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ON TRANSLATION PLANES OF ORDER q^3 WITH AN ORBIT OF LENGTH q^3-1 ON l_∞

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1. Introduction

In his paper [7], Suetake constructed a class of translation planes of cubic order q^3 and he improved his results for each prime power q such that 2 is a nonsquare element of $GF(q)$. Any plane of the class admits a collineation group G of the linear translation complement such that

(*) G has orbits of length 2 and q^3-1 on l_∞.

In this paper we construct a new class of translation planes of order q^3 with the property (*) for each prime power q with $q\equiv 1 \pmod 2$ (§2). If π is any translation plane of order q^3 with the property (*) and if π is not an Andre plane, then we have either (i) the linear translation complement $LC(\pi)$ is of order $3^i(q^3-1)(q-1)$ with $0\leq i\leq 2$, or (ii) $q=3$ and $LC(\pi)$ is isomorphic to $SL(2, 13)$ (§3). In §4, we present a certain characterization of the class of planes with the property (*). Throughout the paper all sets, planes and groups are finite. Our notation is largely standard and taken from [3] and [5].

2. Description of the class of planes Π

Let $F=GF(q^3)$ be a field order of q^3, where $q=p^n$ and p is an odd prime. Let K be a subfield of F order q. Throughout the paper we consider the translation plane π of order q^3 which is a 6 dimensional vector space over K.

To represent spread sets and collineation groups of π, we use a method of [6]; Let $w\in F-K$ and put a 3×3 matrix $W=\begin{pmatrix}1 & 1 & 1\\ w & \bar{w} & \bar{\bar{w}}\\ w^2 & \bar{w}^2 & \bar{\bar{w}}^2\end{pmatrix}$, where $\bar{w}=w^q$ and $\bar{\bar{w}}=w^{q^2}$. Let $M(3, q_1)$ be the set of all matrices over $GF(q_1)$ for a prime power q_1. Set $M(3, q)^*=W^{-1}M(3, q)W\ (\subset M(3, q^3))$, $GL(3, q)^*=W^{-1}GL(3, q)W$ and $V^*=V(3, q)W=\{(v_1, v_2, v_3)W \mid v_1, v_2, v_3\in K\}$. Then $M(3, q)^*=\left\{\begin{pmatrix}a & \bar{c} & \bar{\bar{b}}\\ b & \bar{a} & \bar{\bar{c}}\\ c & \bar{b} & \bar{\bar{a}}\end{pmatrix} \,\middle|\, a, b, c\in F\right\}$, $GL(3, q)^*=\{P\in M(3, q)^* \mid \det(P)\neq 0\}$ and $V^*=\{(a, \bar{a}, \bar{\bar{a}}) \mid a\in F\}$

(cf. [6]). Here "*dei*" means the determinant of a matrix. Clearly $GL(3, q)^*$ acts naturally on V^* as the general linear group of the vector space over K. The notations $\begin{pmatrix} a & \bar{c} & \bar{\bar{b}} \\ b & \bar{a} & \bar{\bar{c}} \\ c & \bar{b} & \bar{\bar{a}} \end{pmatrix}$ and $(a, \bar{a}, \bar{\bar{a}})$ are abbreviated to $[a, b, c]$ and to $[a]$, respectively. For $k \in K$, we have $k[a, b, c]=[ka, kb, kc]$ and $k[a]=[ka]$. Set $I(x)=[x, 0, 0]$, $I=I(1)$, $O=I(0)$ and $J=[0, 1, 0]$.

Under these conditions a set $\Sigma \subset GL(3, q)^* \cup \{O\}$ is defined to be *a spread set* if $O \in \Sigma$, $|\Sigma|=q^3$ and $\det(M-N) \neq 0$ for any two distinct $M, N \in \Sigma$.

The translation plane π which corresponds to Σ is defined in the usual manner ([5]); the set of points of π is $V^* \times V^*$ and the set of lines passing through the origin is $\mathcal{L}=\{L(M) \mid M \in \Sigma\} \cup \{L(\infty)\}$, where $L(M)=\{([x], [x]M) \mid x \in F\}$ and $L(\infty)=\{([0], [x]) \mid x \in F\}$.

Throughout the rest of this paper let u be a nonsquare element of K. As $q^2+q+1 \equiv 1 \pmod 2$, u is also a nonsquare element of F. Let $\bar{F}$ be the algebraic closure of F and set $\tilde{F}=F-\{\pm 1\}$, $F^{\#}=F-\{0\}$, $K^{\#}=K-\{0\}$.

We now define Π_K a class of translation planes of order q^3. Let Φ_K be the set of all ordered triples $(a, b, c) \in K \times K \times K$ such that

$$\Sigma_{(a,b,c)} = \{I(x)\,[a, b, c]\,I(x) \mid x \in F\} \cup \{I(x)\,u[a, b, c]^{-1}\,I(x) \mid x \in F\}$$

is a spread set. We denote by $\pi_{(a,b,c)}$ the translation plane corresponding to the spread set $\Sigma_{(a,b,c)}$ and set $\Pi_K=\{\pi_{(a,b,c)} \mid (a, b, c) \in \Phi_K\}$. Furthermore set $\Pi=\bigcup_K \Pi_K$, where K runs through all finite fields of odd characteristic.

Clearly the set of matrices $I(x)$ $(x \in F^{\#})$ forms an abelian group of order q^3-1. Hence an ordered triple $(a, b, c) \in K \times K \times K$ is contained in Φ_K if and only if

(i) $f(x)=\det(I(x)\,[a, b, c]\,I(x)-[a, b, c]) \neq 0$ for any $x \in \tilde{F}$,

(ii) $g(x)=\det(I(x)\,[i, j, k]\,I(x)-[i, j, k]) \neq 0$ for any $x \in \tilde{F}$, where $[i, j, k]=u[a, b, c]^{-1}$, and

(iii) $h(x)=\det(I(x)[a, b, c]\,I(x)-[i, j, k]) \neq 0$ for any $x \in F$.

Using these we can show that $\Phi_K=\Phi_K^{(1)} \cup \Phi_K^{(2)}$, where $\Phi_K^{(1)}=\{(a, b, c) \in K \times K \times K \mid a(a^2-bc)=0,\ b^3+c^3-2abc \neq 0\}$ and $\Phi_K^{(2)}=\{(a, b, c) \in K \times K \times K \mid a(a^2-bc) \neq 0,\ b^3+c^3-2abc=0\}$ (Proposition 1). As a corollary, we have

Theorem 1. $\Pi_K=\{\pi_{(a,b,c)} \mid (a, b, c) \in \Phi_K^{(1)} \cup \Phi_K^{(2)}\}$.

In the rest of this section we prove $\Phi_K=\Phi_K^{(1)} \cup \Phi_K^{(2)}$. Let $(a, b, c) \in K \times K \times K$ and set $A=a$, $B=a^2-bc$, $C=b^3+c^3-2abc$ and $D=a^3+b^3+c^3-3abc$. Then $AB+C=D=\det[a, b, c]$.

Lemma 2.1. *Assume* $D \neq 0$. *Then*

(i) $f(x)=ABN(x^2-1)+CN(x^{q+1}-1)$ *and* $g(x)=u^3 D^{-2} f(x)$. *Here* $N(z)=z^{q^2+q+1}$ *for* $z \in F$.

(ii) $h(x)=D^{-1}\det(([a, b, c]I(x))^2-uI)$.

Proof. By direct calculation we have (i) and $g(x)=u^3\det(I(x)[a, b, c]^{-1}I(x))\det([a, b, c]-I(x)^{-1}[a, b, c]I(x)^{-1})\det[a, b, c]^{-1}=u^3(\det[a, b, c])^{-2}f(x)$, hence (i) holds. Similarly we have (ii).

Lemma 2.2. *Let $r(t, x)=\det(xI-[a, b, c]I(t))$ be a characteristic polynomial of $[a, b, c]I(t)$ with $t\in F$. Then*

(i) $r(t, x)=x^3-AT(t)x^2+BT(t^{q+1})x-DN(t)$. *(Here $T(z)=z+z^q+z^{q^2}$ is the trace map.)*

(ii) *Let $t\in F$. Then $h(t)=0$ if and only if $u=-BT(t^{q+1})$ and $uAT(t)=-DN(t)$.*

Proof. By direct calculation we have (i). Suppose $u=-BT(t^{q+1})$ and $uAT(t)=-DN(t)$ for some $t\in F$. Then, by (i), $r(t, x)=x^3-kx^2-ux+uk=(x-k)(x^2-u)$, where $k=AT(t)\in K$. Let v be a root of x^2-u in the algebraic closure $\bar{F}$. Then v is an eigenvalue of $[a, b, c]I(t)$. Hence $h(t)=0$. Conversely, assume $h(t)=0$ for some $t\in F$. Let z_1, z_2, z_3 be the eigenvalues of $[a, b, c]I(t)$. Then, by Lemma 2.1, $z_i^2=u$ for some i. As $r(t, x)$ is a cubic polynomial over K and $z_i\in\bar{F}-F$, $r(t, x)=(x-k)(x^2-u)$ for some $k\in K$. Hence $AT(t)=k$, $BT(t^{q+1})=-u$ and $-DN(t)=ku$. Thus $u=-BT(t^{q+1})$ and $uAT(t)=-DN(t)$.

In Lemmas 2.4–2.7, we assume the following.

Hypothesis 2.3. $(a, b, c)\in\Phi_K$ *and* $ABC\neq 0$.

Lemma 2.4. *Set $i=uA/D$, $j=u/B$ and $w(x)=(x^3-jx)/(x^2-i)$. Then,*

(i) *i is nonsquare in K.*

(ii) *$w(x)$ is a bijection from K onto itself.*

Proof. Since $C\neq 0$, $i(i-j)\neq 0$ and so $(x^3-jx, x^2-i)=((i-j)x, x^2-i)=1$. Deny (ii) and let $e\in K-\{w(t)\mid t\in K, t^2-i\neq 0\}$. Then $x^3-ex^2-jx+ie$ is an irreducible polynomial over K. Let t be a root of this polynomial in $\bar{F}$. Then $t\in F-K$ and so $x^3-ex^2-jx+ie=(x-t)(x-t^q)(x-t^{q^2})$. Hence $T(t)=e$, $T(t^{q+1})=-j$ and $N(t)=-ie$ and so $u=-BT(t^{q+1})$ and $-DN(t)=uAT(t)$. This contradicts (ii) of Lemma 2.2. Thus (ii) holds and (i) follows from (ii).

Lemma 2.5. *Set $k=3i^2+6ij-j^2$. Then, either $9i=j$ or $F(y)=4iy^4-ky^2+4i^2j$ is nonsquare in K for each $y\in K$.*

Proof. We have $w(x)=w(y)$ if and only if $(x-y)((y^2-i)x^2-(i-j)yx-i(y^2-j))=0$. By (i) of Lemma 2.4, $y^2-i\neq 0$. Assume $F(y)$ is square in K for some y and set $v=\sqrt{F(y)}\in K$. Then $w(x)=w(y)=w(x')$, where $\{x, x'\}=\{((i-j)y\pm v)/2(y^2-i)\}$. By Lemma 2.4 (ii), $y=x=x'$. Hence $v=0$ and $y(2y^2-3i+j)=0$. As $ij\neq 0$, $y\neq 0$ and so $0=v^2=4i((3i-j)/2)^2-(3i^2+6ij-j^2)$

$(3i-j)/2+4i^2j=(i-j)^2(9i-j)/2$. Therefore $9i=j$.

Lemma 2.6 $9i\neq j$.

Proof. Assume $9i=j$. Then $9AB=D$. As $D\neq 0$, $\mathrm{char}K\neq 3$. Let $e\in K-R$, where $R=\{27(x^3-3x^2-2)/(3x+1)\mid x\in K,\ x\neq -3^{-1}\}$. Since $3x+1\nmid 27(x^3-3x^2-2)=(3x+1)(9x^2-30x+10)-64$, $S(x)=27(x^3-3x^2-2)-e(3x+1)$ is irreducible over K. Hence $S(x)=27(x-t)(x-t^q)(x-t^{q^2})$ for some $t\in F-K$. Therefore $T(t)=3$, $T(t^{q+1})=-e/9$ and $N(t)=2+e/27$. In particular $T(t)-T(t^{q+1})-3N(t)+3=0$. However, by Lemma 2.1(i), $f(t)=AB(N(t^2-1)+8N(t^{q+1}-1))=AB(T(t)-T(t^{q+1})-3N(t)+3)(T(t)+T(t^{q+1})-3N(t)-3)=0$, a contradiction.

Lemma 2.7. $q\leq 13$.

Proof. By Lemmas 2.5 and 2.6, $F(y)$ is nonsquare in K for any $y\in K$. Applying Lemma of [8], either $q\leq 13$ or $k^2-4\times 4i\times 4i^2j=(i-j)^3(9i-j)=0$. Again, by Lemma 2.6, $9i-j\neq 0$ and therefore $q\leq 13$.

Lemma 2.8. *Let $E(y)=dy^4+ey^2+f$, $d, e, f\in K$ and assume that d is nonsquare in K. If $q\leq 13$ and $E(y)$ is nonsquare for each $y\in K$, then $e^2-4df=0$.*

Proof. Let K_1 be the set of nonzero square elements of K. The $E(y)$'s satisfying the conditions above are as follows, which we obtained by using a computer;

(1) $K=GF(13)$: $(d, e, f)=(2m, 2m, 7m), (2m, 5m, 8m), (2m, 6m, 11m), (2m, 7m, 11m), (2m, 8m, 8m), (2m, 11m, 7m)$, $m\in K_1$.

(2) $K=GF(11)$: $(d, e, f)=(2m, m, 7m), (2m, 3m, 8m), (2m, 4m, 2m), (2m, 5m, 11m), (2m, 9m, 6m)$, $m\in K_1$.

(3) $K=GF(9)=\langle w\rangle GF(3)$, where $w^2=-1$: $(d, e, f)=((w+1)m, m, (w+2)m), ((w+1)m, 2m, (w+2)m), ((w+1)m, wm, (2w+1)m), ((w+1)m, 2wm, (2w+1)m)$, $m\in K_1$.

(4) $K=GF(7)$: $(d, e, f)=(3m, 3m, 6m), (3m, 5m, 5m), (3m, 6m, 3m)$, $m\in K_1$.

(5) $K=GF(5)$: $(d, e, f)=(2m, 2m, 3m), (2m, 3m, 3m)$, $m\in K_1$.

(6) $K=GF(3)$: $(d, e, f)=(2m, m, 2m)$, $m\in K_1$.

Using these, we can verify that $e^2-4df=0$ for each case.

Proposition 1. $\Phi_K=\Phi_K^{(1)}\cup\Phi_K^{(2)}$.

Proof. Assume $D\neq 0$ and $AB=0$. Then $C=D\neq 0$. Hence, by Lemma 2.1, $f(x)\neq 0$ and $g(x)\neq 0$ for any $x\in\widetilde{F}$. By Lemma 2.2, $h(t)\neq 0$ for any $t\in F$. Therefore $\Phi_K^{(1)}\subset\Phi_K$.

Assume $D\neq 0$ and $C=0$. Then $AB=D\neq 0$ and so, by Lemma 2.1, $f(x)\neq 0$ and $g(x)\neq 0$ for any $x\in\widetilde{F}$. If $h(t)=0$ for some $t\in F$, then $t\neq 0$ and $T(t)\,T(t^{q+1})-N(t)=0$ by Lemma 2.2 (ii). Since $T(t)\,T(t^{q+1})-N(t)=N(t+t^q)$, it follows that $t^{q-1}=-1$. However, this implies $2|q^2+q+1$, a contradiction. Therefore $\Phi_K^{(2)}\subset\Phi_K$.

Assume $D\neq 0$ and $ABC\neq 0$. Then, by Lemmas 2.5–2.8, $k^2-4\times 4i\times 4i^2=0$. As we have seen in the proof of Lemma 2.7, this is a contradiction. Therefore $\Phi_K=\Phi_K^{(1)}\cup\Phi_K^{(2)}$.

REMARK 2.9. We can easily verify that the planes constructed in [7] are contained in $\{\pi_{(a,b,c)}\,|\,(a, b, c)\in\Phi_K^{(1)}\}$ $(\subset\Pi_K)$.

3. The planes with the orbits of length 2 and q^3-1

Throughout this section we assume the following.

Hypothesis 3.1. (i) *π is a translation plane of order q^3 with kern $K=GF(q)$, where q is a power of an odd prime p.*

(ii) *A subgroup G of the linear translation complement of π has orbits Γ and Δ of length 2 and q^3-1, respectively, on l_∞.*

(iii) *π is not an Andre plane.*

Let Σ be a spread set corresponding to π and let $C(\pi)$ denote the translation complement of π. *The linear translation complement $LC(\pi)$* of π is defined by the set of all nonsingular 6×6 matrices $g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}$ such that

(i) $A, B, C, D\in M(3, q)^*$.

(ii) If C is nonsingular, then $C^{-1}D\in\Sigma$. (In this case $L(\infty)g=L(C^{-1}D)$.)

(iii) If C is singular, then $C=0$ and D is nonsingular. (In this case $L(\infty)g=L(\infty)$.)

(iv) Given $M\in\Sigma$, if $A+MC$ is nonsingular, then $(A+MC)^{-1}(B+MD)\in\Sigma$. (In this case $L(M)g=L(M_1)$, where $M_1=(A+MC)^{-1}(B+MD)$.)

(v) Given $M\in\Sigma$, if $A+MC$ is singular, then $A+MC=0$. (In this case $L(M)g=L(\infty)$.)

Set $\mathcal{L}=\{L(M)\,|\,M\in\Sigma\}\cup\{L(\infty)\}$. Then, since the restriction $LC(\pi)^{\mathcal{L}}$ is isomorphic to $LC(\pi)^{l_\infty}$, we often identify $\mathcal{L}$ with l_∞.

By Lemma 2.1 of [5], without loss of generality we may assume $\Gamma=\{L(\infty), L(O)\}$ and $G=LC(\pi)_\Gamma$, the global stabilizer of Γ in $LC(\pi)$. Set $H=G_{L(\infty),L(0)}$, the stabilizer of $L(\infty)$ and $L(O)$ in G. Then $|G\colon H|=|\Gamma|=2$ and $\{\begin{pmatrix} A & O\\ O & B\end{pmatrix}|\,A, B\in GL(3, q)^*\}\geq H\geq\{a\begin{pmatrix} I & O\\ O & I\end{pmatrix}|\,a\in K^\sharp\}$. Moreover $|\Delta|=q^3-1\,|\,|G|$ and so $(q-1)(q^2+q+1)/2\,|\,|H|$.

Set $U_1=\{I(a)\,|\,0\neq a\in F\}$ and $U=U_1\langle J\rangle$, $J=[0, 1, 0]$. Furthermore set $\tilde{U}_1=\{\begin{pmatrix} A & O\\ O & B\end{pmatrix}|\,A, B\in U_1\}$ and $\tilde{U}=\{\begin{pmatrix} A & O\\ O & B\end{pmatrix}|\,A, B\in U\}$. Then we have

Lemma 3.2. *Let K_1 or K_2 be the group of homologies in H with axis $L(\infty)$*

or $L(O)$, respectively. Let f_1 or f_2 be a homomorphism from H to $GL(3, q)^*$ defined by $f_1\begin{pmatrix} A & O \\ O & B \end{pmatrix}=B$ or $f_2\begin{pmatrix} A & O \\ O & B \end{pmatrix}=A$, respectively. Then, a basis may be choosen for $\pi(=V^*\times V^*)$ so that $H\leq \tilde{U}$ and $f_i(H)\leq U$ for $i=1, 2$. In particular, $|H/K_i|$ $|3(q^3-1)$.

Proof. Clearly K_1 and K_2 are normal subgroups of H and $K_1\cap K_2=1$. Hence $H/K_1\simeq f_1(H)\subset GL(3, q)^*$ and H/K_1 has a normal subgroup K_1K_2/K_1 $(\simeq K_2)$. By Hypothesis 3.1 (ii), K_1 and K_2 are G-conjugate and $|K_1|=|K_2|$ $|q^3-1$. In particular $p\not| \, |K_2|$.

Assume K_2 is nonsolvable. Then H/K_1 has a normal subgroup isomorphic to $SL(2, 5)$ by Corollary 3.5 of [5] and $p\not| \, |SL(2, 5)|$. In particular K_1 has a characteristic subgroup isomorphic to $SL(2, 5)$ and so $O_p(H/K_i)=1$ by the structure of Aut$(PG(2, q))$, $1\leq i\leq 2$. Let g be a natural homomorphism from $GL(3, q)^*$ into $PGL(3, q)$. Applying the results of [2], $|g(f_1(H))|\,|3$ $|PSL(2, 5)|(q-1)$. Hence $|H| \mid |K_1|\times(q-1)^2\times 180$. On the other hand $|K_1K_2|=|K_1|^2 \mid |H|$ and so $|H| \mid (q-1)^4\,(180)^2$. However, since $(q^2+q+1, 2\times 5)=1$, we have $(q^2+q+1, |H|)\leq 3$, contrary to $(q^3-1)/2 \mid |H|$. Therefore K_2 is solvable.

Assume $(|K_2|, q^2+q+1)>3$. Then $(|g(f_1(K_2))|, q^2+q+1)>3$. By [2], $|g(f_1(H))| \mid 3(q^2+q+1)\,(3, q-1)$ and $f_1(H)$ is $GL(3, q)^*$-conjugate to a subgroup of U.

Next assume $(|K_2|, q^2+q+1)\leq 3$. Then $g(f_1(H))$ is a subgroup of $GL(3, q)^*$ such that $(q^2+q+1)/(3, q-1) \mid |g(f_1(H))|$. By [2], we have either $SL(3, q)^*\leq f_1(H)$ or $f_1(H)$ is $GL(3, q)^*$-conjugate to a subgroup of U. Suppose $SL(3, q)^*\leq f_1(H)$ and let z be an element of order p such that $Wf_1(z)W^{-1}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then z fixes exactly q^2 vectors in $L(\infty)$. Therefore the fixed structure of z is a subplane of π of order q^2, contrary to Bruck's Theorem [4]. Thus, choosing a suitable basis of V^*, we may assume $f_1(H)\subset U$. By considering the mapping f_2, similarly we may assume $f_2(H)\subset U$. Thus the lemma holds

Lemma 3.3. $G-H\subset\{\begin{pmatrix} O & A \\ B & O \end{pmatrix} | A, B\in U\}$.

Proof. Let $z=\begin{pmatrix} O & A \\ B & O \end{pmatrix}\in G-H$ and let $h=\begin{pmatrix} X & O \\ O & Y \end{pmatrix}\in H$. Then $z^{-1}hz=\begin{pmatrix} B^{-1}YB & O \\ O & A^{-1}XA \end{pmatrix}\in H$ for any $h\in H$. Since X, $Y\in U$, $A^{-1}f_1(H)\,A$, $B^{-1}f_2(H)$ $B\subset U$. Hence A and B normalize the cyclic subgroup of U of order t, where t is a prime with $t \mid (q^2+q+1)/(3, q-1)$. As U is a maximal subgroup of $GL(3, q)^*$ by [2], A and B are contained in U.

Lemma 3.4. $\Sigma \cap U = \phi$.

Proof. Suppose false. By Lemmas 3.2 and 3.3, together with the transitivity of G on Δ, Σ is contained in $U \cup \{O\}$. Let $M \in \Sigma - \{O\}$. By considering $M^{-1}\Sigma$, we may assume that Σ contains I. Since $I(x)^{-1}[a, b, c]\, I(x) = [a, bx^{1-q}, cx^{1-q^2}]$ for $x \in F$ and $N(x^{1-q}) = N(x^{1-q^2}) = 1$, we have $(LC(\pi))_{(L(I))} \geq \{\begin{pmatrix} I(x) & O \\ O & I(x) \end{pmatrix} \mid x \in F^{\#}\}$. Hence π is an Andre plane by Corollary 12.2 of [5], contrary to Hypothesis 3.1 (iii).

Lemma 3.5. *Set* $K_0 = \{\begin{pmatrix} kI & O \\ O & kI \end{pmatrix} \mid k \in K^{\#}\}$. *Let P be a point on* Δ. *Then,*

(i) *If* $s = \begin{pmatrix} A & O \\ O & B \end{pmatrix} \in H - K_0$ *fixes a point P, then either A or B is contained in* $U - U_1$.

(ii) H_P/K_0 *is isomorphic to a subgroup of* $Z_3 \times Z_3$.

Proof. Assume A, $B \in U_1$ and set $A = I(a)$ and $B = I(b)$. Let $L(M) \in \mathcal{L}$ be a component corresponding to the line OP. Since s fixes OP, $I(a)^{-1}[x, y, z]\, I(b) = [x, y, z]$, where $M = [x, y, z]$. Then $[ax, \bar{a}y, \bar{\bar{a}}z] = [bx, by, bz]$ and so $x(a-b) = y(\bar{a}-b) = z(\bar{\bar{a}}-b) = 0$. If $x = 0$, then $yz \neq 0$ by Lemma 3.4. From this $\bar{a} = b = \bar{\bar{a}}$ and hence $a = b \in K$. Similarly, if either $y = 0$ or $z = 0$, $a = b \in K$. On the other hand, if $xyz \neq 0$, $a = \bar{a} = \bar{\bar{a}} = b$, which also implies $a = b \in K$. Therefore $s \in K_0$ and (i) holds.

Set $W = \{\begin{pmatrix} A & O \\ O & B \end{pmatrix} \in H_P \mid A, B \in U_1\}$. Then $H/W_P \leq Z_3 \times Z_3$ and by (i) $W = K_0$. Thus (ii) holds.

Lemma 3.6. $(q^3-1)(q-1) \mid |G|$ *and* $|G| \mid 2 \cdot 3^2 (q-1)(q^3-1)$.

Proof. Since $|G| = |G: G_P| \times |G_P: H_P|\, |H_P: K_0|\, |K_0| = (q^3-1)(q-1)\, |G_P\ H: H| \times |H_P: K_0|$, the lemma follows from the previous lemma.

We denote by m_t the highest power of a prime t dividing a positive integer m.

Lemma 3.7. $|G|_2 = ((q-1)^2)_2$.

Proof. By Lemma 3.6, $(q-1)^2 \mid |G|$. Set $2^r = (q-1)_2$ and let S be a Sylow 2-subgroup of G. Then $2^{2r} \mid |S|$. Assume $2^{2r+1} \mid |S|$. Clearly S acts on the set of points $V_0 = \{([a], [b]) \mid a, b \in F^{\#}\}$. Since $V_0 = q^6 - 2q^3 + 1 = (q-1)^2 (q^2+q+1)^2$ and $2 \nmid q^2+q+1$, S is not semiregular on V_0. Therefore some involution $s \in S$ fixes a point $Q \in V_0$. As G contains no Baer involutions, s is a homology with axis OQ by a Baer's theorem. Applying Lemma 3.3, $s = \begin{pmatrix} O & A \\ A^{-1} & O \end{pmatrix}$ for some $A \in U$. Set $L(M) = OQ$, where $M \in \Sigma$. Then $([x], [x]M)s = ([x], [x]M)$ for any $x \in F$. Hence $[x] = [x]MA^{-1}$ for any $x \in F$. Therefore

$M=A$. However, this contradicts Lemma 3.4. Thus the lemma holds.

Lemma 3.8. *Either* (i) $|G|=(q-1)(q^3-1)$ *or* $3(q-1)(q^3-1)$ *or* (ii) $|G|=3^2(q-1)(q^3-1)$, $q^3-1 \mid 2|K_1|^2$ *and a Sylow* 3*-subgroup of* $K_1\times K_2$ *is not contained in* $\tilde{U}_1$. *Moreover* G *contains an abelian normal Hall subgroup of order* $(q^2+q+1)/(3, q-1)$.

Proof. Deny (i). Then, by Lemmas 3.6 and 3.7, $|G|=3^2(q-1)(q^3-1)$ and so it suffices to show that $q^3-1 \mid 2|K_1|^2$.

Let R be a Hall subgroup of G such that $|R|=(q^2+q+1)/(3, q-1)$. By Lemma 3.2, R is an abelian normal subgroup of G. Since K_1 and K_2 are Frobenius complements, a Sylow 3-subgroup S of $K_1\times K_2$ is an abelian subgroup of rank 2. Therefore $N_R(S)=C_R(S)$ by Theorem 5.2.4 of [3]. Since $|H/K_1| \mid 3(q^3-1)$, $3(q-1)/2 \mid |K_1|$. Hence $|K_1|_3=3(q-1)_3$ and $|S|=3^2((q-1)_3)^2$ as $K_1\times K_2\leq H$.

Assume $S\not\leq\tilde{U}_1$. Then $S\cap K_1$, $S\cap K_2\not\leq\tilde{U}_1$ since $S=(S\cap K_1)(S\cap K_2)$. Hence, there exist $b, c\in F^{\#}$, $\begin{pmatrix}[0, b, 0] & O \\ O & [0, c, 0]\end{pmatrix}\in S$. Since $[0, a, 0]^{-1} I(x) [0, a, 0] =I(\bar{x})$ for any $a, x\in F^{\#}$, $N_R(S)=C_R(S)=1$. Therefore either $N_R(S)=1$ or $S\leq \tilde{U}_1$.

If $N_R(S)=1$, then $R\leq K_1K_2$ since $RK_1K_2=N_{RK_1K_2}(S)K_1K_2$. Therefore $(q^2+q+1)/(3, q-1) \mid |K_1|^2=|K_1K_2|$ and $3(q-1)/2 \mid |K_1|$ as $(q^2+q+1, q-1)=(3, q-1)$. Hence $q^3-1=(q-1)(q^2+q+1) \mid 2|K_1|^2$.

If $S\leq\tilde{U}_1$, then $S_P=S\cap K_0$ for any $P\in l_\infty$-$\{L(O), L(\infty)\}$ by Lemma 3.5. Therefore $|S/S\cap K_0|=|S|/(q-1)_3 \mid q^3-1$. However, $|S|/(q-1)_2=9(q-1)_3$ and so $9(q-1)_3 \mid (q-1)(q^2+q+1)$, a contradiction. Thus the lemma holds.

Using the lemmas above, together with Theorem 1 of [1], we now prove the following.

Theorem 2. *Under Hypothesis* 3.1, *either* (i) $LC(\pi)$ *is a solvable group of order* $3^i(q^3-1)(q-1)$ *with* $0\leq i\leq 2$ *or* (ii) $q=3$ *and* $LC(\pi)$ *is isomorphic to* $SL(2, 13)$.

Proof. Set $L=LC(\pi)$. If $L=G$ $(=L_{\{L(\infty),L(0)\}})$, then (i) follows from Lemmas 3.2, 3.3 and 3.8.

Suppose $L\neq G$. Then L is transitive on l_∞. Since L contains no Baer involutions, from Theorem 39.3 of [5], L is not 2-transitive on l_∞. In particular $L_P=H$, $P=L(\infty)$, for otherwise $1+|\Delta|/2=(q+1)(q^2-q+1) \mid |L_P|$, contrary to $(L_P)^{OP}\leq GL(3, q)$. Hence $\Gamma=P^G$ is a block of L.

Let Ω be a complete block system of L which contains Γ. Since $L_\Gamma=G$ and G is transitive on Δ, L acts doubly transitively on Ω. Since $|\Omega-\{\Gamma\}|=(q^3-1)/2$ and $|G/K_0|=3^i(q^3-1)$ with $0\leq i\leq 2$, we have $(L_{\Gamma,\Gamma'})^{\Omega} \mid 18$, $\Gamma\neq\Gamma'\in\Omega$. By Theorem 1 of [1], $L^{\Omega}=PSL(2, 13)$ and $|\Omega|=14$. Therefore $q=3$,

$K_0 \simeq Z_2$ and $L/K_0 \simeq PSL(2, 13)$. Thus (ii) holds in this case.

4. A characterization of the class Π

In this section we continue Hypothesis 3.1 and notations used in the previous section. Let Λ denote the set of primes dividing $(q^2+q+1)/(3, q-1)$ and $\bar{X}$ the restriction of X $(\leq G)$ on the line l_∞. Furthermore we assume the following.

Hypothesis 4.1. (0) *G contains K_0, the group of kern homologies.*
(i) *There exists a 2-element $\bar{z} \in \bar{G}$ such that $C_{\bar{G}}(\bar{z})$ is a Λ'-group.*
(ii) *G contains a nontrivial planar collineation.*

Lemma 4.2. $|G|=3(q^3-1)(q-1)$ *and* $|K_1|=|K_2|=(q-1)/2$. *Moreover* $|G_P/K_0|=3$ *for any* $P \in \Delta$ *and* $K_1 K_2 \leq Z(H)$.

Proof. Set $m=(|K_1|, (q^2+q+1)/(3, q-1))$ and assume t is a prime with $t \mid m$. Then, as $K_1 \simeq K_2$, $K_2 \times K_2$ contains a noncyclic subgroup T of order t^2 and $(K_1 \cap T)^Z = K_2 \cap T$. Hence $C_T(z) \neq 1$, contrary to Hypothesis 4.1. Thus $m=1$ and $|K_1| \mid 3(q-1)$. In particular $|G|=3^i(q^3-1)(q-1)$, $i \leq 1$. Let $P \in \Delta$. By Hypothesis 4.1 (ii), $G_P \neq K_0$ and therefore $|G|=3(q^3-1)(q-1)$ and $|G_P/K_0|=3$ by Theorem 2.

Since $K_1 \times K_2 \leq H$ and $|H|_2=((q-1)_2)^2/2$, we have $|K_1|_2 \leq (q-1)_2/2$. By Lemma 3.2, $|H/K_1| \mid 3(q^3-1)$ and so $(q-1)/2 \mid |K_1|$. On the other hand $|K_1| \mid q^3-1$. Hence, either $|K_1|=(q-1)/2$ or $|K_1|=3(q-1)/2$ and $q \equiv 1$ (mod 3). Let X be a Hall subgroup of G of order $(q^2+q+1)/(3, q-1)$. Then $[X, K_1 K_2] \leq X \cap K_1 K_2 = 1$ by Lemma 3.8. From this $K_1 K_2 \leq C_{\tilde{U}}(X) \leq \tilde{U}_1$. If $|K_1 K_2|_3 = |G|_3$, then $K_1 \times K_2$ $(\leq \tilde{U}_1)$ contains a planar collineation of order 3. This is a contradiction. Therefore $|K_1 K_2|_3 < |G|_3$ and we have $|K_1| = (q-1)/2$. In particular $K_1 \leq Z(H)$. Similarly $K_2 \leq Z(H)$.

Lemma 4.3. *Let s_1 be a nontrivial planar element of G. Then a basis for π may be choosen so that* $\langle s_1 \rangle = \left\langle \begin{pmatrix} J & O \\ O & J \end{pmatrix} \right\rangle$ *and* $H \leq \tilde{U}$.

Proof. By Lemma 4.2, s_1 is an element of order 3. By Lemma 3.2 $H \leq \tilde{U}$ and since s_1 is not semiregular on the lines $L(\infty)$ and $L(O)$, we may assume either (i) $s_1 = \begin{pmatrix} [0, a, 0] & O \\ O & [0, b, 0] \end{pmatrix}$ or (ii) $s_1 = \begin{pmatrix} [0, a, 0] & O \\ O & [0, 0, b] \end{pmatrix}$ for some $a, b \in F^\#$. As $(s_1)^3=1$, $N(a)=N(b)=1$. There exist elements $c, d \in F^\#$ such that $c^{q-1}=a$ and $d^{q-1}=b$, respectively. Then $\begin{pmatrix} I(c) & O \\ O & I(d) \end{pmatrix}^{-1} \begin{pmatrix} JI(a) & O \\ O & JI(b) \end{pmatrix} \begin{pmatrix} I(c) & O \\ O & I(d) \end{pmatrix} = \begin{pmatrix} J & O \\ O & J \end{pmatrix}$ and $\begin{pmatrix} I(c) & O \\ O & I(d^{-q}) \end{pmatrix}^{-1} \begin{pmatrix} JI(a) & O \\ O & J^2 I(b) \end{pmatrix} \begin{pmatrix} I(c) & O \\ O & I(d^{-q}) \end{pmatrix} = \begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}$. Therefore, to prove the lemma it suffices to show that $s_1 \neq \begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}$.

Assume $s_1=\begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}$. Since $\begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}^{-1}\begin{pmatrix} I(x) & O \\ O & I(y) \end{pmatrix}\begin{pmatrix} J & O \\ O & J^2 \end{pmatrix}=\begin{pmatrix} I(\bar{x}) & O \\ O & I(\bar{\bar{y}}) \end{pmatrix}$ if $\begin{pmatrix} I(x) & O \\ O & I(y) \end{pmatrix}\in H$, then $\begin{pmatrix} I(x^{q-1}) & O \\ O & I \end{pmatrix}\in K_1$, contrary to Lemma 4.2. Thus the lemma holds.

Lemma 4.4. *There exists a 2-element t_1 of $G-H$ which centralizes s_1.*

Proof. Set $N=N_G(\langle s_1\rangle K_0)$. By a Witt's theorem, $|N\colon \langle s_1\rangle K_0|=q-1$. In particular $N\not\leq H$. Hence there is a 2-element $t_1\in N$ such that $t_1\notin H$. Then t_1 normalizes $\langle s_1\rangle\times\langle kI\rangle$, where $|\langle kI\rangle|=(q-1)_3$. Since $g^{-1}Jg=[0, x^{1-q}, 0]$ for any $g\in\{[x, 0, 0], [0, x, 0], [0, 0, x]\,|\,x\in F^{\#}\}$, $t_1^{-1}s_1t_1=s_1 \pmod{\langle kI\rangle}$. From Theorem 5.3.2 of [3], t_1 centralizes $\langle s_1\rangle\times\langle kI\rangle$. Therefore t_1 centralies s_1.

Lemma 4.5. *A basis for π can be choosen so that $\begin{pmatrix} O & uI \\ I & O \end{pmatrix}$, $\begin{pmatrix} J & O \\ O & J \end{pmatrix}\in G$ and $H\leq\tilde{U}$.*

Proof. Let t_1 be as in Lemma 4.4 and set $t_1=\begin{pmatrix} O & A \\ B & O \end{pmatrix}$. By Lemma 3.3, $A, B\in U$ and as $(t_1)^2$ is a 2-element of G, we have $t_1=g_1$, g_2 or g_3, where $g_1=\begin{pmatrix} O & I(a) \\ I(b) & O \end{pmatrix}$, $g_2=\begin{pmatrix} O & JI(a) \\ J^2I(b) & O \end{pmatrix}$ and $g_3=\begin{pmatrix} O & J^2I(a) \\ JI(b) & O \end{pmatrix}$. Here $a, b\in K^{\#}$ as t_1 centralizes s_1. Since $\begin{pmatrix} O & J \\ I & O \end{pmatrix}^{-1}g_2\begin{pmatrix} O & J \\ I & O \end{pmatrix}=\begin{pmatrix} O & I(b) \\ I(a) & O \end{pmatrix}$, $\begin{pmatrix} I & O \\ O & J \end{pmatrix}^{-1}g_3\begin{pmatrix} I & O \\ O & J \end{pmatrix}=\begin{pmatrix} O & I(a) \\ I(b) & O \end{pmatrix}$ and $\begin{pmatrix} I & O \\ O & I(b) \end{pmatrix}^{-1}g_1\begin{pmatrix} I & O \\ O & I(b) \end{pmatrix}=\begin{pmatrix} O & I(ab) \\ I & O \end{pmatrix}$, by choosing a suitable basis for π we may assume that $t_1=\begin{pmatrix} O & I(u_1) \\ I & O \end{pmatrix}$ for some 2-element u_1 of $K^{\#}$.

Suppose $u_1=v^2$ for some $v\in K$. Then $\left(v^{-1}\begin{pmatrix} I & O \\ O & I \end{pmatrix}\begin{pmatrix} O & I(u_1) \\ I & O \end{pmatrix}\right)^2=\begin{pmatrix} I & O \\ O & I \end{pmatrix}$. Hence G contains an involution which interchanges $L(\infty)$ and $L(O)$ and so it is a homology with axis $L(M)$ for some $M\in\Sigma-\{O\}$, contrary to Lemma 4.2. Thus u_1 is a nonsquare 2-element of $K^{\#}$. From Lemma 4.2, $G\geq K_2=\left\{\begin{pmatrix} I & O \\ O & x^2I \end{pmatrix}\,\middle|\,x\in K^{\#}\right\}$, so that $\begin{pmatrix} O & uI \\ I & O \end{pmatrix}\in G$.

From now on we put $t_1=\begin{pmatrix} O & uI \\ I & O \end{pmatrix}$ and $s_1=\begin{pmatrix} J & O \\ O & J \end{pmatrix}$.

Lemma 4.6. (i) *Let $L(M)$ $(M\in\Sigma)$ be a line fixed by s_1. Then $M=[a, b, c]$ for some $a, b, c\in K$.*

(ii) *Let $L(M)$ $(M\in\Sigma-\{O\})$ be a line fixed by s_1. Set $\Omega_1=\{L(k^2M)\,|\,k\in K^{\#}\}$ and $\Omega_2=\{L(uK^2M^{-1})\,|\,k\in K^{\#}\}$. Then $\Omega_1\cup\Omega_2\cup\{L(O), L(\infty)\}$ is the set of lines of $\mathcal{L}$ fixed by s_1.*

Proof. Assume $L(M)s_1=L(M)$ and set $M=[a, b, c]$. Then $J^{-1}[a, b, c]J=$

$[a, b, c]$, so that $[\bar{a}, \bar{b}, \bar{c}]=[a, b, c]$. Thus $a, b, c\in K$ and (i) holds. Moreover, since $H\geq K_1, K_2$ and $L(M)t_1=L(uM^{-1})$, (ii) holds.

Lemma 4.7. *H contains an abelian normal subgroup X of G of order* $(q^3-1)(q-1)/2$ *such that* $K_0K_1K_2\leq X\leq \tilde{U}_1$ *and* $H=X\langle s_1\rangle$.

Proof. By Lemma 4.2, H/K_i contains a unique cyclic subgroup X_i/K_i of order q^3-1 such that $H/K_i=(X_i/K_i)(\langle s_1\rangle K_i/K_i)$, $i\in\{1, 2\}$. As K_i is contained in the center of H, X_i is an abelian normal subgroup of G of order $(q^3-1)(q-1)/2$.

Assume $X_1\neq X_2$. Then $H=X_1X_2$ and hence $|H/(X_1\cap X_2)|=9$ and $X_1\cap X_2$ is in the center of H. This contradicts the fact that $s_1\in H$. Therefore $X_1=X_2$. Set $X=X_1=X_2$. Then X has the desired properties.

Lemma 4.8. *X contains a cyclic normal Hall* Λ*-subgroup Z of order* $(q^2+q+1)/(3, q-1)$.

Proof. Let Z be a subgroup of X of order $(q^2+q+1)/(3, q-1)$. From Lemma 4.2, $Z\cap K_1=1$. Since $ZK_1/K_1\leq GL(3, q)^*$, Z is cyclic.

Lemma 4.9. *Let Y be a Sylow 3-subgroup of X. Then* YK_0/K_0 *is cyclic.*

Proof. Suppose false and set $3^m=(q-1)_3$. Then $|Y|=3^{2m}(3, q-1)$. Since $K_1\leq X$ and $K_1\cap K_2=1$, we have $q\equiv 1 \pmod 3$ and $YK_0/K_0\simeq Z_{3m}\times Z_3$. As $\tilde{U}_1\simeq Z_{q^3-1}\times Z_{q^3-1}$, $\{g\in\tilde{U}_1 | g^3\in K_0\}\leq YK_0/K_0$. In particular $f=\begin{pmatrix} I(r) & O \\ O & I(r)\end{pmatrix}\in Y$, where r is an element of $F^\sharp$ of order 3^{m+1}. Let $L(M)$ $(M\in\Sigma-\{O\})$ be a line fixed by s_1 and put $M=[a, b, c]$. Let Ω_1 and Ω_2 be as in Lemma 4.6. Since $L(M)f=L(I(r)^{-1}MI(r))=L(([a, br^{1-q}, cr^{1-q^2}])$ and $3|1-q$, $L([a, br^{1-q}, cr^{1-q^2}])$ is a line fixed by s_1. As $f\in H$, $L(a, br^{1-q}, cr^{1-q^2}])\in\Omega_1$. Hence $r^{1-q}=1$, a contradiction. Thus the lemma holds.

Lemma 4.10. H/K_0 *contains a cyclic normal subgroup* X/K_0 *of order* $(q^3-1)/2$ *which is inverted by* t_1.

Proof. From Lemmas 4.7–4.9, together with the fact that K_1K_0/K_0 is cyclic of order $(q-1)/2$, H/K_0 contains a cyclic normal subgroup X/K_0 of order $(q^3-1)/2$. Clearly t_1 inverts K_1K_0/K_0. Since $t_1^2\in K_0$ and $[Z, X]\equiv 1 \pmod{K_0}$, t_1 inverts ZK_0/K_0. Moreover t_1 inverts a Sylow 3-subgroup of X/K_0 by Lemma 4.9. Therefore t_1 inverts X/K_0.

Lemma 4.11. *There exists an element* $g\in X$ *such that* $g=\begin{pmatrix} I(x^{-1}) & O \\ O & I(x)\end{pmatrix}$ *and* $F^\sharp=\langle x\rangle$.

Proof. Let $g_1=\begin{pmatrix} I(y) & O \\ O & I(z)\end{pmatrix}$ be an element of X such that g_1K_0 is a generator of X/K_0. Since $|X/K_1K_0|\equiv 1 \pmod 2$, we may assume y and z are square elements of $F^{\#}$. Since t_1 inverts $g_1 \pmod{K_0}$, $\begin{pmatrix} I(y^{-1}) & O \\ O & I(z^{-1}) \end{pmatrix} \equiv g_1^{t_1} \equiv \begin{pmatrix} I(z) & O \\ O & I(y) \end{pmatrix}$ $\pmod{K_0}$. Hence $yz=j^2$ for some $j\in K^{\#}$ and so $g_1=g_2\begin{pmatrix} I & O \\ O & I(j^2) \end{pmatrix}$, where $g_2=$ $\begin{pmatrix} I(y) & O \\ O & I(y^{-1}) \end{pmatrix}$. On the other hand $g_3=\begin{pmatrix} I(k) & O \\ O & I(k^{-1}) \end{pmatrix}=\begin{pmatrix} I(k^2) & O \\ O & I \end{pmatrix}\begin{pmatrix} I(k^{-1}) & O \\ O & I(k^{-1}) \end{pmatrix}$ $\in K_1K_0=K_2K_0$, $|\langle k\rangle|=(q-1)/2$. Therefore $X=\langle g_1K_0\rangle K_1K_0=\langle g_2, g_3\rangle K_0$. This, together with $t_1^2\in K_0$, implies the lemma.

Lemma 4.12. *$H=\langle\begin{pmatrix} I(x^{-1}) & O \\ O & I(x) \end{pmatrix} | x\in F^{\#}\rangle K_0\langle s_1\rangle$ and $G=H\langle t_1\rangle$.*

Proof. From Lemma 4.11, the lemma holds.

We now present a characterization of the class Π.

Theorem 3. *Let π be a translation plane of order q^3 with kern $K=GF(q)$, where $q\equiv 1 \pmod 2$ and assume π is not an Andre plane. Then π is contained in the class Π if and only if the following three conditions are satisfied:*

(i) *A subgroup G of $LC(\pi)$ has orbits of length 2 and q^3-1 on l_∞.*

(ii) *The centralizer of a 2-element $z^{l_\infty}\in G^{l_\infty}$ in G^{l_∞} is a Λ'-group, where Λ is the set of primes dividing $(q^2+q+1)/(3, q-1)$.*

(iii) *G contains a nontrivial planar element.*

Proof. Suppose $\pi\in\Pi_K$, $K=GF(q)$. Then it can be easily verified that $LC(\pi)$ contains the group described in Lemma 4.12. Therefore we have "only if" part of the theorem.

Conversely, let π be a plane with the properties (i)–(iii). By Lemmas 4.6 and 4.12, $\Sigma=\{I(x)[a, b, c]I(x) | x\in F\}\cup\{I(x)u[a, b, c]^{-1}I(x) | x\in F\}$, where $[a, b, c]\in GL(3, q)$. By definition of Π, π is contained in Π. Thus the theorem holds.

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