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ON THE DARBOUX TRANSFORMATION OF THE SECOND ORDER DIFFERENTIAL OPERATOR OF FUCHSIAN TYPE ON THE RIEMANN SPHERE

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The main purpose of the present paper is to clarify some analytical properties of the *Darboux transformation* of the second order ordinary differential operator

$$L(P) = D^2 - P(x)\,, \quad D = d/dx$$

of Fuchsian type on the Riemann sphere $\boldsymbol{P}_1$. Throughout the paper, for brevity, we assume that $P(x)$ is of the form

$$P(x) = \sum_{j=1}^{n} \alpha_j (x-a_j)^{-2}\,. \tag{1}$$

The Darboux transformation of $L(P)$ is defined as follows: Let $Y(x) = {}^t(y_1(x), y_2(x))$ be the fundamental system of solutions of

$$L(P)y = y'' - P(x)y = 0\,, \quad ' = d/dx \tag{2}$$

such that $W(Y(x))=1$, where

$$W(Y(x)) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

is the Wronskian. Put

$$q(x; \zeta) = (\partial/\partial x) \log \zeta \times Y(x) \tag{3}$$

and

$$A_\pm(\zeta) = D \pm q(x; \zeta)$$

respectively, where $\zeta = [\xi_1 : \xi_2]$ is the homogeneous coordinate of $\boldsymbol{P}_1$ and

$$\zeta \times Y(x) = \xi_1 y_1(x) + \xi_2 y_2(x)\,.$$

Then $L(P)$ is decomposed into the product of the first order operators

$$L(P) = A_+(\zeta) A_-(\zeta)\,.$$

By exchanging the role of $A_\pm(\zeta)$, we obtain the another second order operators

$$L^*(P;\zeta,Y)=A_-(\zeta)A_+(\zeta)$$

parametrized by $\zeta\in\boldsymbol{P}_1$. We call $L^*(P;\zeta,Y)$ the Darboux transformation of $L(P)$ by $Y(x)$. Put

$$P^*(x;\zeta)=P(x)-2\partial q(x;\zeta)/\partial x \tag{4}$$

then we have

$$L^*(P;\zeta,Y)=D^2-P^*(x;\zeta)\,.$$

J.L. Burchnall and T.W. Chaundy [5] treated such an operation in their research on the commutative ordinary differential operator and called it transference. Moreover, M.M. Crum [6] studied an analogous method as an algorithm of adding or removing eigenvalues of Sturm-Liouville operators on a finite interval. Recently, many authors extended these methods in various cases in connection with the soliton theory. See e.g,. P. Deift [7], P. Deift and E. Trubowitz [8] and F. Ehlers and H. Knörrer [9]. The aim of the present work is to explore an analogue of these methods for the differential operator of Fuchsian type on $\boldsymbol{P}_1$.

We briefly sketch our results obtained in this paper in the following. Let $\boldsymbol{F}$ be the set of all $L(P)$ such that $P(x)$ is of the form (1) and the ratio $y_2(x)/y_1(x)$ of the fundamental system of solutions of (2) is a rational function. Then we can show easily that the Darboux transformation $L^*(P;\zeta,Y)$ of $L(P)$ is of Fuchsian type on $\boldsymbol{P}_1$ for any $\zeta\in\boldsymbol{P}_1$ if and only if $L(P)\in\boldsymbol{F}$. Moreover put

$$\chi(L(P))=\#\{\zeta\,|\,L^*(P;\zeta,Y)\in\boldsymbol{F}\} \tag{5}$$

for $L(P)\in\boldsymbol{F}$, where $\#$ denotes cardinal. Then $\chi(L(P))$ turns out to be equal to 0, 1 or ∞, where $\chi(L(P))=\infty$ refer to that $L^*(P;\zeta,Y)\in\boldsymbol{F}$ for all $\zeta\in\boldsymbol{P}_1$. Thus $\boldsymbol{F}$ is decomposed into the disjoint union

$$\boldsymbol{F}=\boldsymbol{F}_0\cup\boldsymbol{F}_1\cup\boldsymbol{F}_\infty\,, \tag{6}$$

where

$$\boldsymbol{F}_\nu=\{L(P)\in\boldsymbol{F}\,|\,\chi(L(P))=\nu\},\quad \nu=0,1,\infty\,.$$

According to the classification (6), the characterizatoin of $L^*(P;\zeta,Y)$ in connection with the isomonodromic deformation is obtained. Next we define $Z_n(P)$ and $X_n(P)=DZ_n(P)$ by the recursion formula called the *Lenard relation* (cf. [12].)

$$X_n(P)=2^{-1}Z_{n-1}(P)P'+X_{n-1}(P)P-4^{-1}D^2X_{n-1}(P)\,,\quad n=1,2,\cdots \tag{7}$$

with $Z_0(P)=P$. Then it is known that $Z_n(P)$ and $X_n(P)$ are the polynomials in $P^{(s)}$ $(s=0, 1, \cdots, 2n+1)$ with constant coefficients, where $P^{(s)}$ is the s-th derivative of P. Now let Λ_n be all of rational functions $P(x)$ of the form (1) such that $L(P)\in \boldsymbol{F}$, $X_j(P)\neq 0$ for $j<n$ and $X_n(P)=0$. Then we can construct easily the solution $u(x; \xi)$, which is rational in x, of the n-th KdV equation

$$\partial u(x; \xi)/\partial \xi = X_n(u(x; \xi)) \tag{8}$$

from $P^*(x; \zeta)$ for $P(x)\in\Lambda_n$, where ξ is the complex variable. Put

$$\Lambda^* = \cup_{n=0}^{\infty} \Lambda_n$$

and

$$\boldsymbol{F}^* = \{L(P) \,|\, P\in\Lambda^*\}$$

then, combining the characterization of $L^*(P; \zeta, Y)$ in connection with the isomonodromic deformation and the fact that $P^*(x; \zeta)$ gives rise the rational function solution of the n-th KdV equation (8), $\boldsymbol{F}^*$ turns out to be the subclass of $\boldsymbol{F}_\infty$ closed under the Darboux transformation. Moreover we show that for every $P\in\Lambda^*$ there exists the n-th KdV flow on Λ^* passing through P.

The isomonodromic deformation theory, which offers the basic tools for above considerations, originates in classical works by R. Fuchs [11], L. Schlesinger [21] and R. Garnier [13]. Recently, these are generalized by many authors and turn out to be deeply related to many problems in mathematical physics. See e.g., [10], [14], [15], [16], [17], [18], [19] and [20]. However we need only elementary part of the theory so far as for the present work.

On the other hand, rational function solutions of the KdV equation have been obtained by various methods: M.J. Ablowitz and J. Satsuma [1] constructed them by taking long wave limit of the soliton solutions obtained by Hirota's method; M. Adler and J. Moser [2] constructed them by using the Darboux transformation, which is called the Crum transformation by them. See also H. Airault, H.P. McKean and J. Moser [3].

While our results are deeply related to that of Adler-Moser, our viewpoint and method are somewhat different from those of them. More precisely, our aim is not only to construct rational function solutions of the n-th KdV equation but also to characterize the space Λ^* of rational function solutions generated by the Darboux transformation in connection with the isomonodromic deformation. Moreover, while Adler-Moser constructed rational function solutions by using the polynomial τ-function originated by M. Sato and his colleagues, our method is essentially based on the Lenard relation (7).

The contents of the present paper are as follows. Section 1 is devoted for preliminary considerations about the Darboux transformation. In section 2, the Darboux transformation is investigated in connection with the isomonodromic deformation. In section 3, the relation between the Darboux transformation and the Lenard relation is discussed. In section 4, the space Λ^* of rational function solutions of the higher order *KdV* equation is constructed.

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1. Preliminary

In this section we first give a necessary and sufficient condition on $L(P)$ for $L^*(P;\zeta, Y)$ to be of Fuchsian type on $\boldsymbol{P}_1$ for all $\zeta\in\boldsymbol{P}_1$. One readily verifies

$$P^*(x;\zeta) = -P(x)+2q(x;\zeta)^2\,. \tag{1.1}$$

Hence, one can see that $L^*(P;\zeta, Y)$ is of Fuchsian type on $\boldsymbol{P}_1$ if and only if $q(x;\zeta)$ is rational in x. Moreover we have

Lemma 1.1 (cf. Baldassarri and Dwork [4]). *$L^*(P;\zeta, Y)$ is of Fuchsian type on $\boldsymbol{P}_1$ for all $\zeta\in\boldsymbol{P}_1$ if and only if $L(P)\in\boldsymbol{F}$.*

Proof. Put $f(x)=y_2(x)/y_1(x)$ then we have

$$y_1(x) = f'(x)^{-1/2}\,,\quad y_2(x) = f'(x)^{-1/2}f(x)\,. \tag{1.2}$$

Hence

$$q(x;\zeta) = (\partial/\partial x)\log f'(x)^{-1/2}(\xi_1+\xi_2 f(x)) \tag{1.3}$$

follows. Therefore, if $L(P)\in\boldsymbol{F}$, i.e., $f(x)$ is a rational function then $q(x;\zeta)$ is also rational in x for any $\zeta\in\boldsymbol{P}_1$. Thus $L^*(P;\zeta, Y)$ is of Fuchsian type for all $\zeta\in\boldsymbol{P}_1$. Conversely, suppose that $q(x;\zeta)$ is rational in x for all $\zeta\in\boldsymbol{P}_1$. Let $\zeta_i=[\xi_{i1}:\xi_{i2}]\in\boldsymbol{P}_1$ ($i=1, 2, 3$; $\zeta_j\neq\zeta_k$ if $j\neq k$). Hence, if $1\leqslant j<k\leqslant 3$ then we can choose ξ_{is} ($i=1, 2, 3$; $s=1, 2$) such that

$$\xi_{j1}\xi_{k2}-\xi_{j2}\xi_{k1} = 1\,. \tag{1.4}$$

Then

$$q(x;\zeta_j)-q(x;\zeta_k) = \frac{-f'(x)}{(\xi_{j1}+\xi_{j2}f(x))\,(\xi_{k1}+\xi_{k2}f(x))}\,,\quad j<k$$

follows from (1.3). Hence

$$\frac{q(x;\zeta_1)-q(x;\zeta_3)}{q(x;\zeta_1)-q(x;\zeta_2)} = \frac{\xi_{21}+\xi_{22}f(x)}{\xi_{31}+\xi_{32}f(x)} \tag{1.5}$$

follows. Since the left hand side of (1.5) is rational in x and, by (1.4), the right hand side of (1.5) is the nondegenerate fractional transformation of $f(x)$, $f(x)$ is a rational function, i.e., $L(P)\in\boldsymbol{F}$.

Let $\phi(x)$ be meromorphic at $x=a$. Suppose that

$$\phi(x)=\sum_\nu c_\nu(x-a)^\nu$$

is the Laurent expansion of $\phi(x)$ at $x=a$. Put

$$D_a^{-1}\phi(x)=\sum_{\nu\neq-1}(\nu+1)^{-1}c_\nu(x-a)^{\nu+1}+c_{-1}\log(x-a).$$

Note

$$DD_a^{-1}\phi(x)=\phi(x), \quad D_a^{-1}D\phi(x)=\phi(x)-c_0.$$

The fundamental system $Y(x; a)$ of solutions of (2) normlized at $x=a$ is defined by using D_a^{-1} as follows. In what follows, we assume $L(P)\in\boldsymbol{F}$. Since $P(x)$ is of the form (1), $P(x)$ is expanded at $x=a$ as

$$P(x)=\sum_{\nu=-2}^{\infty}c_\nu(a)(x-a)^\nu.$$

Let $\lambda_\pm(a)$ be solutions of

(1.6) $$\lambda(\lambda-1)-c_{-2}(a)=0.$$

Namely, $\lambda_\pm(a_j)$ are the exponents of (2) at the sigular points $x=a_j$ and, moreover, if $P(x)$ is holomorphic at $x=a$ then we can set

$$\lambda_+(a)=1, \quad \lambda_-(a)=0.$$

Since $L(P)\in\boldsymbol{F}$, i.e., the ratio $y_2(x)/y_1(x)$ of the fundamental system of solutions of (2) is the rational function, we can assume that the exponent difference

$$n(a_j)=\lambda_+(a_j)-\lambda_-(a_j)$$

is a nonnegative integer. We have

(1.7) $$\lambda_\pm(a)=2^{-1}(1\pm n(a)).$$

If $n(a_j)=0$ then $x=a$ is the logarithmic singular point. Therefore we can assume

$$n(a_j)\geqslant 2.$$

By Frobenius method, we obtain the unique solution

(1.8+) $$y_2(x; a)=(x-a)^{\lambda_+(a)}\sum_{\nu=0}^{\infty}k_\nu^+(a)(x-a)^\nu, \quad k_0^+(a)=1$$

of the equation (2). Put

$$y_1(x; a)=-y_2(x; a)D_a^{-1}(y_2(x; a)^{-2})$$

and

$$Y(x; a)={}^t(y_1(x; a), \quad y_2(x; a)).$$

Then $Y(x; a)$ is the fundamental system of solutions of (2) such that $W(Y(x; a))=1$. We call $Y(x; a)$ the *normalized fundamental system* of solutions of (2). Since $L(P)\in \boldsymbol{F}$, we have

$$y_1(x;\, a)=(x-a)^{\lambda_-(a)}\sum_{\nu=0}^{\infty}k_\nu^-(a)(x-a)^\nu \tag{1.8$-$}$$

with $k_0^-(a)\neq 0$. For brevity, we often adopt the following conventions; Y_0 stands for $Y(x; 0)$ and Y_j for $Y(x; a_j)$ $(j=1, 2, \cdots, n)$ respectively. In what follows, we investigate mainly the Darboux transformation $L^*(P; \zeta, Y_0)$ by $Y(x; 0)$. Therefore, the notation $P^*(x; \zeta)$ refers to $P(x)-2(\partial/\partial x)^2\log\zeta\times Y(x; 0)$ in what follows;

$$P^*(x;\, \zeta)=P(x)-2(\partial/\partial x)^2\log\zeta\times Y(x;\, 0)\,.$$

The Darboux transformation $L^*(P; \zeta, Z)$ by another solution $Z(x)$ is obtained by

$$L^*(P;\, \zeta,\, Z)=L^*(P;\, \zeta\times C,\, Y_0)\,,$$

where $C=(c_{ij})\in SL(2, \boldsymbol{C})$ such that $Z(x)=CY(x; 0)$ and $\zeta\times C=[\xi_1c_{11}+\xi_2c_{21}: \xi_1c_{12}+\xi_2c_{22}]\in \boldsymbol{P}_1$.

Suppose $\zeta\in\boldsymbol{P}_1$; $\zeta=[1: t_0]$ for $\zeta\neq\infty$ and $\zeta=[t_\infty: 1]$ for $\zeta\neq 0$. Now put

$$\begin{aligned} y_{10}^*(x;\, \zeta)&=(y_1(x;\, 0)+t_0y_2(x;\, 0))^{-1} \quad \text{for}\quad \zeta=[1:\, t_0]\neq\infty\\ y_{1\infty}^*(x;\, \zeta)&=(t_\infty y_1(x;\, 0)+y_2(x;\, 0))^{-1} \quad \text{for}\quad \zeta=[t_\infty:\, 1]\neq 0\end{aligned} \tag{1.9}$$

and

$$y_{2\mu}^*(x;\, \zeta)=y_{1\mu}^*(x;\, \zeta)D_0^{-1}(y_{1\mu}^*(x;\, \zeta)^{-2})\,,\quad \mu=0,\, \infty\,, \tag{1.10}$$

then one can see readily that

$$Y_\mu^*(x;\, \zeta)={}^t(y_{1\mu}^*(x;\, \zeta)\,,\ \ y_{2\mu}^*(x;\, \zeta))\,,\quad \mu=0,\, \infty$$

are the fundamental systems of solutions of

$$L^*(P;\, \zeta,\, Y_0)y=y''-P^*(x;\, \zeta)y=0\,,\quad \zeta\in U_\mu,\ \mu=0,\, \infty \tag{1.11}$$

respectively, where $U_0=\boldsymbol{P}_1\backslash\{\infty\}$ and $U_\infty=\boldsymbol{P}_1\backslash\{0\}$.

The following lemma is an elementary fact about the residue of meromorphic function.

Lemma 1.2. *If $f(x)$ is a single valued meromorphic function in a vicinity of $x=a$ such that*

$$\operatorname{Ref} f(x)^k/f'(x)\,|_{x=a}=0$$

holds for $k=0$ or 2 then

$$\operatorname{Res} f(x)/f'(x)\,|_{x=a}=0\,.$$

Proof. The singularities of $f(x)/f'(x)$ consist of the zeros of $f'(x)$ and the poles of $f(x)$. First suppose that $f'(a)=0$ and

$$\text{Res } 1/f'(x)|_{x=a} = 0 .$$

Then we have

$$f'(x) = (x-a)^r g_1(x)$$

and

$$f(x) = c+(x-a)^{r+1}g_2(x) ,$$

where r is a positive integer, $g_j(x)$ $(j=1, 2)$ are holomorphic at $x=a$ and $g_j(a)\neq 0$ $(j=1, 2)$. Hence we have

$$\text{Res} f(x)/f'(x)|_{x=a} = c \text{ Res } 1/f'(x)|_{x=a} = 0 .$$

Moreover if $x=a$ is a pole of $f(x)$ then $x=a$ is a removable sigularity of $f(x)/f'(x)$. The proof in the case of

$$\text{Res} f(x)^2/f'(x)|_{x=a} = 0 ,$$

is parallel to the above.

Next we have

Lemma 1.3. *If $L(P)\in \boldsymbol{F}$ and*

$$\# \{\zeta\in \boldsymbol{P}_1 | L^*(P; \zeta, Y_0)\in \boldsymbol{F}\} \geqslant 2$$

then $L^*(P; \zeta, Y_0)\in \boldsymbol{F}$ for any $\zeta\in \boldsymbol{P}_1$.

Proof. First suppose $L^*(P; \zeta_j, Y_0)\in \boldsymbol{F}$ $(j=1, 2)$, where $\zeta_j=[1: t_j]\neq\infty$ and $\zeta_1\neq\zeta_2$. Put

$$Z(x) = {}^t(z_1(x), z_2(x)) = CY(x; 0) ,$$

where

$$C = (t_2-t_1)^{-1/2} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \end{bmatrix} \in SL(2, \boldsymbol{C}) ,$$

then we have

$$L^*(P; \zeta_1, Y_0) = L^*(P; 0, Z)$$

and

$$L^*(P; \zeta_2, Y_0) = L^*(P; \infty, Z) .$$

Put $f(x)=z_2(x)/z_1(x)$ then, by direct calculation, we have

$$\begin{aligned} (1.12) \qquad y^*_{20}(x; \zeta)/y^*_{10}(x; \zeta) &= D_0^{-1}((y_1(x; 0)+t_0 y_2(x; 0))^2) \\ &= D_0^{-1}((\alpha_1 z_1(x)+\alpha_2 z_2(x))^2) \\ &= D_0^{-1}(f'(x)^{-1}(\alpha_1+\alpha_2 f(x))^2) , \end{aligned}$$

where $\zeta=[1:t_0]\neq\infty$ and $(\alpha_1, \alpha_2)=(1, t_0)C^{-1}$. In particular, since

$$(1, t_1)C^{-1} = ((t_2-t_1)^{1/2}, 0)$$

and

$$(1, t_2)C^{-1} = (0, (t_2-t_1)^{1/2}),$$

we have

$$y_{20}^*(x; \zeta_1)/y_{10}^*(x; \zeta_1) = (t_2-t_1)D_0^{-1}(1/f'(x))$$

and

$$y_{20}^*(x; \zeta_2)/y_{10}^*(x; \zeta_2) = (t_2-t_1)D_0^{-1}(f(x)^2/f'(x)).$$

By the assumption, $D_0^{-1}(f(x)^\nu/f'(x))$ $(\nu=0, 2)$ are rational functions. Therefore, by lemma 1.2, $D_0^{-1}(f(x)/f'(x))$ is also a rational function. Thus, by (1.12), $y_{20}^*(x; \zeta)/y_{10}^*(x; \zeta)$ is rational in x for any $\zeta\neq\infty$. Similarly to the above, we can show that $y_{2\infty}^*(x; \infty)/y_{1\infty}^*(x; \infty)$ is rational in x. Thus $L^*(P; \zeta, Y_0)\in \boldsymbol{F}$ holds for any $\zeta\in \boldsymbol{P}_1$ The proof in case of $L^*(P; \zeta_j, Y_0)\in \boldsymbol{F}$ $(j=1, 2; \zeta_1\neq\infty$ and $\zeta_2=\infty)$ is also parallel to the above.

Put

$$\chi(L(P)) = \#\{\zeta\in \boldsymbol{P}_1 \mid L^*(P; \zeta, Y_0)\in \boldsymbol{F}\}$$

for $L(P)\in \boldsymbol{F}$, where $\chi(L(P))=\infty$ refers to that $L^*(P; \zeta, Y_0)\in \boldsymbol{F}$ for any $\zeta\in \boldsymbol{P}_1$. By lemma 1.3, it suffices to consider in cases of $\chi(L(P))=0$, 1 and ∞. Put

$$\boldsymbol{F}_\nu = \{L(P)\in \boldsymbol{F} \mid \chi(L(P)) = \nu\}, \quad \nu = 0, 1, \infty.$$

Thus $\boldsymbol{F}$ is decomposed into the disjoint union as (6);

$$\boldsymbol{F} = \boldsymbol{F}_0 \cup \boldsymbol{F}_1 \cup \boldsymbol{F}_\infty.$$

Next we investigate the singular points of $L^*(P; \zeta, Y_0)$. Put

$$f(x; a) = y_2(x; a)/y_1(x; a),$$

which is a rational function, then we have

$$y_1(x; a) = f'(x; a)^{-1/2};\ y_2(x; a) = f'(x; a)^{-1/2} f(x; a).$$

We define the connection matrices C_j $(j=1, 2, \cdots, n)$

$$C_j = \widehat{W}(Y(x; 0))\widehat{W}(Y(x; a_j))^{-1}, \tag{1.13}$$

where $\widehat{W}(Y(x))=(Y(x), (d/dx)Y(x))$ is the Wronskian matrix.

First suppose that $b(\zeta)$ is one of the nonsingular zeros of $\zeta\times Y(x; 0)$, i.e., $P(x)$ is holomorphic at $x=b(\zeta)$ and $\zeta\times Y(b(\zeta); 0)=0$. Since $\zeta\times Y(x; 0)$ is a nontrivial solution of the second order differential equation (2), $x=b(\zeta)$ is a simple zero. Therefore one verifies that

$$(\partial/\partial x)q(x;\zeta)+(x-b(\zeta))^{-2}=(\partial/\partial x)^2\log\zeta\times Y(x;0)+(x-b(\zeta))^{-2}$$

is holomorphic at $x=b(\zeta)$, i.e., by (4),

$$P^*(x;\zeta)-2(x-b(\zeta))^{-2}=P(x)-2((\partial/\partial x)q(x;\zeta)+(x-b(\zeta))^{-2})$$

is holomorphic at $x=b(\zeta)$. Hence, it follows that $x=b(\zeta)$ is a regular singular point of equation (1.11) such that its characteristic exponents are equal to 2 and -1. Moreover the integrand of the right hand side of (1.10) turns out to be rational in x and holomorphic at $x=b(\zeta)$, i.e., $x=b(\zeta)$ is a non-logarithmic singular point of (1.11). Thus, it follows that $x=b(\zeta)$ is an apparent singular point of equation (1.11).

Next we investigate $L^*(P;\zeta, Y_0)$ at the singular points $x=a_j$ $(j=1, 2, \cdots, n)$ of $L(P)$ itself. Suppose $\zeta\times C_j\neq\infty$, i.e., $\xi_1c_{11}(j)+\xi_2c_{21}(j)\neq 0$, where $\zeta=[\xi_1:\xi_2]$ and $C_j=(c_{ik}(j))$ is defined by (1.13). Then we have

$$\begin{aligned} q(x:\zeta) &= (\partial/\partial x)\log\zeta\times C_jY(x;a_j) \\ &= (\partial/\partial x)\log(y_1(x;a_j)+\kappa_j(\zeta)y_2(x;a_j)), \end{aligned}$$

where $\kappa_j(\zeta)=(\xi_1c_{12}(j)+\xi_2c_{22}(j))/(\xi_1c_{11}(j)+\xi_2c_{21}(j))$. By (1.8$\pm$), we have

$$y_1(x;a_j)+\kappa_j(\zeta)y_2(x;a_j)=(x-a_j)^{\lambda_-(a_j)}\textstyle\sum_{\nu=0}^{\infty}c_\nu(a_j)(x-a_j)^\nu,$$

where $c_0(a_j)\neq 0$. Hence

$$\partial q(x;\zeta)/\partial x+\lambda_-(a_j)(x-a_j)^{-2}=(\partial/\partial x)^2\log\textstyle\sum_{\nu=0}^{\infty}c_\nu(a_j)(x-a_j)^\nu \tag{1.14}$$

follows and the right hand side of (1.14) is holomorphic at $x=a_j$. This implies that if $\zeta\times C_j\neq\infty$ then $P^*(x;\zeta)-(\alpha_j+2\lambda_-(a_j))(x-a_j)^{-2}$ is holomorphic at $x=a_j$. Similarly, if $\zeta\times C_j=\infty$ then $P^*(x;\zeta)-(\alpha_j+2\lambda_+(a_j))(x-a_j)^{-2}$ is holomorphic at $x=a_j$. By (1.7), we have

$$\alpha_j+2\lambda_\pm(a_j)=4^{-1}(n(a_j)\pm 1)(n(a_j)\pm 3).$$

Thus we have shown

Lemma 1.4. *If $L(P)\in\boldsymbol{F}$ then $P^*(x;\zeta)$ is expressed by the partial fraction*

$$P^*(x;\zeta)=\textstyle\sum_{j=1}^{n}\alpha_j^*(x-a_j)^{-2}+2\sum_{i=1}^{m}(x-b_i(\zeta))^{-2},$$

where $b_i(\zeta)$ $(i=1, 2, \cdots, m)$ are the nonsingular zeros of $\zeta\times Y(x;0)$ and

$$\alpha_j^*=\begin{cases}4^{-1}(n(a_j)-1)(n(a_j)-3), & \text{if } \zeta\times C_j\neq\infty\\ 4^{-1}(n(a_j)+1)(n(a_j)+3), & \text{if } \zeta\times C_j=\infty.\end{cases}$$

Moreover $b_i(\zeta)$ $(i=1, 2, \cdots, m)$ are the apparent singular points of $L^*(P;\zeta, Y_0)$.

Next we classify the singular point $x=a_j$ of $L(P)\in\boldsymbol{F}$ according to whether

$x=a_j$ is the logarithmic singular point of $L^*(P;\zeta, Y_0)$ or not. Put $\zeta_j=\zeta\times C_j^{-1}$, where C_j is defined by (1.13), then we have

Lemma 1.5. (1) *$x=a_j$ is the nonlogarithmic singular point of $L^*(P;\zeta_j, Y_0)$.*
(2) *There are only three possibilities:*
(i) *$P^*(x;\zeta)$ is holomorphic at $x=a_j$ for any $\zeta\neq\zeta_j$.*
(ii) *$x=a_j$ is a non-logarithmic singular point of $L^*(P;\zeta, Y_0)$ for any $\zeta\neq\zeta_j$.*
(iii) *$x=a_j$ is a logarithmic singular point of $L^*(P;\zeta, Y_0)$ for any $\zeta\neq\zeta_j$.*

Proof. Suppose $\zeta=[1:t_0]\neq\infty$. We have

$$(1.15)\qquad (y_1(x;0)+t_0y(x;0))^2=(\rho_{10}(\zeta;j)y_1(x;a_j)+\rho_{20}(\zeta;j)y_2(x;a_j))^2,$$

where $\rho_{k0}=\rho_{k0}(\zeta;j)=c_{1k}(j)+t_0c_{2k}(j)$ $(k=1, 2)$. Since $L(P)\in\boldsymbol{F}$, the both sides of (1.15) are rational in x. Note that

$$g_j(t_0)=D_{a_j}^{-1}(\rho_{10}y_1(x;a_j)+\rho_{20}y_2(x;a_j))^2-D_0^{-1}(y_1(x;0)+t_0y_2(x;0))^2$$

is independent of x. Therefore, by (1.10), we have

$$(1.16)\qquad y_{20}^*(x;\zeta)=\frac{-g_j(t_0)+D_{a_j}^{-1}(\rho_{10}y_1(x;a_j)+\rho_{20}y_2(x;a_j))^2}{\rho_{10}y_1(x;a_j)+\rho_{20}y_2(x;a_j)}.$$

By (1.8$\pm$), one verifies that

$$(\rho_{10}y_1(x;a_j)+\rho_{20}y_2(x;a_j))^2-\rho_{10}^2(x-a_j)^{1-n(a_j)}\phi_j(x)$$

is holomorphic at $x=a_j$, where

$$\begin{aligned}\phi_j(x)&=((x-a_j)^{-(1-n(a_j))/2}y_1(x;a_j))^2\\&=(\textstyle\sum_{\nu=0}^{\infty}k_\nu^-(a_j)(x-a_j)^\nu)^2,\end{aligned}$$

which is holomorphic at $x=a_j$ (cf. (1.8$-$).). Hence we have

$$(1.17)\qquad\begin{aligned}\gamma(\zeta;j)&=\operatorname{Res}(y_1(x;0)+t_0y_2(x;0))^2|_{x=a_j}\\&=\rho_{10}(\zeta;j)^2\operatorname{Res}(x-a_j)^{1-n(a_j)}\phi_j(x)|_{x=a_j}\\&=\rho_{10}(\zeta;j)^2\textstyle\sum_{\nu=0}^{n(a_j)-2}k_\nu^-(a_j)k_{n(a_j)-\nu-2}^-(a_j).\end{aligned}$$

Now we prove (1) in the case $\zeta_1\neq\infty$. By lemma 1.4, $x=a_j$ is the regular singular point of (1.11) for $\zeta=\zeta_j$. Note that $y_{10}^*(x;\zeta)$ has no logarithmic singular point. Moreover, since $\rho_0(\zeta_j;j)=0$, $\gamma(\zeta_j;j)=0$ follows, i.e., $x=a_j$ is the non-logarithmic singular point of $y_{20}^*(x;\zeta_j)$ by (1.16). The proof in case of $\zeta_j=\infty$ is parallel to the above. Thus (1) has been proved. Next we consider $L^*(P;\zeta_0, Y_0)$ for $\zeta\neq\zeta_j$. First we assume $\zeta_j=\infty$. Then $\rho_{10}(\zeta;j)\neq0$ holds for any $\zeta\neq\infty$. By lemma 1.4, $P^*(x;\zeta)$ is holomorphic at $x=a_j$ for any $\zeta\neq\infty$ if and only if $n(a_j)=3$. Now let $n(a_j)\neq3$ then, by lemma 1.4, $x=a_j$ is the regular singular point of (1.11) for any $\zeta \neq \infty$. Moreover, by (1.17), $x=a_j$ is non-logarithmic if and only if

$$\sum_{\nu=0}^{n(a_j)-2} k_\nu^-(a_j) k_{n(a_j)-\nu-2}^-(a_j) = 0 . \tag{1.18}$$

Note that (1.18) is independent of ζ. Hence (2) has been proved in case of $\zeta_j=\infty$. The proof in case of $\zeta_j \neq \infty$ can been obtained in the similar way.

We say that the singular point $x=a_j$ of $L(P) \in \boldsymbol{F}$ is of *L-type* if and only if $x=a_j$ is the logarithmic singular point of (1.11) for any $\zeta \neq \zeta_j$. Next we have

Lemma 1.6. (1) *$L(P) \in \boldsymbol{F}_\infty$ if and only if $L(P) \in \boldsymbol{F}$ has no singular points of L-type.*
(2) *Let $a_{j_1}, a_{j_2}, \cdots, a_{j_k}$ $(1 \leqslant j_1 < j_2 \cdots < j_k \leqslant n)$ be all of the singular points of L-type of $L(P) \in \boldsymbol{F}$. Then $L(P) \in \boldsymbol{F}_1$ if and only if*

$$\# \{\infty \times C_{j_s}^{-1} \mid s = 1, 2, \cdots, k\} = 1 . \tag{1.19}$$

Proof. (1) holds true obviously. Now suppose that (1.19) is valid. Put $\zeta_0 = \infty \times C_{j_s}^{-1}$ $(s=1, 2, \cdots, k)$. Then $L^*(P; \zeta_0, Y_0) \in \boldsymbol{F}$ follows from lemma 1.5 (1). Moreover, if $\zeta \neq \zeta_0$ then, by lemma 1.5 (2), $x=a_{j_s}$ $(s=1, 2, \cdots, k)$ are logarithmic singular points of $y_{2\mu}^*/y_{1\mu}^*$ $(\mu=0, \infty)$, i.e., $L^*(P; \zeta, Y_0) \notin \boldsymbol{F}$. Thus $L(P) \in \boldsymbol{F}_1$ follows. Next suppose

$$\# \{\infty \times C_{j_s}^{-1} \mid s = 1, 2, \cdots, k\} \geqslant 2 .$$

Then we can assume without loss of generality that $x=a_j$ $(j=1, 2)$ are of L-type and $\zeta_1 \neq \zeta_2$, where $\zeta_j = \infty \times C_j^{-1}$ $(j=1, 2)$. By lemma 1.5 (2), $z=a_1$ is the logarithmic singular point of (1.11) for $\zeta=\zeta_2$, i.e., $L^*(P; \zeta_2, Y_0) \notin \boldsymbol{F}$. Moreover, since $x=a_2$ is of L-type, if $\zeta \neq \zeta_2$ then $x=a_2$ is the logarithmic singular point of (1.11) for any $\zeta \neq \zeta_2$, i.e., $L^*(P; \zeta, Y_0) \notin \boldsymbol{F}$ for any $\zeta \neq \zeta_2$. Thus we have shown that $L(P) \in \boldsymbol{F}_0$. Now suppose $L(P) \in \boldsymbol{F}_1$ then, by the above, we have

$$\# \{\infty \times C_{j_s}^{-1} \mid s = 1, 2, \cdots, k\} < 2 .$$

Hence, from (1) of this lemma, (1.19) follows.

2. The monodromy matrices of $Y^*(x; \zeta)$

In this section we investigate how the monodromy matrices of $Y_\mu^*(x; \zeta)$ $(\mu=0, \infty)$ depend on the deformation parameters t_μ $(\mu=0, \infty)$ respectively by calculating them exactly.

Suppose that $L(P) \in \boldsymbol{F}$ and $S=\{a_1, a_2, \cdots, a_n, \infty\}$ is the set of all regular singular points of (2). Let $x_0 \in X = \boldsymbol{P}_1 \backslash S$ and Γ_j $(j=1, 2, \cdots, n)$ be the anticlockwise closed circuit around $x=a_j$ respectively such that $x_0\in\Gamma_j$ and Γ_j does not contain other singular points inside. Note that since $b_i(\zeta)$ $(i=1,2,\cdots,m)$, the non-singular zeros of $\zeta\times Y(x;0)$, are apparent singularities of (1.11), it suffices to investigate the monodromy matrices only for Γ_j $(j=1,2,\cdots,n)$. Let $M_\mu(\Gamma_j;\zeta)$ $(\mu=0,\infty;\zeta\in U_\mu,j=1,2,\cdots,n)$ be the monodromy matrix of $Y^*_\mu(x;\zeta)$ along Γ_j respectively;

$$(2.1)\qquad Y^*_\mu(x_0\Gamma_j;\zeta)=M_\mu(\Gamma_j;\zeta)Y^*_\mu(x_0;\zeta),$$

where $U_0=\boldsymbol{P}_1\backslash\{\infty\}$, $U_\infty=\boldsymbol{P}_1\backslash\{0\}$ and $f(x\Gamma_j)$ is the analytic prolongation of $f(x)$ along Γ_j.

Suppose $\zeta=[1:t_0]\in U_0\backslash\{\zeta_j\}$, where $\zeta_j=\infty\times C_j^{-1}$, i.e., $\zeta\neq\infty$ and $\rho_{10}=\rho_{10}(\zeta;j)=c_{11}(j)+t_0c_{21}(j)\neq 0$. Then, by (1.8±), (1.9) and (1.15), we have

$$y^*_{10}(x;\zeta)=\frac{(x-a_j)^{-\lambda_-(a_j)}}{\rho_{10}\phi_{-j}(x)+\rho_{20}(x-a_j)^{\lambda_+(a_j)-\lambda_-(a_j)}\phi_{+j}(x)},$$

where $\rho_{20}=\rho_{20}(\zeta;j)=c_{12}(j)+t_0c_{22}(j)$,

$$\begin{aligned}\phi_{-j}(x)&=(x-a_j)^{-\lambda_-(a_j)}y_1(x;a_j)\\&=\Sigma_\nu k_\nu^-(a_j)(x-a_j)^\nu\end{aligned}$$

and

$$\begin{aligned}\phi_{+j}(x)&=(x-a_j)^{-\lambda_+(a_j)}y_2(x;a_j)\\&=\Sigma_\nu k_\nu^+(a_j)(x-a_j)^\nu .\end{aligned}$$

Hence, from (1.7),

$$(2.2)\qquad y^*_{10}(x_0\Gamma_j;\zeta)=(-1)^{n(a_j)-1}y^*_{10}(x_0;\zeta)$$

follows. Moreover, by (1.16) and (1.17), we have

$$(2.3)\qquad \begin{aligned}y^*_{20}(x_0\Gamma_j;\zeta)=&(-1)^{n(a_j)-1}2\pi i\rho_{10}(\zeta;j)^2d_jy^*_{10}(x_0;\zeta)\\&+(-1)^{n(a_j)-1}y^*_{20}(x_0;\zeta),\end{aligned}$$

where $d_j=\sum_{\nu=0}^{n(a_j)-2}k_\nu^-(a_j)k^-_{n(a_j)-\nu-2}(a_j)$. Next suppose $\zeta_j=[1:t_{0j}]\neq\infty$. Then, since $\rho_{10}(\zeta_j;j)=0$, we have

$$y^*_{10}(x;\zeta_j)=\pm 1/\rho_{20}(\zeta;j)y_2(x;a_j)$$

and, by (1.16),

$$y^*_{20}(x;\zeta_j)=\{-g_j(t_{0j})+\rho_{20}^2D_{a_j}^{-1}y_2(x;a_j)^2\}/\rho_{20}y_2(x;a_j),$$

where $\rho_{k0}=\rho_{k0}(\zeta_j;j)$. Hence, by (1.8±) and lemma 1.5, we have

$$(2.4)\qquad y^*_{k0}(x_0\Gamma_j;\zeta_j)=(-1)^{n(a_j)+1}y^*_{k0}(x_0;\zeta_j),\quad k=1,2.$$

Combining (2.2), (2.3) and (2.4), one verifies

$$Y_0^*(x_0\Gamma_j;\zeta)=(-1)^{n(a_j)-1}\,{}^t(y_{10}^*(x_0;\zeta),\\ 2\pi i\rho_{10}(\zeta;j)^2d_jy_{10}^*(x_0;\zeta)+y_{20}^*(x_0;\zeta))$$

for any $\zeta\in U_0$. Similarly we can show that

$$Y_\infty^*(x_0\Gamma_j;\zeta)=(-1)^{n(a_j)-1}\,{}^t(y_{1\infty}^*(x_0;\zeta),\\ 2\pi i\rho_{1\infty}(\zeta;j)^2d_jy_{1\infty}^*(x_0;\zeta)+y_{2\infty}^*(x_0;\zeta))$$

holds for any $\zeta=[t_\infty:1]\in U_\infty$, where $\rho_{1\infty}(\zeta;j)=t_\infty c_{11}(j)+c_{21}(j)$. Thus we have shown

$$M_\mu(\Gamma_j;\zeta)=(-1)^{n(aj)-1}\begin{bmatrix}1 & 0\\ 2\pi i\rho_{1\mu}(\zeta;j)^2d_j & 1\end{bmatrix},\quad \zeta\in U_\mu,\ \mu=0,\infty. \tag{2.5}$$

On the other hand, we say that $L^*(P;\zeta,Y_0)$ is isomonodromic with respect to $t_\mu\in\Omega\subset\boldsymbol{C}$ ($\mu=0$ or ∞), where Ω is a connected open set, if and only if there exists the fundamental system $Z_\mu(x;t_\mu)$ of solutions of (1.11) such tha t the monodromy matrices of $Z_\mu(x;t_\mu)$ along Γ_j ($j=1,2,\cdots,n$) are independent of $t_\mu\in\Omega$. Then, one can see easily that $L^*(P;\zeta,Y_0)$ is isomonodromic with respect to $t_\mu\in\Omega$ if and only if there exist $K_\mu(t_\mu)\in GL(2,\boldsymbol{C})$, $t_\mu\in\Omega$ such that $K_\mu(t_\mu)M_\mu(\Gamma_j;\zeta)K_\mu(t_\mu)^{-1}$ ($j=1,2,\cdots,n$) are independent of $t_\mu\in\Omega$ respectively. First we have

Lemma 2.1. *Let $L(P)\in F$. If $L^*(P;\zeta,Y_0)$ is isomonodromic with respect to $t_\mu\in\boldsymbol{C}$ for $\mu=0$ or $\mu=\infty$ then $L(P)\notin\boldsymbol{F}_0$.*

Proof. Assume that $L^*(P;\zeta,Y_0)$ is isomonodromic with respect to $t_0\in\boldsymbol{C}$ and $L(P)\in\boldsymbol{F}_0$. Then, by lemma 1.6, there are at least two singular points $x=a_{j_s}$ ($s=1,2$) of L-type such that

$$\infty\times C_{j_1}^{-1}\neq\infty\times C_{j_2}^{-1},$$

i.e., $[-c_{21}(j_1):c_{11}(j_1)]\neq[-c_{21}(j_2):c_{11}(j_2)]$. This implies

$$c_{11}(j_1)c_{21}(j_2)-c_{21}(j_1)c_{11}(j_2)\neq 0. \tag{2.6}$$

Moreover, there exists $K_0(t_0)\in GL(2,\boldsymbol{C})$ such that $K_0(t_0)M_0(\Gamma_j;\zeta)K_0(t_0)^{-1}$ are independent of $t_0\in\boldsymbol{C}$. By direct calculation, we have

$$K_0(t_0)M_0(\Gamma_j;\zeta)K_0(t_0)^{-1}=(-1)^{n(a_j)-1}\{E+2\pi i\rho_{10}^2d_jA(\zeta)\}, \tag{2.7}$$

where $A(\zeta)=K_0(t_0)\begin{bmatrix}0&0\\1&0\end{bmatrix}K_0(t_0)^{-1}$. Therefore, $B_j=\rho_{10}(\zeta;j)^2A(\zeta)$ are independent of t_0. Suppose $\rho_{10}(\zeta;j_1)\neq 0$ then we have

$$B_{j_2}=(\rho_{10}(\zeta;j_2)/\rho_{10}(\zeta;j_1))^2B_{j_1}.$$

Since $L^*(P;\zeta, Y_0)$ is isomonodromic with respect to t_0,

$$\rho_{10}(\zeta;j_2)/\rho_{10}(\zeta;j_1)=(c_{11}(j_2)+t_0c_{21}(j_2))/(c_{11}(j_1)+t_0c_{21}(j_1))$$

is independent of t_0. This implies

$$c_{11}(j_1)c_{21}(j_2)-c_{21}(j_1)c_{11}(j_2)=0\,.$$

By (2.6), this is contradiction. The proof in the case that $L^*(P;\zeta, Y_0)$ is isomonodromic with respect to t_∞ is parallel to the above.

Next we have

Lemma 2.2. *Let $L(P)\in \boldsymbol{F}_1$. Then there exists one and only one $\zeta_*\in \boldsymbol{P}_1$ such that $L^*(P;\zeta, Y_0)$ is isomonodromic with respect to $t_\mu\in\Omega_\mu$, where*

$$\Omega_0=\{t_0\,|\,[1:t_0]\in U_0\backslash\{\zeta_*\}\}$$

and

$$\Omega_\infty=\{t_\infty\,|\,[t_\infty:1]\in U_\infty\backslash\{\zeta_*\}\}\,,$$

and the monodromy group of (1.11) *for $\zeta=\zeta_*$ is not isomorphic to that of* (1.11) *for $\zeta\neq\zeta_*$.*

Proof. Let $a_{j_1}, a_{j_2}, \cdots, a_{j_k}$ be all of the singular points of L-type of $L(P)\in\boldsymbol{F}_1$. Then, by lemma 1.6, $\infty\times C_{j_s}^{-1}$ $(s=1, 2, \cdots, k)$ coincide with each other and we denote it by ζ_*;

$$\zeta_*=\infty\times C_{j_s}^{-1}\,,\qquad s=1, 2, \cdots, k\,. \tag{2.8}$$

Suppose $t_0\in\Omega_0$, i.e., $\rho_{10}(\zeta;j_s)\neq 0$ $(s=1, 2, \cdots, k)$ for $\zeta=[1:t_0]$. Moreover, from (2.8), it follows that

$$c_s=\rho_{10}(\zeta;j_s)/\rho_{10}(\zeta;j_1)\,,\qquad s=1, 2, \cdots, k$$

are nonzero constants. Put

$$K_0(t_0)=\begin{bmatrix}1 & 0\\ 0 & (2\pi i)^{-1}\rho_{10}(\zeta;j_1)^{-2}\end{bmatrix},\qquad t_0\in\Omega_0$$

then, by (2.5), we have

$$K_0M_0(\Gamma_{j_s};\zeta)K_0^{-1}=(-1)^{n(a_{j_s})-1}\begin{bmatrix}1 & 0\\ c_s^2d_{j_s} & 1\end{bmatrix},\qquad \zeta\in U_0\backslash\{\zeta_*\}\,,$$

which are independent of $t_0\in\Omega_0$. Moreover, if $j\neq i_s$ $(s=1, 2, \cdots, k)$ then we have

$$M_0(\Gamma_j;\zeta)=(-1)^{n(a_j)-1}E\,,\qquad \zeta\in U_0\backslash\{\zeta_*\}\,,$$

where E is the unit matrix of size 2. Hence the monodromy matrix of $K_0Y_0^*(x;\zeta)$

along Γ_j $(j \neq j_s: s=1, 2, \cdots, k)$ coincides with $(-1)^{n(a_j)-1}E$. Therefore the monodromy group of $K_0 Y_0^*(x; \zeta)$ is independent of $t_0 \in \Omega_0$. On the other hand, if $\zeta_* \neq \infty$ then, by (2.5), we have

$$M_0(\Gamma_{j_s}; \zeta_*) = (-1)^{n(a_{j_s})-1}E .$$

Since $c_s^2 d_{j_s} \neq 0$ $(s=1, 2, \cdots, k)$, the monodromy group of $Y_0^*(x; \zeta_*)$ is not isomorphic to that of $Y_0^*(x; \zeta)$, $\zeta \in U_0 \setminus \{\zeta_*\}$. The proof in case of $\zeta_* = \infty$ is parallel to the above. Moreover the proof for $Y_\infty^*(x; \zeta)$ can be obtained in the similar way.

Finally we obtain the characterization of $\boldsymbol{F}_\infty$ in connection with the isomonodromic deformation.

Theorem 2.3. *$L(P) \in \boldsymbol{F}_\infty$ if and only if $L^*(P; \zeta, Y_0)$ is isomonodromic with respect to $t_\mu \in \boldsymbol{C}$ for both $\mu=0$ and $\mu=\infty$.*

Proof. First suppose $L(P) \in \boldsymbol{F}_\infty$. Then, by lemma 1.6, $L(P)$ has no singular points of L-type. Hence, by (2.5), we have

$$M_\mu(\Gamma_j; \zeta) = (-1)^{n(a_j)-1}E , \qquad \mu = 0, \infty .$$

This implies that $L^*(P; \zeta, Y_0)$ is isomonodromic with respect to $t_\mu \in \boldsymbol{C}$ for both $\mu = 0$ and $\mu = \infty$. Conversely, suppose that $L^*(P; \zeta, Y_0)$ is isomonodromic with respect to $t_\mu \in \boldsymbol{C}$ for both $\mu=0$ and $\mu=\infty$. Then, by lemma 2.1, $L(P) \notin \boldsymbol{F}_0$ follows. Next if we assume $L(P) \in \boldsymbol{F}_1$ then, by lemma 2.2, there exists one and only one $\zeta_* \in \boldsymbol{P}_1$ such that the monodromy group of (1.11) for $\zeta = \zeta_*$ is not isomorphic to that of (1.11) for $\zeta \neq \zeta_*$. This is contradiction. Thus, $L(P) \notin \boldsymbol{F}_1$ follows. This completes the proof.

3. Recursion formula

In this section, apart from the preceding sections, we do not necessarily assume that $L(u) = D^2 - u(x)$ is of Fuchsian type, but assume only that $u(x)$ is the single valued meromorphic function of x.

Define $Q_j(x)$ $(j=0, 1, 2, \cdots)$ by the recursion formulae

$$(3.1) \qquad 2Q_n'(x) = Q_{n-1}(x)u'(x) + 2Q_{n-1}'(x)u(x) - 2^{-1}Q_{n-1}'''(x) , \qquad n = 1, 2, \cdots$$

with $Q_0(x) = 1$. The formula (3.1) appears in the theory of commutative differential operators due to Burchnall-Chaundy [5]. Then we have

Lemma 3.1 (cf. Tanaka [22]). *$Q_n(x)$ are the polynomials of $u, u', \cdots, u^{(2n-2)}$ with constant coefficients, where $u^{(m)}$ is the m-th derivative of u.*

Proof. We prove this by induction. Assume that $Q_j(x)$ $(j=0, 1, \cdots, n)$

are polynomials of $u, u', \cdots, u^{(2j-2)}$ with constant coefficients respectively. By (3.1) we have

$$2Q'_{j+1}Q_{n-j} = u'Q_jQ_{n-j} + 2uQ'_jQ_{n-j} - 2^{-1}Q'''_jQ_{n-j}, \qquad j = 0, \cdots, n.$$

Hence

$$2\sum_{j=0}^{n} Q'_{j+1}Q_{n-j} = u'\sum_{j=0}^{n} Q_jQ_{n-j} + 2u\sum_{j=0}^{n} Q'_jQ_{n-j} - 2^{-1}\sum_{j=0}^{n} Q'''_jQ_{n-j}$$

follows. Since $Q_0 = 1$, this implies

$$\begin{aligned} 2Q'_{n+1} &= -2\sum_{j=0}^{n-1} Q'_{j+1}Q_{n-j} + u'\sum_{j=0}^{n} Q_jQ_{n-j} \\ &\qquad + 2u\sum_{j=0}^{n} Q'_jQ_{n-j} - 2^{-1}\sum_{j=0}^{n-1} Q'''_{j+1}Q_{n-j} \\ &= -\Big(\sum_{j=0}^{n-1} Q_{j+1}Q_{n-j}\Big)' + \Big(u\sum_{j=0}^{n} Q_jQ_{n-j}\Big)' \\ &\qquad - 2^{-1}\Big(\sum_{j=0}^{n} Q''_jQ_{n-j}\Big)' + 4^{-1}\Big(\sum_{j=0}^{n} Q'_jQ'_{n-j}\Big)'. \end{aligned}$$

Therefore Q_{n+1} is also a polynomial of $u, u', \cdots, u^{(2n)}$ with constant coefficients.

If we regard $Q'_n(x)$ as the polynomial of $u, u', \cdots, u^{(2n-1)}$ with constant coefficients then the constant term of $Q'_n(x)$ is equal to zero. On the other hand, while an arbitrary additive constant appears when we integrate $Q'_n(x)$ to obtain $Q_n(x)$, we set it zero in what follows. Therefore $Q_n(x)$ $(n=1, 2, \cdots)$ are determined uniquely. Put

$$Z_n(u(x)) = 2Q_{n+1}(x), \qquad n = 0, 1, 2, \cdots \tag{3.2}$$

and

$$X_n(u(x)) = 2Q'_{n+1}(x) = DZ_n(u(x)), \qquad n = 0, 1, 2, \cdots. \tag{3.3}$$

For example, we obtain by direct calculation

$$\begin{aligned} &Z_0(u) = u, \quad X_0(u) = u' \\ &Z_1(u) = 4^{-1}(3u^2 - u''), \quad X_1(u) = 4^{-1}(6uu' - u'''). \end{aligned}$$

Rewriting (3.1) in terms of Z_n and X_n, we obtain the Lenard relation (7);

$$X_n(u) = 2^{-1}Z_{n-1}(u)u' + X_{n-1}(u)u - 4^{-1}D^2X_{n-1}(u). \tag{3.4}$$

Next we investigate the relation between the Darboux transformation and X_n, which plays the crucial role in the following. Let $Y = Y(x) = {}^t(y_1(x), y_2(x))$ be a fundamental system of solutions of

$$L(u)y = 0$$

such that $W(Y) = 1$. Here we consider the Darboux transformation

$$L^*(u; \zeta, Y) = D^2 - u^*(x; \zeta)$$

of $L(u)$ by $Y(x)$, where

$$u^*(x;\zeta)=u(x)-2(\partial/\partial x)^2\log\zeta\times Y(x)\,.$$

Note that

$$(3.5)\qquad \begin{aligned} u(x) &= \partial v(x;\zeta)/\partial x+v(x;\zeta)^2\,,\\ u^*(x;\zeta) &= -\partial v(x;\zeta)/\partial x+v(x;\zeta)^2 \end{aligned}$$

are valid, where

$$v=v(x;\zeta)=(\partial/\partial x)\log\zeta\times Y(x)\,.$$

We have.

Theorem 3.2. *The equality*

$$(3.6)\qquad X_n(u^*)+2vZ_n(u^*)=-X_n(u)+2vZ_n(u)$$

holds.

Proof. By direct calculation,

$$X_0(u^*)+2vZ_0(u^*)=-v_{xx}+2v^3$$

and

$$-X_0(u)+2vZ_0(u)=-v_{xx}+2v^3$$

follow from (3.5). Next assume

$$(3.7)\qquad X_{n-1}(u^*)+2vZ_{n-1}(u^*)=-X_{n-1}(u)+2vZ_{n-1}(u)\,,$$

then, by operating with the linear differential operator

$$-D^3+v_x/vD^2+4v^2D$$

on both sides of (3.7), we have

$$(3.8)\qquad \textstyle\sum_{j=0}^4 K_{-j}(v)D^jZ_{n-1}(u^*)=\sum_{j=0}^4 K_{+j}(v)D^jZ_{n-1}(u)\,,$$

where

$$\begin{aligned} K_{\pm0}(v) &= -2v_{xxx}+2v_{xx}v_x/v+8v^2v_x\,,\\ K_{\pm1}(v) &= -6v_{xx}+4v_x^2/v+8v^3\,,\\ K_{\pm2}(v) &= -4(v_x\pm v^2)\,,\\ K_{\pm4}(v) &= -(2v\pm v_x/v) \end{aligned}$$

and

$$K_{\pm4}(v)=\pm1\,.$$

From the Lenard relation (3.4) and (3.5),

$$(3.9)\qquad D^3Z_{n-1}(u)=4(v_x+v^2)DZ_{n-1}(u)+2(v_{xx}+2vv_x)Z_{n-1}(u)-4X_n(u)$$

follows. Differentiating both sides of (3.9), we have

$$D^4Z_{n-1}(u)=4(v_x+v^2)D^2Z_{n-1}(u)+6(v_{xx}+2vv_x)DZ_{n-1}(u)+2(v_{xxx}+2vv_{xx}+2v_x^2)Z_{n-1}(u)-4DX_n(u)\,. \tag{3.10}$$

Eliminate $D^jZ_{n-1}(u)\,(j=3,4)$ by (3.9) and (3.10) from the right hand side of (3.8) then one verifies that the right hand side of (3.8) coincides with

$$-4DX_n(u)+4(2v+v_x/v)X_n(u)\,.$$

By similar calculation, the left hand side of (3.8) turns out to coincide with

$$4DX_n(u^*)+4(2v-v_x/v)X_n(u^*)\,.$$

Hence we have

$$DX_n(u^*)+(2v-v_x/v)X_n(u^*)=-DX_n(u)+(2v+v_x/v)X_n(u)\,. \tag{3.11}$$

By (3.11) we have

$$2(X_n(u^*)-X_n(u))=v_x(X_n(u^*)+X_n(u))/v^2-(DX_n(u^*)+DX_n(u))/v\,. \tag{3.12}$$

Integrating both side of (3.12), we obtain

$$2(Z_n(u^*)-Z_n(u))=-(X_n(u^*)+X_n(u))/v\,.$$

This completes the proof.

4. Rational solution of the n-th KdV equation

In this section we construct a class of solutions, which are rational in x, of the n-th KdV equation

$$\partial u(x;\xi)/\partial\xi=X_n(u(x;\xi))\,,$$

where ξ is a complex variable. We emphasize here that the operator $L(P)$ investigated in this section is of Fuchsian type on $\boldsymbol{P}_1$.

Now suppose $L(P)\in\boldsymbol{F}$. Then $X_n(P)$ and $Z_n(P)$ vanish at $x=\infty$. Therefore, since we assume that the additive constant which appears on the occasion of integrating $X_n(P)$ to obtain $Z_n(P)$ is zero, $X_n(P)=0$ if and only if $Z_n(P)=0$. Hence, by the Lenard relation (7), if $X_n(P)=0$ then $X_{n+j}(P)=0$ holds for $j>0$. Let Λ_n be the set ofall rat ional functions $P=P(x)$ such that $L(P)\in\boldsymbol{F}$, $X_j(P)\neq 0$ for $j=0,1,\cdots,n-1$ and $X_n(P)=0$.

Naturally, if Λ_n is void, all arguments in what follows are vacuous. However it will be shown at the end of this section that Λ_n actually not void.

First we have

Lemma 4.1. *Suppose* $P(x)\in\Lambda_n$ *and put*

$$\Sigma(P) = \{\zeta \mid X_n(P^*(x;\zeta)) \equiv 0\}$$

then

$$\# \Sigma(P) \leqslant 1$$

is valid.

Proof. Put

$$\begin{aligned} w(x;\zeta) &= X_{n-1}(P^*(x;\zeta)) + 2q(x;\zeta)Z_{n-1}(P^*(x;\zeta)) \\ &= -X_{n-1}(P(x)) + 2q(x;\zeta)Z_{n-1}(P(x)) . \end{aligned}$$

By the Lenard relation (7), we have

$$\begin{aligned} 4^{-1}w_{xx} + 2^{-1}qw_x &= 2^{-1}(2qq_x + q_{xx})Z_{n-1}(P) + (q_x + q^2)X_{n-1}(P) - 4^{-1}D^2X_{n-1}(P) \\ &= 2^{-1}P_xZ_{n-1}(P) + PX_{n-1}(P) - 4^{-1}D^2X_{n-1}(P) \\ &= X_n(P) = 0 . \end{aligned}$$

Similarly we have

$$-4^{-1}w_{xx} + 2^{-1}qw_x = X_n(P^*) .$$

Assume $\#\Sigma(P) \geqslant 2$ and let $\zeta_j \in \Sigma(P)$ $(j=1, 2; \zeta_1 \neq \zeta_2)$ then

$$w_x(x;\zeta_j) \equiv 0 , \quad j = 1, 2 \tag{4.1}$$

follows. On the other hand, since $L(P) \in \boldsymbol{F}$, i.e., $f(x;0) = y_2(x;0)/y_1(x;0)$ is a rational function, by (1.3), $q(x;\zeta)$ is of the form

$$\textstyle\sum_j \beta_j(x-a_j)^{-1} + \sum_i (x-b_i)^{-1} .$$

Moreover $X_{n-1}(P)$ and $Z_{n-1}(P)$ are the polynomials of $P^{(s)}$ $(s=1, 2, \cdots, 2n-3)$ whose constant terms are zero. Hence

$$w(x;\zeta_j)\,|_{x=\infty} = 0 , \quad j = 1, 2$$

follows. Therefore, by (4.1), we have

$$w(x;\zeta_j) \equiv 0 , \quad j = 1, 2 .$$

This implies

$$q(x;\zeta_j) = 2^{-1}X_{n-1}(P(x))/Z_{n-1}(P(x)) , \quad j = 1, 2 ,$$

because $Z_{n-1}(P(x)) \not\equiv 0$. Since $W(Y(x;0)) \equiv 1$ and

$$q(x;\zeta) = (\xi_1 y_1'(x;0) + \xi_2 y_2'(x;0))/(\xi_1 y_1(x;0) + \xi_2 y_2(x;0)) ,$$

it turns out that if $\zeta_1 \neq \zeta_2$ then $q(x;\zeta_1) \not\equiv q(x;\zeta_2)$. This is contradiction.

The following is the one of the main results of the present paper.

Theorem 4.2. *If $P(x)\in\Lambda_n$ then there exist the rational functions $c_\mu(t_\mu)$ $(\mu=0, \infty)$ such that*

$$(4.3)\qquad c_\mu(t_\mu)\partial P^*(x;\zeta)/\partial t_\mu = X_n(P^*(x;\zeta)), \quad \mu=0, \infty,$$

where $\zeta=[1:t_0]$ for $\zeta\neq\infty$; $\zeta=[t_\infty:1]$ for $\zeta\neq 0$.

Proof. By lemma 4.1, if $\zeta\notin\Sigma(P)$ then we have

$$\begin{aligned}2q(x;\zeta) &= -X_n(P^*(x;\zeta))/Z_n(P^*(x;\zeta))\\ &= -(\partial/\partial x)\log Z_n(P^*(x;\zeta)).\end{aligned}$$

Suppose $\zeta=[1:t_0]\in U_0\backslash\Sigma(P)$ then we have

$$(\partial/\partial x)\log g_0(x;t_0)^2Z_n(P^*(x;\zeta))\equiv 0,$$

where

$$g_0(x;t_0)=y_1(x;0)+t_0y_2(x;0).$$

This implies that there exists $c_0(t_0)$ depending on only t_0 such that

$$(4.4)\qquad g_0(x;t_0)^2Z_n(P^*(x;\zeta))=-2c_0(t_0).$$

The left hand side of (4.4) has meaning even at $\zeta\in\Sigma(P)\cap U_0$, that is, $c_0(t_0)=0$ if $[1:t_0]\in\Sigma(P)\cap U_0$. Moreover, $c_0(t_0)$ does not vanish for $\zeta=[1:t_0]\in U_0\backslash\Sigma(P)$, since

$$g_0(x;t_0)\not\equiv 0$$

and

$$Z_n(P^*(x;\zeta))\not\equiv 0.$$

One can see immediately that since $L(P)\in\boldsymbol{F}$, the left hand side of (4.4) is rational in x and t_0. Hence $c_0(t_0)$ is also rational in t_0. From (4.4),

$$4c(_0t_0)\partial g_0(x;t_0)/\partial x/g_0(x;t_0)^3=X_n(P^*(x;\zeta))$$

follows. On the other hand, by direct calculation, we have

$$\partial P^*(x;\zeta)/\partial t_0=4\partial g_0(x;t_0)/\partial x/g_0(x;t_0)^3.$$

Thus we have

$$c_0(t_0)\partial P^*(x;\zeta)/\partial t_0=X_n(P^*(x;\zeta)).$$

Similarly we obtain

$$c_\infty(t_\infty)\partial P^*(x;\zeta)/\partial t_\infty=X_n(P^*(x;\zeta))$$

for some rational function $c_\infty(t_\infty)$.

Note that the equations (4.4) themselves are not the original n-th *KdV* equation (8). By the way, since $t_0=1/t_\infty$ for $t_\infty\neq 0$, we can show readily

$$c_\infty(t_\infty) = -t_\infty^2 c_0(1/t_\infty)\,.$$

Hence, if we put

$$\phi_\mu(t) = \int c_\mu(t)^{-1}dt\,, \quad \mu = 0,\,\infty\,,$$

then

$$\phi_0(t_0) = \int c_0(t_0)^{-1}dt_0 = \int c_\infty(t_\infty)^{-1}dt_\infty = \phi_\infty(t_\infty)$$

holds for $t_\mu\neq 0$ ($\mu=0, \infty$). Therefore, if $P(x)\in\Lambda_n$ and $\zeta\neq 0, \infty$ then $P^*(x; \zeta)$ satisfies the original n-th KdV equation

$$\partial P^*(x;\,\zeta)/\partial\xi = X_n(P^*(x;\,\zeta))\,,$$

where $\xi=\phi_\mu(t_\mu)$ ($\mu=0, \infty$).

Next we reconsider the meaning of Theorem 4.2 in view of the isomonodromic deformation. Suppose $\zeta=[1: t_0]\in U_0\backslash\Sigma(P)$. Then, since $c_0(t_0)\neq 0$,

$$a_0(x;\,t_0) = 2^{-1}c_0(t_0)^{-1}Z_{n-1}(P^*(x;\,\zeta)) \tag{4.5}$$

is rational in x. By Theorem 4.2, we have

$$B(\zeta)a_0(x;\,t_0) = \partial P^*(x;\,\zeta)/\partial t_0\,, \tag{4.6}$$

where

$$B(\zeta) = -2^{-1}D^3+2P^*(x;\,\zeta)D+P_x^*(x;\,\zeta)\,.$$

Let $\zeta_*=[1: t_*]\in\Sigma(P)\cap U_0$. Then, by taking limit of the both sides of (4.6) for $t_0\to t_*$, $a_0(x; t_0)$ turns out to be meaningful even for ζ_* and rational in x. Similarly we can show that

$$a_\infty(x;\,t_\infty) = 2^{-1}c_\infty(t_\infty)^{-1}Z_{n-1}(P^*(x;\,\zeta))$$

is rational in x and satisfies

$$B(\zeta)a_\infty(x;\,t_\infty) = \partial P^*(x;\,\zeta)/\partial t_\infty \tag{4.7}$$

for any $\zeta=[t_\infty: 1]\in U_\infty$. Next define $b_\mu(x; t_\mu)$ ($\mu=0, \infty$) by

$$2\partial b_\mu(x;\,t_\mu)/\partial x+\partial^2 a_\mu(x;\,t_\mu)/\partial x^2 = 0\,, \quad \mu = 0,\,\infty\,\cdot \tag{4.8}$$

Then, $b_\mu(x; t_\mu)$ ($\mu=0, \infty$) are rational in x. By (4.6) and (4.7), we have

$$\partial^2 b_\mu/\partial x^2+2P^*\partial a_\mu/\partial x+a_\mu\partial P^*/\partial x-2\partial P^*/\partial t_\mu = 0\,. \tag{4.9}$$

One verifies that (4.8) and (4.9) are nothing but the integrability conditions for the system

$$(4.10)\qquad \begin{aligned} &L^*(P;\zeta, Y_0)z = \partial^2 z/\partial x^2 - P^*(x;\zeta)z = 0\,, \quad \zeta\in U_\mu \\ &\partial z/\partial t_\mu = a_\mu \partial z/\partial x + b_\mu z\,. \end{aligned}$$

Let $Z_\mu(x; t_\mu) = {}^t(z_1(x; t_\mu), z_2(x; t_\mu))$ be the fundamental system of solutions of system (4.10). Let $N_\mu(\Gamma_j; t_\mu)$ be the monodromy matrix of $Z_\mu(x; t_\mu)$ along Γ_j $(j=1, 2, \cdots, n;\ \mu=0, \infty)$;

$$Z_\mu(x\Gamma_j; t_\mu) = N_\mu(\Gamma_j; t_\mu) Z_\mu(x; t_\mu)\,.$$

Then we have

$$\partial Z_\mu(x\Gamma_j; t_\mu)/\partial t_\mu = N_\mu(\Gamma_j; t_\mu)\partial Z_\mu(x; t_\mu)/\partial t_\mu + \partial N_\mu(\Gamma_j; t_\mu)/\partial t_\mu Z_\mu(x; t_\mu)$$

and

$$\partial Z_\mu(x\Gamma_j; t_\mu)/\partial x = N_\mu(\Gamma_j; t_\mu)\partial Z_\mu(x; t_\mu)/\partial x\,.$$

Therefore, from (4.10),

$$\partial Z_\mu(x\Gamma_j; t_\mu)/\partial t_\mu = N_\mu(\Gamma_j; t_\mu)\partial Z_\mu(x; t_\mu)/\partial t_\mu$$

follows, since $a_\mu(x; t_\mu)$ and $b_\mu(x; t_\mu)$ are rational in x for any $t_\mu \in \boldsymbol{C}$. Hence we have

$$\partial N_\mu(\Gamma_j; t_\mu)/\partial t_\mu Z_\mu(x; t_\mu) = 0\,.$$

Differentiating this with respect to x, we obtain

$$\partial N_\mu(\Gamma_j; t_\mu)/\partial t_\mu (Z_\mu, \partial Z_\mu/\partial x) = 0\,.$$

Since the Wronskian matrix $(Z_\mu, \partial Z_\mu/\partial x)$ is nondegenerate,

$$\partial N_\mu(\Gamma_j; t_\mu)/\partial t_\mu = 0$$

follows, i.e., $L^*(P;\zeta, Y_0)$ is isomonodromic with respect to both t_μ $(\mu=0, \infty)$. Hence, by Theorem 2.3, we have

Theorem 4.3. *If $P\in\Lambda_n$ then $L(P)\in \boldsymbol{F}_\infty$ follows, that is, $L^*(P;\zeta, Y_0)\in\boldsymbol{F}$ for any $\zeta\in\boldsymbol{P}_1$.*

Next we have

Theorem 4.4. *Let $P\in\Lambda_n$. If $\zeta\in\Sigma(P)$ then $P^*(x;\zeta)\in\Lambda_m$ for some $m\leqslant n$ and if $\zeta\notin\Sigma\ (P)$ then $P^*(x;\zeta)\in\Lambda_{n+1}$.*

Proof. From Theorem 4.3, $L(P^*) = L^*(P;\zeta, Y_0)\in\boldsymbol{F}$ follows for any $\zeta\in\boldsymbol{P}_1$. Suppose $\zeta\in\Sigma(P)$ then we have

$$X_n(P^*(x;\zeta))\equiv 0\,.$$

Hence, if $\zeta\in\Sigma(P)$ then $P^*(x;\zeta)\in\Lambda_m$ is valid for some $m\leqslant n$. Next let $\zeta=[1:t_0]$ $\in U_0$. Then, by (4.4), we have

(4.11) $$Z_n(P^*(x;\zeta)) = -2c_0(t_0)/g_0(x;t_0)^2\,.$$

Differentiating the both sides of (4.11) with respect to x, we have the following:

(4.12) $$\begin{aligned} X_n(P^*) &= 4c_0 g_{0x}/g_0^3\,,\\ DX_n(P^*) &= 4c_0(g_{0xx}g_0-3g_{0x}^2)/g_0^4\,,\\ D^2X_n(P^*) &= 4c_0(g_{0xxx}g_0^2-9g_{0xx}g_{0x}g_0+12g_{0x}^3)/g_0^5\,. \end{aligned}$$

On the other hand,

(4.13) $$P^*(x;\zeta) = (-g_{0xx}g_0+2g_{0x}^2)/g_0^2$$

is valid, since $1/g_0(x;t_0)$ solves the equation (1.11) for $\mu=0$. From (4.13)

(4.14) $$\partial P^*(x;\zeta)/\partial x = (-g_{0xxx}g_0^2+5g_{0xx}g_{0x}g_0-4g_{0x}^3)/g_0^3$$

follows. Moreover, by the Lenard relation (7),

(4.15) $$X_{n+1}(P^*) = 2^{-1}Z_n(P^*)P_x^*+X_n(P^*)P^*-4^{-1}D^2X_n(P^*)$$

is valid. Put (4.11), (4.12), (4.13) and (4.14) into the right hand side of (4.15), then, by direct calculation, one verifies that the right hand side of (4.15) vanishes identically. Thus we have

$$X_{n+1}(P^*(x;\zeta)) \equiv 0$$

for any $\zeta\in U_0$. Similarly we can show

$$X_{n+1}(P^*(x;\infty)) \equiv 0\,.$$

On the other hand, by lemma 4.1,

$$X_n(P^*(x;\zeta)) \not\equiv 0$$

holds for $\zeta\notin\Sigma(P)$. Hence, by Theorem 4.3, we have shown that $P^*(x;\zeta)\in\Lambda_{n+1}$ for any $\zeta\notin\Sigma(P)$.

Put

$$\Lambda^* = \cup_{n=0}^{\infty}\Lambda_n$$

and

$$\boldsymbol{F}^* = \{L(P)\,|\,P\in\Lambda^*\}\,.$$

Then we have

Theorem 4.5.
(1) $\boldsymbol{F}^*\subset\boldsymbol{F}_\infty$.
(2) $\Lambda_n\neq\emptyset$, $n=0, 1, 2, \cdots$.

Proof. From Theorem 4.3, (1) follows immediately. On the other hand, one can see easily

$$\Lambda_0 = \{0\}.$$

Now assume that $\Lambda_{n-1} \neq \emptyset$ and let $P \in \Lambda_{n-1}$ then, by Theorem 4.4, $P^*(x; \zeta) \in \Lambda_n$ holds for any $\zeta \notin \Sigma(P)$. Therefore

$$\Lambda_n \neq \phi$$

follows.

Thus we have proved that if $L(P) \in \boldsymbol{F}^*$ then $L^*(P; \zeta, Y_0) \in \boldsymbol{F}^*$ for any $\zeta \in \boldsymbol{P}_1$. In other words, $\boldsymbol{F}^*$ is closed under the Darboux transformation. Moreover we have

Theorem 4.6. *For every $L(P_0) \in \boldsymbol{F}^*$, there exist $L(P) \in \boldsymbol{F}^*$ and $\zeta_0 \in \boldsymbol{P}_1$ such that*

$$L(P_0) = L^*(P; \zeta_0, Y_0).$$

Proof. Put

$$P(x) = P_0(x) - 2(\partial/\partial x)^2 \log [1: 0] \times Z(x; 0),$$

where $Z(x; 0) = {}^t(z_1(x; 0), z_2(x; 0))$ is the normalized fundamental system of solutions of

$$L(P_0)z = 0$$

defined in section 1. Then

$$Y(x) = {}^t(z_1(x; 0)^{-1}, z_1(x; 0)^{-1} D_0^{-1}(z_1(x; 0)^2))$$

is the fundamental system of solutions of

$$L(P)y = 0. \tag{4.16}$$

We have

$$L^*(P; 0, Y) = L(P_0).$$

There exists $C \in SL(2, \boldsymbol{C})$ such that

$$Y(x) = CY(x; 0),$$

where $Y(x; 0)$ is the normalized fundamental system of solutions of (4.16). Hence, put

$$\zeta_0 = [1: 0] \times C$$

then we have

$$L(P_0) = L^*(P; \zeta_0, Y_0).$$

Thus we have shown that there exists the orbit of the n-th KdV flow on Λ^* passing through for every $P \in \Lambda^*$.

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