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Osaka University
ON THE ROBERTELLO INVARIANTS OF PROPER LINKS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

AKIO KAWAUCHI

(Received September 14, 1982)

Robertello’s invariant of a classical knot in [9] was generalized by Gordon in [2] to an invariant of a knot in a $Z$-homology 3-sphere, and by the author in [5] to an invariant, $\delta(k \subset S)$, of a knot $k$ in a $Z_2$-homology 3-sphere $S$. In this paper, we shall generalize this invariant to two mutually related invariants, $\delta_0(L \subset S)$ and $\delta(L \subset S)$, of a proper link $L$ in a $Z_2$-homology 3-sphere $S$. In the case of a classical proper link, this $\delta_0$-invariant can be considered as an invariant suggested by Robertello in [9, Theorem 2]. A difference between $\delta_0(L \subset S)$ and $\delta(L \subset S)$ is that $\delta_0(L \subset S)$ is generally an oriented link type invariant, but $\delta(L \subset S)$ is an unoriented link type invariant. A proper link in a $Z_2$-homology 3-sphere (which is not a $Z$-homology 3-sphere) naturally occurs when considering a branched cyclic covering of a 3-sphere, branched along a certain proper link. (If the number of the components of the link is $\geq 2$, the branched covering space can not be a $Z$-homology 3-sphere by the Smith theory.) So, we consider a proper link $\tilde{L}$ in a $Z_2$-homology 3-sphere $\tilde{S}$, obtained from a proper link $L$ in a $Z_2$-homology 3-sphere $S$ by taking a branched cyclic covering, branched along $L$. When the covering degree is prime, we shall establish a relationship between $\delta(\tilde{L} \subset \tilde{S})$ and $\delta(L \subset S)$ and then a relationship between $\delta_0(\tilde{L} \subset \tilde{S})$ and $\delta_0(L \subset S)$.

In Section 1 we define and discuss the slope of a link in a 3-manifold as a generalization of the slope of a knot in a 3-manifold, introduced in [5]. In Section 2 the $\delta_0$-invariant and the $\delta$-invariant are defined and studied. Section 3 deals with relationships between $\delta(\tilde{L} \subset \tilde{S})$ and $\delta(L \subset S)$ and between $\delta_0(\tilde{L} \subset \tilde{S})$ and $\delta_0(L \subset S)$.

Throughout this paper spaces and maps will be considered in the piecewise linear category, and notations and conventions will be the same as those of [5] unless otherwise stated.

1. The slope of a link in a 3-manifold

Let $M$ be a connected oriented 3-manifold. Let $L$ be an oriented link with $r$ components in the interior of $M$. Let $o(L)$ denote the order ($\geq 1$) of
the homology class $[L] \in H_1(M; \mathbb{Z})$. Let $\tau H_1(M)$ be the torsion part of $H_1(M; \mathbb{Z})$. Let $\phi: \tau H_1(M) \times \tau H_1(M) \to \mathbb{Q}/\mathbb{Z}$ be the linking pairing.

**Definition 1.1.** The **slope** of the link $L$, denoted by $s(L) = s(L \subset M)$, is defined by the identity

$$s(L) = \begin{cases} -\phi([L], [L]) & (o(L) < +\infty), \\ \infty & (o(L) = +\infty). \end{cases}$$

If $s(L) = 0$, then we say that the link $L$ is **flat**.

When $r = 1$, $s(L)$ is the same as the slope defined in [5, Definition 1.4] by [5, Lemma 1.8]. Let $r \geq 2$. Let $B_1, B_2, \ldots, B_{r-1}$ be mutually disjoint oriented bands in the interior of $M$ attaching to $L$ as 1-handles. If we obtain a knot $k$ from $L$ by surgery along such $B_1, B_2, \ldots, B_{r-1}$, then we say that the knot $k$ is obtained from $L$ by a fusion.

**Lemma 1.2.** Let $k$ be a knot obtained from a link $L$ by a fusion. Then $s(L) = s(k)$.

**Proof.** Clearly, $[L] = [k]$ in $H_1(M; \mathbb{Z})$. The result follows from Definition 1.1.

Assume that each component $k_i$ of $L$ is a knot of finite order, i.e., $o(k_i) < +\infty$, $i = 1, 2, \ldots, r$. Then the total $\mathbb{Q}$-linking number $\lambda(L) = \sum_{i \neq j} \text{Link}_{M}(k_i, k_j)$ of the link $L \subset M$ is defined by $\lambda(L) = \lambda(L \subset M) \in \mathbb{Q}$. When $r = 1$, we understand that $\lambda(L) = 0$.

**Lemma 1.3.** In $\mathbb{Q}/\mathbb{Z}$, $s(L) = \sum_{i \neq 1} s(k_i) - 2\lambda(L)$.

**Proof.** Since $o(L) < +\infty$ and $[L] = \sum_{i \neq 1} [k_i]$, $s(L) = -\phi([L], [L]) = \sum_{i \neq 1} -\phi([k_i], [k_i]) - 2\sum_{i \neq j} \phi([k_i], [k_j])$ using that $\phi([k_i], [k_j]) = \text{Link}_{M}(k_i, k_j) \pmod{1}$ for $i \neq j$ and $s(k_i) = -\phi([k_i], [k_i])$, we have a desired congruence.

For each element $s \in \mathbb{Q}/\mathbb{Z}$ we can have coprime positive integers $a, b$ such that $s \equiv a/b \pmod{1}$. This fraction $a/b$ and the denominator $b$ are called a **normal presentation** and the denominator of the element $s \in \mathbb{Q}/\mathbb{Z}$, respectively. Now we assume that the denominator of the slope $s(k_i)$ is odd, $i = 1, 2, \ldots, r$. Then $s(k_i)$ has a normal presentation of type $2a_i/b_i$, $i = 1, 2, \ldots, r$.

**Definition 1.4.** We define

$$s^*(L) = \sum_{i \neq 1} a_i/b_i - \lambda(L)$$

in $\mathbb{Q}/\mathbb{Z}$ and call it the **half-slope** of the link $L \subset S$.

The following is easily proved.
Lemma 1.5. In $Q/Z$ $2s^*(L) = s(L)$, and if $s(L) = 0$, then $s^*(L)$ is 0 or 1/2 according as the denominator of $\lambda(L) \in Q/Z$ is odd or even.

2. The $\delta_0$-invariant and the $\delta$-invariant

We consider an oriented link $L$ with components $k_i$, $i = 1, 2, \ldots, r$, in an oriented $Z_2$-homology 3-sphere $S$.

**Definition 2.1.** The link $L$ is proper if the mod 2 linking number, $\text{Link}_{Z_2}(k_i, L - k_i) = 0$ for all $i$, $1 \leq i \leq r$. (We understand a knot to be a proper link.)

Let $W$ be a compact oriented 4-manifold. Let $F$ be a locally flat, oriented (possibly disconnected) surface of (total) genus 0 in $W$. We say that such a pair $F \subset W$ is admissible for a link $L \subset S$, if $S$ is a component of $\partial W$, $\partial F = L$, $H_i(\partial W; Z_2) = 0$ and $[F^+_2] \in H_2(W; Z_2)$ is characteristic, i.e., $[F^+_2] \cdot x = x^2$ for all $x \in H_2(W; Z_2)$, where $F^+_2$ is a (mod 2) cycle obtained from $F$ by attaching (mod 2) 2-chains $c_i$ in $S$ with $\partial c_i = -k_i$, $i = 1, 2, \ldots, r$.

**Lemma 2.2.** For any proper link $L \subset S$ there exists an admissible pair $F \subset W$.

Proof. Let $T(L) = \bigcup_i T(k_i)$ be a tubular neighborhood of $L = \bigcup_i k_i$ in $S$. Construct a 4-manifold $W = (-S) \times [-1, 1] \cup D^2 \times D^2 \cup \cdots \cup D^2 \times D^2$ identifying $T(k_i) \times 1$ with $(\partial D^2) \times D^2$, $i = 1, \ldots, r$, so that $H_i(\partial W; Z_2) = 0$. Let $D_i = (-k_i) \times [-1, 1] \cup D^2 \times 0_i$ be a disk. Let $F = \bigcup_i D_i$. To show that $F \subset W$ is admissible for $L \subset S$, it suffices to check that $[F^+_2] \in H_2(W; Z_2)$ is characteristic. Note that $[D^+_2], i = 1, \ldots, r$, form a basis for $H_2(W; Z_2)$. Since $[F^+_2] = \sum_i [D^+_2]$, we have

$$[F^+_2] \cdot [D^+_2] = [D^+_2] + \sum_{j \neq i} [D^+_2] \cdot [D^+_2]$$

$$= [D^+_2] + \text{Link}_{Z_2}(L - k_i, k_i)$$

$$= [D^+_2], \ i = 1, \ldots, r.$$  

This implies that $[F^+_2]$ is characteristic. This completes the proof.

The pair $F \subset W$, constructed in the proof of Lemma 2.2 is called a standard admissible pair for the proper link $L \subset S$.

**Definition 2.3.** Let $L \subset S$ be a proper link. Then we define

$$\delta_0(L) = \delta_0(L \subset S) = ([F^+_0] - \text{sign } W) / 16 - \mu(\partial W)$$

in $Q/Z$ for any admissible pair $F \subset W$ for $L \subset S$, where $F^+_0$ is a rational 2-cycle obtained from $F$ by attaching rational 2-chains $c_i^0$ in $S$ with $\partial c_i^0 = -k_i$, $i = 1, \ldots, r$.

**Remark 2.4.** We can define the invariant $\delta_0(L \subset S)$ by using a more gen-
eral pair $F \subset W$, where the (total) genus of $F$ may be positive or $F$ may be non-orientable (cf. Freedman-Kirby [1], Guillou-Marin [3], Matsumoto [7]).

To see the well-definedness of $\delta_0(L)$, consider a standard admissible pair $F^* \subset W^*$ for $L \subset S$. Construct an oriented 4-manifold $\bar{W} = W \cup -W^*$ identifying two copies of $S$. Then $\Sigma = F \cup -F^*$ is the disjoint union of 2-spheres. Since $[F^*]$ and $[F^*]$ are characteristic, we see that the mod 2 homology class $[\Sigma] \in H_2(\bar{W}; \mathbb{Z}_2)$ is characteristic. By the Rochlin theorem ([6], [10]),

$$
\mu(\partial \bar{W}) = ([\Sigma]^2 - \text{sign } \bar{W})/16 \text{ in } \mathbb{Q}/\mathbb{Z}.
$$

But,

$$
\mu(\partial \bar{W}) = \mu(\partial W) - \mu(\partial W^*),
$$

and $\mu(\partial W^*) = (F^*]^2 - (F^*)^2$ and $\text{sign } \bar{W} = \text{sign } W - \text{sign } W^*$, where we count $[-F^*]^2, [F^*]^2$ in $W^*$. It follows that

$$
([F^*]^2 - \text{sign } W)/16 - \mu(\partial W) = ([F^*]^2 - \text{sign } W^*)/16 - \mu(\partial W^*)
$$

in $\mathbb{Q}/\mathbb{Z}$, showing the well-definedness of $\delta_0(L)$.

**Definition 2.5.** Two links $L_i \subset S_i$, $i=0, 1$, are said to be *cobordant in the weak sense* if:

1. There exists a compact oriented 4-manifold $W$ such that $\partial W = S_0 \cup S_1$ and $H_*(W; S_i; \mathbb{Z}_2) = 0, i=0, 1$.

2. There exists a locally flat, compact oriented (possibly disconnected) surface $F$ of (total) genus 0 in $W$ such that $\partial F = -L_0 \cup L_1$ (See Figure 1).

![Figure 1](image)

**Theorem 2.6.** If proper links $L_i \subset S_i$, $i=0, 1$, are cobordant in the weak sense, then $\delta_0(L_0) = \delta_0(L_1)$.

Proof. Let $F \subset W$ be a cobordism pair for $L_i \subset S_i$, $i=0, 1$, stated in Definition 2.5. Construct an oriented 4-manifold $W = W \cup D^3 \times [0, 1]$ identifying a 3-cell in $S_i - L_i$ with $D^3 \times i$ for each $i$, $i=0, 1$. Then $\partial W'$ is a connected sum $(-S_0 \# S_1$, which is a $\mathbb{Z}_2$-homology 3-sphere containing a proper link $L'$, regarded as the union $-L_0 \cup L_1$. Clearly, $\delta_0(L' \subset (-S_0 \# S_1) = -\delta_0(L_0 \subset S_0) +$
Note that $H_2(W;\mathbb{Z}_2)=0$. Then $F\subset W'$ is admissible for $L'\subset (-S_0)\# S_1$, and hence
\[
\delta_0(L'\subset (-S_0)\# S_1) = ([F_\phi]^2 - \text{sign } W')/16 - \mu((-S_0)\# S_1) = 0,
\]
because $W'$ is spin and $H_2(W';\mathbb{Q})=0$. Thus, $\delta_0(L_0\subset S_0) = \delta_0(L_1\subset S_1)$. This completes the proof.

In [5, Definition 2.1] the $\delta$-invariant $\delta(k)$ of a knot $k$ in $S$ was defined so as to be $\delta(k)=\delta_0(k)$.

**Corollary 2.7.** Let $k\subset S$ be a knot obtained from a proper link $L\subset S$ by a fusion. Then $\delta_0(L)=\delta_0(k)=\delta(k)$.

**Proof.** The knot $k\subset S$ and the link $L\subset S$ are cobordant in the weak sense. The result follows from Theorem 2.6.

By a Dehn surgery we obtain from a knot $k\subset S$ a unique (up to homeomorphism), closed, connected, oriented 3-manifold $M$ such that $H_1(M;\mathbb{Z})/\text{odd torsion}\cong \mathbb{Z}$, called a $\mathbb{Z}_2$-homology handle (cf. [5, Remark 1.6 and Corollary 1.7]). In [4] we defined an invariant $\varepsilon(M)$, being 0 or 1, of $M$, calculable from the $\mathbb{Z}_2$-Alexander polynomial of $M$.

**Corollary 2.8.** Let $L\subset S$ be a proper link. Let $M$ be the $\mathbb{Z}_2$-homology handle of a knot $k\subset S$, obtained from $L$ by a fusion. Let $a/b$ be a normal presentation of the slope $s(L\subset S)$ with a odd. Then we have
\[
\delta_0(L) = \varepsilon(M)/2 + (a/b - ab)/16
\]
in $\mathbb{Q}/\mathbb{Z}$.

**Proof.** By Lemma 1.2 $s(L)=s(k)$. By Corollary 2.7 $\delta_0(L)=\delta(k)$. Then the desired congruence follows from [5, Theorem 2.7 and Corollary 3.6].

**Definition 2.9.** For a proper link $L$ in $S$ we define
\[
\delta(L) = \delta(L\subset S) = \delta_0(L\subset S) + \lambda(L\subset S)/8
\]
in $\mathbb{Q}/\mathbb{Z}$.

**Remark 2.10.** Definition 2.9 is analogous to Murasugi's definition of the unoriented link type signature in [7] (cf. [5, Remark 4.8]).

**Theorem 2.11.** The invariant $\delta(L\subset S)$ is an unoriented link type invariant of a proper link $L\subset S$. That is, $\delta(L\subset S)=\delta(L'\subset S')$ for any link $L'\subset S'$ with an orientation-preserving homeomorphism $S\to S'$ sending $L$ to $L'$ setwise.

**Proof.** It suffices to show that $\delta(L)$ does not depend on any particular orientations of the components, $k_i$, of $L$. Let $F=\cup i\ast D_i\subset W$ be a standard admissible pair for $L=\cup i\ast k_i\subset S$. Note that $[F_\phi]^2 = \sum i\ast [D_i\phi]^2 + \frac{1}{|S|} \sum [D_i\phi]$. 

\[ [D_{tQ}] = \sum_i [D_{tQ}]^2 - 2\lambda(L). \]

Then
\[ \delta(L) = \delta_0(L) + \lambda(L)/8 = (\sum_i [D_{tQ}]^2 - \text{sign } W)/16 - \mu(\partial W). \]

Since \([D_{tQ}]^2\) is not altered by changing the orientation of \(D_i\) (that is, \(k_i\)), we have a desired result.

A link \(L \subset S\) is \textit{amphiscirial} if there is an orientation-preserving homeomorphism \(S \to -S\) sending \(L\) to itself setwise. The following is direct from Theorem 2.11.

\textbf{Corollary 2.12.} If a proper link \(L \subset S\) is amphiscirial, then \(2\delta(L) = 0\) in \(\mathbb{Q}/\mathbb{Z}\).

Here is an example of a classical proper link.

\textbf{Example 2.13.} Let \(L_r\) be an \(r\)-component link in a 3-sphere \(S^3\), illustrated in Figure 2, where \(r \geq 2\). The link \(L_r\) is clearly proper. Choosing a suitable orientation of \(S^3\), \(\lambda(L_r) = r\). Since we can have a trivial knot from \(L_r\) by a fusion, we see that \(\delta_0(L_r) = 0\). Therefore, \(\delta(L_r) = r/8\) in \(\mathbb{Q}/\mathbb{Z}\).

3. \textbf{Branched cyclic coverings and the \(\delta\)- and \(\delta_0\)-invariants}

We consider a link \(\bar{L} \subset \bar{S}\) obtained from a link \(L \subset S\) by taking an \(n\)-fold cyclic branched covering of \(S\), branched along \(L\). Namely, \(\bar{S}\) is the branched covering space over \(S\), associated with an epimorphism \(H_1(S - L; \mathbb{Z}) \to \mathbb{Z}_n\) sending each meridian of \(L\) to a unit of \(\mathbb{Z}_n\), and \(\bar{L}\) is the lift of \(L\). We assume that \(\bar{S}\) is a \(\mathbb{Z}_2\)-homology 3-sphere.

First we consider the case \(n = 2\). Then \(L\) and \(\bar{L}\) are knots by the Smith theory. Let \(L = k\) and \(\bar{L} = \bar{k}\).

\textbf{Theorem 3.1.} Let \(2a/b\) be a normal presentation of the slope \(s(\bar{k})\). Then
\[ \delta(\bar{k}) = \delta(k) - (a/b - ab)/8 \]

in \(Q/Z\). In particular, if \(\bar{k}\) is flat, then \(\delta(\bar{k}) = \delta(k)\).

Proof. By [5, Lemma 4.5] \(s(\bar{k}) = 2s(\bar{k}) = 4a/b\). Since \((2a + b)/b\) and \((4a + b)/b\) are normal presentations of \(s(\bar{k})\) and \(s(k)\), respectively, we see from Corollary 2.8 that

\[
\delta(\bar{k}) = \frac{\epsilon(M)}{2} + \left\{ \frac{(2a + b)}{b} - \frac{(2a + b)}{b} \right\} \frac{1}{16}
\]

and

\[
\delta(k) = \frac{\epsilon(M)}{2} + \left\{ \frac{(4a + b)}{b} - \frac{(4a + b)}{b} \right\} \frac{1}{16}
\]

where \(\bar{M}\) and \(M\) are the \(Z_2\)-homology handles of \(\bar{k} \subset \bar{S}\) and \(k \subset S\), respectively. Since \(\bar{M}\) is a 2-fold covering space of \(\bar{S}\), it follows from [4, Lemma 4.2] that \(\epsilon(\bar{M}) = \epsilon(M)\). Now we have a desired congruence. This completes the proof.

Next, to consider the case that the covering degree \(n\) is an odd prime \(p\), we remark the following:

**Lemma 3.2.** \(\lambda(L) = \lambda(L)/n\).

**Corollary 3.3.** \(L\) is proper if and only if \(L\) is proper.

Proof of Lemma 3.2. It suffices to show that for \(i \neq j\) \(\text{Link}_3(k_i, k_j) = \text{Link}_3(k_i, k_j)/n\). Let \(F_i\) be a characteristic surface (cf. [5]) of \(k_i\) in \(S\) such that \(L - k_i\) intersects \(F_i\) transversally. Write \([\partial F_i] = a_i r_i [m_i] + b_i r_i [l_i] \in H_i(\partial T(k_i); Z)\) for a meridian-longitude pair \((m_i, l_i)\) of \(T(k_i)\) such that the lift of \(l_i\) has \(n\) components, where \((a_i, b_i) = 1\) and \(r_i\) is an integer \(> 0\). Let \(\bar{l}_i\) be a component of the lift of \(l_i\). For the lift \(\bar{m}_i\) of \(m_i\), the pair \((\bar{m}_i, \bar{l}_i)\) forms an \(m.l\) pair of a tubular neighborhood \(T(\bar{k}_i)\) of \(\bar{k}_i\) which is the lift of \(T(k_i)\). Note that the lift \(\bar{F}_i\) of \(F_i\) is an oriented surface which is a \(Z_2\)-covering space of \(F_i\) branched over the set \(F_i \cap (L - k_i)\). Clearly \([\partial \bar{F}_i] = a_i r_i [\bar{m}_i] + b_i r_i [\bar{l}_i] \in H_i(\partial T(k_i); Z)\). Since the intersection numbers, \(\text{Int}(\bar{F}_i, \bar{k}_j)\) and \(\text{Int}(F_i, k_j)\) are equal, we have

\[
\text{Link}_3(k_i, \bar{k}_j) = \text{Int}(\bar{F}_i, \bar{k}_j)/b_r m = \text{Int}(F_i, k_j)/b_r m = \text{Link}_3(k_i, k_j)/n.
\]

This completes the proof.

Proof of Corollary 3.3. When \(n\) is even, \(L\) and \(L\) are knots by the Smith theory. So, assume \(n\) is odd. It suffices to show that \(\text{Link}_3(k_i, \bar{k}_j)_2 = \text{Link}_3(k_i, k_j)_2\) for \(i \neq j\). This is obtained by a mod 2 version of the proof of Lemma 3.2, since \(b_r m\) is odd. This completes the proof.

We shall show the following theorem, where note that \((p^2 - 1)/8\) is an in-
Theorem 3.4. Let \( \bar{L} = \bigcup \{ \bar{k}_i \} \subset \bar{S} \) be a proper link and assume that the covering degree is an odd prime \( p \). Let \( 2a_i/b_i \) be a normal presentation of the slope \( s(\bar{k}_i) \), \( i=1, 2, \ldots, r \). Then
\[
\delta(\bar{L}) = p\delta(L) - \{(p^2-1)/8\} \sum_{i=1}^r a_i/b_i
\]
in \( \mathbb{Q}/\mathbb{Z} \).

Proof. Let \( F \subset W \) and \( \bar{F} \subset \bar{W} \) be standard admissible pairs for \( L \subset S \) and \( \bar{L} \subset \bar{S} \), respectively, such that \( \bar{F} \subset \bar{W} \) is obtained from \( F \subset W \) by taking a \( Z^p \)-covering branched along \( F \). [One can see directly or by a transfer argument that such pairs exist.] Let \( \partial W - S = S^* \) and \( \partial \bar{W} - \bar{S} = \bar{S}^* \). By the proof of Theorem 2.11,
\[
\delta(L) = (\sum_{i=1}^r [D_i^*]^2 - \text{sign } W)/16 - \mu(S) - \mu(S^*) + \alpha(Z_p, S^*)/16 + \alpha(Z_p, S)/16,
\]
where \( F = \bigcup \{ \bar{k}_i \}, D_i \), and \( \bar{D}_i \) corresponds to \( D_i \). Then since \( [D_i^*]^2/p = [\bar{D}_i^*]^2 \) (cf. [5, the proof of Lemma 4.9]),
\[
\delta(\bar{L}) - p\delta(L) = (1-p^2) \sum_{i=1}^r [\bar{D}_i^*]^2/16 + (-\text{sign } \bar{W} + p \text{ sign } W)/16
\]
\[
- \{\mu(S) - p\mu(S)\} - \{\mu(S^*) - p\mu(S^*)\} - \{\alpha(Z_p, S^*) - \alpha(Z_p, \bar{S}^*)\}/16.
\]
By the definition of \( \alpha \)-invariant in [5, Section 4],
\[
\alpha(Z_p, S) + \alpha(Z_p, S^*) = -\text{sign } W + p \text{ sign } W - (\sum_{i=1}^r [\bar{D}_i^*]^2) (p^2-1)/3.
\]
Therefore,
\[
\delta(\bar{L}) - p\delta(L) = \{(p^2-1)/3\} \sum_{i=1}^r [\bar{D}_i^*]^2 - \{\mu(S) - p\mu(S)\}
\]
\[
- \{\mu(S^*) - p\mu(S^*)\} - \{\alpha(Z_p, S^*) - \alpha(Z_p, \bar{S}^*)\}/16.
\]
First, let \( p \geq 3 \). Then by [5, Theorems 11.1 and 12.1],
\[
\mu(S^*) = p\mu(S^*) + \alpha(Z_p, S^*)/16,
\]
and
\[
\mu(S) = p\mu(S) + \alpha(Z_p, S)/16 + \{(p^2-1)/24\} \sum_{i=1}^r a_i/b_i,
\]
where note that \( (p^2-1)/24 \) is an integer. Since \( [\bar{D}_i^*]^2 \equiv 2a_i/b_i \) (mod 1) (cf. [5, Lemma 2.6]), it follows that
\[
\delta(\bar{L}) - p\delta(L) = -\{(p^2-1)/24\} \sum_{i=1}^r [\bar{D}_i^*]^2 - \{(p^2-1)/24\} \sum_{i=1}^r a_i/b_i,
\]
\[
= -\{(p^2-1)/8\} \sum_{i=1}^r a_i/b_i.
\]
Now let \( p = 3 \). By [5, Theorems 11.1 and 12.1],
\[
\mu(S^*) = 3\mu(S^*) + 9\alpha(Z_3, S^*)/16,
\]
and
\[ \mu(S) = 3\mu(S) + 9\alpha(Z_3, S)/16 + 3\sum_i t_i a_i/b_i. \]

Directly or by a transfer argument, \( \text{sign} \ W = \text{sign} \ W \). Then
\[ \alpha(Z_3, S)/2 + \alpha(Z_3, S^*)/2 = -\text{sign} \ W/2 + 3 \text{ sign } W/2 - (\sum_i t_i [\bar{D}_{iQ}]^2)4/3 \]
\[ \equiv - (\sum_i t_i [\bar{D}_{iQ}]^2)4/3 \pmod{1}. \]

Then,
\[ \delta(\bar{L}) - 3\delta(L) = - (\sum_i t_i [\bar{D}_{iQ}]^2)/3 - \alpha(Z_3, S)/2 - \alpha(Z_3, S^*)/2 \]
\[ -3 \sum_i t_i a_i/b_i = \sum_i t_i [\bar{D}_{iQ}]^2 - 3 \sum_i t_i a_i/b_i = - \sum_i t_i a_i/b_i \]
in \( Q/Z \), because \( [\bar{D}_{iQ}]^2 \equiv 2a_i/b_i \) (mod 1). This completes the proof.

**Theorem 3.5.** Let \( \bar{L} \subset S \) be proper and assume that the covering degree is an odd prime \( p \). Then we have
\[ \delta(\bar{L}) = p \delta(L) - \{(p^2-1)/8\} s^*(\bar{L}) \]
in \( Q/Z \), where \( s^*(\bar{L}) \) is the half-slope of the link \( \bar{L} \subset S \).

**Proof.** By Theorem 3.4,
\[ \delta(\bar{L}) = \lambda(\bar{L})/8 = p \delta(L) + p \lambda(L)/8 - \{(p^2-1)/8\} \sum_i t_i a_i/b_i. \]
Since \( \lambda(L)/p = \lambda(\bar{L}) \) by Lemma 3.2, we have
\[ \delta(\bar{L}) - p \delta(L) = - \{(p^2-1)/8\} (\sum_i t_i a_i/b_i - \lambda(\bar{L})) = - \{(p^2-1)/8\} s^*(\bar{L}). \]
This completes the proof.

**Corollary 3.6.** If \( \bar{L} \) is flat, then \( \delta(\bar{L}) = \delta(L) \).

**Proof.** By Lemma 1.5 \( s(\bar{L}) = 0 \) implies \( s^*(\bar{L}) = 0 \). So, by Theorem 3.5 \( \delta(\bar{L}) = p \delta(L) \). By Lemmas 1.3, 3.2 and [5, Lemma 4.5], \( s(\bar{L}) = ps(\bar{L}) \), so that \( s(\bar{L}) = 0 \). By Corollary 2.8 \( 2\delta(\bar{L}) = 0 \). Using that \( p \) is odd, the proof is completed.

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**References**


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