of partial integration in λ. Making use of the relation

$$\exp(ih^{-1}t\lambda)=-iht^{-1}\partial_\lambda\exp(ih^{-1}t\lambda),$$

we repeat integration by parts in λ twice to obtain that I_2^1 is majorized by a linear sum of such terms as

$$\int_\tau^\infty t^{-2}\|\varphi_0(|x_a|/t)(h\partial_\lambda)^j u\|_0\|\varphi_0(|x_a|/t)(h\partial_\lambda)^k v\|_0 dt$$

with $0\leq j+k\leq 2$. We may assume that $\nu<2$. Then we have by assumption that

$$\|\varphi_0(|x_a|/t)(h\partial_\lambda)^2 u\|_0=t^{2-\nu}O(h^{\sigma_1-\beta\nu});$$

similarly for v with σ_2. This implies that I_2^1 also obeys the bound $I_2^1=O(h^{\sigma_1+\sigma_2-\beta})$ and the proof of the lemma is complete. $\square$

Proof of Lemma 5.2. We prove (i) only. A similar argument applies to (ii) also. Let $\sigma=\beta(2\rho-5/2)-1$ be as in the lemma. We again write $R_{+a}(\lambda, h)$ for the resolvent $R(\lambda+i0;\ H_a(h))$. As is easily seen, the remainder term w in the lemma is given as $w=-I_aJ_aR_{+a}(\lambda, h)f\varphi_a$. Therefore, to complete the proof, it suffices to show that $w_0=w_0(x_a;\ \lambda, h)$ defined as $w_0=\langle x_a\rangle^{-\rho}R_{+a}(\lambda, h)f\varphi_a$ satisfies the estimate

$$\|\langle x_a\rangle^{\nu} w_0\|_a = O(h^{\sigma-\beta\nu}),\ \|(h\partial_\lambda)w_0\|_a = O(h^{\sigma-\beta}),$$
$$\|\langle x_a\rangle^{\nu-2}(h\partial_\lambda)^2 w_0\|_a = O(h^{\sigma-\beta\nu}), \tag{8.1}$$

for any ν, $0\le\nu<\rho-1$, where $\|\cdot\|_a$ again denotes the L^2 norm in $L^2(X_a)$.

By assumption, f is supported in $\Gamma_+(Mh^{-\beta}, d)$ and satisfies the estimates $|\partial_a^\alpha \partial_\lambda^k f|\le C_{\alpha k}\langle x_a\rangle^{-\rho-|\alpha|}$. We take $M_1<M$ and $d_1<d$, and define the symbol $a_+(x_a, \xi_a; \lambda, h)\in S_{-\rho}$ as

$$a_+(x_a, \xi_a; \lambda, h)=f(x_a; \lambda, h)\varphi_0(2|\xi_a-\eta_a\omega_a|/c),$$

where $c>0$ is chosen so small that the symbol is supported in $\Sigma_+(M_1h^{-\beta}, d_1, c)$. We also write a_+ for the pseudodifferential operator $a_+(x_a, hD_a; \lambda, h)$. Then we have $f\varphi_a=a_+\varphi_a$. By a simple calculus of pseudodifferential operators, we see that there exists $b_0\in OPS_0$ uniformly in h and $\lambda\in\Lambda$ such that

$$f\varphi_a=b_0f_0\varphi_a+r_0$$

with $f=f_0$, where the remainder term $r_0=r_0(x_a; \lambda, h)$ obeys the bound

$$\|\langle x_a\rangle^L r\|_a=O(h^L)\ \text{ for any } L\gg1. \tag{8.2}$$

We may assume that the symbol of b_0 is still supported in $\Sigma_+(M_1h^{-\beta}, d_1, c)$. Similarly we have

$$(h\partial_\lambda)^k f\varphi_a=b_kf_k\varphi_a+r_k,\ 1\le k\le2,$$

with remainder term r_k obeying the bound (8.2), where b_k is of class OPS_0 with symbol supported in $\Sigma_+(M_1h^{-\beta}, d_1, c)$ and $f_k=f_k(x_a; \lambda, h)$ has support in $\Gamma_+(Mh^{-\beta}, d)$ and satisfies the estimate $|f_k|\le C\langle x_a\rangle^{-\rho+k}$.

The remainder terms r_k, $0\le k\le2$, are all negligible. In fact, it follows from Proposition 5.4(i) that

$$\|\langle x_a\rangle^{\nu-\rho}R_{+a}(\lambda, h)r_0\|_a=O(h^L),\ \|\langle x_a\rangle^{-\rho}\{(h\partial_\lambda)^{1-k}R_{+a}(\lambda, h)\}r_k\|_a=O(h^L)$$

for k, $0\le k\le1$, and

$$\|\langle x_a\rangle^{\nu-2-\rho}\{(h\partial_\lambda)^{2-k}R_{+a}(\lambda, h)\}r_k\|_a=O(h^L)$$

for k, $0\le k\le2$. In particular, the third estimate above is obtained, if we note that $\rho+2-\nu>5/2$.

We now set $v_k=b_kf_k\varphi_a$, $0\le k\le2$. Let $\psi_0=\varphi_0(h^\beta|x_a|/m)$ and $\psi_\infty=1-\psi_0$. We take m, $m\ll M_1$, small enough. Then it follows from Proposition 5.4(iv) that

$$\|\psi_0R_{+a}(\lambda, h)v_0\|_a=O(h^L),\ \|\psi_0\{(h\partial_\lambda)^{1-k}R_{+a}(\lambda, h)\}v_k\|_a=O(h^L)$$

for k, $0\le k\le1$, and

$$\|\psi_0\{(h\partial_\lambda)^{2-k}R_{+a}(\lambda, h)\}v_k\|_a=O(h^L)$$

for k, $0\le k\le 2$. The terms cut-off by ψ_∞ are evaluated by making repeated use of Proposition 5.4(ii). We apply this proposition with $\mu=0$ to obtain that

$$\|\psi_\infty\langle x_a\rangle^{\nu-\rho}R_{+a}(\lambda,\ h)v_0\|_a=O(h^{\sigma-\beta\nu}).$$

If we further use the same proposition with $\mu=-k$, $0\le k\le 1$, then it follows that

$$\|\psi_\infty\langle x_a\rangle^{-\rho}\{(h\partial_\lambda)^{1-k}R_{+a}(\lambda,\ h)\}v_k\|_a=O(h^{\sigma-\beta}).$$

Similarly we have

$$\|\psi_\infty\langle x_a\rangle^{\nu-2-\rho}\{(h\partial_\lambda)^{2-k}R_{+a}(\lambda,\ h)\}v_k\|_a=O(h^{\sigma-\beta\nu}).$$

We combine these estimates to obtain (8.1) and the proof of the lemma is complete. □

Proof of Lemma 5.3. The proof again uses the time-dependent representation formula for resolvents. Let $F(\lambda)\in C_0^\infty((0,\ \infty))$. We consider the integral

$$I=\int_0^\infty\int_0^\infty F(\lambda)\exp(ih^{-1}t\lambda)(\exp(-ih^{-1}tH(h))fe_a,\ ge_a)_0dtd\lambda.$$

To prove the lemma, it suffices to show that

$$I=\int_0^\infty\int_0^\infty F(\lambda)(G_t(\lambda,\ h)f,\ g)_a dtd\lambda+o(h^{2-2\gamma}). \tag{8.3}$$

As stated previously, the t-integration above is actually taken only over a finite interval $(0,\ Ch^{-\beta})$ with some $C\gg 1$, because $G_t(\lambda,\ h)f$ vanishes on the support of g for $t>Ch^{-\beta}$. Let $c=(1+4\delta)\gamma>\beta$ for the same $\delta>0$ as in (4.8). We set $\tau=h^{-c}$ and divide the integral I into two parts

$$I=I_1+I_2=\int_0^\infty\left\{\int_0^\tau+\int_\tau^\infty\right\}\cdots dtd\lambda.$$

By partial integration in λ, the integral I_2 over $(\tau,\ \infty)$ is majorized by a linear sum of such terms as

$$\int_\tau^\infty t^{-2}\|(h\partial_\lambda)^j(fe_a)\|_0\|(h\partial_\lambda)^k(ge_a)\|_0dt,\ 0\le j+k\le 2.$$

By assumption, we have

$$\|(h\partial_\lambda)^j(fe_a)\|_0+\|(h\partial_\lambda)^j(ge_a)\|_0=O(h^{1-\beta(j+1/2)}).$$

Hence I_2 is shown to behave like $I_2=O(h^{2+c-3\beta})=o(h^{2-2\gamma})$ by choice of c.

To control the integral I_1 over $(0,\ \tau)$, we represent this as a sum of two integrals by decomposing f into two terms with small and large imapact parameters. We set $\varkappa=(1-5\delta)\gamma$ for the same $\delta>0$ as in (4.8) and decompose f into

$$f=f_s+f_l=\varphi_0(h^\kappa|y_a|)f+\varphi_\infty(h^\kappa|y_a|)f.$$

According to this decomposition, I_1 is split into a sum of two integrals. Since $\|f_s e_\alpha\|_0 = O(h^{1+\beta/2-\kappa})$, the integral I_{1s} with cut-off φ_0 is evaluated as $I_{1s} = O(h^{2-c-\kappa}) = o(h^{2-2\gamma})$ by choice of c and $\varkappa$. Similarly we have

$$\int_0^\infty \int_0^\tau F(\lambda)(G_t(\lambda, h) f_s, g)_\alpha dt d\lambda = o(h^{2-2\gamma}). \tag{8.4}$$

The leading term comes from the integral I_{1l} associated with cut-off φ_∞. To see this, we construct an approximate representation for the solution

$$v(t) = v(t\,;\,\lambda, h) = \exp(ih^{-1}t\lambda)\exp(-ih^{-1}tH(h)) f_l e_\alpha$$

to the equation

$$ih\partial_t v = (H(h) - \lambda)v, \quad v|_{t=0} = f_l e_\alpha = J_\alpha f_l \varphi_\alpha.$$

As an approximate solution to this equation, we define

$$u(t) = u(t\,;\,\lambda, h) = J_\alpha \varphi_\alpha G_t(\lambda, h) f_l = J_\alpha \varphi_\alpha \exp(-ih^{-1}tA(\lambda, h)) f_l.$$

Recall the definition (5.2) of $A(\lambda, h)$. Since the relation

$$J_\alpha \varphi_\alpha A(\lambda, h) - (H(h) - \lambda) J_\alpha \varphi_\alpha = -J_\alpha \varphi_\alpha T_\alpha(h) - (I_\alpha - I_{\alpha 0}) J_\alpha \varphi_\alpha$$

holds as an operator from $L^2(X_a)$ into $L^2(X)$, we see that $u(t)$ solves the equation

$$ih\partial_t u = (H(h) - \lambda)u - r_1(t) - r_2(t),$$

where $r_j(t) = r_j(t\,;\,\lambda, h)$, $1 \le j \le 2$, are given as

$$r_1(t) = J_\alpha \varphi_\alpha (T_\alpha(h) G_t(\lambda, h) f_l), \quad r_2(t) = (I_\alpha - I_{\alpha 0}) J_\alpha \varphi_\alpha G_t(\lambda, h) f_l.$$

Hence the Duhamel principle yields that

$$v(t) = u(t) - ih^{-1} \int_0^t \exp(-ih^{-1}(t-s)(H(h) - \lambda))(r_1(s) + r_2(s)) ds.$$

Both the remainder terms r_1 and r_2 have support in $\{y_a : h^{-\kappa} < |y_a| < Ch^{-\beta}\}$ for some $C > 0$ as a function of $y_a \in \Pi_\omega$, so that $h \le C\langle y_a \rangle^{-(\rho-1)/2}$ on their supports, provided that $\delta > 0$ is chosen small enough. By the assumption of the lemma, we have

$$|T_\alpha(h) G_t f_l| \le Ch\langle y_a \rangle^{-\rho-1} \langle y_a + (z_a - t)\eta_a \omega_a \rangle^{-1}$$

and hence $r_1(t)$ obeys the bound

$$\|r_1(t)\|_0^2 = O(h^2) \int_{|y_a| > h^{-\kappa}} |y_a|^{-2\rho-3} dy_a = O(h^{2+\kappa(2\rho+1)})$$

uniformly in t, $0 < t < \tau$. A similar estimate remains true for $r_2(t)$ also. Thus it follows that

$$\int_0^\tau\int_0^t(\|r_1(s)\|_0+\|r_2(s)\|_0)\|ge_a\|_0dsdt=O(h^\mu)O(h^{3-2\gamma}),$$

where $\mu=1+\varkappa(\rho+1/2)+1-\beta/2-2c+2\gamma-3=\gamma(1+O(\delta))$. We can choose $\delta>0$ so small that $\mu>0$ and hence

$$I_{1l}=\int_0^\infty\int_0^\tau F(\lambda)(G_t(\lambda,\ h)f_l,\ g)_a dtd\lambda+o(h^{2-2\gamma}).$$

This, together with (8.4), proves (8.3), because $(G_t(\lambda,\ h)f,\ g)_a=0$ for $t>\tau$ and hence the proof is complete. $\square$

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Hiroshi T. Ito
Department of Mathematics
Kyoto University
Kyoto 606, Japan

Hideo Tamura
Department of Mathematics
Ibaraki University
Mito Ibaraki 310, Japan