Title	Dade's conjecture for tame blocks
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Citation	Osaka Journal of Mathematics. 1994, 31(4), p. 747-772
Version Type	VoR
URL	https://doi.org/10.18910/4165
rights	
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Uno, K.
Osaka J. Math.
31 (1994), 747–772

DADE'S CONJECTURE FOR TAME BLOCKS

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(Received April 12, 1993)

0. Introduction

Let G be a finite group, B a p-block of G, where p is a prime. In [5], Dade conjectured that the alternating sum of the numbers of irreducible characters of certain heights in some blocks of subgroups of G related to B vanishes. (See Section 1 below.) Moreover, he showed that the conjecture holds for blocks with cyclic defect groups and for any blocks of the first Janko group and the smallest Mathieu group. (See Sections 9, 10 and 11 of [5].) The cyclic defect group case can be handled since the structure of such blocks is well known by Dade's work. So, the answer to the conjecture in this case is completely due to him. On the other hand, by virtue of [4], [8] and [6], the structure of tame blocks, that is, 2-blocks whose defect groups are dihedral, quaternion or quasidihedral, is also well known. Thus, one could expect that the conjecture can also be proved in these cases. In fact, the purpose of the present paper is to show that one form of Dade's conjectures, whose concern is extended to the number of ordinary irreducible characters invariant under the action of given automorphisms, holds for tame blocks. (See Section 1.) Thus, for example, the principal 2-block of the smallest Mathieu group, which is treated concretely in Section 11 of [5], is just an example of our case.

Notations and terminologies are standard. See for example [7] and [2]. For any fixed p-block B of a finite group G, and any subgroup H of G, we denote by $\mathrm{Bl}(H, B)$ the set of those p-blocks b of H which satisfy $b^G = B$. The sets of ordinary irreducible characters and Brauer irreducible characters in B are denoted by $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$, respectively. The cardinalities of $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$ are denoted by $k(B)$ and $l(B)$ as usual. For $\chi \in \mathrm{Irr}(B)$, we denote by $d(\chi)$ the biggest integer m such that p^m divides $|G|/\chi(1)$. Thus the sum of $d(\chi)$ and the height of χ gives the defect $d(B)$ of B. In this paper, a p-chain means a chain C: $P_0 < P_1 < \cdots < P_n$ of p-subgroups of G with $P_0 = O_p(G)$. The above n is called the lenth of C. If all P_i's are elementary abelian, then it is called an elementary chain.

This paper is organized as follows. After stating Dade's conjecture in Section 1, we review several results on tame blocks in Section 2. Most of them are found in [4] or [8], or just easy exercises. Using the results in Section 2 and some local analysis, we determine, in Section 3, the related blocks which contribute the alternating sum. Section 4 is devoted to studying some actions of automorphisms on $\mathrm{Irr}(B)$ and on the column index set of the generalized decomposition matrix of B. Through Brauer's permutation lemma, one can see the relation between these two actions. This may give a device for counting the number of invariant elements in $\mathrm{Irr}(B)$ in terms of the action on local objects such as subsections. However, we must also consider the action on $\mathrm{IBr}(B)$ since irreducible Brauer characters appear as the column indices as well. This is analyzed by, with Erdmann's classification of tame algebras, looking at the actions on the ordinary decomposition matrices, on the Cartan matrices or on the stable Auslander-Reiten quivers of the modular block algebras. Eventually, the number of invariant ordinary irreducible characters is given completely in terms of the actions on local objects. These are done in Section 5. We complete the proof of our main result in Section 6.

ACKNOWLEDGEMENT. The first attempt to the subject in this paper was to prove that the ordinary conjecture (the simplest form) holds for tame blocks. It was Professor Dade who suggested the author that the extended conjecture for tame blocks should also be handled. Thus, the author would like to express his heartfelt gratitude to Professor Dade for his suggestion. The result of this paper was obtained during the stay at University of Essen. The author is also grateful to Professor Michler and the Institute for Experimental Mathematics, University of Essen for their hospitality, and to the Alexander von Humboldt Foundation for its financial support.

1. Alternating sums

Let G be a normal subgroup of a finite group E, and let B be a p-block of G. Here p is a prime. Assume that B is E-invariant. For any p-chain C of G, let $N_E(C)$ denote the intersection of the normalizers in E of the subgroups appearing in C. Then $N_G(C)=N_E(C)\cap G$ is a normal subgroup of $N_E(C)$ and we have

$$N_E(C)G/G\simeq N_E(C)/N_G(C).$$

For any subgroup F of E with $G\leq F$ and any integer d, we denote by

$$k(N_G(C),\ B,\ d,\ F)$$

the number of those irreducible characters ψ of $N_G(C)$ which lie in a block b of $N_G(C)$ with $b^G=B$, $d(\psi)=d$ and their inertia subgroups I in $N_E(C)$ satisfy $IG=F$. Notice that if F is not contained in $N_E(C)G$, then the above number is zero. Also, if two p-chains C and C' satisfy $N_E(C)=N_E(C')$, then the above numbers for C and C'coinside (for fixed B, F and d). Moreover, it is clear that this number is invariant under the action of G.

Take the family $\mathscr{E}$ of all the elementary p-chains in G. Let $\mathscr{E}/G$ denote a set of representatives of G-conjugacy classes of $\mathscr{E}$. Then the alternating sum

$$S=\sum_{C\in\mathscr{E}/G}(-1)^{|C|}k(N_G(C),\ B,\ d,\ F) \tag{1.1}$$

is well defined, where $|C|$ denotes the length of C. The extended conjecture can be stated as follows.

Conjecture 1.2. *If $O_p(G)=1$ and $d(B)>0$, then $S=0$ for all F and d.*

By (3.6) and Proposition 3.7 of [5], the family $\mathscr{E}$ can be replaced by some other natural families. As is mentioned in the introduction, our main theorem is the following.

Theorem. *Conjecture* 1.2 *holds for tame blocks.*

In the case of $G=E=F$, we use $k(N_G(C),\ B,\ d)$ instead of $k(N_G(C),\ B,\ d,\ F)$. Conjecture 1.2 in this case is called the ordinary conjecture in [5]. (See Section 6 of [5].)

We now make the following easy remark which may be helpful.

(1.3) *Let F, I and E' be subgroups of E such that $G\le F\le E'G$ and $E'\cap G\le I\le E'$. Then $IG=F$ if and only if $I=F\cap E'$.*

Proof. Notice first that $F=(F\cap E')G$. Conversely, suppose that $IG=F$. Then $I\le F\cap E'$. Let $g\in F\cap E'$. Then we may write $g=hh'$ for $h\in I$ and $h'\in G$. Now $h'=h^{-1}g$ lies in E'. Thus h' lies in $G\cap E'$ and hence in I. Hence g must lie in I. Therefore, we have $I=F\cap E'$. $\square$

Let C be a p-chain of G and F a subgroup of E with $G\le F$. Also, let $b\in \mathrm{Bl}(N_G(C),\ B)$ and $\phi\in\mathrm{Irr}(b)$. If $F\le N_E(C)G$, then apply (1.3) to $E'=N_E(C)$ and the inertia group I of ϕ in $N_E(C)$. Consequently, we may compute the number of characters in b whose inertia groups I satisfy $I=F\cap N_E(C)$. Recall also that if $F\leq N_E(C)G$ is not the case, then $k(N_G(C), B, d, F)=0$.

2. Preliminaries for tame blocks

In this section, we summarize results on tame blocks. All of them come from [4], [8] and [6]. First of all, we introduce several notations, which will be used throughout this paper.

Let G be a finite group and B a tame block of G. We fix a defect group D of B, and write $|D|=2^n$. Then D can be expressed as one of the following.

(I) dihedral ($n\geq 2$): $D=<x, y|x^{2^{n-1}}=y^2=1, yxy=x^{-1}>$

(II) quaternion ($n\geq 3$): $D=<x, y|x^{2^{n-2}}=y^2, y^4=1, y^{-1}xy=x^{-1}>$

(III) quasidihedral ($n\geq 4$): $D=<x, y|x^{2^{n-1}}=y^2=1, yxy=x^{-1+2^{n-2}}>$

Let z denote $x^{2^{n-2}}$. If $n\geq 3$, then z is the unique central involution of D. Also, let $\mathscr{S}$ be the set of D-conjugacy classes in $<x>\backslash<z>$. (If $n=2$, then $\mathscr{S}$ is empty.) Thus, for example, if D is dihedral or quaternion, then we can take $\{x^i|1\leq i\leq 2^{n-2}-1\}$ as a set of representatives of $\mathscr{S}$. However, certainly this set does not work in the case of quasidihedral! Moreover, we fix several automorphisms of D as follows. In case of D dihedral or quaternion, the automorphism σ sends y to xy and fixes x. If $n\geq 3$, the automorphisms τ and ι fix y and send x to x^5 and x^j, respectively, where j is -1 if D is dihedral or quaternion and $-1+2^{n-2}$ if D is quasidihedral.

The above three cases are further divided into, in total, ten situations. (See [4], [8] and p.152-p.155 of [6].) Here we remark that it can be described in terms of some local informations. Define subgroups Q_1 and Q_2 of D by:

$$Q_1=\begin{cases}<z, y>, \text{ if } D \text{ is dihedral or quasidihedral}\\ <x^{2^{n-2}}, y>, \text{ if } D \text{ is quaternion.}\end{cases}$$

$$Q_2=\begin{cases}<z, xy>, \text{ if } D \text{ is dihedral}\\ <x^{2^{n-2}}, xy>, \text{ if } D \text{ is quaternion or quasidihedral.}\end{cases}$$

Note that they are four-groups or quaternion groups of order eight. Moreover, if $n=2$ or if D is quaternion and $n=3$, then $Q_1=Q_2=D$. Let us fix a Sylow B-subpair (D, b) for a moment, and for each Q' of the above subgroups, take a block $b_{Q'}$ of $C_G(Q')$ such that

$(Q', b_{Q'})$ is contained in (D, b). Let $N(b_{Q'})$ denote the stabilizer of $b_{Q'}$ in $N_G(Q')$. Then we say that;

(2.1) (i) *B satisfies (aa) if $N(b_{Q_i})\backslash C_G(Q_i)$ has an element of order three for $i=1$, 2.*

(ii) *B satisfies (ab) if $N(b_{Q_i})\backslash C_G(Q_i)$ has an element of order three for exactly one i when D is dihedral or quaternion, and for only $i=1$ when D is quasidihedral.*

(iii) *When D is quasidihedral, B satisfies (ba) if $N(b_{Q_i})\backslash C_G(Q_i)$ has an element of order three for only $i=2$.*

(iv) *B satisfies (bb) if $N(b_{Q_i})\backslash C_G(Q_i)$ does not have an element of order three $(i=1, 2)$.*

It should be noticed that if $n=2$ or if D is quaternion and $n=3$, then $Q_1=Q_2$ and we are concerned with only (aa) and (bb) of the above. The above notation is the same as the one used in [4] and [8]. We usually write or refer to, for example, (IIaa) to indicate the case where D is quaternion and B satisfies (aa). Also, sometimes n must be restricted. For instance, (Iaa, 2) means the case (Iaa) with $n=2$ while (Iaa, ≥ 3) means the case (Iaa) with $n\geq 3$. Moreover, if we write (I), for instance, then it means that we are treating the cases (Iaa) (Iab) and (Ibb) simultaniously. The argument in this paper will be given in such a way that, as long as they are pararell in certain cases, we discuss just once indicating which cases are concerned.

Now we give representatives of G-conjugacy classes of B-subsections. In the following, we only give conjugacy classes of G since the associate block is uniquely determined. (See [1].) For the proof, see (4.A) of [4] and Proposition 2.10 of [8].

Lemma 2.2. (i) *Two elements in $<x>\backslash<z>$ are D-conjugate if and only if they are G-*conjugate.

(ii) *The following and representatives of $\mathscr{S}$ give representatives of G-conjugacy classes of B-subsections.*

$$\begin{array}{rl} & \textit{case} \\ (Iaa)(IIaa)(IIIaa): & z \\ (Iab)(IIab)(IIIba): & z,\ y \\ (Ibb)(IIbb)(IIIbb): & z,\ y,\ xy \\ (IIIab): & z,\ xy \end{array}$$

REMARK. In the cases of (Iab) and (IIab), one possibly has to take

xy instead of y. However, it is essentially the same as above and we consider only the above cases.

Now we give formulae of the number of irreducible characters in B of given height. See p.231 of [8].

Lemma 2.3. *In any case, $k(G, B, n)=4$, and if $n\neq 3$, we have $k(G, B, n-1)=2^{n-2}-1$. For all the other values of d, $k(G, B, d)=0$ except for the following cases.*

$$k(G, B, 2)=\begin{cases}1, & \text{if } (IIab)\ (IIbb,\ 3)\ (IIIaa) \text{ or } (IIIba)\\ 2, & \text{if } (IIaa,\ \geq 4)\\ 3, & \text{if } (IIaa,\ 3)\end{cases}$$

The number of irreducible Brauer characters in B is also known.

Lemma 2.4. *We have the following.*

$$l(B)=\begin{array}{r}1,\\ 2,\\ 3,\end{array}\begin{cases}\text{if } (Ibb)\ (IIbb) \text{ or } (IIIbb)\\ \text{if } (Iab)\ (IIab)\ (IIIab) \text{ or } (IIIba)\\ \text{if } (Iaa)\ (IIaa) \text{ or } (IIIaa)\end{cases}$$

Now, we give the results on automorphisms and centralizers of some subgroups, which will be used in the paper.

Lemma 2.5. (i) *If $(I, 2)$ or $(II, 3)$ is the case, then* $\mathrm{Out}(D)$ *is isomorphic to the symmetric group of degree three. Otherwise,* $\mathrm{Out}(D)$ *is an abelian 2-group, and in fact,* $\mathrm{Out}(D)=<\bar{\sigma}>\times<\bar{\tau}>$ *in cases of (I) and (II), and* $\mathrm{Out}(D)=<\bar{\tau}>$ *in case of (III), where the bars indicate the natural images in* $\mathrm{Out}(D)$.

(ii) *Suppose that $(I, 2)$ and $(II, 3)$ are not the case. Then the restriction to $<x>$ gives a homomorphism from* $\mathrm{Aut}(D)$ *to* $\mathrm{Aut}(<x>)$. *Moreover, it follows that* $\mathrm{Aut}(<x>)=<\tau'>\times<\iota'>$, *where τ' and ι' are the restrictions of τ and ι to $<x>$, respectively.*

Lemma 2.6. *Suppose that $n\geq 3$ and let u be a non-central involution of D. Then $C_D(u)=<z, u>$ which is an elementary abelian group of order four.*

Finally, we determine G-conjugacy classes of elementary 2-chains. (See Sect. 1.) As is remarked in [5, Lemma 6.9], for our purpose it suffices to consider only those in D. In view of Lemma 2.2, we have the following. For the proof, remark that any non-central involution u of D is G-conjugate to z if and only if $N_G(<z, u>)\backslash C_G(<z, u>)$ contains an element of order three. (See also p.152-p.155 of [6] and (2.1).)

Lemma 2.7. *Assume that* $O_2(G)=1$. *The following give representatives of G-conjugacy classes of elementary 2-chains whose final subgroups lie in D. (In each case, we omit the trivial chain* 1.)

(Iaa, ≥ 3*):* $1<<z>$, $1<<z><<z, u>$ *and* $1<<z, u>$, *where* u *is in* $\{y, xy\}$.

(Iab): Those in (Iaa, ≥ 3*) above and* $1<<y>$ *and* $1<<y><<z, y>$.

(Ibb, ≥ 3*): Those in (Iaa,* ≥ 3*) above and* $1<<u>$ *and* $1<<u><<z, u>$, *where* u *is in* $\{y, xy\}$.

(Iaa,2): $1<D$, $1<<x>$ *and* $1<<x><D$.

(Ibb,2): Those in (Iaa,2) above and $1<<u>$ *and* $1<<u><<D$, *where* u *is in* $\{y, xy\}$.

(II): $1<<z>$.

(IIIaa) and (IIIab): $1<<z>$, $1<<z><<z,y>$ *and* $1<<z,y>$.

(IIIba) and (IIIbb): Those in (IIIaa) and (IIIab) above and $1<<y>$ *and* $1<<y><<z, y>$.

REMARK. Notice that in any case, the number of the conjugacy classes of elementary 2-chains is even. We will put them into pairs so that two 2-chains in every pair have length of opposite parity and give the same number of characters which have to be taken into account.

3. Local blocks

We first consider the normalizer of $1 <<z>$ or $1<D$.

Proposition 3.1. *Let* $H=C_G(z)$ *if* $n\geq 3$ *and* $H=N_G(D)$ *if* $n=2$. *Then,* Bl(H, B) *consists of the unique block* B_1 *which has a defect group* D. *Moreover, the following hold.*

If B *satisfies (I,* ≥ 3*), then* B_1 *satisfies (Ibb).*

If B *satisfies (I, 2) or (II), then* B and B_1 *satisfy the same property.*

If B *satisfies (IIIaa) or (IIIba), then* B_1 *satisfies (IIIba).*

If B *satisfies (IIIab) or (IIIbb), then* B_1 *satisfies (IIIbb).*

Proof. The first statement is clear from standard block theory. Notice that, if $n\geq 3$, z is central in H. Thus if (I, ≥ 3) is the case, then z is not H-conjugate to y nor xy. Hence (Ibb) holds for B_1. If $n=2$, then the two cases are distinguished by the existance of an element of order three in $N_G(D)\backslash C_G(D)$. Hence the result holds. In cases of (II) and (III), those properties are determined by the existance of an element of order three in the normalizers of certain subgroups (see (2.1)) and H contains such normalizers. Moreover, z is not H-conjugate to y. Therefore, the conclusions on B_1 follow. (See also Lemma 3.1 of [8]. There is a misprint in its statement. (ab) must be (ba) there. See its proof.) □

The next two results concern some involutions.

Proposition 3.2. *Let u be an involution in D. If $n \geq 3$, suppose that u is not G-conjugate to z. Let $Q = <z, u>$ if $n \geq 3$ and $Q = D$ if $n = 2$, and let $H = C_G(u)$ and $N = H \cap N_G(Q)$. Then every block in* Bl(H, B) *or* Bl(N, B) *has defect group Q. In particular, the first main theorem of Brauer gives a bijection between* Bl(H, B) *and* Bl(N, B). *Moreover, those blocks satisfy* (*Ibb*).

Proof. Let b be a block in Bl(H, B), and let Q' be a defect group of b. Remark that u lies in $Q \cap Q'$ and that the centralizer of $u$in any G-conjugate of D containing Q' has order four. Thus since $b^G = B$, there is some g in G such that $(Q')^g = Q$ and thus $u^g \in Q$. If $u^g = u$, then$g \in H$ and Q is a defect group of b. So, assume that $u^g \neq u$. If $n \geq 3$, then we must have $u^g = uz$. Thus, there is some v in $N_D(Q)$ such that $u^{gv} = u$. If $n = 2$, then since u and u^g are G-conjugate, b satisfies (Iaa) and thus there is an element v of $N_G(D)$ such that $u^{gv} = u$. In either case, gv lies in H and we have $(Q')^{gv} = Q^v = Q$. Hence Q is a defect group of b. For a block b_1 in Bl(N, B), the block b_1^H lies in Bl(H, B) and Q is contained in a defect group of b_1. Hence Q must be the defect group of b_1. Finally, since u is central in H, the last statement holds. This completesthe proof. □

Remark. It is known that Bl(H, B), for H in the above proposition, consists of a single element.

Proposition 3.3. *Let u be a non-central involution in D, and let $H = N_G(<z, u>)$ and $N = H \cap C_G(z)$. (Note: $n \geq 3$.) Assume that u is not G-conjugate to z. Then, $H = N$.*

Proof. Let $g \in H$. Then, z^g lies in $<z, u>$. However, since both u and zu are not G-conjugate to z (note: u and zu are D-conjugate), we must have $z^g = z$. Hence $g \in N$. □

Remark. It is easy to check, without assuming that u is not G-conjugate to z, that a block lying in Bl(H, B) for the above H has a defect group isomorphic to the dihedral group of order eight.

4. The action of E

From now on we assume that G is a normal subgroup of a finite group E and B is a tame block of G fixed by E. First remark that the above yields $E=N_E(D)G$. Moreover, we have

$$\text{(4.1.a)} \quad E/C_E(D)G \cong N_E(D)/N_E(D)\cap C_E(D)G = N_E(D)/C_E(D)N_G(D).$$

Also, if $n\geq 3$, then $E=C_E(z)G$ and $N_E(D)C_E(D)C_G(z)=C_E(z)$, which implies

$$\text{(4.1.b)} \qquad C_E(z)/C_E(D)C_G(z) \cong N_E(D)/C_E(D)N_G(D).$$

In this section, we consider the E-actions on several sets. Recall that in view of Lemma 2.2, the set of all the conjugacy classes of G intersecting with $\langle x\rangle\backslash\langle z\rangle$ can be indentified with $\mathscr{S}$. If (I,2) and (II,3) are not the case, then every automorphism of D must send x to some odd power of x, and if (IIaa,3) is the case, then x is G-conjugate to y and to xy. Moreover, in case of (I,2), the set $\mathscr{S}$ is empty. Thus we remark that;

Lemma 4.2. *Unless (IIbb,3) is the case, the set of conjugacy classes in G intersecting with $\langle x\rangle\backslash\langle z\rangle$ is E-stable.*

In the rest of the paper, we identify $\mathscr{S}$ with the set of conjugacy classes in G intersecting with $\langle x\rangle\backslash\langle z\rangle$. Moreover, it will be identified with a certain subset of the column index set of the generalized decomposition matrix of B.

Now we show that E is naturally related to $\mathrm{Aut}(D)$ as follows.

Lemma 4.3. (i) *There is a natural homomorphism μ from E to K/K' for some subgroups K and K' of* $\mathrm{Aut}(D)$ *with $K\trianglerighteq K'\geq \mathrm{Inn}(D)$ such that* $\mathrm{Ker}\,\mu=C_E(D)G$.

(ii) *If (I,2) and (II,3) are not the case, then the above μ induces a homomorphism μ' from E* to $\mathrm{Aut}(\langle x\rangle)/\langle\iota'\rangle$ *with* $\mathrm{Ker}\,\mu'\geq C_E(D)G$.

Proof. (i) Consider the natural homomorphism from $N_E(D)/C_E(D)$ to $\mathrm{Aut}(D)$. Let K and K' be the images of $N_E(D)/C_E(D)$ and $C_E(D)N_G(D)/C_E(D)$, respectively, under this homomorphism. In view of (4.1.a), this gives a homomorphism μ from E to K/K' with $\mathrm{Ker}\,\mu=C_E(D)G$. Since

$$C_E(D)N_G(D)/C_E(D)\cong N_G(D)/C_G(D)\geq DC_G(D)/C_G(D)\cong D/Z(D),$$

it follows that $K' \geq \mathrm{Inn}(D)$.

(ii) Assume that (I,2) and (II,3) are not the case. Then the restriction to $<x>$ gives a natural homomorphism from $\mathrm{Aut}(D)$ to $\mathrm{Aut}(<x>)$. Let v be in $N_G(D)$. Then x^v is some power of x. However, in view of Lemma 2.2, x^v must be x or $\iota(x)$. Thus K' is mapped into $<\iota'>$ by the above homomorphism. Moreover, the element y gives the automorphism ι'. Hence $<\iota'>$ is exactly the image of K'. Therefore, the result follows. This completes the proof. $\square$

REMARK. By some local analysis one can also show that, if (I,2) and (II,3) are not the case, then the above $K' = \mathrm{Inn}(D)$. Thus, $E/C_E(D)G$ is isomorphic to a subgroup of $\mathrm{Out}(D)$.

In the rest of this section, we assume that $n \geq 4$ and study the relation between the E-actions on $\mathscr{S}$ and on the set of height one characters in B. First, we introduce some Galois actions on characters. Although this is, in fact, not absolutely necessary, it might help us to understand the situation.

Let L be the field extension of $\mathbf{Q}$ generated by a primitive $|G|_{2'}$-th root of unity over $\mathbf{Q}$. Let ε be 1 if D is dihedral or quaternion and -1 if D is quasidihedral. Moreover, let ζ be a primitive 2^{n-1}-th root of unity. Then it follows from (5.A) of [4] and Proposition 4.1 of [8] that all the values of irreducible characters in B lie in $L(\zeta+\varepsilon\zeta^{-1})$. In particular, the Galois group Γ of $L(\zeta+\varepsilon\zeta^{-1})$ over L acts on $\mathrm{Irr}(B)$. Let us describe Γ. Let Γ^* be the Galois group of $L(\zeta)$ over L. Then, it is isomorphic to the ring R of units of $\mathbf{Z}/2^{n-1}\mathbf{Z}$, and in fact, there is a natural isomorphism ρ from R to Γ^* such that

$$\zeta^{\rho(\overline{m})} = \zeta^m$$

for all odd m in $\mathbf{Z}$, where $\overline{m}$ means the image of m in R. Moreover, Γ is a factor group of Γ^* and the above ρ induces an isomorphism

$$\rho' \colon \; R/R' \to \Gamma.$$

Here, R' is the subgroup of R generated by $\overline{-1}$ if D is dihedral or quaternion, and by $\overline{-1+2^{n-2}}$ if D is quasidihedral. In particular, Γ is a cyclic group of order 2^{n-3}. Now Theorem 3 of [4] and Propositions 4.2 and 4.5 of [8] assert that;

Lemma 4.4. *Let $n \geq 4$. Then, all the height zero and height $n-2$ characters in B are Γ-invariant. Moreover, the set of height one characters in B has $n-2$ Γ-orbits $F_0, F_1, F_2, \cdots, F_{n-3}$ such that $|F_i| = 2^i$ for all i with*

$0 \leq i \leq n-3$.

On the other hand, $\mathrm{Aut}(<x>)/<\iota'>$ is also naturally isomorphic to R/R', and hence to Γ. Thus by Lemma 4.3 (ii), the composite of

$$E \xrightarrow{\mu'} \mathrm{Aut}(<x>)/<\iota'> \cong R/R' \xrightarrow{\rho'} \Gamma$$

gives a homomorphism

$$\nu: \quad E \to \Gamma$$

such that $\mathrm{Ker}\,\nu \geq C_E(D)G$. Now we have the following.

Lemma 4.5. *Suppose that $n \geq 4$. Then $\chi^{\alpha^{-1}} = \chi^{\nu(\alpha)}$ for all α in E and all height one irreducible characters χ in B.*

Proof. Let $\alpha \in E$, and χ a height one character in B. Let m be an integer with $\rho'(\overline{m}R') = \nu(\alpha)$, and j denote -1 if D is dihedral or quaternion and $-1+2^{n-2}$ if D is quasidihedral. Then for any integer k, we have $\chi^{\alpha^{-1}}(x^k) = \chi((x^k)^\alpha)$, which is equal to $\chi(x^{km})$ or $\chi(x^{kmj})$. Those are equal to $\chi^{\rho(\overline{m})}(x^k)$ and $\chi^{\rho(\overline{mj})}(x^k)$, respectively. Thus it follows that $\chi^{\alpha^{-1}}(x^k) = \chi^{\rho'(\overline{m}R')}(x^k) = \chi^{\nu(\alpha)}(x^k)$. Hence the entries of the generalized decomposition matrix for $\chi^{\alpha^{-1}}$ and $\chi^{\nu(\alpha)}$ coincide on the column corresponding to x^k. Then noticing the difference of these entries corresponding to height one characters, which can be found in (6C) of [4] and Proposition 4.6 of [8], it follows that $\chi^{\alpha^{-1}}$ and $\chi^{\nu(\alpha)}$ must be Γ-conjugate. (Namely, they lie in the same family F_r. See Lemma 4.4.) Thus as is shown in Lemma 4.3 of [8], the entries for $\chi^{\alpha^{-1}}$ and $\chi^{\nu(\alpha)}$ also coincide on the column indices not corresponding to any x^k. Therefore, they must be equal and this completes the proof. □

The above implies the following.

Corollary 4.6. *Suppose that $n \geq 4$. Let F be a subgroup of E with* $\mathrm{Ker}\,\nu \leq F$.

(i) *Write $|F/\mathrm{Ker}\,\nu| = 2^t$. Then $k(G,B,n-1,F) = 2^{n-3-t}$ if $E \neq F$ and $k(G,B,n-1,F) = 2^{n-2-t}-1$ otherwise.*

(ii) *The E-actions on $\mathcal{S}$ and on the set of height one characters in B are permutation isomorphic.*

Proof. (i) The subgroup $\mathrm{Ker}\,\nu$ acts trivially on the height one characters by Lemma 4.5. Moreover, since Γ is cyclic, $\nu(F)$ is the unique subgroup of Γ of order 2^t. On the other hand, if $\chi \in F_r$, then the inertia group of χ in Γ is the unique subgroup of order 2^{n-3-r}. Thus, if $E \neq F$, then $k(G,B,n-1,F)$ coincides with the number of characteres in F_{n-3-t}, which is 2^{n-3-t}. If $E=F$, then the characters in F_0, $F_1,\cdots$, F_{n-3-t} are exactly those height one characters which are E-invariant. Hence we get

$$k(G,B,n-1,E)=1+2+\cdots+2^{n-3-t}=2^{n-2-t}-1.$$

(ii) Since $\operatorname{Ker}\nu=\operatorname{Ker}\mu'$, where μ' is the same as in Lemma 4.3 (ii), $\operatorname{Ker}\nu$ acts trivially on $\mathscr{S}$, too. Recall again that $E/\operatorname{Ker}\nu$ is cyclic by Lemma 4.3 (ii). On the other hand, through the isomorphisms $\operatorname{Aut}(<x>)/<\iota'>\cong\Gamma\cong R/R'$, the R/R'-actions on $\mathscr{S}$ and on the set of height one characters are permutation isomorphic. (See Lemma 4.4 and the paragraph following it.) Since E-actions are determined by ν: $E\rightarrow\Gamma$, the result follows. $\square$

Now we turn to the E-action on the column index set of the generalized decomposition matrix of B. Recall that the columns are indexed by the G-conjugacy classes of (u,η)'s, where $u\in D$ and $\eta\in\operatorname{IBr}(b_u)$ for $b_u\in\operatorname{Bl}(C_G(u),B)$. Since B is E-invariant, E can also act on this index set. Notice also that for tame blocks, $\operatorname{IBr}(b_u)$ consists of a single element η_u if u is not 1 nor z. Thus, in view of Lemma 2.2, we can and will identify $\mathscr{S}$ with a certain subset of the column index set. Let $\mathscr{S}'$ denote the complement of $\mathscr{S}$ in the set of column indices of the generalized decomposition matrix. In general, $|\mathscr{S}'|=k(B)-2^{n-2}+1$ since $|\mathscr{S}|=2^{n-2}-1$. For convenience, we give representatives of $\mathscr{S}'$ below. Here and in the rest of this paper, η_i's and η'_i's denote elements of $\operatorname{IBr}(B)$ and $\operatorname{IBr}(B_1)$, respectively, where B_1 is the unique block of $C_G(z)$ with $B_1^G=B$.

$$\mathscr{S}'=\begin{cases}\{(1,\eta_1)(1,\eta_2)(1,\eta_3)(z,\eta'_1)\}, & \text{if (Iaa)}\\ \{(1,\eta_1)(1,\eta_2)(z,\eta'_1)(y,\eta_y)\}, & \text{if (Iab)}\\ \{(1,\eta_1)(z,\eta'_1)(y,\eta_y)(xy,\eta_{xy})\}, & \text{if (Ibb)(IIbb) or (IIIbb)}\\ \{(1,\eta_1)(1,\eta_2)(1,\eta_3)(z,\eta'_1)(z,\eta'_2)(z,\eta'_3)\}, & \text{if (IIaa)}\\ \{(1,\eta_1)(1,\eta_2)(z,\eta'_1)(z,\eta'_2)(y,\eta_y)\}, & \text{if (IIab) or (IIIba)}\\ \{(1,\eta_1)(1,\eta_2)(1,\eta_3)(z,\eta'_1)(z,\eta'_2)\}, & \text{if (IIIaa)}\\ \{(1,\eta_1)(1,\eta_2)(z,\eta'_1)(xy,\eta_{xy})\}, & \text{if (IIIab)}\end{cases}$$

The notation $\mathscr{S}'$ is also applied even when n is 2 or 3. Finally, notice that $C_E(D)G$ acts on $\mathscr{S}$ trivially.

The final result in this section is as follows.

Corollary 4.7. *Suppose that $n \geq 4$. Then the numbers of E-orbits in $\{\chi \in \mathrm{Irr}(B) | d(\chi) \neq n-1\}$ and in $\mathscr{S}'$ coincide.*

Proof. Remark that the set of height one characters in B and $\mathscr{S}$ are E-stable. Since Corollary 4.6 (ii) implies that the numbers of E-orbits in the set of height one characters and in $\mathscr{S}$ coincide, Brauer's permutation lemma yields the result. □

5. The action of E on irreducible characters

Now we consider the E-action on height zero or $n-2$ irreducible characters. In some cases E must fix height zero characters. The following gives which cases are such.

Proposition 5.1. *In the following cases the four height zero irreducible characters in B* are *E-invariant : (Iab), (IIab), (III).*

Proof. First consider the cases (IIab) and (IIIba). In these cases, $l(B)=2$, there is only one character of height $n-2$ and $|\mathscr{S}'|=5$. Then by the table of possible algebras of tame type in [6], the Cartan matrix must be one of the following.

$$\begin{pmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 8 & 4 \\ 4 & 2^{n-2}+2 \end{pmatrix}$$

Thus E must fix each element of $\mathrm{IBr}(B)$. Moreover, by the same reason, the column indices corresponding to (z, η_1') and (z, η_2') are E-invariant. (Note that the unique block B_1 in $\mathrm{Bl}(C_G(z), B)$ satisfies (IIab) or (IIIba) by Proposition 3.1.) Also, of course, the column index corresponding to y must be fixed by E. Hence the number of E-orbits in $\mathscr{S}'$ is five and therefore the result follows from Corollary 4.7.

Next we treat the cases (Iab), (IIIab) and (IIIbb). Note that in these cases, there are only characters of height zero and one in B and $|\mathscr{S}'|=4$. Again by Corollary 4.7, it suffices to show that E fixes each element in $\mathscr{S}'$. (Note: In case of (Iab,3), $k(G, B, 2)=1$ and $\mathscr{S}$ consists of a single element corresponding to x. Thus it suffices to prove that E fixes each element of $\mathscr{S}'$, in this case, too. See Lemma 4.2.) Moreover, in cases of (Iab) and (IIIbb), $E=C_E(z)G$ and the fact that y and z are not G-conjugate imply that y is not E-conjugate to z. Also, any automorphism of D can not send xy to any power of x. Thus looking at the elements in $\mathscr{S}'$ case by case, it then suffices to show that each irreducible Brauer character in B are E-invariant. This is clear in case of (IIIbb) since $l(B)=1$. In the case of (Iab), it follows from the table of possible algebras of dihedral type in [6], the Cartan matrix must be

$$\begin{pmatrix} 2^n & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 4 & 2 \\ 2 & 2^{n-2}+1 \end{pmatrix}.$$

Hence the two characters in $\mathrm{IBr}(B)$ are E-invariant. In the case of (IIab), there is one more possibility for the Cartan matrix, namely:

$$\begin{pmatrix} 2^{n-2}+1 & 2^{n-2}-1 \\ 2^{n-2}-1 & 2^{n-2}+1 \end{pmatrix}$$

(However, as far as the author knows, no example of a block having this Cartan matrix is known at the present.) In this case, $\mathscr{S}'$ has four elements corresponding to $(1, \eta_1)$, $(1, \eta_2)$, (z, η'_1), (xy, η_{xy}) with $\mathrm{IBr}(B)=\{\eta_1, \eta_2\}$, $\mathrm{IBr}(B_1)=\{\eta'_1\}$ and $\mathrm{IBr}(b_{xy})=\{\eta_{xy}\}$. Thus the number of E-orbits in $\mathscr{S}'$ is at least three and we can conclude by Corollary 4.7 that at least two of height zero characters are E-invariant. On the other hand, if the above is the Cartan matrix, then each height zero character is modularly irreducible. (See the table in [6].) Hence at least one irreducible Brauer character in B is E-invariant. However since $l(B)=2$, the both must be E-invariant. This completes the proof of these cases.

Finally, consider the case of (IIIaa). Then $l(B)=3$, $l(B_1)=2$ and $|\mathscr{S}'|=5$. Here B_1 is the same as in the previous paragraphs. First of all, since B_1 satisfies (IIIba) by Proposition 3.1, it follows from the first paragraph that E fixes each column index corresponding to (z, η'_1) or (z, η'_2). (Note that B_1 is also $C_E(z)$-invariant.) Again by the table in [6], there are six possibilities for Cartan matrix. However, four of them have the following diagonal entries.

$$(2^n,\ 2^{n-2}+1,\ 2^{n-2}+2),\quad (4,\ 2^{n-2}+1,\ 2^{n-2}+2),$$
$$(8,\ 2^{n-2}+2,\ 2^{n-2}+1),\quad (3,\ 2^{n-2}+2,\ 2^{n-2}+1).$$

Hence if the Cartan matrix is one of the above four, then E fixes every elemant in $\mathrm{IBr}(B)$. The remaining two are the following.

$$\begin{pmatrix} 2^{n-2}+2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2^{n-2}+2 & 2^{n-2} & 2^{n-2} \\ 2^{n-2} & 2^{n-2}+1 & 2^{n-2}-1 \\ 2^{n-2} & 2^{n-2}-1 & 2^{n-2}+1 \end{pmatrix}$$

The corresponding decomposition matrices are the transposes of the following, respectively.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & \dots & 1 \end{pmatrix}$$

Let us number the elements of $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$ as $\chi_1, \chi_2 \cdots$ and η_1, η_2 and η_3, respectively, according to the above matrices. Notice that χ_i with $1 \leq i \leq 4$ are height zero characters. Consider the first decomposition matrix. Clearly η_1 is E-invariant. Thus the number of E-orbits in $\mathscr{S}'$ is at least four, which yields that at least two height zero characters are fixed by E by Corollary 4.7. If χ_1 or χ_3 is E-fixed, then since η_1 is E-fixed, E must fix η_2 and η_3, too. On the other hand, if χ_2 or χ_4 is E-fixed, then η_2 or η_3 is so because the restrictions of χ_2 and χ_4 to 2-regular elements give η_2 and η_3, respectively. This implies that E fixes η_2 and η_3. Hence in any case, all the irreducible Brauer characters are E-invariant, which together with Corollary 4.7 implies the result. Now if the second decomposition matrix is the case, then consider the number of height zero characters which have η_i as a constituent upon the restriction to 2-regular elements. These numbers are different for η_1, η_2 and η_3. Thus those must be fixed by E. Therefore, Corollary 4.7 again gives the result in this case. This completes the proof. □

In the cases which are not covered in the above proposition, in fact there possibly exist characters which are not E-invariant. Those are found for instance in the case where B is the principal block of D, $PSL_2(q)$ or $SL_2(q)$ on which a suitable automorphism acts. Before we consider those actions, we look at irreducible Brauer characters.

In the following lemma, we use some information on the stable Auslander-Reiten quivers of the modular block algebras, which are found in [6]. In this paper, modular block algebras mean those over some algebraically closed field of characteristic 2. It is known that, if (Iaa) is the case, then the stable Auslander-Reiten quiver has two 3-tubes, which are stable under the action of Ω. Here Ω is the Heller operator. (See V.4.1 and V.5.6.1 of [6].) In this case, we denote 3-tubes by T_1 and T_2.

Lemma 5.2. *Suppose that B satisfies $(Iaa, \geq 3)$. Let T_1 and T_2 be the 3-tubes in the stable Auslander-Reiten quiver of the modular block algebra* of B. *Then, E fixes exactly one irreducible Brauer character in B if T_1 and T_2 are E-conjugate, and E fixes all the irreducible Brauer characters otherwise.*

Proof. We use several results in [6]. First of all, the possible decomposition matrices are as follows. (Since we only need the rows corresponding to height zero characters, we give only their first four rows.)

$$(1)\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

If the first one is the case, E must fix at least one irreducible Brauer character. Moreover, even if the second one is the case, the same is true when $n \geq 3$ since the diagonal entries of the Cartan matrix are $(2, 2^{n-2}+1, 2^{n-2}+1)$. Thus our assumption yields that E fixes at least one element in $\mathrm{IBr}(B)$. Let S_1, S_2 and S_3 be non-ismorphic simple modules over the modular block algebra of B, and let P_i be the projective cover of S_i and J_i the Jacobson radical of P_i, $1 \leq i \leq 3$. We may assume that S_1 is E-invariant. Then the modular block algebra of B is of dihedral type in the sense of [6]. Thus J_i/S_i has at most two indecomposable direct summands. For each i, let U_i be an indecomposable direct summand of J_i/S_i and $\tilde{U}_i$ an extension of S_i by U_i. (See p.110 of [6].) Then in the stable Auslander-Reiten quiver of the modular block algebra, we may assume that all $\tilde{U}_i$'s lie at the end of some 3-tubes and have Ω-period three. (For the proof of these facts, see IV.4, IV.5 and V.3.1 of [6]. See also p.116, p.277 and p.293 of [6].)

Suppose that T_1 and T_2 are E-conjugate. Then some α in E interchanges T_1 and T_2. Since $(P_1/S_1)^\alpha \cong P_1/S_1$, the module J_1/S_1 is not indecomposable. Write $J_1/S_1 = U_1 \oplus V_1$. Then by VI.4.3 of [6], the top factors of U_1 and V_1 are simple. Moreover, since we must bave $U_1^\alpha \cong V_1$, the element α interchanges their top factors. If α fixes S_2 and S_3, then the top factor of J_1 has some simple module with multiplicity two. This implies that the quiver which gives the modular block algebra of B has a doudle arrow. However, in the list of algebras of dihedral type, there is no such. Hence we can conclude that $S_2^\alpha \cong S_3$.

Conversely, suppose that S_1 is E-invariant and $S_2^\alpha \cong S_3$ for some α in E. Then we may assume that $\tilde{U}_2^\alpha \cong \tilde{U}_3$. If T_1 and T_2 are α-invariant, then since $\tilde{U}_2 \not\cong \tilde{U}_3$ and since T_i's are Ω-invariant, we must have $\tilde{U}_2 \cong \Omega\tilde{U}_3$ and $\tilde{U}_3 \cong \Omega\tilde{U}_2$. (Note: $\tilde{U}_2$, $\tilde{U}_3$, $\Omega\tilde{U}_2$ and $\Omega\tilde{U}_3$ lie in the same component, and $\tilde{U}_2 \cong \Omega \tilde{U}_3$ is equivalent to $\tilde{U}_3 \cong \Omega \tilde{U}_2$.) Hence it follows that $\tilde{U}_3 \cong \Omega \tilde{U}_2 \cong \Omega^2 \tilde{U}_3$, which gives a contradiction since the Ω-period of $\tilde{U}_3$ is three. Therefore, α must interchange T_1 and T_2. This completes the proof. □

A similar assertion holds for the local block in the case of (IIaa, ≥ 4). In the following lemma, we assume that B satisfies (IIaa, ≥ 4) and let B_1 be the unique block of $C_G(z)$ with $B_1^G = B$. Consider the block $\bar{B}_1$ of $C_G(z)/<z>$ corresponding to B_1. Its defect group is $D/<z>$, a dihedral group of order 2^{n-1} and $\bar{B}_1$ satisfies (Iaa). (Note also that $l(B_1) = l(\bar{B}_1) = 3$.) In particular, the stable Auslander-Reiten quiver of the modular block algebra of $\bar{B}_1$ has two 3-tubes. Say $\bar{T}_1$ and $\bar{T}_2$.

Lemma 5.3. *Assume that B satisfies (IIaa, ≥ 4). In the same notation as above, $C_E(z)$ fixes exactly one irreducible Brauer character in B_1 if $\bar{T}_1$ and $\bar{T}_2$ are $C_E(z)$-conjugate, and $C_E(z)$ fixes all the irreducible Brauer characters otherwise.*

Proof. Applying Lemma 5.2 to $\bar{B}_1$, we get the result. □

In several arguments thereafter we will again use the notations T_1, T_2, $\bar{T}_1$ and $\bar{T}_2$ appeared in the previous lemmas.

Proposition 5.4. (i) *In the cases of (Iaa, ≥ 3), it follows that $k(G, B, n, E) = 2$ if T_1 and T_2 are E-conjugate and $k(G, B, n, E) = 4$ otherwise.*

(ii) *In the case of (Iaa,2), letting s be the number of $N_E(D)$-invariant elements in* $\mathrm{IBr}(B_1)$, *where B_1 is the unique element in* $\mathrm{Bl}(N_G(D), B)$, *we have $k(G, B, n, E) = s+1$.*

Proof. We have $|\mathscr{S}'| = 4$. If $n \geq 4$, then by Corollary 4.7, it suffices to consider the action of E on $\mathscr{S}'$. Also, in case of (I,3), notice that $k(G, B, 2) = 1$ and that $\mathscr{S}$ consists of one element corresponding to x. Hence it also suffices to look at the action of E on $\mathscr{S}'$. (See also Lemma 4.2.) Furthermore, if $n = 2$, then certainly the E-action on $\mathscr{S}'$ is enough to look at. Also, the element of $\mathscr{S}'$ correspoding to z is E-invariant. Hence in any case, we have to look at the E-action on $\mathrm{IBr}(B)$. If $n \geq 3$, then by Lemma 5.2, the number of E-orbits in $\mathscr{S}'$ is either 3 or 4 according as T_1 and T_2 are E-conjugate or not. Therefore, (i) holds.

(ii) Assume that $n = 2$. First note that D is a defect group of B_1, $l(B_1) = 3$ and its decomposition matrix is (2) in the proof of Lemma 5.2. (In fact, it is known that the modular block algebra of B_1 is Morita equivalnet to that of the principal block of the alternating group of degree four. See V.2.14 of [6].) In particular, its stable Auslander-Reiten quiver has also two 3-tubes. Say T'_1 and T'_2. Moreover, since modules in 3-tubes have D as their vertex, the Green correspondence gives a graph isomorphism between $T_1 \cup T_2$ and $T'_1 \cup T'_2$, which commutes with the action of $N_E(D)$. The structure of T'_i's is well known and can be found in [3]. In particular, if we let S'_1, S'_2 and S'_3 be non-isomorphic simple modules over the modular block algebra of B_1 and let $\tilde{U}'_i$ be an extension of S'_i by S_{i+1}, $1 \leq i \leq 3$, where the indices are taken modulo three, then it is known that $\tilde{U}'_i$'s lie in the end of one 3-tube. We distinguish two cases.

Suppose that the number s in the statement is zero. In this case $N_E(D)$ must rotate both T'_1 and T'_2 and hence T_1 and T_2. Now the modular block algebra of B is also known to be Morita equivalent to the block algebra of the principal block of the alternating group of degree four or five, and thus its Auslander-Reiten quiver is well known. (See 6.6 of [3].) By those observations, one can conclude that the decomposition matrix of B must be the same as that of B_1 (namely, (2) in the proof of Lemma 5.2), and no element in $\mathrm{IBr}(B)$ is E-invariant. Thus we get $k(G, B, n, E)=1$.

Suppose now that $s \geq 1$. Then s is 1 or 3. In this case, the argument in the proof of Lemma 5.2 works, and it follows that $s=1$ if and only if T'_1 and T'_2 are $N_E(D)$-conjugate. Moreover, this holds if and only if T_1 and T_2 are E-conjugate. Thus by the structure of 3-tubes we can determine the number of E-invariant irreducible Brauer characters in B in each case. Namely, this number is 1 if $s=1$ and is 3 if $s=3$. This yields the result. □

A similar consequence can be proved in the case of (IIaa).

Proposition 5.5. *Suppose that B satisfies (IIaa).*

(i) *Assume $n \geq 4$. Then $k(G, B, n, E)=2$ and $k(G, B, 2, E)=0$ if $\bar{T}_1$ and $\bar{T}_2$ are $C_E(z)$-conjugate, and $k(G, B, n, E)=4$ and $k(G, B, 2, E)=2$ otherwise.*

(ii) *Assume that $n=3$. Let s be the number of $C_E(z)$-invariant elements in* $\mathrm{IBr}(B_1)$. *(B_1 is the unique block in* $\mathrm{Bl}(C_G(z), B)$.*) Then we have $k(G, B, n, E)=s+1$ and $k(G, B, 2, E)=s$.*

Proof. Let $\mathscr{C}$ denote $\{\chi \in \mathrm{Irr}(B) \mid d(\chi)=n \text{ or } 2\}$. Consider the E-actions on $\mathscr{C}$ and $\mathscr{S}'$. We first list the possible Cartan matrices for B which are found in [6].

$$\begin{pmatrix} 2^n & 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-2}+2 & 2^{n-2} \\ 2^{n-1} & 2^{n-2} & 2^{n-2}+2 \end{pmatrix}, \begin{pmatrix} 8 & 4 & 4 \\ 4 & 2^{n-2}+2 & 2 \\ 4 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2^{n-2}+2 & 2^{n-2} \\ 2 & 2^{n-2} & 2^{n-2}+2 \end{pmatrix}$$

Note that E must fix at least one element of $\mathrm{IBr}(B)$ unless the third one is the case and $n=3$.

We first assume that $n \geq 4$. Then $|\mathscr{C}|=|\mathscr{S}'|=6$. According to the above Cartan matrices, the first six rows of the decomposition matrices, corresponding to the elements of $\mathscr{C}$, are as follows.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Recall that in any case, E must fix at least one element of $\mathrm{IBr}(B)$. Also, if the remaining two are E-conjugate, then we must have $k(G, B, n, E)=2$ and $k(G, B, 2, E)=0$, and $k(G, B, n, E)=4$ and $k(G, B, 2, E)=2$ otherwise.

Suppose first that $\bar{T}_1$ and $\bar{T}_2$ are $C_E(z)$-conjugate. Then Lemma 5.3 yields that the number m of E-orbits in $\mathscr{S}'$ is either 4 or 5. If $m=5$, then E fixes every element in $\mathrm{IBr}(B)$ and the above argument shows that the number of E-ordits in $\mathscr{C}$ is six, contradicting Corollary 4.7. Hence $m=4$ and we obtain the desired result.

If $\bar{T}_1$ and $\bar{T}_2$ are not $C_E(z)$-conjugate, then the above number m is either 5 or 6 by Lemma 5.3. If $m=5$, then two elements in $\mathrm{IBr}(B)$ are not E-invariant and the above argument implies that the number of E-orbits in $\mathscr{C}$ is four, a contradiction. Hence $m=6$ and we get the result.

Now assume that $n=3$. Then $\mathscr{C}=\mathrm{Irr}(B)$ and hence $|\mathscr{C}|=7$. The block B_1 satisfies (IIaa) and thus $l(B)=3$. The possibilities of decomposition matrices are as follows.

$$(1)\quad \begin{pmatrix} 1&0&0\\ 1&1&0\\ 1&0&1\\ 1&1&1\\ 0&1&0\\ 0&0&1\\ 2&1&1 \end{pmatrix},\qquad (2)\quad \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&1\\ 1&1&1\\ 1&1&0\\ 1&0&1\\ 0&1&1 \end{pmatrix}$$

(If $n=3$, then there is no block with the second Cartan matrix in the first paragraph of this proof.) If the above (1) is the Cartan matrix, then since E fixes at least one element of $\mathrm{IBr}(B)$, we have $k(G, B, 3, E)=2$ and $k(G, B, 2, E)=1$ if the remaining two are E-conjugate and $k(G, B, 3, E)=4$ and $k(G, B, 2, E)=3$ otherwise. On the other hand, if (2) is the Cartan matrix, then we have one more possibility, namely, $k(G, B, 3, E)=1$ and $k(G, B, 2, E)=0$ if the three elements in $\mathrm{IBr}(B)$ are E-conjutate.

Let m' be the number of $C_E(z)$-orbits in $\mathrm{IBr}(B_1)$. ($1\leq m'\leq 3$.) We regard m' simultaniously as the number of E-orbits in the subset $\{(z, \eta'_1), (z, \eta'_2), (z, \eta'_3)\}$ of $\mathscr{S}$. In the following m denotes the number of E-orbits in the entire column index set of the generalized decomposition matrix. Since the index corresponding to x is E-invariant, we have $m'+2\leq m\leq m'+4$. For each possibility of m, the above argument yields the following.

m	$m'+2$	$m'+3$	$m'+4$
$k(G, B, 3, E)$	1	2	4
$k(G, B, 2, E)$	0	1	3

The numbers of E-orbits in $\mathscr{C}$ are 3, 5 and 7, respectively. Thus by Brauer's permutation lemma, for each m' with $1\leq m'\leq 3$, there is only one possibility for m. Namely, one of the following holds.

m'	1	2	3
m	3	5	7
$k(G, B, 3, E)$	1	2	4
$k(G, B, 2, E)$	0	1	3

Since $m'=1$, 2, 3 are equivalent to $s=0$, 1, 3, repectively, the result follows. $\square$

In cases of (Ibb) and (IIbb) we have the following propositions.

Proposition 5.6. *Suppose that B satisfies $(Ibb, \geq 3)$ or $(IIbb, \geq 4)$. Then $k(G, B, n, E)=2$ if y and xy are E-conjugate, and $k(G, B, n, E)=4$ otherwise.*

Proof. If $n \geq 4$, then by Corollary 4.7, it suffices to consider the action of E on $\mathscr{S}'$. Also, in case of $(I, 3)$, notice that $k(G, B, 2)=1$ and that $\mathscr{S}$ consists of one element corresponding to x. Hence it also suffices to look at the action of E on $\mathscr{S}'$. (See also Lemma 4.2.) In both cases, $|\mathscr{S}'|=4$ and $l(B)=1$.

Since $E=C_E(z)G$ and since z is not G-conjugate to y or xy, it follows that z is E-conjugate neither to y nor to xy. Thus the number of E-orbits in $\mathscr{S}'$ is 4 if y and xy are not E-conjugate, and is 3 otherwise. Hence the result follows. $\square$

Proposition 5.7. *Let A denote $E/C_E(D)G$ and suppose that B satisfies $(Ibb,2)$ or $(IIbb,3)$. Then A is isomorphic to some subgroup of the symmetric group of degree three. Moreover, we have the following.*

$$k(G, B, n, E)= \begin{cases} 1, & \text{if } |A|=3 \text{ or } 6 \\ 2, & \text{if } |A|=2 \\ 4, & \text{if } |A|=1 \end{cases}$$

Proof. Since $\mathrm{Out}(D)$ is isomorphic to the symmetric group of degree three, the first statement holds from Lemma 4.3.

In both cases, $l(B)=1$ Moreover, if (IIbb,3) is the case, $l(B_1)=1$, where B_1 is the unique block of $C_G(z)$ with $B_1^G=B$, and $k(G, B, 2)=1$. All the other column indices correspond to x, y and xy. Hence in order to determine the number m of E-orbits in the index set it suffices to consider the action of A on the G-conjugacy classes containing x, y, or xy. Notice also that m is at least n in either case. Thus the number of E-orbits in the set of height zero characters is at least two. If $|A|$ is 3 or 6, then some element of E permutes the classes corresponding to x, y and xy cyclically. Thus $k(G, B, n, E)$ must be 1. If $|A|=2$, then the number of E-orbits in the set of height zero characters in B is 3 and thus $k(G, B, n, E)=2$. If $|A|=1$, we certainly obtain $k(G, B, n, E)=4$. $\square$

REMARK. The possible values of $k(G, B, d, E)$ which appear in the statements of this section can actually be realized in some tame blocks.

6. The extended conjecture

As in the previous sections, we assume that G is a normal subgroup of E and B is a tame block of G fixed by E. In this section, we prove our main theorem, namely, that the extended conjecture holds for tame blocks. Thus we also assume that $O_2(G)=1$. First we show that in order to prove the extended conjecture it suffices to look at the block B_1 appearing in Proposition 3.1;

Lemma 6.1. *Let H be $C_G(z)$ if $n\geq 3$ and $N_G(D)$ if $n=2$. If*

$$(*)\qquad k(G, B, d, F)=k(H, B, d, F)$$

holds for all d and all F with $G\leq F\leq E$, then the extended conjecture holds.

Proof. Fix F as above. Let u be an involution in D. Suppose that u is not G-conjutate to z if $n\geq 3$. Let C_1 and C_2 be $1<<u>$ and $1<<u><Q$, respectively, where Q is the same as in Proposition 3.2, and let $H_1=N_G(C_1)$ and $N_1=N_G(C_2)$. Then by the remark following Proposition 3.2, each of $\mathrm{Bl}(H_1, B)$ and $\mathrm{Bl}(N_1, B)$ consists of a single element. Say b and b', respectively. In particular, b (resp. b') is $N_E(C_1)$(resp. $N_E(C_2)$)-invariant. Moreover, we have $N_F(C_1)=N_F(C_2)H_1$. Now, since b is a tame block, applying (*) to b (with G, B and F being replaced by H_1, b and $N_F(C_1)$, respectively) we obtain

$$k(H_1, b, d, N_F(C_1))=k(N_1, b, d, N_F(C_1))$$

for all d. Now, if F is not contained in $N_E(C_1)G$, then $k(H_1, B, d, F)=0$, and otherwise using (1.3) we have

$$k(H_1, b, d, N_F(C_1))=k(H_1, B, d, F)$$

for all d. On the other hand, since $N_E(C_2)H_1=N_E(C_1)$, we have $N_E(C_1)G=N_E(C_2)G$. Thus, if F is not contained in $N_E(C_1)G$, then $k(N_1, B, d, F)=0$, and otherwise we have

$$k(N_1, b, d, N_F(C_1))=k(N_1, B, d, F)$$

for all d. Hence we obtain

$$(**)\qquad k(H_1, B, d, F)=k(N_1, B, d, F)$$

for all d and F. Finally, let u' be a non-central involution in D. Let $H'=N_E(1<<z, u'>)$ and $N'=N_E(1<<z><<z, u'>)$. Then since

$n \geq 3$ and $E = C_E(z)G$, the same arguments as above and as in the proof of Proposition 3.3 impliy that

$$(***) \qquad k(H' \cap G, B, d, F) = k(N' \cap G, B, d, F)$$

for all F and d. Notice that in each of (*), (**) and (***), the two chains in the both sides have length of opposite parity. Therefore, in view of Lemma 2.7, the alternating sum (1.1) vanishes. This completes the proof. □

By the above lemma we can now concentrate on the blocks B and B_1. Here B_1 is the unique block of H, where H is in Lemma 6.1, such that $B_1^G = B$. In the rest of this paper, we fix these notations. In addition, we let E' be $C_E(z)$ if $n \geq 3$ and $N_E(D)$ if $n = 2$. Remark that since B is E-invariant, the Frattini argument yields that $GE' = E$. In particular, $E = F$ if and only if $E' = F \cap E'$. When we check (*) in Lemma 6.1, we distinguish the cases assording to the value d. First consider the case where $d = n-1$ and $n \geq 4$ (Lemma 6.2), then $d = n$ (Lemma 6.3) and finally $d = 2$ (Lemma 6.4). Recall that for all the other values of d, the number $k(G, B, d, F)$ is zero. Also, recall that B_1 is E'-invariant and $k(H, B, d, F)$ is equal to $k(H, B_1, d, F \cap E')$.

Lemma 6.2. *Suppose that $n \geq 4$. Then $k(G, B, n-1, F) = k(H, B, n-1, F)$ for all F.*

Proof. Through the natural homomorphism from $N_E(D)/C_E(D)$ to $\mathrm{Aut}(D)$, we can define a homomorphism $\nu_{E'}\colon E' \to \Gamma$ in a way similar to ν, and $E/C_E(D)G \cong E'/C_E(D)H$ yields that $E/\mathrm{Ker}\,\nu = E'/\mathrm{Ker}\,\nu_{E'}$ and $\mathrm{Ker}\,\nu \cap E' = \mathrm{Ker}\,\nu_{E'}$. (See (4.1.b).) Moreover, the conclusions similar to Lemma 4.5 and Corollary 4.6 hold for $\nu_{E'}$ and $k(H, B_1, n-1, F \cap E')$. If F does not contain $\mathrm{Ker}\,\nu$, then it follows from Lemma 4.5 that both $k(G, B, n-1, F)$ and $k(H, B, n-1, F)$ are zero. On the other hand, if $F \geq \mathrm{Ker}\,\nu$, then since $F/\mathrm{Ker}\,\nu \cong E' \cap F/\mathrm{Ker}\,\nu_{E'}$, the result follows from Corollary 4.6. □

Lemma 6.3. *$k(G, B, n, F) = k(H, B, n, F)$ for all F.*

Proof. We distinguish several conditions which the block B satisfies. (For example, case (IIab) below means that we treat the case where B satisfies (IIab).)

Cases (IIab) and (III). Note that by Proposition 3.1, B_1 also satisfies (IIab) or (III) accordingly. It follows from Proposition 5.1 that $k(G, B, n, F) = 0$ if $E \neq F$, and $k(G, B, n, E) = 4$. Since $FE' = E$, applying

Proposition 5.1 to B_1, it also follows that $k(H, B, n, F)=0$ if $F \neq E$, and $k(H, B, n, E)=4$. Thus the result follows.

Case (Iab). Recall that B_1 satisfies (Ibb), and notice that $n \geq 3$. By Proposition 5.1 and the argument given in the cases of (IIab) and (III), it suffices to show that all the height zero characters in B_1 are E'-invariant. Thus by using Proposition 5.6, it then suffices to prove that y and xy are not E'-conjugate. Suppose to the contrary that y and xy are E'-conjugate. Then since z and xy are G-conjugate, we can conclude that z and y are E-conjugate. However, since $E=C_E(z)G$ and since y is not G-conjugate to z, this derives a contradiction. Therefore, the result holds.

Cases (Iaa,2) and (IIaa,3). Recall that B_1 satisfies (Iaa,2) or (IIaa,3) accordingly. By Propositions 5.4 and 5.5, the values $k(G, B, n, F)$ and $k(H, B, n, F)$ are both equal to $s+1$, where s is the number of irreducible Brauer characters in B_1 whose inertia subgroups in E' are $F \cap E'$. Therefore, the result follows.

Cases (Ibb,2) and (IIbb,3). B_1 also satisfies (Ibb,2) or (IIbb,3) accordingly. If F does not contain $C_E(D)G$, then both $k(G,B,n,F)$ and $k(H, B, n, F)$ are zero by Proposition 5.7. Assume that $F \geq C_E(D)G$. Note that $F/C_E(D)G \cong F \cap E'/C_E(D)H$. We set $s=|F/C_E(D)G|$ and $t=|E/C_E(D)G|$. Then it follows from Proposition 5.7 that

$$k(G,B,n,F)=k(H,B,n,F)=\begin{cases} 0, & \text{if } (s,t)=(1,6) \text{ or } (3,6) \\ 1, & \text{if } (s,t)=(2,6),\ (6,6) \text{ or } (3,3) \\ 2, & \text{if } (s,t)=(1,2) \text{ or } (2,2) \\ 3, & \text{if } (s,t)=(1,3) \\ 4, & \text{if } (s,t)=(1,1). \end{cases}$$

Thus the result holds.

Cases (Iaa, ≥ 3), (Ibb, ≥ 3), (IIaa, ≥ 4) and (IIbb, ≥ 4). Recall that B_1 satisfies (Ibb, ≥ 3) if B satisfies (Iaa, ≥ 3), and B and B_1 satisfy the same property otherwise. Define a certain subgroup in each individual case in the following way.

In case of (Iaa, ≥ 3), I_1 is the stabilizer of T_1 (and of T_2) in E. (See Lemma 5.2.)

In case of (IIaa, ≥ 4), $I_2=IG$, where I is the stabilizer of $\bar{T}_1$ (and of $\bar{T}_2$) in E'. (See the paragraph preceding Lemma 5.3.)

In cases of (Ibb, ≥ 3) and (IIbb, ≥ 4), $I_3=C_E(y)G$.

In cases of (Iaa, ≥ 3), (Ibb, ≥ 3) and (IIbb, ≥ 4), $I_4=C_{E'}(y)G$.

Then by using Propositions 5.4, 5.5 and 5.6, we can conclude that

$$k(\tilde{G},B,n,F)=\begin{cases}0, & \text{if } I_i\neq F\neq E\neq I_i \text{ or } I_i=E\neq F\\ 2, & \text{if } I_i=F\neq E \text{ or } I_i\neq F=E\\ 4, & \text{if } I_i=E=F\end{cases}$$

holds for the following $\tilde{G}$'s and I_i's.

Case for B	$\tilde{G}$	I_i
$(Iaa,\geq 3)$	G	I_1
$(IIaa,\geq 4)$	G and H	I_2
$(Ibb,\geq 3),(IIbb,\geq 4)$	G	I_3
$(Iaa,\geq 3),(Ibb,\geq 3),(IIbb,\geq 4)$	H	I_4

The above shows that the result follows in case of (IIaa, ≥ 4). In case of (Iaa, ≥ 3), it suffices to prove that $I_1=I_4$. For each $i,i=1,2$, vertices of all the modules in T_i coincide and are equal to Q_1 or Q_2. (See [6,V.4.1].) However, by [6,V.5.13], Q_1 and Q_2 are not G-conjugate. Thus we may assume that for each $i,i=1,2$, all the modules in T_i have Q_i as their vertex. Hence we have $I_1=N_E(Q_1)G$, and thus $I_4\leq I_1$. To see that $I_1\leq I_4$, let $\alpha\in I_1$. Write $\alpha=\alpha'g$, where $\alpha'\in E'$ and $g\in G$. Since Q_1 and Q_2 are not G-conjugate, $Q_1^{\alpha'}$ is not G-conjugate to Q_2. Then, since $\alpha'\in E'=N_E(D)H$, the group $Q_1^{\alpha'}$ must be H-conjugate to Q_1. Hence $\alpha'=\alpha_1 h$ for some $\alpha_1\in N_{E'}(Q_1)$ and $h\in H$. Therefore, $\alpha=\alpha'g=\alpha_1 hg$ lies in $N_{E'}(Q_1)G$, and we get $I_1\leq N_{E'}(Q_1)G=I_4$.

In the cases of (Ibb, ≥ 3) and (IIbb, ≥ 4), we must show that $I_3=I_4$. Clearly, $I_4\leq I_3$. Now, $I_3\leq I_4$ can be proved by the same argument as above replacing Q_1, Q_2 and $N_{E'}(Q_1)$ by y, xy and $C_{E'}(y)$, respectively. (Note that in these cases, y and xy are not G-conjugate.) This completes the proof of the lemma. □

Lemma 6.4. *$k(G, B, 2, F)=k(H, B, 2, F)$ for all F.*

Proof. By Lemma 2.3 and Proposition 3.1, the result is clear unless B satisfies (IIaa). However, in the case of (IIaa), B_1 also satisfies (IIaa). Using Proposition 5.5, $k(G, B, 2, F)$ and $k(H, B, 2, F)$ coincide by an argument similar to the one given in the cases of (IIaa,3) and (IIaa, ≥ 4) in Lemma 6.3. Therefore, the result holds. □

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