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ON THE ROBERTELLO INVARIANTS OF PROPER LINKS

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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Robertello's invariant of a classical knot in [9] was generalized by Gordon in [2] to an invariant of a knot in a Z-homology 3-sphere, and by the author in [5] to an invariant, $\delta(k\subset S)$, of a knot k in a Z_2-homology 3-sphere S. In this paper, we shall generalize this invariant to two mutually related invariants, $\delta_0(L\subset S)$ and $\delta(L\subset S)$, of a proper link L in a Z_2-homology 3-sphere S. In the case of a classical proper link, this δ_0-invariant can be considered as an invariant suggested by Robertello in [9, Theorem 2]. A difference between $\delta_0(L\subset S)$ and $\delta(L\subset S)$ is that $\delta_0(L\subset S)$ is generally an oriented link type invariant, but $\delta(L\subset S)$ is an unoriented link type invariant. A proper link in a Z_2-homology 3-sphere (which is not a Z-homology 3-sphere) naturally occurs when considering a branched cyclic covering of a 3-sphere, branched along a certain proper link. (If the number of the components of the link is ≥ 2, the branched covering space can not be a Z-homology 3-sphere by the Smith theory.) So, we consider a proper link $\tilde{L}$ in a Z_2-homology 3-sphere $\tilde{S}$, obtained from a proper link L in a Z_2-homology 3-sphere S by taking a branched cyclic covering, branched along L. When the covering degree is prime, we shall establish a relationship between $\delta(\tilde{L}\subset\tilde{S})$ and $\delta(L\subset S)$ and then a relationship between $\delta_0(\tilde{L}\subset\tilde{S})$ and $\delta_0(L\subset S)$.

In Section 1 we define and discuss the slope of a link in a 3-manifold as a generalization of the slope of a knot in a 3-manifold, introduced in [5]. In Section 2 the δ_0-invariant and the δ-invariant are defined and studied. Section 3 deals with relationships between $\delta(\tilde{L}\subset\tilde{S})$ and $\delta(L\subset S)$ and between $\delta_0(\tilde{L}\subset\tilde{S})$ and $\delta_0(L\subset S)$.

Throughout this paper spaces and maps will be considered in the piecewise linear category, and notations and conventions will be the same as those of [5] unless otherwise stated.

1. The slope of a link in a 3-manifold

Let M be a connected oriented 3-manifold. Let L be an oriented link with r components in the interior of M. Let $o(L)$ denote the order (≥ 1) of the homology class $[L] \in H_1(M; Z)$. Let $\tau H_1(M)$ be the torsion part of $H_1(M; Z)$. Let $\phi: \tau H_1(M) \times \tau H_1(M) \rightarrow Q/Z$ be the linking pairing.

DEFINITION 1.1. The *slope* of the link L, denoted by $s(L)=s(L \subset M)$ is defined by the identity

$$s(L) = \begin{cases} -\phi([L], [L]) & (o(L)<+\infty)\,, \\ \infty & (o(L)=+\infty)\,. \end{cases}$$

If $s(L)=0$, then we say that the link L is *flat*.

When $r=1$, $s(L)$ is the same as the slope defined in [5, Definition 1.4] by [5, Lemma 1.8]. Let $r \geq 2$. Let $B_1, B_2, \cdots, B_{r-1}$ be mutually disjoint oriented bands in the interior of M attaching to L as 1-handles. If we obtain a knot k from L by surgery along such $B_1, B_2, \cdots, B_{r-1}$, then we say that the knot k is *obtained from L by a fusion.*

Lemma 1.2. *Let k be a knot obtained from a link L by a fusion. Then* $s(L)=s(k)$.

Proof. Clearly, $[L]=[k]$ in $H_1(M; Z)$. The result follows from Definition 1.1.

Assume that each component k_i of L is a knot of finite order, i.e., $o(k_i)<+\infty$, $i=1, 2, \cdots, r$. Then the total Q-linking number $\lambda(L)=\lambda(L \subset M) \in Q$ of the link $L \subset M$ is defined by $\lambda(L)=\sum_{i>j} \mathrm{Link}_M(k_i, k_j)$. When $r=1$, we understand that $\lambda(L)=0$.

Lemma 1.3. *In* Q/Z $s(L)=\sum_{i=1}^{r} s(k_i)-2\lambda(L)$.

Proof. Since $o(L)<+\infty$ and $[L]=\sum_{i=1}^{r} [k_i]$, $s(L)=-\phi([L], [L])=\sum_{i=1}^{r} -\phi([k_i], [k_i])-2\sum_{i>j} \phi([k_i], [k_j])$. Using that $\phi([k_i], [k_j]) \equiv \mathrm{Link}_M(k_i, k_j) \pmod 1$ for $i \neq j$ and $s(k_i)=-\phi([k_i], [k_i])$, we have a desired congruence.

For each element $s \in Q/Z$ we can have coprime positive integers a, b such that $s \equiv a/b \pmod 1$. This fraction a/b and the denominator b are called a *normal presentation* and the *denominator* of the element $s \in Q/Z$, respectively. Now we assume that the denominator of the slope $s(k_i)$ is odd, $i=1, 2, \cdots, r$. Then $s(k_i)$ has a normal presentation of type $2a_i/b_i$, $i=1, 2, \cdots, r$.

DEFINITION 1.4. We define

$$s^*(L) = \sum_{i=1}^{r} a_i/b_i - \lambda(L)$$

in Q/Z and call it the *half-slope* of the link $L \subset S$.

The following is easily proved.

Lemma 1.5. *In Q/Z $2s^*(L)=s(L)$, and if $s(L)=0$, then $s^*(L)$ is* 0 *or* 1/2 *according as the denominator of $\lambda(L)\in Q/Z$ is odd or even.*

2. The ∂_0-invariant and the ∂-invariant

We consider an oriented link L with components k_i, $i=1, 2, \cdots, r$, in an oriented Z_2-homology 3-sphere S.

DEFINITION 2.1. The link L is *proper* if the mod 2 linking number, $\text{Link}_S(k_i, L-k_i)_2=0$ for all i, $1\leq i\leq r$. (We understand a knot to be a proper link.)

Let W be a compact oriented 4-manifold. Let F be a locally flat, oriented (possibly disconnected) surface of (total) genus 0 in W. We say that such a pair $F\subset W$ is *admissible* for a link $L\subset S$, if S is a component of ∂W, $\partial F=L$, $H_1(\partial W; Z_2)=0$ and $[F_2^+]\in H_2(W; Z_2)$ is characteristic, i.e., $[F_2^+]\cdot x=x^2$ for all $x\in H_2(W; Z_2)$, where F_2^+ is a (mod 2) cycle obtained from F by attaching (mod 2) 2-chains c_i in S with $\partial c_i=-k_i$, $i=1, 2, \cdots, r$.

Lemma 2.2. *For any proper link $L\subset S$ there exists an admissible pair $F\subset W$.*

Proof. Let $T(L)=\cup_{i=1}^r T(k_i)$ be a tubular neighborhood of $L=\cup_{i=1}^r k_i$ in S. Construct a 4-manifold $W=(-S)\times[-1, 1]\cup D^2\times D_1^2\cup\cdots\cup D^2\times D_r^2$ identifying $T(k_i)\times 1$ with $(\partial D^2)\times D_i^2$, $i=1, \cdots, r$, so that $H_1(\partial W; Z_2)=0$. Let $D_i=(-k_i)\times[-1, 1]\cup D^2\times 0_i$ be a disk. Let $F=\cup_{i=1}^r D_i$. To show that $F\subset W$ is admissible for $L\subset S$, it suffices to check that $[F_2^+]\in H_2(W; Z_2)$ is characteristic. Note that $[D_{i2}^+]$, $i=1, \cdots, r$, form a basis for $H_2(W; Z_2)$. Since $[F_2^+]=\sum_{i=1}^r [D_{i2}^+]$, we have

$$\begin{aligned}[F_2^+]\cdot[D_{i2}^+] &= [D_{i2}^+]^2+\sum_{j(\neq i)}[D_{j2}^+]\cdot[D_{i2}^+]\\ &= [D_{i2}^+]^2+\text{Link}_S(L-k_i, k_i)_2\\ &= [D_{i2}^+]^2,\ i=1, \cdots, r\,.\end{aligned}$$

This implies that $[F_2^+]$ is characteristic. This completes the proof.

The pair $F\subset W$, constructed in the proof of Lemma 2.2 is called a *standard admissible pair* for the proper link $L\subset S$.

DEFINITION 2.3. Let $L\subset S$ be a proper link. Then we define

$$\delta_0(L)=\delta_0(L\subset S)=([F_Q^+]^2-\text{sign } W)/16-\mu(\partial W)$$

in Q/Z for any admissible pair $F\subset W$ for $L\subset S$, where F_Q^+ is a rational 2-cycle obtained from F by attaching rational 2-chains c_i^Q in S with $\partial c_i^Q=-k_i$, $i=1, \cdots, r$.

REMARK 2.4. We can define the invariant $\delta_0(L\subset S)$ by using a more general pair $F\subset W$, where the (total) genus of F may be positive or F mac be non-orientable (cf. Freedman-Kirby [1], Guillou-Marin [3], Matsumoto [7]).

To see the well-definwsness of $\delta_0(L)$, consider a standard admissible pair $F^*\subset W^*$ for $L\subset S$. Construct an oriented 4-manifold $\bar{W}=W\cup -W^*$ identifying two copies of S. Then $\sum=F\cup -F^*$ is the disjoint union of 2-spheres. Since $[F_2^+]$ and $[F_2^{*+}]$ are characteristic, we see that the mod 2 homology class $[\sum]_2\in H_2(\bar{W}; Z_2)$ is characteristic. By the Rochlin theorem ([6], [10]), $\mu(\partial\bar{W})=([\sum]^2-\text{sign }\bar{W})/16$ in Q/Z. But, $\mu(\partial\bar{W})=\mu(\partial W)-\mu(\partial W^*)$, $[\sum]^2=[F_Q^+]^2-[-F_Q^{*+}]^2=[F_Q^+]^2-[F_Q^{*+}]^2$ and sign $\bar{W}=$sign $W-$sign W^*, where we count $[-F_Q^{*+}]^2$, $[F_Q^{*+}]^2$ in W^*. It follows that

$$([F_Q^+]^2-\text{sign }W)/16-\mu(\partial W)=([F_Q^{*+}]^2-\text{sign }W^*)/16-\mu(\partial W^*)$$

in Q/Z, showing the well-definedness of $\delta_0(L)$.

DEFINITION 2.5. Two links $L_i\subset S_i$, $i=0, 1$, are said to be *cobordant in the weak sense* if:

(1) There exists a compact oriented 4-manifold W such that $\partial W=-S_0\cup S_1$ and $H_*(W, S_i; Z_2)=0$, $i=0, 1.$,

(2) There exists a locally flat, compact oriented (possibly disconnected) surface F of (total) genus 0 in W such that $\partial F=-L_0\cup L_1$ (See Figure 1).

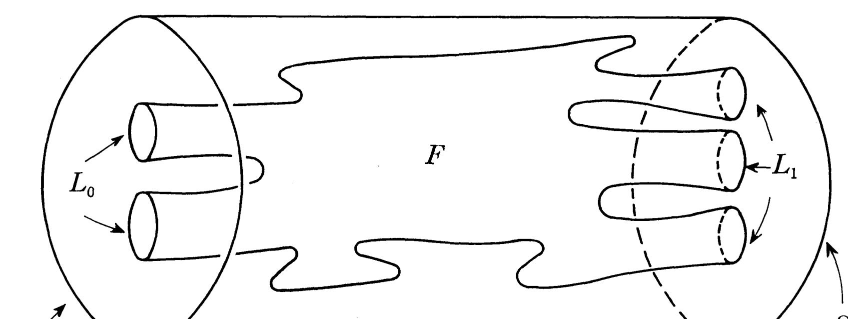

Figure 1.

Theorem 2.6. *If proper links $L_i\subset S_i$, $i=0, 1$, are cobordant in the weak sense, then $\delta_0(L_0)=\delta_0(L_1)$.*

Proof. Let $F\subset W$ be a cobordism pair for $L_i\subset S_i$, $i=0, 1$, stated in Definition 2.5. Construct an oriented 4-manifold $W'=W\cup D^3\times[0, 1]$ identifying a 3-cell in S_i-L_i with $D^3\times i$ for each i, $i=0, 1$. Then $\partial W'$ is a connected sum $(-S_0)\#S_1$, which is a Z_2-homology 3-sphere containing a proper link L', regarded as the union $-L_0\cup L_1$. Clearly, $\delta_0(L'\subset(-S_0)\#S_1)=-\delta_0(L_0\subset S_0)+$

$\delta_0(L_1\subset S_1)$. Note that $H_2(W'; Z_2)=0$. Then $F\subset W'$ is admissible for $L'\subset (-S_0)\# S_1$, and hence

$$\delta_0(L'\subset(-S_0)\# S_1) = ([F_Q^+]^2-\text{sign } W')/16-\mu((-S_0)\# S_1) = 0\,,$$

because W' is spin and $H_2(W'; Q)=0$. Thus, $\delta_0(L_0\subset S_0)=\delta_0(L_1\subset S_1)$. This completes the proof.

In [5, Definition 2.1] the δ-invariant $\delta(k)$ of a knot k in S was defined so as to be $\delta(k)=\delta_0(k)$.

Corollary 2.7. *Let $k\subset S$ be a knot obtained from a proper link $L\subset S$ by a fusion. Then $\delta_0(L)=\delta_0(k)=\delta(k)$.*

Proof. The knot $k\subset S$ and the link $L\subset S$ are cobordant in the weak sense. The result follows from Theorem 2.6.

By a Dehn surgery we obtain from a knot $k\subset S$ a unique (up to homeomorphism), closed, connected, oriented 3-manifold M such that $H_1(M; Z)/\text{odd torsion}\cong Z$, called a Z_2-homology handle (cf. [5, Remark 1.6 and Corollary 1.7]). In [4] we defined an invariant $\in(M)$, being 0 or 1, of M, calculable from the Z_2-Alexander polynomial of M.

Corollary 2.8. *Let $L\subset S$ be a proper link. Let M be the Z_2-homology handle of a knot $k\subset S$, obtained from L by a fusion. Let a/b be a normal presentation of the slope $s(L\subset S)$ with a odd. Then we have*

$$\delta_0(L) = \in(M)/2+(a/b-ab)/16$$

in Q/Z.

Proof. By Lemma 1.2 $s(L)=s(k)$. By Corollary 2.7 $\delta_0(L)=\delta(k)$. Then the desired congruence follows from [5, Theorem 2.7 and Corollary 3.6].

DEFINITION 2.9. For a proper link L in S we define

$$\delta(L) = \delta(L\subset S) = \delta_0(L\subset S)+\lambda(L\subset S)/8$$

in Q/Z.

REMARK 2.10. Definition 2.9 is analogous to Murasugi's definition of the unoriented link type signature in [7] (cf. [5, Remark 4.8]).

Theorem 2.11. *The invariant $\delta(L\subset S)$ is an unoriented link type invariant of a proper link $L\subset S$. That is, $\delta(L\subset S)=\delta(L'\subset S')$ for any link $L'\subset S'$ with an orientation-preserving homeomorphism $S\to S'$ sending L to L' setwise.*

Proof. It suffices to show that $\delta(L)$ does not depend on any particular orientations of the components, k_i, of L. Let $F=\cup_{i=1}^r D_i\subset W$ be a standard admissible pair for $L=\cup_{i=1}^r k_i\subset S$. Note that $[F_Q^+]^2=\sum_{i=1}^r [D_{iQ}^+]^2+2\sum_{i>j}[D_{iQ}^+]\cdot$

$[D_{jQ}^{+}]=\sum_{i=1}^{r}[D_{iQ}^{+}]^2-2\lambda(L)$. Then

$$\delta(L)=\delta_0(L)+\lambda(L)/8=(\sum_{i=1}^{r}[D_{iQ}^{+}]^2-\text{sign } W)/16-\mu(\partial W)\,.$$

Since $[D_{iQ}^{+}]^2$ is not altered by changing the orientation of D_i (that is, k_i), we have a desired result.

A link $L\subset S$ is *amphicheiral* if there is an orientation-preserving homeomorphism $S\to -S$ sending L to itself setwise. The following is direct from Theorem 2.11.

Corollary 2.12. *If a proper link $L\subset S$ is amphicheiral, then* $2\delta(L)=0$ *in Q/Z.*

Here is an example of a classical proper link.

EXAMPLE 2.13. Let L_r be an r-component link in a 3-sphere S^3, illustrated in Figure 2, where $r\geq 2$. The link

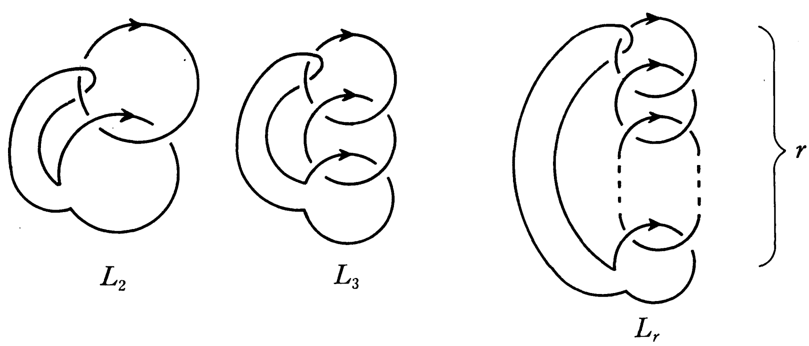

Figure 2.

L_r is clearly proper. Choosing a suitable orientation of S^3, $\lambda(L_r)=r$. Since we can have a trivial knot from L_r by a fusion, we see that $\delta_0(L_r)=0$. Therefore, $\delta(L_r)=r/8$ in Q/Z.

3. Branched cyclic coverings and the δ- and δ_0-invariants

We consider a link $\tilde{L}\subset\tilde{S}$ obtained from a link $L\subset S$ by taking an n-fold cyclic branched covering of S, branched along L. Namely, $\tilde{S}$ is the branched covering space over S, associated with an epimorphism $H_1(S-L;Z)\to Z_n$ sending each meridian of L to a unit of Z_n, and $\tilde{L}$ is the lift of L. We assume that $\tilde{S}$ is a Z_2-homology 3-sphere.

First we consider the case $n=2$. Then L and $\tilde{L}$ are knots by the Smith theory. Let $L=k$ and $\tilde{L}=\tilde{k}$.

Theorem 3.1. *Let $2a/b$ be a normal presentation of the slope $s(\tilde{k})$. Then*

$$\delta(\tilde{k}) = \delta(k) - (a/b - ab)/8$$

in Q/Z. *In particular, if* $\tilde{k}$ *is flat, then* $\delta(\tilde{k}) = \delta(k)$.

Proof. By [5, Lemma 4.5] $s(k)=2s(\tilde{k})=4a/b$. Since $(2a+b)/b$ and $(4a+b)/b$ are normal presentations of $s(\tilde{k})$ and $s(k)$, respectively, we see from Corollary 2.8 that

$$\delta(\tilde{k}) = \in(\tilde{M})/2 + \{(2a+b)/b - (2a+b)b\}/16 = \in(\tilde{M})/2 + (a/b - ab)/8 + (1-b^2)/16\,,$$

and

$$\delta(k) = \in(M)/2 + \{(4a+b)/b - (4a+b)b\}/16 = \in(M)/2 + (a/b - ab)/4 + (1-b^2)/16\,,$$

where $\tilde{M}$ and M are the Z_2-homology handles of $\tilde{k} \subset \tilde{S}$ and $k \subset S$, respectively. Since $\tilde{M}$ is a 2-fold covering space of M, it follows from [4, Lemma 4.2] that $\in(\tilde{M}) = \in(M)$. Now we have a desired congruence. This completes the proof.

Next, to consider the case that the covering degree n is an odd prime p, we remark the following:

Lemma 3.2. $\lambda(\tilde{L}) = \lambda(L)/n$.

Corollary 3.3. *$\tilde{L}$ is proper if and only if L is proper.*

Proof of Lemma 3.2. It suffices to show that for $i \neq j$ $\mathrm{Link}_{\tilde{S}}(\tilde{k}_i, \tilde{k}_j) = \mathrm{Link}_S(k_i, k_j)/n$. Let F_i be a characteristic surface (cf. [5]) of k_i in S such that $L - k_i$ intersects F_i transversally. Write $[\partial F_i] = a_i r_i [m_i] + b_i r_i [l_i]$ in $H_1(\partial T(k_i); Z)$ for a meridian-longitude pair (m_i, l_i) of $T(k_i)$ such that the lift of l_i has n components, where $(a_i, b_i) = 1$ and r_i is an integer >0. Let $\tilde{l}_i$ be a component of the lift of l_i. For the lift $\tilde{m}_i$ of m_i, the pair $(\tilde{m}_i, \tilde{l}_i)$ forms an *m.l.* pair of a tubular neighborhood $T(\tilde{k}_i)$ of $\tilde{k}_i$ which is the lift of $T(k_i)$. Note that the lift $\tilde{F}_i$ of F_i is an oriented surface which is a branched Z_n-covering space of F_i branched over the set $F_i \cap (L - k_i)$. Clearly $[\partial \tilde{F}_i] = a_i r_i [\tilde{m}_1] + b_i r_i n [\tilde{l}_i]$ in $H_1(\partial T(\tilde{k}_i); Z)$. Since the intersection numbers, $\mathrm{Int}(\tilde{F}_i, \tilde{k}_j)$ and $\mathrm{Int}(F_i, k_j)$ are equal, we have

$$\mathrm{Link}_{\tilde{S}}(\tilde{k}_i, \tilde{k}_j) = \mathrm{Int}(\tilde{F}_i, \tilde{k}_j)/b_i r_i n = \mathrm{Int}(F_i, k_j)/b_i r_i n = \mathrm{Link}_S(k_i, k_j)/n\,.$$

This completes the proof.

Proof of Corollary 3.3. When n is even, L and $\tilde{L}$ are knots by the Smith theory. So, assume n is odd. It suffices to show that $\mathrm{Link}_{\tilde{S}}(\tilde{k}_i, \tilde{k}_j)_2 = \mathrm{Link}_S(k_i, k_j)_2$ for $i \neq j$. This is obtained by a mod 2 version of the proof of Lemma 3.2, since $b_i r_i n$ is odd. This completes the proof.

We shall show the following theorem, where note that $(p^2-1)/8$ is an integer.

Theorem 3.4. *Let $\tilde{L}=\cup_{i=1}^r \tilde{k}_i \subset \tilde{S}$ be a proper link and assume that the covering degree is an odd prime p. Let $2a_i/b_i$ be a normal presentation of the slope $s(\tilde{k}_i)$, $i=1, 2, \cdots, r$. Then*

$$\delta(\tilde{L}) = p\delta(L)-\{(p^2-1)/8\}\textstyle\sum_{i=1}^r a_i/b_i$$

in Q/Z.

Proof. Let $F\subset W$ and $\tilde{F}\subset \tilde{W}$ be standard admissible pairs for $L\subset S$ and $\tilde{L}\subset \tilde{S}$, respectively, such that $\tilde{F}\subset \tilde{W}$ is obtained from $F\subset W$ by taking a Z_p-covering branched along F. [One can see directly or by a transfer argument that such pairs exist.] Let $\partial W-S=S^*$ and $\partial \tilde{W}-\tilde{S}=\tilde{S}^*$. By the proof of Theorem 2.11,

$$\delta(L) = (\textstyle\sum_{i=1}^r [D_{iQ}^+]^2-\text{sign } W)/16-\mu(S)-\mu(S^*)\,, \text{ and}$$
$$\delta(\tilde{L}) = (\textstyle\sum_{i=1}^r [\tilde{D}_{iQ}^+]^2-\text{sign } \tilde{W})/16-\mu(\tilde{S})-\mu(\tilde{S}^*)\,,$$

where $F=\cup_{i=1}^r D_i$, $\tilde{F}=\cup_{i=1}^r \tilde{D}_i$, and $\tilde{D}_i$ corresponds to D_i. Then since $[D_{iQ}^+]^2/p=[\tilde{D}_{iQ}^+]^2$ (cf. [5, the proof of Lemma 4.9]),

$$\delta(\tilde{L})-p\delta(L) = (1-p^2)\textstyle\sum_{i=1}^r [\tilde{D}_{iQ}^+]^2/16+(-\text{sign } \tilde{W}+p \text{ sign } W)/16$$
$$-(\mu(\tilde{S})-p\mu(S))-(\mu(\tilde{S}^*)-p\mu(S^*))\,.$$

By the definition of α-invariant in [5, Section 4],

$$\alpha(Z_p, \tilde{S})+\alpha(Z_p, \tilde{S}^*) = -\text{sign } \tilde{W}+p \text{ sign } W-(\textstyle\sum_{i=1}^r [\tilde{D}_{iQ}^+]^2)(p^2-1)/3\,.$$

Therefore,

$$\delta(\tilde{L})-p\delta(L) = \{1-p^2+(p^2-1)/3\}(\textstyle\sum_{i=1}^r [\tilde{D}_{iQ}^+]^2)/16-(\mu(\tilde{S})-p\mu(S)$$
$$-\alpha(Z_p, \tilde{S})/16)-(\mu(\tilde{S}^*)-p\mu(S^*)-\alpha(Z_p, \tilde{S}^*)/16)\,.$$

First, let $p>3$. Then by [5, Theorems 11.1 and 12.1],

$$\mu(\tilde{S}^*) = p\mu(S^*)+\alpha(Z_p, \tilde{S}^*)/16\,, \text{ and}$$
$$\mu(\tilde{S}) = p\mu(S)+\alpha(Z_p, \tilde{S})/16+\{(p^2-1)/24\}\textstyle\sum_{i=1}^r a_i/b_i\,,$$

where note that $(p^2-1)/24$ is an integer. Since $[\tilde{D}_{iQ}^+]^2\equiv 2a_i/b_i \pmod 1$ (cf. [5, Lemma 2.6]), it follows that

$$\delta(\tilde{L})-p\delta(L) = -\{(p^2-1)/24\}\textstyle\sum_{i=1}^r [\tilde{D}_{iQ}^+]^2-\{(p^2-1)/24\}\sum_{i=1}^r a_i/b_i$$
$$= -\{(p^2-1)/8\}\textstyle\sum_{i=1}^r a_i/b_i\,.$$

Now let $p=3$. By [5, Theorems 11.1 and 12.1],

$$\mu(\tilde{S}^*) = 3\mu(S^*)+9\alpha(Z_3, \tilde{S}^*)/16\,, \text{ and}$$

$$\mu(\tilde{S}) = 3\mu(S) + 9\alpha(Z_3, \tilde{S})/16 + 3\sum_{i=1}^{r} a_i/b_i .$$

Directly or by a transfer argument, sign $\hat{W}$=sign W. Then

$$\begin{aligned}\alpha(Z_3, \tilde{S})/2 + \alpha(Z_3, \tilde{S}^*)/2 &= -\text{sign } \tilde{W}/2 + 3 \text{ sign } W/2 - (\textstyle\sum_{i=1}^{r} [\tilde{D}^+_{iQ}]^2)4/3 \\ &\equiv -(\textstyle\sum_{i=1}^{r} [\tilde{D}^+_{iQ}]^2)4/3 \pmod 1.\end{aligned}$$

Then,

$$\begin{aligned}\delta(\tilde{L}) - 3\delta(L) &= -(\textstyle\sum_{i=1}^{r} [\tilde{D}^+_{iQ}]^2)/3 - \alpha(Z_3, \tilde{S})/2 - \alpha(Z_3, \tilde{S}^*)/2 \\ &-3\textstyle\sum_{i=1}^{r} a_i/b_i = \sum_{i=1}^{r} [\tilde{D}^+_{iQ}]^2 - 3\sum_{i=1}^{r} a_i/b_i = -\sum_{i=1}^{r} a_i/b_i\end{aligned}$$

in Q/Z, because $[\tilde{D}^+_{iQ}]^2 \equiv 2a_i/b_i \pmod 1$. This completes the proof.

Theorem 3.5. *Let $\tilde{L} \subset \tilde{S}$ be proper and assume that the covering degree is an odd prime p. Then we have*

$$\delta_0(\tilde{L}) = p\delta_0(L) - \{(p^2-1)/8\} s^*(\tilde{L})$$

in Q/Z, where $s^*(\tilde{L})$ is the half-slope of the link $\tilde{L} \subset \tilde{S}$.

Proof. By Theorem 3.4,

$$\delta_0(\tilde{L}) + \lambda(\tilde{L})/8 = p\delta_0(L) + p\lambda(L)/8 - \{(p^2-1)/8\} \textstyle\sum_{i=1}^{r} a_i/b_i .$$

Since $\lambda(L)/p = \lambda(\tilde{L})$ by Lemma 3.2, we have

$$\delta_0(\tilde{L}) - p\delta_0(L) = -\{(p^2-1)/8\} (\textstyle\sum_{i=1}^{r} a_i/b_i - \lambda(\tilde{L})) = -\{(p^2-1)/8\} s^*(\tilde{L}) .$$

This completes the proof.

Corollary 3.6. *If $\tilde{L}$ is flat, then $\delta_0(\tilde{L}) = \delta_0(L)$.*

Proof. By Lemma 1.5 $s(\tilde{L})=0$ implies $s^*(\tilde{L})=0$. So, by Theorem 3.5 $\delta_0(\tilde{L}) = p\delta_0(L)$. By Lemmas 1.3, 3.2 and [5, Lemma 4.5], $s(L) = ps(\tilde{L})$, so that $s(L)=0$. By Corollary 2.8 $2\delta_0(L)=0$. Using that p is odd, the proof is completed.

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