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# ON SCATTERING THEORY OF A FIRST ORDER ORDINARY DIFFERENTIAL OPERATOR

# ETSURO DATE

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In this paper we study the scattering theory of the operator

(0.1) 
$$L_{u} = iID - i \begin{bmatrix} 0 & u_{1} & u_{2} \\ u_{1}^{*} & 0 & 0 \\ u_{2}^{*} & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$
$$D = d/dx, \quad u = {}^{t}(u_{1}, u_{2})$$

where  $u_1$  and  $u_2$  are complex-valued integrable functions, and \* denotes complexconjugate.

The operator  $L_u$  has been introduced by Manakov [4] in order to solve the system of non-linear Schrödinger equations

$$(0.2) iu_t + 2^{-1}u_{xx} + ||u||^2 u = 0, ||u||^2 = |u_1|^2 + |u_2|^2$$

in terms of the scattering data of (0.1).

In [4], Manakov has described the scattering theory of (0.1) formally. The scattering theory of (0.1) resembles in many respects to that of the operator

$$(0.3) L_{u} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D - i \begin{bmatrix} 0 & u \\ u^{*} & 0 \end{bmatrix}$$

which has been introduced by Zakharov and Shabat [7] in order to solve the scalar non-linear Schrödinger equation. The scattering theory of (0.3) has been treated by Tanaka [6] in detail.

The main differences between (0.1) and (0.3) appear in the structure of the scattering matrix (S-matrix) (§2) and in the part of the inverse problem (§5). The S-matrix is the operator which relates the asymptotic behavior of solutions of eigenvalue problem  $L_{u}f = \zeta f$  as  $x \to -\infty$  to the asymptotic behavior as  $x \to \infty$ . As the suitable base of this eigenspace, we take Jost solutions. In the case of (0.3) the S-matrix is  $2 \times 2$  type and has the form

$$\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$$

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with analogous properties as in the case of one-dimensional Schrödinger operator [2]. In our case the S-matrix is a  $3 \times 3$  unitary matrix  $(a_{\alpha\beta})_{1 \le \alpha,\beta \le 3}$ . The element  $a_{11}(\zeta)$  is holomorhic in the upper half plane Im  $\zeta > 0$  ( $\zeta$  is the parameter of the eigenvalue problem  $L_{\alpha}f=\zeta f$ ), and  $a_{\alpha\beta}(\zeta)$  ( $\alpha,\beta=2,3$ ) are holomorphic in the lower half plane Im  $\zeta < 0$ . Let  $\zeta_j$  ( $j=1, \dots, N$ ) be the zeros of  $a_{11}(\zeta)$  in the upper half plane. The elements  $a_{\alpha\beta}(\zeta)$  ( $\alpha,\beta=2,3$ ) and their values at  $\zeta_j^*$  play important roles in deriving the system of integral equations of the inverse problem. At  $\zeta=\zeta_j, \zeta_j^*$ , Jost solutions are linearly dependent and coefficients of the linear relation are related to  $a_{\alpha\beta}(\zeta_j^*)$  ( $\alpha, \beta=2, 3$ ). These relations do not appear explicitly in [4]. Whole S-matrix is determined from its first row and pair of complex numbers associated to each zero  $\zeta_j$ . Actual construction is much more complicated than the analogous one for the case of (0.3).

In solving the inverse problem, we meet a new problem which does not arise in the cases of one-dimensional Schrödinger operator and (0.3). In our formulation of the problem we must show certain relations among the solutions of the fundamental integral equations of the inverse problem. Though we have no complete proof, we can show that these relations hold for the case of reflectionless potentials which correspond to the S-matrix of the case  $a_{12}(\xi) \equiv a_{13}(\xi) \equiv 0$ .

In  $\S$  and 2 we treat the direct scattering theory. In \$ we reconstruct the S-matrix from the scattering data. In \$ we derive the fundamental integral equations of the inverse problem. In \$ we study the inverse problem.

Throughout the paper integrations are taken over  $(-\infty, \infty)$  unless explicitly indicated. For a complex matrix  $M, M^{\dagger}$  denotes its adjoint matrix *i.e.*  $M^{\dagger} = {}^{t}M^{*}$ .

The author expresses his hearty gratitude to Professor S. Tanaka for suggesting this problem and for his attention to this work.

#### 1. Jost solutions

We consider on the whole real line the eigenvalue problem

(1.1) 
$$L_{u}f = \zeta f, \quad \zeta = \xi + i\eta, \quad \xi, \eta \in \mathbf{R}$$

where the operator  $L_{\mu}$  is defined by (0.1).

In case of  $u_1 = u_2 = 0$ , the functions

$$\begin{aligned} \theta_1(x,\,\zeta) &= {}^t(1,\,0,\,0) \exp\left(-i\zeta x\right) \\ \theta_2(x,\,\zeta) &= {}^t(0,\,1,\,0) \exp\left(i\zeta x\right) \\ \theta_3(x,\,\zeta) &= {}^t(0,\,0,\,1) \exp\left(i\zeta x\right) \end{aligned}$$

form a complete system of solutions of (1.1).

Put  $v(x) = |u_1(x)| + |u_2(x)|$ , and throughout the paper we assume that  $v(x) \in L^1(-\infty, \infty)$ . Then put

$$\sigma_{\pm}(x) = \pm \int_{x}^{\pm\infty} v(y) dy$$

For a three-dimensional vector  $y = {}^{t}(y_1, y_2, y_3)$  we denote

$$|y| = \max(|y_1|, |y_2|, |y_3|).$$

First we have

**Theorem 1.1.** There exist unique solutions  $\phi_{\alpha}$ ,  $\psi_{\beta}$  ( $\alpha$ ,  $\beta=1, 2, 3$ ) of (1.1) such that

$$\begin{aligned} |\phi_1(x,\,\zeta) - \theta_1(x,\,\zeta)| &\leq C\sigma_+(x) \exp(\eta x) & \eta \leq 0 \\ |\phi_\alpha(x,\,\zeta) - \theta_\alpha(x,\,\zeta)| &\leq C\sigma_+(x) \exp(-\eta x) & \eta \geq 0 \quad \alpha = 2, 3 \end{aligned}$$

as  $x \rightarrow \infty$  and

$$\begin{aligned} |\psi_{1}(x,\,\zeta) - \theta_{1}(x,\,\zeta)| &\leq C\sigma_{-}(x) \exp(\eta x) \qquad \eta \geq 0 \\ |\psi_{\alpha}(x,\,\zeta) - \theta_{\alpha}(x,\,\zeta)| &\leq C\sigma_{-}(x) \exp(-\eta x) \qquad \eta \leq 0 \quad \alpha = 2, 3 \end{aligned}$$

as  $x \to -\infty$ .  $\psi_1$ ,  $\phi_2$  and  $\phi_3$  are holomorphic in  $\zeta$  in the upper half plane and  $\phi_1$ ,  $\psi_2$  and  $\psi_3$  are holomorphic in  $\zeta$  in the lower half plane.

Proof. Substituting the expression

$$f(x) = lpha(x) heta_1(x,\,\zeta) + eta(x) heta_2(x,\,\zeta) + \gamma(x) heta_3(x,\,\zeta)$$

into (1.1), we have

$$\begin{aligned} \alpha'(x) &= (u_1(x)f_2(x) + u_2(x)f_3(x)) \exp(i\zeta x) \\ \beta'(x) &= -u_1^*(x)f_1(x) \exp(-i\zeta x) \\ \gamma'(x) &= -u_2^*(x)f_1(x) \exp(-i\zeta x) . \end{aligned}$$

From these relations we obtain the integral equations for  $\phi_{\alpha}$ :

(1.2) 
$$\phi_{a}(x, \zeta) - \theta_{a}(x, \zeta)$$

$$= \begin{pmatrix} -\int_{x}^{\infty} (u_{1}(y) \phi_{a2}(y, \zeta) + u_{2}(y) \phi_{a3}(y, \zeta)) \exp(i\zeta(y-x)) dy \\ \int_{x}^{\infty} u_{1}^{*}(y) \phi_{a1}(y, \zeta) \exp(i\zeta(x-y)) dy \\ \int_{x}^{\infty} u_{2}^{*}(y) \phi_{a1}(y, \zeta) \exp(i\zeta(x-y)) dy \end{pmatrix}$$

These integral equations can be solved by the method of successive approximation yielding the required estimates as in [1, p. 22]. Q.E.D.

We call the solutions  $\phi_{\alpha}$ ,  $\psi_{\beta}$  Jost solutions of (1.1).

Put

(1.3)  

$$\begin{aligned}
\phi_1(x,\,\zeta) &= \exp\left(-i\zeta x\right)k_1(x,\,\zeta) \\
\phi_{\alpha}(x,\,\zeta) &= \exp\left(i\zeta x\right)k_{\alpha}(x,\,\zeta) & \alpha = 2, 3 \\
\psi_1(x,\,\zeta) &= \exp\left(-i\zeta x\right)h_1(x,\,\zeta) \\
\psi_{\alpha}(x,\,\zeta) &= \exp\left(i\zeta x\right)h_{\alpha}(x,\,\zeta) & \alpha = 2, 3.
\end{aligned}$$

For  $k_{\alpha}$ ,  $h_{\beta}$ , assume that the integral representations

$$k_{1}(x, \zeta) = {}^{t}(1, 0, 0) + \int_{0}^{\infty} K_{1}(x, y) \exp(-2i\zeta y) dy$$

$$k_{2}(x, \zeta) = {}^{t}(0, 1, 0) + \int_{0}^{\infty} K_{2}(x, y) \exp(2i\zeta y) dy$$

$$k_{3}(x, \zeta) = {}^{t}(0, 0, 1) + \int_{0}^{\infty} K_{3}(x, y) \exp(2i\zeta y) dy$$

$$h_{1}(x, \zeta) = {}^{t}(1, 0, 0) + \int_{-\infty}^{0} H_{1}(x, y) \exp(-2i\zeta y) dy$$

$$h_{2}(x, \zeta) = {}^{t}(0, 1, 0) + \int_{-\infty}^{0} H_{2}(x, y) \exp(2i\zeta y) dy$$

$$h_{3}(x, \zeta) = {}^{t}(0, 0, 1) + \int_{-\infty}^{0} H_{3}(x, y) \exp(2i\zeta y) dy$$

hold. Putting them into (1.2), we obtain the integral equations for  $K_{\alpha}$ :

$$\begin{split} K_{11}(x, y) + \int_{x}^{\infty} (u_{1}(z) K_{12}(z, y) + u_{2}(z) K_{13}(z, y)) dz &= 0 \\ K_{12}(x, y) - \int_{x}^{x+y} u_{1}^{*}(z) K_{11}(z, x+y-z) dz &= u_{1}^{*}(x+y) \\ K_{13}(x, y) - \int_{x}^{x+y} u_{2}^{*}(z) K_{11}(z, x+y-z) dz &= u_{2}^{*}(x+y) \\ K_{\alpha 1}(x, y) + \int_{x}^{x+y} (u_{1}(z) K_{\alpha 2}(z, x+y-z) + u_{2}(z) K_{\alpha 3}(z, x+y-z)) dz \\ &= -u_{\alpha - 1}(x+y) \\ K_{\alpha 2}(x, y) - \int_{x}^{\infty} u_{1}^{*}(z) K_{\alpha 1}(z, y) dz = 0 \\ K_{\alpha 3}(x, y) - \int_{x}^{\infty} u_{2}^{*}(z) K_{\alpha 1}(z, y) dz = 0 \quad \alpha = 2, 3. \end{split}$$

Let v(x) be in  $L^{1,\infty} = L^1 \cap L^{\infty}$  (in the following we use this notation) and put

$$m_{\pm}(x) = \mathrm{ess.} \sup_{\pm y \ge \pm x} v(y)$$
.

Then the integral equations for kernels  $K_{\sigma}$  can be solved by the method of successive approximation and the solutions are estimated as

$$|K_{\alpha}(x, y)| \leq Cm_{+}(x+y) \exp(\sigma_{+}(x)) \qquad \alpha = 1, 2, 3.$$

We also have the integral equations for kernels  $H_{\alpha}$  and the estimates

$$|H_{\alpha}(x, y)| \leq Cm_{-}(x+y) \exp(\sigma_{-}(x)) \qquad \alpha = 1, 2, 3$$

hold. Thus we have

**Theorem 1.2.** Suppose that v is in  $L^{1,\infty}$  and  $m_{\pm}(\pm x)$  are in  $L^{1,\infty}(a, \infty)$  for some a. Then Jost solutions have the integral representations (1.3). And the functions  $u_1$  and  $u_2$  are reconstructed by

$$u_1(x) = K_{12}^*(x, 0) = -K_{21}(x, 0) = -H_{12}^*(x, 0) = H_{21}(x, 0)$$
  
$$u_2(x) = K_{13}^*(x, 0) = -K_{31}(x, 0) = -H_{13}^*(x, 0) = H_{31}(x, 0).$$

### 2. Scattering matrix and scattering data

In this section we define the scattering matrix and the scattering data of the operator (0.1), and investigate the properties of them.

Let y(x), z(x) and w(x) be solutions of (1.1), then as is well known the wronskian  $|y, z, w| = \det(y, z, w)$  satisfies the differential equation  $f' = i\zeta f$ . From this fact and the asymptotic behavior of Jost solutions, we have

$$(2.1) \qquad |\phi_1(x,\,\xi),\,\phi_2(x,\,\xi),\,\phi_3(x,\,\xi)| = |\psi_1(x,\,\xi),\,\psi_2(x,\,\xi),\,\psi_3(x,\,\xi)| = \exp(i\xi x)\,.$$

Hence  $\psi_{\alpha}(x, \xi)$  are expressed by means of  $\phi_{\alpha}(x, \xi)$ :

(2.2) 
$$\psi_{\alpha}(x, \xi) = \sum_{\beta=1}^{3} a_{\alpha\beta}(\xi) \phi_{\beta}(x, \xi) \qquad \alpha = 1, 2, 3$$

i.e.

$$(\psi_1(x,\,\xi),\,\psi_2(x,\,\xi),\,\psi_3(x,\,\xi))=(\phi_1(x,\,\xi),\,\phi_2(x,\,\xi),\,\phi_3(x,\,\xi))^t(a_{lphaeta}(\xi))$$

On the other hand, let  $L_{u}y = \zeta_{1}y$ ,  $L_{u}z = \zeta_{2}z$ . Then we can show

$$(z'^{\dagger}y + z^{\dagger}y') + i(\zeta_1 - \zeta_2^*)z^{\dagger}Iy = 0$$

where I is the matrix in (0.1). Therefore in case of  $\zeta_1 = \zeta_2^*$  we have  $(d/dx)(z^{\dagger}y) = 0$ . Using this fact, we have

(2.3) 
$$\phi_{a}^{\dagger}(x,\xi)\phi_{\beta}(x,\xi) = \psi_{a}^{\dagger}(x,\xi)\psi_{\beta}(x,\xi) = \delta_{\alpha\beta}$$

(2.4) 
$$a_{\alpha\beta}(\xi) = \phi_{\beta}^{\dagger}(x, \xi) \psi_{\alpha}(x, \xi)$$

i.e.

$$(2.5) \qquad {}^t\!(a_{\alpha\beta}(\xi)) = (\phi_1(x,\,\xi)\,,\,\phi_2(x,\,\xi)\,,\,\phi_3(x,\,\xi))^\dagger(\psi_1(x,\,\xi)\,,\,\psi_2(x,\,\xi)\,,\,\psi_3(x,\,\xi))$$

and

$$\sum_{\gamma=1}^{3} a^*_{lpha\gamma}(\xi) a_{eta\gamma}(\xi) = \delta_{lphaeta}$$
 .

These relations and (2.1) imply that the matrix  $(a_{\alpha\beta}(\xi))$  is unitary and unimodular (*i.e.* det  $(a_{\alpha\beta}(\xi))=1$ ), and the matrices  $(\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))$ ,  $(\psi_1(x, \xi), \psi_2(x, \xi), \psi_3(x, \xi))$  are unitary ([4]).

Putting the integral representations for  $\phi_{\alpha}$ ,  $\psi_{\beta}$  into (2.4), we obtain the integral representations for  $a_{\alpha\beta}$ :

(2.6)  
$$a_{11}(\xi) = 1 + \int_{0}^{\infty} A_{11}(y) \exp(2i\xi y) dy$$
$$a_{\alpha\beta}(\xi) = \delta_{\alpha\beta} + \int_{0}^{\infty} A_{\alpha\beta}(y) \exp(-2i\xi y) dy \qquad \alpha, \beta = 2, 3$$
$$a_{10}(\xi) = \int A_{10}(y) \exp(-2i\xi y) dy \qquad \alpha = 2, 3$$
$$a_{\alpha1}(\xi) = \int A_{\alpha1}(y) \exp(2i\xi y) dy \qquad \alpha = 2, 3.$$

We write explicite formula only for  $A_{11}$ :

$$A_{11}(y) = H_{11}(x, -y) + K_{11}^*(x, y) + \sum_{\beta=1}^3 \int_0^y K_{1\beta}^*(x, y-z) H_{1\beta}(x, -z) dz.$$

The functions  $A_{\alpha\beta}$  are bounded integrable functions estimated as

$$|A_{11}(y)|, |A_{\alpha\beta}(y)| \leq C(m_{-}(-2^{-1}y) + m_{+}(2^{-1}y)) \qquad y > 0, \ \alpha, \ \beta = 2, \ 3$$
$$|A_{10}(y)|, |A_{\alpha1}(y)| \leq Cm_{\pm}(\pm y) \qquad \pm y > 0 \quad \alpha = 2, \ 3.$$

The coefficient  $a_{11}(\xi)$  is the boundary value of the holomorphic function in the upper half plane and  $a_{\alpha\beta}(\xi)$  ( $\alpha$ ,  $\beta=2$ , 3) are the boundary values of the holomorphic functions in the lower half plane. These holomorphic functions are defined by

(2.7) 
$$\begin{aligned} a_{11}(\zeta) &= \phi_1^{\dagger}(x,\,\zeta^*)\psi_1(x,\,\zeta) & \text{Im } \zeta \ge 0\\ a_{\alpha\beta}(\zeta) &= \phi_{\beta}^{\dagger}(x,\,\zeta^*)\psi_{\alpha}(x,\,\zeta) & \text{Im } \zeta \le 0\,, \quad \alpha,\,\beta = 2,\,3\,. \end{aligned}$$

Next we derive the relations among Jost solutions. For this purpose we begin with following lemma, which is obtained by direct calculation.

**Lemma 2.1.** Let y(x) and z(x) be solutions of (1.1), then the function  $w(x) = \exp(i\zeta^*x)(y^*(x)\times z^*(x))$  satisfies the equation  $L_uw = \zeta^*w$ , where the expression  $y \times z$  denotes the vector product of three-dimensional vectors y, z.

For Jost solutions we have

Lemma 2.2. The relations

(2.8) 
$$\phi_1(x,\,\zeta^*) = \exp\left(i\zeta^*x\right)(\phi_2^*(x,\,\zeta) \times \phi_3^*(x,\,\zeta)) \qquad \text{Im } \zeta \ge 0$$

hold.

Proof. As we have seen at the beginning of this section, the matrix  $(\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))$  is unitary *i.e.* 

$$(\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))^{-1} = (\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))^{\dagger}$$

Calculating the left-hand side according to the formula for the inverse matrix, we have (2.8) for real  $\xi$ :

$$\phi_1(x,\,\xi) = \exp\left(i\xi x\right) \left(\phi_2^*(x,\,\xi) \times \phi_2^*(x,\,\xi)\right).$$

From this relation on the real line we obtain relations among integral kernels:

$$egin{aligned} K_{11}(x,\,y) &= \int_{0}^{y} (K^*_{22}(x,\,y-z) K^*_{33}(x,\,z) - K^*_{23}(x,\,y-z) K^*_{32}(x,\,z)) \, dz \ &+ K^*_{22}(x,\,y) + K^*_{33}(x,\,y) \end{aligned}$$

and similar relations for  $K_{12}$  and  $K_{13}$ . Using these relations, we have (2.8). Q.E,D.

REMARK 1. In case of  $u_2 \equiv 0$ , we have

$$\phi_{\mathfrak{z}}(x,\,\zeta)=\psi_{\mathfrak{z}}(x,\,\zeta)=\theta_{\mathfrak{z}}(x,\,\zeta)$$

and

$$\exp(i\zeta^*x)(y^*(x)\times\theta_3^*(x,\,\zeta))={}^t(y_2^*(x),\,-y_1^*(x),\,0)$$

Thus the operation  $\exp(i\zeta^*x)(y^*(x)\times\theta_3^*(x,\zeta))$  corresponds to the operation # in [6], [7].

From (2.7), (2.8) and (2.9) we obtain another expressions of  $a_{11}(\zeta)$ :

$$\begin{aligned} a_{11}(\zeta) &= \exp\left(-i\zeta x\right) |\psi_1(x,\,\zeta),\,\phi_2(x,\,\zeta),\,\phi_3(x,\,\zeta)| \\ a_{11}^*(\zeta^*) &= \exp\left(-i\zeta^* x\right) |\phi_1(x,\,\zeta^*),\,\psi_2(x,\,\zeta^*),\,\psi_3(x,\,\zeta^*)| \qquad \text{Im } \zeta \geqq 0 \,. \end{aligned}$$

Therefore if  $\zeta_0$  is a zero of  $a_{11}(\zeta)$  in the upper half plane,  $\psi_1$ ,  $\phi_2$ ,  $\phi_3$  are linearly dependent at  $\zeta_0$  and  $\phi_1$ ,  $\psi_2$ ,  $\psi_3$  are linearly dependent at  $\zeta_0^*$ :

(2.10) 
$$\begin{aligned} \psi_1(x,\,\zeta_0) &= c_1\phi_2(x,\,\zeta_0) + c_2\phi_3(x,\,\zeta_0) \\ \phi_1(x,\,\zeta_0^*) &= d_1\psi_2(x,\,\zeta_0^*) + d_2\psi_3(x,\,\zeta_0^*) \,. \end{aligned}$$

There exist relations among  $c_{\alpha}$  and  $d_{\beta}$ , which we need later. The following theorem gives the relations.

**Theorem 2.3.** The relations

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(2.11) 
$$\begin{array}{c} c_1^*d_1 = -a_{33}(\zeta_0^*) & c_1^*d_2 = a_{23}(\zeta_0^*) \\ c_2^*d_1 = a_{32}(\zeta_0^*) & c_2^*d_2 = -a_{22}(\zeta_0^*) \end{array}$$

hold.

Proof. By lemma 2.2 and (2.10), we have

$$(2.12) \quad \exp(i\zeta_0 x)(\psi_2^*(x,\,\zeta_0^*)\times\psi_3^*(x,\,\zeta_0^*)) = c_1\phi_2(x,\,\zeta_0) + c_2\phi_3(x,\,\zeta_0)$$

$$(2.13) \quad \exp((i\zeta_0^*x)(\phi_2^*(x,\zeta_0)\times\phi_3^*(x,\zeta_0)) = d_1\psi_2(x,\zeta_0^*) + d_2\psi_3(x,\zeta_0^*).$$

Taking the vector product of  $\phi_2(x, \zeta_0)$  and (2.12), we have

$$\phi_2^*(x,\,\zeta_0)\times(\psi_2(x,\,\zeta_0^*)\times\psi_3(x,\,\zeta_0^*))=c_2^*\exp\left(i\zeta_0^*x\right)(\phi_2^*(x,\,\zeta_0)\times\phi_3^*(x,\,\zeta_0)).$$

From this and (2.13), we have

$$(2.14) \quad \phi_2^*(x,\,\zeta_0) \times (\psi_2(x,\,\zeta_0^*) \times \psi_3(x,\,\zeta_0^*)) = c_2^* d_1 \psi_2(x,\,\zeta_0^*) + c_2^* d_2 \psi_3(x,\,\zeta_0^*).$$

Using the identity for vector products

$$y \times (z \times w) = ({}^{t}wy)z - ({}^{t}yz)w$$

and the relations (2.7), we conclude that the left-hand side of (2.14) is equal to

$$a_{32}(\zeta_0^*)\psi_2(x, \zeta_0^*) - a_{22}(\zeta_0^*)\psi_3(x, \zeta_0^*).$$
  
Q.E.D.

We now assume that

$$(2.15) \quad a_{11}(\xi) \neq 0$$

for any real  $\xi$ . As the integral representation (2.6) holds with integral kernel  $A_{11}, a_{11}(\zeta) \rightarrow 1$  as  $|\zeta| \rightarrow \infty$ . By this property and (2.15)  $a_{11}(\zeta)$  has only finite number of zeros in the upper half plane. We assume further that those zeros are simple. We denote them by  $\zeta_j$   $(j=1, \dots, N)$  and denote by  $c_{j\alpha}, d_{j\alpha}$   $(j=1, \dots, N, \alpha=1, 2)$  the number associated with  $\zeta_j$  by the relations (2.10).

We define  $r_{\pm \alpha}(\xi)$  ( $\alpha = 1, 2$ ) by

$$\begin{aligned} r_{+\omega}(\xi) &= a_{11}(\xi)^{-1}a_{1,\omega+1}(\xi) \\ r_{-\omega}(\xi) &= a_{11}^*(\xi)^{-1}a_{\omega+1,1}^*(\xi) \,. \end{aligned}$$

We call the matrix  $(a_{\alpha\beta}(\xi))$  the scattering matrix and the collection  $\{r_{+\alpha}(\xi), \zeta_j, c_j(\alpha=1, 2, j=1, \dots, N)\}$  (resp.  $\{r_{-\alpha}(\xi), \zeta_j^*, d_{j\alpha}(\alpha=1, 2, j=1, \dots, N)\}$ ) the right (resp. left) scattering data of the operator  $L_u$  or of the potential  $u=t(u_1, u_2)$ .

We note here that the functions  $r_{\pm a}$  have the expressions

(2.16) 
$$r_{\pm a}(\xi) = \int F_{\pm 0a}(x) \exp\left(-2i\xi x\right) dx , \quad F_{\pm 0a} \in L^{1,\infty} .$$

For, by (2.15) and the Wiener's theorem [3, Theorem W], there exists a function  $g \in L^1$  which satisfies

$$a_{11}(\xi)^{-1} = 1 + \int g(x) \exp((2i\xi x)) dx$$
,

and in view of the fact  $A_{11} \in L^{1,\infty}$ , we can show that  $g \in L^{1,\infty}$ .

REMARK 2. In case of  $u_2 \equiv 0$  the scattering matrix turns out to be

$$\begin{bmatrix} a(\xi) & b(\xi) & 0 \\ -b^*(\xi) & a^*(\xi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where  $a(\xi)$  and  $b(\xi)$  are the coefficients which appear in [6], [7].

REMARK 3. Our arguments in this section work for the operator

(2.17) 
$$L_{\boldsymbol{u}} = i \begin{bmatrix} 1 & 0 \\ 0 & -I_{\boldsymbol{n}} \end{bmatrix} D - i \begin{bmatrix} 0 & {}^{t}\boldsymbol{u} \\ \boldsymbol{u}^{*} & 0 \end{bmatrix}, \quad \boldsymbol{u} = {}^{t}(\boldsymbol{u}_{1}, \cdots, \boldsymbol{u}_{\boldsymbol{n}}),$$

where  $I_n$  is the identity matrix of degree n, by suitable modifications.

## 3. Reconstruction of the scattering matrix from the scattering data

Let the collection  $\{r_{\alpha}(\xi), \zeta_{j}, c_{j\alpha} \ (\alpha=1, 2, j=1, \dots, N)\}$  have the following properties: the functions  $r_{\alpha}(\xi)$  have the expressions

$$r_{a}(\xi) = \int R_{a}(x) \exp((2i\xi x)) dx \quad R_{a} \in L^{1,\infty}$$

 $\zeta_j$  are complex numbers, which satisfy  $\operatorname{Im} \zeta_j > 0$ ,  $\zeta_j \neq \zeta_k$   $(j \neq k)$ , and  $c_{jo}$  are complex numbers, which satisfy  $\prod_{j=1}^{N} (|c_{j1}| + |c_{j2}|) \neq 0$ .

Our purpose in this section is to determine the unitary and unimodular matrix  $(a_{\alpha\beta}(\xi))$  ( $\alpha$ ,  $\beta=1$ , 2, 3) and complex numbers  $d_{j\alpha}$  from this collection so that they satisfy following requirements:

i) the elements  $a_{\alpha\beta}$  have the integral representations (2.6) with bounded integrable kernels  $A_{\alpha\beta}$ ,

ii) the numbers  $\zeta_j$  are the zeros of  $a_{11}(\zeta)$  in the upper half plane and the zeros  $\zeta_j$  of  $a_{11}(\zeta)$  are simple,

iii) the numbers  $c_{j\alpha}$  and  $d_{j\alpha}$  satisfy the relations

$$egin{array}{ll} c_{j\,1}^{*}d_{j1} = -a_{33}(\zeta_{j}^{*}) & c_{j\,1}^{*}d_{j2} = a_{23}(\zeta_{j}^{*}) \ c_{j\,2}^{*}d_{j1} = a_{32}(\zeta_{j}^{*}) & c_{j\,2}^{*}d_{j2} = -a_{22}(\zeta_{j}^{*}) \,, \end{array}$$

iv) the functions  $r_{\alpha}$  are reconstructed by

$$r_{\omega}(\xi) = a_{11}(\xi)^{-1}a_{1,\omega+1}(\xi)$$
.

At first by virtue of Wiener's theorem, there exists a function  $g \in L^{1,\infty}$  which satisfies

$$1+\int g(x) \exp (2i\xi x) dx = (1+|r_1(\xi)|^2+|r_2(\xi)|^2)^{-1}$$

We denote the left-hand side of the above relation by  $f(\boldsymbol{\xi})$ . Since the matrix-valued function

$$F(\xi) = \begin{bmatrix} 1 - f(\xi) |r_1(\xi)|^2 & -f(\xi)r_1^*(\xi)r_2(\xi) \\ -f(\xi)r_1(\xi)r_2^*(\xi) & 1 - f(\xi) |r_2(\xi)|^2 \end{bmatrix}$$

is positive definite, by the factorization theorem of positive definite matrix function [3, Theorem 8.2], there exists a matrix-valued function

$$G_{1}(\zeta) = \begin{bmatrix} 1 + \int_{0}^{\infty} g_{22}(x) \exp(-2i\zeta x) dx & \int_{0}^{\infty} g_{23}(x) \exp(-2i\zeta x) dx \\ \int_{0}^{\infty} g_{32}(x) \exp(-2i\zeta x) dx & 1 + \int_{0}^{\infty} g_{33}(x) \exp(-2i\zeta x) dx \end{bmatrix}$$

where  $g_{\alpha\beta}(x)$  ( $\alpha$ ,  $\beta=2$ , 3) belong to  $L^1$ , which satisfies

 $G_1^{\dagger}(\xi)G_1(\xi) = F(\xi)$ 

and det  $G_1(\zeta) \neq 0$  for Im  $\zeta \leq 0$ . The matrix function  $G_1(\zeta)$  is uniquely determined. By the proof of the theorem we quoted above and the fact  $g_1, R_1, R_2 \in L^{1,\infty}$ , we can show that the function  $g_{\alpha\beta}$  are in  $L^{1,\infty}$ .

Put

$$b_{11}(\zeta) = \prod_{j=1}^{N} (\zeta - \zeta_j) (\zeta - \zeta_j^*)^{-1}, \quad b_j = \dot{b}_{11}(\zeta_j)^{-1}$$

where  $\dot{f} = df/d\zeta$ . We consider the system of linear equations for the unknowns  $d_{j\sigma}$ 

(3.1) 
$$-c_{k_2}^{\prime*}d_{k_2} = 1 - \sum_{j=1}^{N} b_j c_{j1}^{\prime} d_{j1}^* (\zeta_k^* - \zeta_j)^{-1}$$

(3.2) 
$$c_{k_1}^{\prime*}d_{k_2} = -\sum_{j=1}^{N} b_j c_{j2}^{\prime} d_{j1}^{\ast} (\zeta_k^* - \zeta_j)^{-1}$$

(3.3) 
$$c_{k_2}^{\prime*}d_{k_1} = -\sum_{j=1}^N b_j c_{j1}^\prime d_{j2}^* (\zeta_k^* - \zeta_j)^{-1}$$

(3.4) 
$$-c_{k_1}'^* d_{k_1} = 1 - \sum_{j=1}^N b_j c_{j2}' d_j^* (\zeta_k^* - \zeta_j)^{-1}$$

where  $c'_{j\alpha}$  are defined by

(3.5) 
$${}^{t}(c'_{j1}, c'_{j2}) = (\det G_{1}^{*}(\zeta_{j}^{*}))^{-1}G_{1}^{*}(\zeta_{j}^{*})^{t}(c_{j1}, c_{j2}),$$

or in matrix notation

- $(3.1)' \quad -C_2 * d_2 = J ABC_1 d_1 *$
- $(3.2)' \qquad C_1^* d_2 = -ABC_2 d_1^*$

$$(3.3)' \quad C_2^* d_1 = -ABC_1 d_1^*$$

$$(3.4)' \qquad -C_1 * d_1 = J - ABC_2 d_2 *$$

where  $A = ((\zeta_j^* - \zeta_k)^{-1})$ ,  $J = {}^t(1, \dots, 1) d_a = {}^t(d_{1a}, \dots, d_{Na})$  and B,  $C_a$  are diagonal matrices with the entries  $(b_j)$ ,  $(c'_{ja})$  respectively.

This system consists of 4N-equations for 2N-unknowns. However, by virtue of the relations  $(AB)^{-1} = (AB)^*$  and ABJ = -J (see [7, Appendix]), we conclude that (3.1)' is equivalent to (3.4)' and (3.2)' is equivalent to (3.3)'. Thus this system reduces to 2N-equations. Next by elimination, we have

$$(C_1A^*C_1^*+C_2A^*C_2^*)B^*d_{\alpha}=C_{\alpha}J \quad \alpha=1,2.$$

Since

$$i(C_1A^*C_1^*+C_2A^*C_2^*)=(i(c'_{j1}c'_{k1}^*+c'_{j2}c'_{k2}^*)(\zeta_j-\zeta_k^*)^{-1}),$$

we have

$$-i((f_j, f_k))B^*d_{\alpha} = C_{\alpha}J$$

where  $((f_j, f_k))$  denotes the Gram matrix of vector-valued functions  $f_j$ ;

$$f_{j}(x) = {}^{t}(c'_{j1}, c'_{j2}) \exp(i\zeta_{j}x) \quad x \ge 0.$$

Therefore under our assumptions the numbers  $d_{ja}$  are uniquely determined and satisfy the equations (3.1)-(3.4).

Let  $d_{i\alpha}$  be determined as above, and put

$$\begin{split} b_{22}(\zeta) &= 1 - \sum_{j=1}^{N} b_j c'_{j1} d^*_{j1} (\zeta - \zeta_j)^{-1} \\ b_{23}(\zeta) &= - \sum_{j=1}^{N} b_j c'_{j2} d^*_{j1} (\zeta - \zeta_j)^{-1} \\ b_{32}(\zeta) &= - \sum_{j=1}^{N} b_j c'_{j1} d^*_{j2} (\zeta - \zeta_j)^{-1} \\ b_{33}(\zeta) &= 1 - \sum_{j=1}^{N} b_j c'_{j2} d^*_{j2} (\zeta - \zeta_j)^{-1} . \end{split}$$

Consider the function  $f(\zeta) = b_{22}(\zeta) b_{33}(\zeta) - b_{23}(\zeta) b_{32}(\zeta)$ . This function is a rational function of  $\zeta$ , and by the relations  $(3.1) - (3.4) f(\zeta)$  vanishes at  $\zeta_j^*$   $(j=1, \dots, N)$  and has poles of at most first order at  $\zeta$ . Therefore the function  $f(\zeta) b_{11}^{*}(\zeta^*)^{-1}$  is a bounded entire function which tends to 1 as  $|\zeta| \to \infty$ . Thus we have

$$(3.6) b_{11}^*(\zeta^*) = b_{22}(\zeta)b_{33}(\zeta) - b_{23}(\zeta)b_{32}(\zeta).$$

Using (3.1)–(3.4) and another expression for  $b_{11}(\zeta)$ 

$$b_{11}(\zeta) = 1 + \sum_{j=1}^{N} b_j^{*} (\zeta - \zeta_j^{*})^{-1}$$
,

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we have, by direct calculation,

$$(3.7) b_{22}(\zeta) = b_{11}^*(\zeta^*)b_{33}^*(\zeta^*) b_{32}(\zeta) = -b_{11}^*(\zeta^*)b_{23}^*(\zeta^*).$$

From (3.6) and (3.7) we conclude that the matrix

$$G_2(\xi) = (b_{\alpha\beta}(\xi)) \quad \alpha, \beta = 2, 3$$

is unitary.

Put  $G(\zeta) = G_2(\zeta) G_1(\zeta)$ , then the matrix-valued function  $G(\zeta)$  is holomorphic in the lower half plane and by (3.5)

$$G(\zeta_j^*) = \begin{bmatrix} -c_{j_2}^*d_{j_2} & c_{j_1}^*d_{j_2} \\ c_{j_2}^*d_{j_1} & -c_{j_1}^*d_{j_1} \end{bmatrix}$$

holds.

Define  $a_{\alpha\beta}(\zeta)$  ( $\alpha$ ,  $\beta=2$ , 3) by

$$(a_{\alpha\beta}(\zeta)) = G(\zeta).$$

By the argument of [3, p. 230], the functions  $a_{\alpha\beta}(\zeta)$  ( $\alpha, \beta = 2, 3$ ) satisfy our requirements i), iii).

Next define  $a_{11}(\zeta)$ ,  $a_{12}(\xi)$ ,  $a_{13}(\xi)$ ,  $a_{21}(\xi)$  and  $a_{31}(\xi)$  by the following:

$$\begin{aligned} a_{11}(\zeta) &= a_{22}^*(\zeta^*)a_{33}^*(\zeta^*) - a_{23}^*(\zeta^*)a_{32}^*(\xi^*) \\ a_{1a}(\xi) &= a_{11}(\xi)r_{a-1}(\xi) \quad \alpha = 2, 3 \\ a_{21}(\xi) &= a_{32}^*(\xi)a_{13}^*(\xi) - a_{12}^*(\xi)a_{33}^*(\xi) \\ a_{31}(\xi) &= a_{12}^*(\xi)a_{23}^*(\xi) - a_{22}^*(\xi)a_{13}^*(\xi) . \end{aligned}$$

It is easy to see that the matrix  $(a_{\alpha\beta})$  which we have constructed satisfies i), ii), iii), iv) and (2.15).

REMARK. In the general case (2.17) the determination of  $G(\zeta)$  is done by a quite analogous way. But on the determination of  $d_{j\alpha}$  there appears a system of higher order algebraic equations. We do not know the solvability and the uniqueness of the solution of that system.

## 4. Fundamental integral equations

In this section we derive the system of integral equations which connects kernels  $K_{\alpha}$ ,  $H_{\alpha}$  with the scattering data.

From (2.16) we have

$$F_{\pm 0\alpha}(x) = \pi^{-1} \int r_{\pm \alpha}(\xi) \exp((2i\xi x)) d\xi \quad \alpha = 1, 2.$$

Now on the identity

$$a_{11}(\xi)^{-1}h_1(x,\,\xi) - {}^t(1,\,0,\,0)$$
  
=  $(k_1(x,\,\xi) - {}^t(1,\,0,\,0)) + r_{+1}(\xi) \exp(2i\xi x)k_2(x,\,\xi)$   
+ $r_{+2}(\xi) \exp(2i\xi x)k_3(x,\,\xi)$ 

multiply  $\pi^{-1} \exp (2i\xi y)$  and integrate over  $(-\infty, \infty)$ . The right-hand side gives

(4.1) 
$$K_{1}(x, y) + {}^{t}(0, F_{+01}(x+y), F_{+02}(x+y)) + \int_{0}^{\infty} F_{+01}(x+y+z) K_{2}(x, z) dz + \int_{0}^{\infty} F_{+02}(x+y+z) K_{3}(x, z) dz .$$

For y > 0 the left-hand side gives

(4.2) 
$$2i \sum_{j=1}^{N} \dot{a}_{11}(\zeta_j)^{-1} h_1(x, \zeta_j) \exp(2i\zeta_j y) \\ = 2i \sum_{j=1}^{N} \dot{a}_{11}(\zeta_j)^{-1} (c_{j1}k_2(x, \zeta_j) + c_{j2}k_3(x, \zeta_j)) \exp(2i\zeta_j(x+y)).$$

Put

$$\begin{split} f_{+j\alpha}(x) &= -2i\dot{a}_{11}(\zeta_j)^{-1}c_{j\alpha}\exp\left(2i\zeta_j x\right) \quad j = 1, \, \cdots, \, N, \quad \alpha = 1, 2, \\ F_{+\alpha}(x) &= F_{+\alpha}(x) + \sum_{j=1}^N f_{+j\alpha}(x) \, . \end{split}$$

Putting these into the equality (4.1)=(4.2), we have

(4.3) 
$$K_{1}(x, y) + \int_{0}^{\infty} F_{+1}(x+y+z) K_{2}(x, z) dz + \int_{0}^{\infty} F_{+2}(x+y+z) K_{3}(x, z) dz + {}^{t}(0, F_{+1}(x+y), F_{+2}(x+y)) = 0.$$

Analogously

$$egin{aligned} &H_1(x,\,y) + \int_{-\infty}^0 F_{-1}(x+y+z) H_2(x,\,z) \, dz + \int_{-\infty}^0 F_{-2}(x+y+z) H_3(x,\,z) \, dz \ &+ t(0,\,F_{-1}(x+y),\,F_{-2}(x+y)) = 0 \end{aligned}$$

holds where

$$F_{-a}(x) = F_{-0a}(x) + \sum_{j=1}^{N} f_{-ja}(x)$$
  
$$f_{-ja}(x) = -2id_{11}^{*}(\zeta_{j})^{-1}d_{ja} \exp(2i\zeta_{j}^{*}x).$$

The derivation of (4.3) is similar to that in the case (0.3). The derivation of the following equation (4.5) does not appear in the case of (0.3).

We denote  $(a_{\alpha\beta}(\xi))^{-1} = (b_{\alpha\beta}(\xi))(\alpha, \beta=2, 3)$ . Since the scattering matrix  $(a_{\alpha\beta}(\xi))(\alpha, \beta=1, 2, 3)$  is unitary and unimodular, it follows that det  $(a_{\alpha\beta}(\xi); \alpha, \beta=2, 3)=a_{11}^*(\xi)$ . Accordingly the coefficients  $b_{\alpha\beta}$  are the boundary values of the functions meromorphic in the lower half plane. These functions are defined by the relations

(4.4) 
$$\begin{array}{c} b_{22}(\zeta) = a_{11}^*(\zeta^*)^{-1}a_{33}(\zeta) & b_{23}(\zeta) = -a_{11}^*(\zeta^*)^{-1}a_{23}(\zeta) \\ b_{32}(\zeta) = -a_{11}^*(\zeta^*)^{-1}a_{32}(\zeta) & b_{33}(\zeta) = a_{11}^*(\zeta^*)^{-1}a_{22}(\zeta) \,. \end{array}$$

From the identities

$$egin{aligned} h_{a}(x,\,\xi) &= a_{a1}(\xi) \, k_1(x,\,\xi) \, \exp \left(-2i\xi x\right) + a_{a2}(\xi) \, k_2(x,\,\xi) \ &+ a_{a3}(\xi) \, k_3(x,\,\xi) \quad lpha &= 2, 3 \ , \end{aligned}$$

by using the matrix  $(b_{\alpha\beta}(\xi))$ , we obtain

$$b_{22}(\xi)h_2(x,\xi)+b_{23}(\xi)h_3(x,\xi)-t(0,1,0) =k_2(x,\xi)-t(0,1,0)-r_{+1}^*(\xi)k_1(x,\xi)\exp(-2i\xi x).$$

Multiply  $\pi^{-1} \exp(-2i\xi y)$  on this identity and integrate over  $(-\infty, \infty)$ . Then the right-hand side gives

$$K_{2}(x, y) - \int_{0}^{\infty} F_{+01}^{*}(x+y+z) K_{1}(x, z) dz - {}^{t}(F_{+01}^{*}(x+y), 0, 0) dz$$

By (4.4), for y > 0 the left-hand side gives

$$2i \sum_{j=1}^{N} \dot{a}_{11}^{*}(\zeta_{j})^{-1}(a_{33}(\zeta_{j}^{*})h_{2}(x,\zeta_{j}^{*})-a_{23}(\zeta_{j}^{*})h_{3}(x,\zeta_{j}^{*})) \exp(-2i\zeta_{j}^{*}y)$$

using the relations (2.11)

$$=2i\sum_{j=1}^{N} d_{11}^{*}(\zeta_{j})^{-1}(-c_{j1}^{*}d_{j1}h_{2}(x,\zeta_{j}^{*})-c_{j1}^{*}d_{j2}h_{3}(x,\zeta_{j}^{*}))\exp\left(-2i\zeta_{j}^{*}y\right)$$

by (2.10)

$$= -2i \sum_{j=1}^{N} d_{11}^{*}(\zeta_j)^{-1} c_{j1}^{*} k_1(x, \zeta_j^{*}) \exp\left(-2i \zeta_j^{*}(x+y)\right).$$

Thus we have

(4.5) 
$$K_2(x, y) - \int_0^\infty F_{+1}^*(x+y+z) K_1(x, z) dz - t(F_{+1}^*(x+y), 0, 0) = 0.$$

Analogously we have

$$\begin{split} K_3(x, y) &- \int_0^\infty F_{+2}^*(x+y+z) K_1(x, z) dz - {}^t(F_{+2}^*(x+y), 0, 0) = 0 \\ H_2(x, y) &- \int_{-\infty}^0 F_{-1}^*(x+y+z) H_1(x, z) dz - {}^t(F_{-1}^*(x+y), 0, 0) = 0 \\ H_3(x, y) &- \int_{-\infty}^0 F_{-2}^*(x+y+z) H_1(x, z) dz - {}^t(F_{-2}^*(x+y), 0, 0) = 0 . \end{split}$$

By the standard arguments as in [1], [5], we can show

 $|F_{+a}(x)| \leq C_{+}(x)m_{+}(x), \quad |F_{-a}(x)| \leq C_{-}(x)m_{-}(x) \quad \alpha = 1, 2$ 

where  $C_{\pm}(x)$  are some non-increasing functions as  $x \rightarrow \pm \infty$  respectively.

#### 5. Inverse problem

In this section we study the inverse problem for the operator  $L_{\mu}$ .

Let  $\{r_{\pm a}(\xi), \zeta_j, c_{ja}, (\alpha=1, 2, j=1, \dots, N)\}$  be the right scattering data. We construct the scattering matrix according to the construction in §3. Then the functions

$$F_{\pm a}(x) = \pi^{-1} \int r_{\pm a}(\xi) \exp(2i\xi x) d\xi$$

belong to  $L^{1,\infty}$ . Here we assume that the functions  $m_{\pm a}(\pm x) = \sup_{\pm y \ge \pm x} |F_{\pm 0a}(\pm y)|$  belong to  $L^{1}(a, \infty)$  for some a.

The solvability of the system of integral equations in §4 for kernels  $F_{\pm a}$ , the uniqueness of the solutions and the estimates for the solutions can be shown by an analogous way which has been stated in [6]. We denote its solutions by  $K_{a}$ ,  $H_{a}$  and define  $k_{a}$ ,  $h_{a}$ ,  $\phi_{a}$ ,  $\psi_{a}$  by (1.4), (1.3). If for these solutions the equalities

(5.1) 
$$\begin{aligned} K_{12}^{*}(x, 0) &= -K_{21}(x, 0) \quad K_{13}^{*}(x, 0) &= -K_{31}(x, 0) \\ -H_{12}^{*}(x, 0) &= H_{21}(x, 0) \quad -H_{13}^{*}(x, 0) &= H_{31}(x, 0) \end{aligned}$$

hold, we put

$$u_{+1}(x) = K_{12}^*(x, 0) \qquad u_{+2}(x) = K_{13}^*(x, 0) u_{-1}(x) = -H_{12}^*(x, 0) \qquad u_{-2}(x) = -H_{13}^*(x, 0) .$$

In that case by analogous arguments as in [1], [5], we can show that  $\phi_{\alpha}$ ,  $\psi_{\alpha}$  satisfy (1.1) with  $u=u_{\pm}=t(u_{\pm 1}, u_{\pm 2})$  respectively, and that under certain additional analytical conditions on  $F_{\pm 0,\alpha} u_{\pm}(x)$  coincide.

At present we have no proof for (5.1). Here we show that for reflectionless case  $(r_{\pm 1}(\xi) = r_{\pm 2}(\xi) = 0)$  (5.1) can be verified. In that case as the kernels  $F_{\pm \infty}$  degenerate, the system of integral equations reduces to system of linear equations. We seek the solutions  $K_{\infty}$  in the form

$$K_{1}(x, y) = \sum_{l=1}^{N} f_{1l}(x) \exp(2i\zeta_{l} y)$$
  

$$K_{j}(x, y) = \sum_{l=1}^{N} f_{jl}(x) \exp(-2i\zeta_{l} * y) \quad j = 2, 3$$

where

$$f_{jl}(x) = {}^{t}(f_{jl}^{(1)}(x), f_{jl}^{(2)}(x), f_{jl}^{(3)}(x)) \quad j = 1, 2, 3, \quad l = 1, \dots, N.$$

Put

$$g_{jk}(x) = {}^{t}(f_{j1}^{(k)}(x), \cdots, f_{jN}^{(k)}(x)) \quad j, k = 1, 2, 3.$$

Then the system of integral equations takes the following form, which is the system of linear equations for unknowns  $f_{jl}^{(k)}$ :

$$g_{11}+BC_{1}EA^{*}g_{21}+BC_{2}EA^{*}g_{31} = 0$$
  

$$g_{12}+BC_{1}EA^{*}g_{22}+BC_{2}EA^{*}g_{32}-2iBC_{1}EJ = 0$$
  

$$g_{13}+BC_{1}EA^{*}g_{23}+BC_{2}EA^{*}g_{33}-2iBC_{2}EJ = 0$$
  

$$g_{j1}-B^{*}C_{j-1}^{*}E^{*}Ag_{11}-2iB^{*}C_{j-1}^{*}E^{*}J = 0$$
  

$$g_{j2}-B^{*}C_{j-1}^{*}E^{*}Ag_{12} = 0$$
  

$$g_{j3}-B^{*}C_{j-1}^{*}E^{*}Ag_{13} = 0 \quad j = 2, 3$$

where A, B are defined in §3 and  $C_j$ , E are the diagonal matrices with the entries  $(c_{I_j})$ ,  $(\exp(2i\zeta_I x))$  respectively.

From above equations we obtain equations for  $g_{12}$ ,  $g_{13}$ ,  $g_{21}$ ,  $g_{31}$ 

$$D_1^* g_{1j} = 2iBC_{j-1}EJ \quad j = 2, 3$$
$$D_2 \begin{bmatrix} g_{21} \\ g_{31} \end{bmatrix} = 2i \begin{bmatrix} B^*C_1^*E^*J \\ B^*C_2^*E^*J \end{bmatrix}$$

where

$$D_{1} = I_{N} + E^{*}C_{1}^{*}B^{*}AEC_{1}BA^{*} + E^{*}C_{2}^{*}B^{*}AEC_{2}BA^{*}$$
$$D_{2} = \begin{bmatrix} I_{N} + E^{*}C_{1}^{*}B^{*}AEC_{1}BA^{*} & E^{*}C_{1}^{*}B^{*}AEC_{2}BA^{*} \\ E^{*}C_{2}^{*}B^{*}AEC_{1}BA^{*} & I_{N} + E^{*}C_{2}^{*}B^{*}AEC_{2}BA^{*} \end{bmatrix}$$

 $(I_N \text{ is the identity matrix of degree } N).$ 

We calculate the determinants and the cofactors of matrices  $D_1$  and  $D_2$ , by using the formula

$$\det (\delta_{jk} + x_{jk}) = 1 + \sum_{l=1}^{m} X_{l} \quad j, k = 1, \dots, m$$

where  $X_l$  is the sum of the principal minors of matrix  $(x_{jk})$  of degree l and the modification of this formula. We denote the  $(\alpha, \beta)$ -cofactors of the matrices  $D_j$  by  $d_{\alpha\beta}^{(j)}$  respectively. Then the expressions

$$\begin{split} K_{1j}^{*}(x, 0) &= -2i(\det D_{1})^{-1} \sum_{\alpha,\beta=1}^{N} d_{\alpha\beta}^{(1)} c_{\sigma,j-1}^{*} b_{\sigma}^{*} \exp\left(-2i\zeta_{\sigma}^{*}x\right) \\ K_{21}(x, 0) &= 2i(\det D_{2})^{-1} \sum_{\alpha,\beta=1}^{N} \left(d_{\alpha\beta}^{(2)} c_{\alpha}^{*} + d_{\sigma+N,\beta}^{(2)} c_{\sigma2}^{*}\right) b_{\sigma}^{*} \exp\left(-2i\zeta_{\sigma}^{*}x\right) \\ K_{31}(x, 0) &= 2i(\det D_{2})^{-1} \sum_{\alpha,\beta=1}^{N} \left(d_{\alpha,\beta+N}^{(2)} c_{\sigma1}^{*} + d_{\alpha+N,\alpha+N}^{(2)} c_{\beta2}^{*}\right) b_{\sigma}^{*} \exp\left(-2i\zeta_{\sigma}^{*}x\right) \end{split}$$

hold.

After direct calculation, we have

$$det D_{1} = det D_{2}$$

$$= \sum_{0 \leq j \leq N, \ 1 \leq k_{1} < \dots < k_{j} \leq N, \ 1 \leq l_{1} < \dots < l_{j} \leq N} b_{k_{1}}^{*} \cdots b_{k_{j}}^{*} b_{l_{1}} \cdots b_{l_{j}} \times \exp \left(2i(\zeta_{l_{1}} + \dots + \zeta_{l_{j}} - \zeta_{k_{1}}^{*} - \dots - \zeta_{k_{j}}^{*})x\right) \times det \left((\zeta_{r}^{*} - \zeta_{s}^{*})^{-1}\right) det \left((c_{s,1}^{*} c_{r,1} + c_{s,2}^{*} c_{r,2})(\zeta_{s}^{*} - \zeta_{r})^{-1}\right)$$

where  $r = l_1, \dots, l_j, s = k_1, \dots, k_j$ .

And for cofactors we obtain the relations

 $\begin{aligned} & d_{\alpha\beta}^{(1)} c_{\alpha1}^* b_{\alpha}^* \exp\left(-2i\zeta_{\alpha}^* x\right) = (d_{\beta\alpha}^{(2)} c_{\beta1}^* + d_{\beta+N,\alpha}^{(2)} c_{\beta2}^*) b_{\beta}^* \exp\left(-2i\zeta_{\beta}^* x\right) \\ & d_{\alpha\beta}^{(1)} c_{\beta2}^* b_{\alpha}^* \exp\left(-2i\zeta_{\alpha}^* x\right) = (d_{\beta,\alpha+N}^{(2)} c_{\beta1}^* + d_{\beta+N,\alpha+N}^{(2)} c_{\beta2}^*) b_{\beta}^* \exp\left(-2i\zeta_{\beta} x\right) \\ & \alpha, \beta = 1, \dots, N. \end{aligned}$ 

Thus we have (5.1).

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