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ON TRANSLATION PLANES OF ORDER q^3 WITH AN ORBIT OF LENGTH q^3 —1 ON I_{∞}

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1. Introduction

In his paper [7], Suetake constructed a class of translation planes of cubic order q^3 and he improved his results for each prime power q such that 2 is a nonsquare element of GF(q). Any plane of the class admits a collineation group G of the linear translation complement such that

(*) G has orbits of length 2 and q^3-1 on l_{∞} .

In this paper we construct a new class of translation planes of order q^3 with the property (*) for each prime power q with $q \equiv 1 \pmod{2}$ (§2). If π is any translation plane of order q^3 with the property (*) and if π is not an Andre plane, then we have either (i) the linear translation complement $LC(\pi)$ is of order $3^i(q^3-1)(q-1)$ with $0 \le i \le 2$, or (ii) q=3 and $LC(\pi)$ is isomorphic to SL(2, 13) (§3). In §4, we present a certain characterization of the class of planes with the property (*). Throughout the paper all sets, planes and groups are finite. Our notation is largely standard and taken from [3] and [5].

2. Description of the class of planes Π

Let $F=GF(q^3)$ be a field order of q^3 , where $q=p^n$ and p is an odd prime. Let K be a subfield of F order q. Throughout the paper we consider the translation plane π of order q^3 which is a 6 dimensional vector space over K.

To represent spread sets and collineation groups of π , we use a method of [6]; Let $w \in F - K$ and put a 3×3 matrix $W = \begin{pmatrix} 1 & 1 & 1 \\ w & \overline{w} & \overline{w} \\ w^2 & \overline{w}^2 & \overline{w}^2 \end{pmatrix}$, where $\overline{w} = w^q$ and $\overline{w} = w^{q^2}$. Let $M(3, q_1)$ be the set of all matrices over $GF(q_1)$ for a prime power q_1 . Set $M(3, q)^* = W^{-1}M(3, q)W(\subset M(3, q^3))$, $GL(3, q)^* = W^{-1}GL(3, q)W$ and $V^* = V(3, q)W = \{(v_1, v_2, v_3)W | v_1, v_2, v_3 \in K\}$. Then $M(3, q)^* = \begin{cases} a & \overline{c} & \overline{b} \\ b & a & \overline{c} \\ c & \overline{b} & \overline{a} \end{cases}$ $a, b, c \in F \end{cases}$, $GL(3, q)^* = \{P \in M(3, q)^* | \det(P) \neq 0\}$ and $V^* = \{(a, \overline{a}, \overline{a}) | a \in F\}$

(cf. [6]). Here "det" means the determinant of a matrix. Clearly $GL(3, q)^*$ acts naturally on V^* as the general linear group of the vector space over K. The

notations $\begin{pmatrix} a & \overline{c} & \overline{b} \\ b & a & \overline{c} \\ c & \overline{b} & \overline{a} \end{pmatrix}$ and $(a, \overline{a}, \overline{a})$ are abbreviated to [a, b, c] and to [a], respectively.

For $k \in K$, we have k[a, b, c] = [ka, kb, kc] and k[a] = [ka]. Set I(x) = [x, 0, 0], I = I(1), O = I(0) and J = [0, 1, 0].

Under these conditions a set $\Sigma \subset GL(3, q)^* \cup \{O\}$ is defined to be a spread set if $O \in \Sigma$, $|\Sigma| = q^3$ and det $(M - N) \neq 0$ for any two distinct $M, N \in \Sigma$.

The translation plane π which corresponds to Σ is defined in the usual manner ([5]); the set of points of π is $V^* \times V^*$ and the set of lines passing through the origin is $\mathcal{L} = \{L(M) | M \in \Sigma\} \cup \{L(\infty)\}$, where $L(M) = \{([x], [x]M) | x \in F\}$ and $L(\infty) = \{([0], [x]) | x \in F\}$.

Throughout the rest of this paper let u be a nonsquare element of K. As $q^2+q+1\equiv 1 \pmod{2}$, u is also a nonsquare element of F. Let \overline{F} be the algebraic closure of F and set $\widetilde{F}=F-\{\pm 1\}$, $F^{\ddagger}=F-\{0\}$, $K^{\ddagger}=K-\{0\}$.

We now define Π_K a class of translation planes of order q^3 . Let Φ_K be the set of all ordered triples $(a, b, c) \in K \times K \times K$ such that

 $\Sigma_{(a,b,c)} = \{I(x) [a, b, c] I(x) | x \in F\} \cup \{I(x) u[a, b, c]^{-1} I(x) | x \in F\}$

is a spread set. We denote by $\pi_{(a,b,c)}$ the translation plane corresponding to the spread set $\Sigma_{(a,b,c)}$ and set $\Pi_K = {\pi_{(a,b,c)} | (a, b, c) \in \Phi_K}$. Furthermore set $\Pi = \bigcup_{k} \Pi_K$, where K runs through all finite fields of odd characteristic.

Clearly the set of matrices I(x) $(x \in F^*)$ forms an abelian group of order q^3-1 . Hence an ordered triple $(a, b, c) \in K \times K \times K$ is contained in Φ_K if and only if

(i) $f(x) = \det (I(x) [a, b, c] I(x) - [a, b, c]) \neq 0$ for any $x \in \widetilde{F}$,

(ii) $g(x) = \det(I(x)[i, j, k] I(x) - [i, j, k]) \neq 0$ for any $x \in \tilde{F}$, where $[i, j, k] = u[a, b, c]^{-1}$, and

(iii) $h(x) = \det (I(x)[a, b, c] I(x) - [i, j, k]) \neq 0$ for any $x \in F$. Using these we can show that $\Phi_K = \Phi_K^{(1)} \cup_K^{(2)}$, where $\Phi_K^{(1)} = \{(a, b, c) \in K \times K \times K \mid a(a^2 - bc) = 0, b^3 + c^3 - 2abc = 0\}$ and $\Phi_K^{(2)} = \{(a, b, c) \in K \times K \times K \mid a(a^2 - bc) \neq 0, b^3 + c^3 - 2abc = 0\}$ (Proposition 1). As a corollary, we have

Theorem 1. $\Pi_{K} = \{\pi_{(a,b,c)} | (a, b, c) \in \Phi_{K}^{(1)} \cup \Phi_{K}^{(2)} \}.$

In the rest of this section we prove $\Phi_K = \Phi_K^{(1)} \cup \Phi_K^{(2)}$. Let $(a, b, c) \in K \times K \times K$ and set A=a, $B=a^2-bc$, $C=b^3+c^3-2abc$ and $D=a^3+b^3+c^3-3abc$. Then $AB+C=D=\det[a, b, c]$.

Lemma 2.1. Assume $D \neq 0$. Then

(i) $f(x) = ABN(x^2-1) + CN(x^{q+1}-1)$ and $g(x) = u^3 D^{-2} f(x)$. Here $N(x) = x^{q^2+q+1}$ for $x \in F$.

(ii)
$$h(x) = D^{-1} \det (([a, b, c] I(x))^2 - uI).$$

Proof. By direct calculation we have (i) and $g(x) = u^3 \det (I(x) [a, b, c]^{-1} I(x)) \det ([a, b, c] - I(x)^{-1} [a, b, c] I(x)^{-1}) \det [a, b, c]^{-1} = u^3 (\det [a, b, c])^{-2} f(x)$, hence (i) holds. Similarly we have (ii).

Lemma 2.2. Let $r(t, x) = \det(xI - [a, b, c] I(t))$ be a characteristic polynomial of [a, b, c] I(t) with $t \in F$. Then

(i) $r(t, x) = x^3 - AT(t)x^2 + BT(t^{q+1})x - DN(t)$. (Here $T(z) = z + z^q + z^{q^2}$ is the trace map.)

(ii) Let $t \in F$. Then h(t)=0 if and only if $u=-BT(t^{q+1})$ and uAT(t) = -DN(t).

Proof. By direct calculation we have (i). Suppose $u=-BT(t^{q+1})$ and uAT(t)=-DN(t) for some $t \in F$. Then, by (i), $r(t, x)=x^3-kx^2-ux+uk=(x-k)(x^2-u)$, where $k=AT(t)\in K$. Let v be a root of x^2-u in the algebraic closure \overline{F} . Then v is an eigenvalue of [a, b, c] I(t). Hence h(t)=0. Conversely, assume h(t)=0 for some $t\in F$. Let z_1, z_2, z_3 be the eigenvalues of [a, b, c] I(t). Then, by Lemma 2.1, $z_i^2=u$ for some i. As r(t, x) is a cubic polynomial over K and $z_i\in\overline{F}-F$, $r(t, x)=(x-k)(x^2-u)$ for some $k\in K$. Hence AT(t)=k, $BT(t^{q+1})=-u$ and -DN(t)=ku. Thus $u=-BT(t^{q+1})$ and uAT(t)=-DN(t).

In Lemmas 2.4–2.7, we assume the following.

Hypothesis 2.3. $(a, b, c) \in \Phi_K$ and $ABC \neq 0$.

Lemma 2.4. Set i=uA/D, j=u/B and $w(x)=(x^3-jx)/(x^2-i)$. Then,

- (i) *i* is nonsquare in K.
- (ii) w(x) is a bijection from K onto itself.

Proof. Since $C \neq 0$, $i(i-j) \neq 0$ and so $(x^3-jx, x^2-i) = ((i-j)x, x^2-i) = 1$. Deny (ii) and let $e \in K - \{w(t) | t \in K, t^2-i \neq 0\}$. Then $x^3-ex^2-jx+ie$ is an irreducible polynomial over K. Let t be a root of this polynomial in \overline{F} . Then $t \in F - K$ and so $x^3 - ex^2 - jx + ie = (x-t) (x-t^q) (x-t^{q^2})$. Hence T(t) = e, $T(t^{q+1}) = -j$ and N(t) = -ie and so $u = -BT(t^{q+1})$ and -DN(t) = uAT(t). This contradicts (ii) of Lemma 2.2. Thus (ii) holds and (i) follows from (ii).

Lemma 2.5. Set $k=3i^2+6ij-j^2$. Then, either 9i=j or $F(y)=4iy^4-ky^2+4i^2j$ is nonsquare in K for each $y \in K$.

Proof. We have w(x) = w(y) if and only if $(x-y)((y^2-i)x^2-(i-j)yx-i(y^2-j)) = 0$. By (i) of Lemma 2.4, $y^2 - i \neq 0$. Assume F(y) is square in K for some y and set $v = \sqrt{F(y)} \in K$. Then w(x) = w(y) = w(x'), where $\{x, x'\} = \{((i-j)y \pm v)/2(y^2-i)\}$. By Lemma 2.4 (ii), y = x = x'. Hence v = 0 and $y(2y^2-3i+j)=0$. As $ij \neq 0$, $y \neq 0$ and so $0 = v^2 = 4i((3i-j)/2)^2 - (3i^2+6ij-j^2)$

 $(3i-j)/2+4i^2j=(i-j)^2(9i-j)/2$. Therefore 9i=j.

Lemma 2.6 $9i \pm j$.

Proof. Assume 9i=j. Then 9AB=D. As $D \neq 0$, char $K \neq 3$. Let $e \in K$ -R, where $R = \{27(x^3-3x^2-2)/(3x+1) | x \in K, x \neq -3^{-1}\}$. Since $3x+1 \not/ 27$ $(x^3-3x^2-2)=(3x+1) (9x^2-30x+10)-64$, $S(x)=27(x^3-3x^2-2)-e(3x+1)$ is irreducible over K. Hence $S(x)=27(x-t) (x-t^q) (x-t^{q^2})$ for some $t \in F-K$. Therefore T(t)=3, $T(t^{q+1})=-e/9$ and N(t)=2+e/27. In particular $T(t)-T(t^{q+1})-3N(t)+3=0$. However, by Lemma 2.1(i), $f(t)=AB(N(t^2-1)+8N(t^{q+1}-1))=AB(T(t)-T(t^{q+1})-3N(t)+3) (T(t)+T(t^{q+1})-3N(t)-3)=0$, a contradiction.

Lemma 2.7. $q \le 13$.

Proof. By Lemmas 2.5 and 2.6, F(y) is nonsquare in K for any $y \in K$. Applying Lemma of [8], either $q \le 13$ or $k^2 - 4 \times 4i \times 4i^2 j = (i-j)^3(9i-j) = 0$. Again, by Lemma 2.6, $9i-j \ne 0$ and therefore $q \le 13$.

Lemma 2.8. Let $E(y)=dy^4+ey^2+f$, d, e, $f \in K$ and assume that d is non-square in K. If $q \leq 13$ and E(y) is nonsquare for each $y \in K$, then $e^2-4df=0$.

Proof. Let K_1 be the set of nonzero square elements of K. The E(y)'s satisfying the conditions above are as follows, which we obtained by using a computer;

(1) K=GF(13): $(d, e, f)=(2m, 2m, 7m), (2m, 5m, 8m), (2m, 6m, 11m), (2m, 7m, 11m), (2m, 8m, 8m), (2m, 11m, 7m), m \in K_1$.

(2) K = GF(11): (d, e, f) = (2m, m, 7m), (2m, 3m, 8m), (2m, 4m, 2m), (2m, 5m, 11m), (2m, 9m, 6m), $m \in K_1$.

(3) $K=GF(9)=\langle w \rangle GF(3)$, where $w^2=-1$: (d, e, f)=((w+1)m, m, (w+2)m), ((w+1)m, 2m, (w+2)m), ((w+1)m, wm, (2w+1)m), ((w+1)m, 2wm, (2w+1)m), $m \in K_1$.

(4) K = GF(7): $(d, e, f) = (3m, 3m, 6m), (3m, 5m, 5m), (3m, 6m, 3m), m \in K_1$.

- (5) K = GF(5): $(d, e, f) = (2m, 2m, 3m), (2m, 3m, 3m), m \in K_1$.
- (6) K = GF(3): $(d, e, f) = (2m, m, 2m), m \in K_1$.

Using these, we can verify that $e^2 - 4df = 0$ for each case.

Proposition 1. $\Phi_{K} = \Phi_{K}^{(1)} \cup \Phi_{K}^{(2)}$.

Proof. Assume $D \neq 0$ and AB = 0. Then $C = D \neq 0$. Hence, by Lemma 2.1, $f(x) \neq 0$ and $g(x) \neq 0$ for any $x \in \tilde{F}$. By Lemma 2.2, $h(t) \neq 0$ for any $t \in F$. Therefore $\Phi_K^{(1)} \subset \Phi_K$.

Assume $D \neq 0$ and C=0. Then $AB=D \neq 0$ and so, by Lemma 2.1, $f(x) \neq 0$ and $g(x) \neq 0$ for any $x \in \tilde{F}$. If h(t)=0 for some $t \in F$, then $t \neq 0$ and T(t) $T(t^{q+1})$ -N(t)=0 by Lemma 2.2 (ii). Since T(t) $T(t^{q+1})-N(t)=N(t+t^{q})$, it follows

that $t^{q-1} = -1$. However, this implies $2|q^2+q+1$, a contradiction. Therefore $\Phi_{\kappa}^{(2)} \subset \Phi_{\kappa}$.

Assume $D \neq 0$ and $ABC \neq 0$. Then, by Lemmas 2.5-2.8, $k^2 - 4 \times 4i \times 4i^2 = 0$. As we have seen in the proof of Lemma 2.7, this is a contradiction. Therefore $\Phi_K = \Phi_K^{(1)} \cup \Phi_K^{(2)}$.

REMARK 2.9. We can easily verify that the planes constructed in [7] are contained in $\{\pi_{(a,b,c)} | (a, b, c) \in \Phi_{K}^{(1)}\}$ ($\subset \Pi_{K}$).

3. The planes with the orbits of length 2 and q^3-1

Throughout this section we assume the following.

Hypothesis 3.1. (i) π is a translation plane of order q^3 with kern K = GF(q), where q is a power of an odd prime p.

(ii) A subgroup G of the linear translation complement of π has orbits Γ and Δ of length 2 and q^3-1 , respectively, on l_{∞} .

(iii) π is not an Andre plane.

Let Σ be a spread set corresponding to π and let $C(\pi)$ denote the translation complement of π . The linear translation complement $LC(\pi)$ of π is defined

by the set of all nonsingular 6×6 matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that (i) $A, B, C, D \in M(3, q)^*$.

(ii) If C is nonsingular, then $C^{-1}D \in \Sigma$. (In this case $L(\infty)g = L(C^{-1}D)$.)

(iii) If C is singular, then C=0 and D is nonsingular. (In this case $L(\infty)$ $g=L(\infty)$.)

(iv) Given $M \in \Sigma$, if A + MC is nonsingular, then $(A + MC)^{-1}(B + MD) \in \Sigma$. (In this case $L(M)g = L(M_1)$, where $M_1 = (A + MC)^{-1}(B + MD)$.)

(v) Given $M \in \Sigma$, if A + MC is singular, then A + MC = 0. (In this case $L(M)g = L(\infty)$.)

Set $\mathcal{L} = \{L(M) \mid M \in \Sigma\} \cup \{L(\infty)\}$. Then, since the restriction $LC(\pi)^{\mathcal{L}}$ is isomorphic to $LC(\pi)^{l_{\infty}}$, we often identify \mathcal{L} with l_{∞} .

By Lemma 2.1 of [5], without loss of generality we may assume $\Gamma = \{L(\infty), L(O)\}$ and $G = LC(\pi)_{\Gamma}$, the global stabilizer of Γ in $LC(\pi)$. Set $H = G_{L(\infty), L(0)}$, the stabilizer of $L(\infty)$ and L(O) in G. Then $|G: H| = |\Gamma| = 2$ and $\{\begin{pmatrix} A & O \\ O & B \end{pmatrix}| A, B \in GL(3, q)^*\} \ge H \ge \{a \begin{pmatrix} I & O \\ O & I \end{pmatrix}| a \in K^*\}$. Moreover $|\Delta| = q^3 - 1||G|$ and so $(q-1)(q^2+q+1)/2||H|$. Set $U_1 = \{I(a)|0 \neq a \in F\}$ and $U = U_1 \le J >$, J = [0, 1, 0]. Furthermore set

 $\widetilde{U}_1 = \{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} | A, B \in U_1 \}$ and $\widetilde{U} = \{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} | A, B \in U \}$. Then we have

Lemma 3.2. Let K_1 or K_2 be the group of homologies in H with axis $L(\infty)$

or L(O), respectively. Let f_1 or f_2 be a homomorphism from H to $GL(3, q)^*$ defined by $f_1\begin{pmatrix} A & O \\ O & B \end{pmatrix} = B$ or $f_2\begin{pmatrix} A & O \\ O & B \end{pmatrix} = A$, respectively. Then, a basis may be choosen for $\pi(=V^* \times V^*)$ so that $H \leq \tilde{U}$ and $f_i(H) \leq U$ for i=1, 2. In particular, $|H/K_i| |3(q^3-1)$.

Proof. Clearly K_1 and K_2 are normal subgroups of H and $K_1 \cap K_2 = 1$. Hence $H/K_1 \simeq f_1(H) \subset GL(3, q)^*$ and H/K_1 has a normal subgroup K_1K_2/K_1 $(\simeq K_2)$. By Hypothesis 3.1 (ii), K_1 and K_2 are G-conjugate and $|K_1| = |K_2|$ $|q^3 - 1$. In particular $p \not| |K_2|$.

Assume K_2 is nonsolvable. Then H/K_1 has a normal subgroup isomorphic to SL(2, 5) by Corollary 3.5 of [5] and $p \not\prec |SL(2, 5)|$. In particular K_1 has a characteristic subgroup isomorphic to SL(2, 5) and so $O_p(H/K_i)=1$ by the structure of Aut(PG(2, q)), $1 \le i \le 2$. Let g be a natural homomorphism from $GL(3, q)^*$ into PGL(3, q). Applying the results of [2], $|g(f_1(H))||3$ |PSL(2, 5)|(q-1). Hence $|H| \mid |K_1| \times (q-1)^2 \times 180$. On the other hand $|K_1K_2| = |K_1|^2 \mid |H|$ and so $|H| \mid (q-1)^4 (180)^2$. However, since $(q^2+q+1, 2 \times 5)=1$, we have $(q^2+q+1, |H|) \le 3$, contrary to $(q^3-1)/2 \mid |H|$. Therefore K_2 is solvable.

Assume $(|K_2|, q^2+q+1)>3$. Then $(|g(f_1(K_2))|, q^2+q+1)>3$. By [2], $|g(f_1(H))| | 3(q^2+q+1)(3, q-1)$ and $f_1(H)$ is $GL(3, q)^*$ -conjugate to a subgroup of U.

Next assume $(|K_2|, q^2+q+1) \leq 3$. Then $g(f_1(H))$ is a subgroup of $GL(3, q)^*$ such that $(q^2+q+1)/(3, q-1) | |g(f_1(H))|$. By [2], we have either $SL(3, q)^* \leq f_1(H)$ or $f_1(H)$ is $GL(3, q)^*$ -conjugate to a subgroup of U. Suppose $SL(3, q)^* \leq f(H)$ or $f_1(H)$ is d let u be an element of order t such that $Wf(u)W^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then

 $f_1(H)$ and let z be an element of order p such that $Wf_1(z)W^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

z fixes exactly q^2 vectors in $L(\infty)$. Therefore the fixed structure of z is a subplane of π of order q^2 , contrary to Bruck's Theorem [4]. Thus, choosing a suitable basis of V^* , we may assume $f_1(H) \subset U$. By considering the mapping f_2 , similarly we may assume $f_2(H) \subset U$. Thus the lemma holds

Lemma 3.3.
$$G-H \subset \{ \begin{pmatrix} O & A \\ B & O \end{pmatrix} | A, B \in U \}.$$

Proof. Let $z = \begin{pmatrix} O & A \\ B & O \end{pmatrix} \in G - H$ and let $h = \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \in H$. Then $z^{-1}hz = \begin{pmatrix} B^{-1} & YB & O \\ O & A^{-1} & XA \end{pmatrix} \in H$ for any $h \in H$. Since $X, Y \in U, A^{-1}f_1(H) A, B^{-1}f_2(H) B \subset U$. Hence A and B normalize the cyclic subgroup of U of order t, where t is a prime with $t \mid (q^2 + q + 1)/(3, q - 1)$. As U is a maximal subgroup of GL $(3, q)^*$ by [2], A and B are contained in U.

Lemma 3.4. $\Sigma \cap U = \phi$.

Proof. Suppose false. By Lemmas 3.2 and 3.3, together with the transitivity of G on Δ , Σ is contained in $U \cup \{O\}$. Let $M \in \Sigma - \{O\}$. By considering $M^{-1}\Sigma$, we may assume that Σ contains I. Since $I(x)^{-1}[a, b, c] I(x) = [a, bx^{1-q}, cx^{1-q^2}]$ for $x \in F$ and $N(x^{1-q}) = N(x^{1-q^2}) = 1$, we have $(LC(\pi))_{(L(I))} \ge \{ \begin{pmatrix} I(x) & O \\ O & I(x) \end{pmatrix} | x \in F^* \}$. Hence π is an Andre plane by Corollary 12.2 of [5], contrary to Hypothesis 3.1 (iii).

Lemma 3.5. Set $K_0 = \{ \begin{pmatrix} kI & O \\ O & kI \end{pmatrix} | k \in K^{\sharp} \}$. Let P be a point on Δ . Then, (i) If $s = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \in H - K_0$ fixes a point P, then either A or B is contained in $U - U_1$.

(ii) H_P/K_0 is isomorphic to a subgroup of $Z_3 \times Z_3$.

Proof. Assume A, $B \in U_1$ and set A = I(a) and B = I(b). Let $L(M) \in \mathcal{L}$ be a component corresponding to the line OP. Since s fixes OP, $I(a)^{-1}[x, y, z]$ I(b) = [x, y, z], where M = [x, y, z]. Then $[ax, \overline{a}y, \overline{a}z] = [bx, by, bz]$ and so $x(a-b) = y(\overline{a}-b) = z(\overline{a}-b) = 0$. If x=0, then $yz \neq 0$ by Lemma 3.4. From this $\overline{a} = b = \overline{a}$ and hence $a = b \in K$. Similarly, if either y=0 or z=0, $a=b \in K$. On the other hand, if $xyz \neq 0$, $a=\overline{a} = \overline{a} = b$, which also implies $a=b \in K$. Therefore $s \in K_0$ and (i) holds.

Set $W = \{ \begin{pmatrix} A & O \\ O & B \end{pmatrix} \in H_P | A, B \in U_1 \}$. Then $H/W_P \leq Z_3 \times Z_3$ and by (i) $W = K_0$. Thus (ii) holds.

Lemma 3.6.
$$(q^3-1)(q-1)||G|$$
 and $|G|||2 \cdot 3^2(q-1)(q^3-1)$.

Proof. Since $|G| = |G: G_P| \times |G_P: H_P| |H_P: K_0| |K_0| = (q^3 - 1) (q - 1) |G_P|$ $H: H| \times |H_P: K_0|$, the lemma follows from the previous lemma.

We denote by m_t the highest power of a prime t dividing a positive integer m.

Lemma 3.7. $|G|_2 = ((q-1)^2)_2$.

Proof. By Lemma 3.6, $(q-1)^2 | |G|$. Set $2^r = (q-1)_2$ and let S be a Sylow 2-subgroup of G. Then $2^{2r} | |S|$. Assume $2^{2r+1} | |S|$. Clearly S acts on the set of points $V_0 = \{([a], [b]) | a, b \in F^*\}$. Since $V_0 = q^6 - 2q^3 + 1 = (q-1)^2$ $(q^2 + q + 1)^2$ and $2 \not/ q^2 + q + 1$, S is not semiregular on V_0 . Therefore some involution $s \in S$ fixes a point $Q \in V_0$. As G contains no Baer involutions, s is a homology with axis OQ by a Baer's theorem. Applying Lemma 3.3, $s = \begin{pmatrix} O & A \\ A^{-1} & O \end{pmatrix}$ for some $A \in U$. Set L(M) = OQ, where $M \in \Sigma$. Then ([x], [x]M)s =([x], [x]M) for any $x \in F$. Hence $[x] = [x]MA^{-1}$ for any $x \in F$. Therefore

M=A. However, this contradicts Lemma 3.4. Thus the lemma holds.

Lemma 3.8. Either (i) $|G| = (q-1)(q^3-1)$ or $3(q-1)(q^3-1)$ or (ii) $|G| = 3^2(q-1)(q^3-1)$, $q^3-1 |2|K_1|^2$ and a Sylow 3-subgroup of $K_1 \times K_2$ is not contained in \tilde{U}_1 . Moreover G contains an abelian normal Hall subgroup of order $(q^2+q+1)/(3, q-1)$.

Proof. Deny (i). Then, by Lemmas 3.6 and 3.7, $|G| = 3^2(q-1)(q^3-1)$ and so it suffices to show that $q^3-1 |2|K_1|^2$.

Let R be a Hall subgroup of G such that $|R| = (q^2+q+1)/(3, q-1)$. By Lemma 3.2, R is an abelian normal subgroup cf G. Since K_1 and K_2 are Frobenius complements, a Sylow 3-subgroup S of $K_1 \times K_2$ is an abelian subgroup of rank 2. Therefore $N_R(S) = C_R(S)$ by Theorem 5.2.4 of [3]. Since $|H/K_1|$ $|3(q^3-1), 3(q-1)/2| |K_1|$. Hence $|K_1|_3 = 3(q-1)_3$ and $|S| = 3^2((q-1)_3)^2$ as $K_1 \times K_2 \le H$.

Assume $S \not \equiv \widetilde{U}_1$. Then $S \cap K_1$, $S \cap K_2 \not \equiv \widetilde{U}_1$ since $S = (S \cap K_1) (S \cap K_2)$. Hence, there exist $b, c \in F^{\ddagger}, \begin{pmatrix} [0, b, 0] & O \\ O & [0, c, 0] \end{pmatrix} \in S$. Since $[0, a, 0]^{-1} I(x) [0, a, 0] = I(\mathbf{x})$ for any $a, x \in F^{\ddagger}, N_R(S) = C_R(S) = 1$. Therefore either $N_R(S) = 1$ or $S \leq \widetilde{U}_1$.

If $N_R(S)=1$, then $R \leq K_1K_2$ since $RK_1K_2=N_{RK_1K_2}(S)K_1K_2$. Therefore $(q^2+q+1)/(3, q-1) \mid |K_1|^2 = |K_1K_2|$ and $3(q-1)/2 \mid |K_1|$ as $(q^2+q+1, q-1) = (3, q-1)$. Hence $q^3-1=(q-1)(q^2+q+1) \mid 2 \mid K_1 \mid^2$.

If $S \le \tilde{U}_1$, then $S_P = S \cap K_0$ for any $P \in l_{\infty} - \{L(O), L(\infty)\}$ by Lemma 3.5. Therefore $|S|/S \cap K_0| = |S|/(q-1)_3 | q^3 - 1$. However, $|S|/(q-1)_2 = 9(q-1)_3$ and so $9(q-1)_3 | (q-1)(q^2+q+1)$, a contradiction. Thus the lemma holds.

Using the lemmas above, together with Theorem 1 of [1], we now prove the following.

Theorem 2. Under Hypothesis 3.1, either (i) $LC(\pi)$ is a solvable group of order $3^i(q^3-1)(q-1)$ with $0 \le i \le 2$ or (ii) q=3 and $LC(\pi)$ is isomorphic to SL(2, 13).

Proof. Set $L=LC(\pi)$. If L=G $(=L_{(L(\infty),L(0))})$, then (i) follows from Lemmas 3.2, 3.3 and 3.8.

Suppose $L \neq G$. Then *L* is transitive on l_{∞} . Since *L* contains no Baer involutions, from Theorem 39.3 of [5], *L* is not 2-transitive on l_{∞} . In particular $L_P = H$, $P = L(\infty)$, for otherwise $1 + |\Delta|/2 = (q+1)(q^2 - q + 1) ||L_P|$, contrary to $(L_P)^{oP} \leq GL(3, q)$. Hence $\Gamma = P^G$ is a block of *L*.

Let Ω be a complete block system of L which contains Γ . Since $L_{\Gamma}=G$ and G is transitive on Δ , L acts doubly transitively on Ω . Since $|\Omega - {\Gamma}| = (q^3-1)/2$ and $|G/K_0| = 3^i(q^3-1)$ with $0 \le i \le 2$, we have $(L_{\Gamma,\Gamma'})^{\alpha} | 18, \Gamma \neq \Gamma' \in \Omega$. By Theorem 1 of [1], $L^{\alpha} = PSL(2, 13)$ and $|\Omega| = 14$. Therefore q=3,

 $K_0 \simeq Z_2$ and $L/K_0 \simeq PSL(2, 13)$. Thus (ii) holds in this case.

4. A characterization of the class Π

In this section we continue Hypothesis 3.1 and notations used in the previous section. Let Λ denote the set of primes dividing $(q^2+q+1)/(3, q-1)$ and \overline{X} the restriction of $X (\leq G)$ on the line l_{∞} . Furthermore we assume the following.

Hypothesis 4.1. (0) G contains K_0 , the group of kern homologies.

(i) There exists a 2-element $\bar{z} \in \bar{G}$ such that $C_{\bar{G}}(\bar{z})$ is a Λ' -group.

(ii) G contains a nontrivial planar collineation.

Lemma 4.2. $|G| = 3(q^3 - 1)(q - 1)$ and $|K_1| = |K_2| = (q - 1)/2$. Moreover $|G_P/K_0| = 3$ for any $P \in \Delta$ and $K_1 K_2 \leq Z(H)$.

Proof. Set $m=(|K_1|, (q^2+q+1)/(3, q-1))$ and assume t is a prime with $t \mid m$. Then, as $K_1 \simeq K_2$, $K_2 \times K_2$ contains a noncyclic subgroup T of order t^2 and $(K_1 \cap T)^z = K_2 \cap T$. Hence $C_T(z) = 1$, contrary to Hypothesis 4.1. Thus m=1 and $|K_1| \mid 3(q-1)$. In particular $|G| = 3^i(q^3-1)(q-1)$, $i \le 1$. Let $P \in \Delta$. By Hypothesis 4.1 (ii), $G_P \neq K_0$ and therefore $|G| = 3(q^3-1)(q-1)$ and $|G_P/K_0| = 3$ by Theorem 2.

Since $K_1 \times K_2 \le H$ and $|H|_2 = ((q-1)_2)^2/2$, we have $|K_1|_2 \le (q-1)_2/2$. By Lemma 3.2, $|H/K_1| \mid 3(q^3-1)$ and so $(q-1)/2 \mid |K_1|$. On the other hand $|K_1| \mid q^3-1$. Hence, either $|K_1| = (q-1)/2$ or $|K_1| = 3(q-1)/2$ and $q \equiv 1$ (mod 3). Let X be a Hall subgroup of G of order $(q^2+q+1)/(3, q-1)$. Then $[X, K_1K_2] \le X \cap K_1K_2 = 1$ by Lemma 3.8. From this $K_1K_2 \le C_{\widetilde{U}}(X) \le \widetilde{U}_1$. If $|K_1K_2|_3 = |G|_3$, then $K_1 \times K_2$ $(\le \widetilde{U}_1)$ contains a planar collineation of order 3. This is a contradiction. Therefore $|K_1K_2|_3 < |G|_3$ and we have $|K_1| = (q-1)/2$. In particular $K_1 \le Z(H)$.

Lemma 4.3. Let s_1 be a nontrivial planar element of G. Then a basis for π may be choosen so that $\langle s_1 \rangle = \langle \begin{pmatrix} J & O \\ O & J \end{pmatrix} \rangle$ and $H \leq \tilde{U}$.

Proof. By Lemma 4.2, s_1 is an element of order 3. By Lemma 3.2 $H \le \widehat{U}$ and since s_1 is not semiregular on the lines $L(\infty)$ and L(O), we may assume either (i) $s_1 = \begin{pmatrix} [0, a, 0] & O \\ O & [0, b, 0] \end{pmatrix}$ or (ii) $s_1 = \begin{pmatrix} [0, a, 0] & O \\ O & [0, 0, b] \end{pmatrix}$ for some $a, b \in$ F^{\ddagger} . As $(s_1)^3 = 1$, N(o) = N(b) = 1. There exist elements $c, d \in F^{\ddagger}$ such that $c^{q-1} = a$ and $d^{q-1} = b$, respectively. Then $\begin{pmatrix} I(c) & O \\ O & I(d) \end{pmatrix}^{-1} \begin{pmatrix} JI(a) & O \\ O & JI(b) \end{pmatrix} \begin{pmatrix} I(c) & O \\ O & I(d) \end{pmatrix}$ $= \begin{pmatrix} J & O \\ O & J \end{pmatrix}$ and $\begin{pmatrix} I(c) & O \\ O & I(d^{-q}) \end{pmatrix}^{-1} \begin{pmatrix} JI(a) & O \\ O & J^2I(b) \end{pmatrix} \begin{pmatrix} I(c) & O \\ O & I(d^{-q}) \end{pmatrix} = \begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}$. Therefore, to prove the lemma it suffices to show that $s_1 \neq \begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}$.

Assume
$$s_1 = \begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}$$
. Since $\begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}^{-1} \begin{pmatrix} I(x) & O \\ O & I(y) \end{pmatrix} \begin{pmatrix} J & O \\ O & J^2 \end{pmatrix} = \begin{pmatrix} I(\bar{x}) & O \\ O & I(\bar{y}) \end{pmatrix}$ if $\begin{pmatrix} I(x) & O \\ O & I(y) \end{pmatrix} \in H$, then $\begin{pmatrix} I(x^{q-1}) & O \\ O & I \end{pmatrix} \in K_1$, contrary to Lemma 4.2. Thus the lemma holds

lemma holds.

Lemma 4.4. There exists a 2-element t_1 of G-H which centralizes s_1 .

Proof. Set $N=N_G(\langle s_1 \rangle K_0)$. By a Witt's theorem, $|N: \langle s_1 \rangle K_0|=q-1$. In particular $N \leq H$. Hence there is a 2-element $t_1 \in N$ such that $t_1 \notin H$. Then t_1 normalizes $\langle s_1 \times kI \rangle$, where $|\langle kI \rangle| = (q-1)_3$. Since $g^{-1}Jg = [0, x^{1-q}, 0]$ for any $g \in \{[x, 0, 0], [0, x, 0], [0, 0, x] | x \in F^*\}, t_1^{-1} s_1 t_1 = s_1 \pmod{\langle kI \rangle}$. From Theorem 5.3.2 of [3], t_1 centralizes $\langle s_1 \times kI \rangle$. Therefore t_1 centralies s_1 .

Lemma 4.5. A basis for π can be choosen so that $\begin{pmatrix} O & uI \\ I & O \end{pmatrix}$, $\begin{pmatrix} J & O \\ O & I \end{pmatrix} \in G$ and $H \leq \tilde{U}.$

Proof. Let t_1 be as in Lemma 4.4 and set $t_1 = \begin{pmatrix} O & A \\ R & O \end{pmatrix}$. By Lemma 3.3, A, $B \in U$ and as $(t_1)^2$ is a 2-element of G, we have $t_1 = g_1, g_2$ or g_3 , where $g_1 =$ $\begin{pmatrix} O & I(a) \\ I(b) & O \end{pmatrix}$, $g_2 = \begin{pmatrix} O & JI(a) \\ J^2I(b) & O \end{pmatrix}$ and $g_3 = \begin{pmatrix} O & J^2I(a) \\ JI(b) & O \end{pmatrix}$. Here $a, b \in K^*$ as t_1 centralizes s_1 . Since $\begin{pmatrix} O & J \\ I & O \end{pmatrix}^{-1} g_2 \begin{pmatrix} O & J \\ I & O \end{pmatrix} = \begin{pmatrix} O & I(b) \\ I(a) & O \end{pmatrix}, \begin{pmatrix} I & O \\ O & J \end{pmatrix}^{-1} g_3 \begin{pmatrix} I & O \\ O & J \end{pmatrix} = \begin{pmatrix} O \\ I(b) \end{pmatrix}$ $\begin{pmatrix} I(a) \\ O & I(b) \end{pmatrix}^{-1} g_1 \begin{pmatrix} I & O \\ O & I(b) \end{pmatrix} = \begin{pmatrix} O & I(ab) \\ I & O \end{pmatrix}$, by choosing a suitable basis for $\pi \text{ we may assume that } t_1 = \begin{pmatrix} O & I(u_1) \\ I & O \end{pmatrix} \text{ for some 2-element } u_1 \text{ of } K^{\sharp}.$ Suppose $u_1 = v^2$ for some $v \in K$. Then $(v^{-1} \begin{pmatrix} I & O \\ O & I \end{pmatrix} \begin{pmatrix} O & I(u_1) \\ I & O \end{pmatrix})^2 = \begin{pmatrix} I & O \\ O & I \end{pmatrix}.$ Hence G contains an involution which interchanges $L(\infty)$ and L(O) and so it is a homology with axis L(M) for some $M \in \Sigma - \{O\}$, contrary to Lemma 4.2.

Thus u_1 is a nonsquare 2-element of K^* . From Lemma 4.2, $G \ge K_2 = \{ \begin{pmatrix} I & O \\ O & r^2 I \end{pmatrix} \}$ $|x \in K^*$, so that $\begin{pmatrix} O & uI \\ I & O \end{pmatrix} \in G$.

From now on we put $t_1 = \begin{pmatrix} O & uI \\ I & O \end{pmatrix}$ and $s_1 = \begin{pmatrix} J & O \\ O & I \end{pmatrix}$.

Lemma 4.6. (i) Let L(M) $(M \in \Sigma)$ be a line fixed by s_1 . Then M =[a, b, c] for some $a, b, c \in K$.

(ii) Let L(M) $(M \in \Sigma - \{O\})$ be a line fixed by s_1 . Set $\Omega_1 = \{L(k^2M) | k \in I\}$ K^{*}} and $\Omega_2 = \{L(uK^2M^{-1}) | k \in K^*\}$. Then $\Omega_1 \cup \Omega_2 \cup \{L(O), L(\infty)\}$ is the set of lines of \mathcal{L} fixed by s_1 .

Proof. Assume $L(M)s_1=L(M)$ and set M=[a, b, c]. Then $J^{-1}[a, b, c] J=$

[a, b, c], so that $[\overline{a}, \overline{b}, \overline{c}] = [a, b, c]$. Thus $a, b, c \in K$ and (i) holds. Moreover, since $H \ge K_1$, K_2 and $L(M)t_1 = L(uM^{-1})$, (ii) holds.

Lemma 4.7. *H* contains an abelian normal subgroup X of G of order $(q^3-1)(q-1)/2$ such that $K_0K_1K_2 \le X \le \tilde{U}_1$ and $H = X \le s_1 \ge$.

Proof. By Lemma 4.2, H/K_i contains a unique cyclic subgroup X_i/K_i of order q^3-1 such that $H/K_i = (X_i/K_i)$ ($\langle s_i \rangle K_i/K_i$), $i \in \{1, 2\}$. As K_i is contained in the center of H, X_i is an abelian normal subgroup of G of order (q^3-1) (q-1)/2.

Assume $X_1 \neq X_2$. Then $H = X_1 X_2$ and hence $|H|(X_1 \cap X_2)| = 9$ and $X_1 \cap X_2$ is in the center of H. This contradicts the fact that $s_1 \in H$. Therefore $X_1 = X_2$. Set $X = X_1 = X_2$. Then X has the desired properties.

Lemma 4.8. X contains a cyclic normal Hall Λ -subgroup Z of order $(q^2+q+1)/(3, q-1)$.

Proof. Let Z be a subgroup of X of order $(q^2+q+1)/(3, q-1)$. From Lemma 4.2, $Z \cap K_1 = 1$. Since $ZK_1/K_1 \leq GL(3, q)^*$, Z is cyclic.

Lemma 4.9. Let Y be a Sylow 3-subgroup of X. Then YK_0/K_0 is cyclic.

Proof. Suppose false and set $3^m = (q-1)_3$. Then $|Y| = 3^{2m}(3, q-1)$. Since $K_1 \leq X$ and $K_1 \cap K_2 = 1$, we have $q \equiv 1 \pmod{3}$ and $YK_0/K_0 \simeq Z_{3m} \times Z_3$. As $\tilde{U}_1 \simeq Z_{q^3-1} \times Z_{q^3-1}$, $\{g \in \tilde{U}_1 | g^3 \in K_0\} \leq YK_0/K_0$. In particular $f = \begin{pmatrix} I(r) & O \\ O & I(r) \end{pmatrix} \in Y$, where r is an element of F^{\ddagger} of order 3^{m+1} . Let L(M) $(M \in \Sigma - \{O\})$ be a line fixed by s_1 and put M = [a, b, c]. Let Ω_1 and Ω_2 be as in Lemma 4.6. Since L(M) $f = L(I(r)^{-1} MI(r)) = L(([a, br^{1-q}, cr^{1-q^2}])$ and $3 | 1-q, L([a, br^{1-q}, cr^{1-q^2}]) \in \Omega_1$. Hence $r^{1-q} = 1$, a contradiction. Thus the lemma holds.

Lemma 4.10. H/K_0 contains a cyclic normal subgroup X/K_0 of order $(q^3-1)/2$ which is inverted by t_1 .

Proof. From Lemmas 4.7-4.9, together with the fact that K_1K_0/K_0 is cyclic of order (q-1)/2, H/K_0 contains a cyclic normal subgroup X/K_0 of order $(q^3-1)/2$. Clearly t_1 inverts K_1K_0/K_0 . Since $t_1^2 \in K_0$ and $[Z, X] \equiv 1 \pmod{K_0}$, t_1 inverts ZK_0/K_0 . Moreover t_1 inverts a Sylow 3-subgroup of X/K_0 by Lemma 4.9. Therefore t_1 inverts X/K_0 .

Lemma 4.11. There exists an element $g \in X$ such that $g = \begin{pmatrix} I(x^{-1}) & O \\ O & I(x) \end{pmatrix}$ and $F^{\ddagger} = \langle x \rangle$.

Proof. Let $g_1 = \begin{pmatrix} I(y) & O \\ O & I(z) \end{pmatrix}$ be an element of X such that $g_1 K_0$ is a genera-

tor of X/K_0 . Since $|X/K_1K_0| \equiv 1 \pmod{2}$, we may assume y and z are square elements of F^* . Since t_1 inverts $g_1 \pmod{K_0}$, $\binom{I(y^{-1}) \ O}{O \ I(z^{-1})} \equiv g_1^{t_1} \equiv \binom{I(z) \ O}{O \ I(y)}$ (mod K_0). Hence $yz=j^2$ for some $j \in K^*$ and so $g_1=g_2\begin{pmatrix}I & O\\ O \ I(j^2)\end{pmatrix}$, where $g_2=\binom{I(y) \ O}{O \ I(y^{-1})}$. On the other hand $g_3=\binom{I(k) \ O}{O \ I(k^{-1})} = \binom{I(k^2) \ O}{O \ I} \binom{I(k^{-1}) \ O}{O \ I(k^{-1})} \in K_1K_0=K_2K_0$, $|\langle k \rangle| = (q-1)/2$. Therefore $X=\langle g_1K_0 \rangle K_1K_0=\langle g_2, \ g_3 \rangle K_0$. This, together with $t_1^2 \in K_0$, implies the lemma.

Lemma 4.12.
$$H = \langle \begin{pmatrix} I(x^{-1}) & O \\ O & I(x) \end{pmatrix} | x \in F^* \rangle K_0 \langle s_1 \rangle \text{ and } G = H \langle t_1 \rangle.$$

Proof. From Lemma 4.11, the lemma holds.

We now present a characterization of the class Π .

Theorem 3. Let π be a translation plane of order q^3 with kerr K=GF(q), where $q \equiv 1 \pmod{2}$ and assume π is not an Andre plane. Then π is contained in the class Π if and only if the following three conditions are satisfied:

(i) A subgroup G of $LC(\pi)$ has orbits of length 2 and q^3-1 on l_{∞} .

(ii) The centralizer of a 2-element $z^{l_{\infty}} \in G^{l_{\infty}}$ in $G^{l_{\infty}}$ is a Λ' -group, where Λ is the set of primes dividing $(q^2+q+1)/(3, q-1)$.

(iii) G contains a nontrivial planar element.

Proof. Suppose $\pi \in \Pi_K$, K = GF(q). Then it can be easily verified that $LC(\pi)$ contains the group described in Lemma 4.12. Therefore we have "only if" part of the theorem.

Conversely, let π be a plane with the properties (i)-(iii). By Lemmas 4.6 and 4.12, $\Sigma = \{I(x) \mid a, b, c] \mid I(x) \mid x \in F\} \cup \{I(x) \mid a, b, c]^{-1} \mid I(x) \mid x \in F\}$, where $[a, b, c] \in GL(3, q)$. By definition of Π , π is contained in Π . Thus the theorem holds.

References

- [1] M. Aschbacher: 2-transitive groups whose 2-point stabilizer has 2-rank 1, J. Algebra 36 (1975), 98-127.
- [2] D.M. Bloom: The subgroups of PSL(3, q) for odd q, Trans. Amer. Math. Soc. 127 (1967), 150–178.
- [3] D. Gorenstein: Finite groups, Harper and Row, New York, 1968.
- [4] D.R. Hughes and F.C. Piper: Projective planes, Springer-Verlag, Berlin-Heidelberg-New York, 1973.
- [5] H. Lüneburg: Translation planes, Springer-Verlag, Berlin-Heidelberg-New

York, 1980.

- [6] T. Oyama: On quasifields, Osaka J. Math. 22 (1985), 35-54.
- T. Suetake: A new class of translation planes of order q³, Osaka J. Math. 22 (1985), 773-786.
- [8] S.D. Cohen: Likeable functions in finite fields, Israel J. Math. 46 (1983), 123-126.

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