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Unbiasedness in the Test of Goodness of Fit

By Masashi OKAMOTO

1. Introduction. Let X_1, \dots, X_N be a random sample from the population with the *d.f.* $F(x)$. We are asked to test the hypothesis H_0 that $F(x)$ is identical with a specified continuous *d.f.* $F_0(x)$ against all alternatives. For this purpose we shall use the multinomial distribution, dividing the real line into n intervals $(a_{i-1}, a_i]$, $i = 1, \dots, n$, where $a_0 = -\infty$ and $a_n = +\infty$, so that $F_0(a_i) - F_0(a_{i-1}) = 1/n$, $i = 1, \dots, n$. If a_i are not determined uniquely, we may take any values satisfying the conditions. Put $p_i = F(a_i) - F(a_{i-1})$ and denote by N_i the number of X 's that fall into the interval $(a_{i-1}, a_i]$. Then, of course, $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n N_i = N$. Denote, further, by W the space consisting of n -dimensional lattice points (k_1, \dots, k_n) , where k_i is regarded as the observed value of the random variable N_i (therefore, $\sum_{i=1}^n k_i = N$).

The test is equivalent with determining the set (acceptance region) in the space W . The set S in W will be called symmetric provided that, if S contains the point (k_1, \dots, k_n) , then S contains also all its permutations (k_1', \dots, k_n') . We shall say, finally, that S satisfies condition O when, if S contains (k_1, \dots, k_n) such as $k_j \geq k_i + 2$, then S contains also $(k_1, \dots, k_i + 1, \dots, k_j - 1, \dots, k_n)$. It is easily verified that if S is symmetric the convexity implies the condition O . The converse, however, is not necessarily true. For example, we shall consider, in the case $N = 12$, $n = 3$, the set S consisting of nine points shown in Fig. 1 and their permutations. S is symmetric and satisfies the condition O , but is not convex, since the middle point $(7, 4, 1)$ of the points $(8, 2, 2)$, $(6, 6, 0)$ does not belong to S .

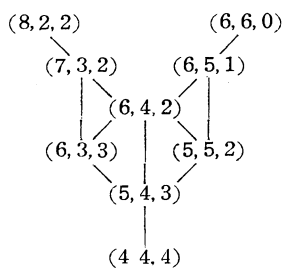


Fig. 1

2. Theorem of unbiasedness.

Theorem. *If the acceptance region R of the test is symmetric and satisfies the condition O , the test of H_0 is unbiased against any alternative.*

Proof. Putting

$$P = \sum_{(k_1, \dots, k_n) \in R} \frac{N!}{k_1! \dots k_n!} p_1^{k_1} \dots p_n^{k_n},$$

we have to prove that P is maximum when $p_1 = \dots = p_n$.

Since R is symmetric, P is a symmetric function in p_1, \dots, p_n . Thus we have only to prove that, if $p_1 < p_2$,

$$P(x) = \sum_{(k_1, \dots, k_n) \in R} \frac{N!}{k_1! \dots k_n!} (p_1 + x)^{k_1} (p_2 - x)^{k_2} p_3^{k_3} \dots p_n^{k_n}$$

is monotonically increasing in x , when $0 \leq x \leq (p_2 - p_1)/2$. In the sequel we shall consider x only in this range.

For any $(n-1)$ -tuple (k, k_3, \dots, k_n) such that $k + k_3 + \dots + k_n = N$, let $R_{kk_3 \dots k_n}$ be the subset of R consisting of (k_1, \dots, k_n) such that $k_1 + k_2 = k$. (Some may be null set.) Then $R_{kk_3 \dots k_n}$ are disjoint and exhaust R . Therefore

$$P(x) = \sum_{(k, k_1, \dots, k_n)} P_{kk_3 \dots k_n}(x) \quad (1)$$

where

$$P_{kk_3 \dots k_n}(x) = \sum \frac{N!}{k_1! \dots k_n!} (p_1 + x)^{k_1} (p_2 - x)^{k_2} p_3^{k_3} \dots p_n^{k_n},$$

\sum extending over all n -tuples (k_1, \dots, k_n) belonging to $R_{kk_3 \dots k_n}$.

Since R is symmetric and satisfies the condition O , all $R_{kk_3 \dots k_n}$ are symmetric and satisfy the condition O with respect to k_1, k_2 . Thus, if not null set,

$$R_{kk_3 \dots k_n} = \left\{ (i, k-i, k_3, \dots, k_n) : j \leq i \leq k-j \right\},$$

where j is a non-negative integer $\leq k/2$, depending on k, k_3, \dots, k_n , and so

$$\begin{aligned} P_{kk_3 \dots k_n}(x) &= \sum_{i=j}^{k-j} \frac{N!}{i!(k-i)!k_3! \dots k_n!} (p_1 + x)^i (p_2 - x)^{k-i} p_3^{k_3} \dots p_n^{k_n} \\ &= \frac{N!}{k!k_3! \dots k_n!} p_3^{k_3} \dots p_n^{k_n} \sum_{i=j}^{k-j} \binom{k}{i} (p_1 + x)^i (p_2 - x)^{k-i}. \end{aligned} \quad (2)$$

Put

$$B_j(x) = \sum_{i=j}^{k-i} \binom{k}{i} (p_1 + x)^i (p_2 - x)^{k-i}. \quad (3)$$

If $j = 0$,

$$B_0(x) = \sum_{i=0}^k \binom{k}{i} (p_1 + x)^i (p_2 - x)^{k-i} = (p_1 + p_2)^k. \quad (4)$$

If $1 \leq j \leq k/2$, denoting by the prime the derivative with respect to x .

$$B'_j(x) = \frac{k!}{(j-1)!(k-j)!} \left\{ (p_1 + x)^{j-1} (p_2 - x)^{k-j} - (p_1 + x)^{k-j} (p_2 - x)^{j-1} \right\}.$$

Since $j-1 < k-j$, $p_1+x \leq p_2-x$, we have

$$B_j(x) \geq 0, \text{ for } 1 \leq j \leq k/2. \quad (5)$$

With (2), (3), (4) and (5), we have

$$P'_{kk_3 \dots k_n}(x) \geq 0.$$

This and (1) imply

$$P'(x) \geq 0,$$

and this completes the proof.

3. Applications.

(1) The χ^2 -test. The acceptance region R of the χ^2 -test consists of the points (k_1, \dots, k_n) such that

$$\sum_{i=1}^n (k_i - N/n)^2 \leq c^2,$$

where c is a constant depending on the level of significance of the test. R is readily seen to be symmetric. In order to verify the condition O , we have only to show that

$$(k_1+1-N/n)^2 + (k_2-1-N/n)^2 < (k_1-N/n)^2 + (k_2-N/n)^2,$$

when $k_2 \geq k_1+2$. This inequality follows from the relations

$$\begin{aligned} (k_1+1-a)^2 - (k_1-a)^2 &= 2k_1+1-2a \\ &< 2k_2-1-2a = (k_2-a)^2 - (k_2-1-a)^2. \end{aligned}$$

Thus, by the theorem of the preceding section, the χ^2 -test of H_0 is unbiased. This fact was mentioned by H. B. Mann and A. Wald [1], but as they used the Taylor expansion of the power, it is only the local unbiasedness that they proved.

(2) David's test. The acceptance region R of David's test [2] consists of (k_1, \dots, k_n) such that at most c k 's are zero, where c is again a constant depending on the level of significance.

R is symmetric. As for the condition O , let $A = (k_1, \dots, k_n) \in R$ and $k_j \geq k_i+2$. If $k_i = 0$, the number of zeroes in $B = (k_1, \dots, k_i+1, \dots, k_j-1, \dots, k_n)$ is smaller by one than that in A . If $k_i > 0$, both are equal. Therefore $B \in R$, and the condition O is satisfied.

Thus, David's test is also unbiased. The author proved it in his recent paper [3], but the proof was lacking in generality and simpleness.

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