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THE TAIL ESTIMATION OF THE QUADRATIC VARIATION OF A QUASI LEFT CONTINUOUS LOCAL MARTINGALE

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Abstract

We discuss some estimates of the tail distributions of the supremum and the quadratic variation of a local martingale. The assumption made so far in the literature on exponential moments involving a quasi left continuous local martingale is improved.

1. Introduction and main result

There have been a number of works on tail distributions of the supremum and the quadratic variation of a local martingale. On the other hand, in the paper [7] Kotani gave a necessary and sufficient condition for one-dimensional diffusion processes to be martingales. In Azéma, Gundy, and Yor [1], the uniform integrability of a continuous martingale in terms of tails of its supremum and quadratic variation was first characterized. The existence of the limits of the tails was considered by Galtchouk and Novikov [5] (for a discrete time martingale), Novikov [10], Elworthy, Li, and Yor [2], [3], Madan and Yor [9] (for a continuous local martingale), Liptser and Novikov [8], and Kaji [6] (for a càdlàg local martingale) by using the Tauberian theorem. In the statements on the quadratic variation of a local martingale, the existence of some exponential moments involving a local martingale is assumed, but Takaoka [11] relaxed its assumption for a continuous local martingale. In this paper we also do so for a càdlàg local martingale.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbf{R}_+}, P)$ be a filtered probability space with usual conditions, where $\mathbf{R}_+ = [0, \infty)$, and $M = \{M_t\}_{t\in\mathbf{R}_+}$ is a càdlàg local martingale with $M_0 = 0$ defined on it. We denote by μ the random measure on $\mathbf{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbf{R}_+$ and Borel subsets U of $\mathbf{X}$

$$\mu(\,\cdot\,, (0, t] \times U) = \sum_{0<s\leq t} 1_U(\Delta M_s),$$

where $\mathbf{X} = \mathbf{R} - \{0\}$ and $\Delta M_t = M_t - M_{t-}$, $t > 0$. That is, μ is the counting measure of jumps of M. Then we denote by $\hat{\mu}$ its predictable compensator. If M is a locally square integrable martingale, then it is well-known that we can define a predictable

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quadratic variation process $\langle M\rangle = \{\langle M\rangle_t\}_{t\in\mathbf{R}_+}$ and an optional quadratic variation process $[M] = \{[M]_t\}_{t\in\mathbf{R}_+}$ and the canonical decomposition

$$M = M^c + M^d$$

holds, where M^c is a continuous local martingale with $M_0^c = 0$ and M^d is a stochastic integral process with respect to $\mu - \hat{\mu}$ defined as

$$M_t^d = \int_{(0,t]\times\mathbf{X}} x\{\mu(\,\cdot\,, ds\,dx) - \hat{\mu}(\,\cdot\,, ds\,dx)\}, \quad t \in \mathbf{R}_+.$$

Moreover recall that

$$\langle M^d\rangle_t = \int_{(0,t]\times\mathbf{X}} x^2\hat{\mu}(\,\cdot\,, ds\,dx), \quad t \in \mathbf{R}_+.$$

First, we recall the result by Liptser and Novikov [8].

Theorem 1.1. *Assume that M is a locally square integrable martingale, $\langle M\rangle_\infty = \lim_{t\to\infty}\langle M\rangle_t < \infty$ a.s., and $\{M_\tau^+\}_{\tau\in\mathcal{T}}$ is uniformly integrable, where $\mathcal{T}$ is the set of stopping times τ. Then*
(i) $0 \le E[M_\infty] \le E[M_\infty^+] < \infty$.
Besides,
(ii) *if $\{\Delta M_\tau\}_{\tau\in\mathcal{T}}$ is uniformly integrable, then*

$$\lim_{\lambda\to\infty} \lambda P\left(\sup_{t\in\mathbf{R}_+}(M_t^-) > \lambda\right) = E[M_\infty];$$

(iii) *if $|\Delta M| \le K$ and $E[e^{\epsilon M_\infty}] < \infty$ for some $K > 0$, and $\epsilon > 0$, then*

$$\lim_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) = \lim_{\lambda\to\infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = \sqrt{\frac{2}{\pi}}E[M_\infty].$$

Here we notice that the uniform boundedness for jumps is assumed in the above result. But Kaji [6] gave the following improvement.

Theorem 1.2. *Assume the existence of the random variable M_∞ such that $\lim_{t\to\infty} M_t = M_\infty < \infty$ a.s. and that $\{M_\tau^-\}_{\tau\in\mathcal{T}}$ is uniformly integrable. Then*
(i) $-\infty < -E[M_\infty^-] \le E[M_\infty] \le 0$
holds. Besides, if $\{\Delta M_\tau\}_{\tau\in\mathcal{T}}$ is uniformly integrable, then
(ii) $\lim_{\lambda\to\infty} \lambda P(\sup_{t\in\mathbf{R}_+} M_t > \lambda) = -E[M_\infty]$.

Theorem 1.3. *Assume that M is a locally square integrable martingale and that $\langle M\rangle_\infty < \infty$ a.s., $\{M_\tau^-\}_{\tau\in\mathcal{T}}$ is uniformly integrable, and there exists $\lambda_0 > 0$ such that*

$$E\left[\exp\left\{\lambda_0 M_\infty^- + \int_{\mathbf{R}_+\times\{|x|>K\}} |\phi_{\lambda_0}(x)|\hat{\mu}(\,\cdot\,, ds\,dx)\right\}\right] < \infty \tag{1}$$

for some $K > 0$, where $\phi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x - (\lambda^2/2)x^2$. Then
(i) $\lim_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) = -\sqrt{2/\pi}\,E[M_\infty]$,
(ii) $\lim_{\lambda\to\infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = -\sqrt{2/\pi}\,E[M_\infty]$.

As a remark, we note that the condition (1) refines the conditions "$|\Delta M| \le K$ and $E[e^{\lambda_0 M_\infty^-}] < \infty$ for some $\lambda_0, K > 0$".

Finally, we introduce our main result:

Theorem 1.4. *Assume that M is a locally square integrable martingale and quasi left continuous, $\langle M\rangle_\infty < \infty$ a.s., $\{M_\tau^-\}_{\tau\in\mathcal{T}}$ is uniformly integrable.*
(i) *Assume moreover that there exists $\lambda_0 > 0$ such that*

$$E\left[\int_{\mathbf{R}_+\times\{|x|>K\}} |\phi_{\lambda_0}(x)|\hat{\mu}(\,\cdot\,, ds\,dx)\right] < \infty \tag{2}$$

for some $K > 0$. Then

$$\lim_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}}E[M_\infty].$$

(ii) *On the other hand, if we assume that there exists $\lambda_0 > 0$ such that*

$$E\left[\left\{\int_{\mathbf{R}_+\times\{|x|>K\}} |\phi_{\lambda_0}(x)|\hat{\mu}(\,\cdot\,, ds\,dx)\right\}^{2+\alpha}\right] < \infty$$

for some $K > 0$, $\alpha > 0$. Then

$$\lim_{\lambda\to\infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}}E[M_\infty].$$

The proof of the above shall be divided in three steps. As a first step, we will relax the assumption involving the finiteness of some exponential moment of a local martingale in Theorem 1.3, but we assume its quasi left continuity:

Theorem 1.5. *Assume that M is a locally square integrable martingale and quasi left continuous, $\langle M\rangle_\infty < \infty$ a.s., $\{M_\tau^-\}_{\tau\in\mathcal{T}}$ is uniformly integrable, and there exists $\lambda_0 > 0$ such that*

$$E\biggl[\exp\biggl\{-\lambda_0 M_\infty + \int_{\mathbf{R}_+\times\{|x|>K\}} |\phi_{\lambda_0}(x)|\hat{\mu}(\,\cdot\,, ds\,dx)\biggr\}\biggr] < \infty \tag{3}$$

for some $K > 0$. Then

$$\lim_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}}E[M_\infty].$$

As a second step, in Subsection 3.2 we will describe the proof of (i) from Theorem 1.5 by Takaoka's method [10]. Finally, we can obtain (ii) from (i). This proof is the same as in Subsection 6.4 of Kaji [6] and is omitted.

2. Proof of Theorem 1.5

2.1. Two lemmas. First, it is known that

$$\int_{\mathbf{R}_+\times\mathbf{X}} |\phi_{\lambda_0}(x)|\hat{\mu}(\,\cdot\,, ds\,dx) < \infty \quad \text{a.s.} \tag{4}$$

and

$$\int_{\mathbf{R}_+\times\mathbf{X}} |\psi_{\lambda_0}(x)|\hat{\mu}(\,\cdot\,, ds\,dx) < \infty \quad \text{a.s.,} \tag{5}$$

where $\psi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x$. See Subsection 5.1 in Kaji [6].

Lemma 2.1.

$$E\bigl[e^{-\lambda M_\infty - (\lambda^2/2)\langle M^c\rangle_\infty - \int_{\mathbf{R}_+\times\mathbf{X}} \psi_\lambda(x)\hat{\mu}(\cdot, ds\,dx)}\bigr] = 1, \quad 0 < \forall\lambda < \lambda_0.$$

Proof. According to Lemma 5.2 of Kaji [6], the condition $E[e^{\lambda_0 M_\infty^-}] < \infty$ implies the desired conclusion. In fact, we can see

$$E[e^{\lambda_0 M_\infty^-}] \le E[e^{-\lambda_0 M_\infty}] + 1,$$

where the right hand side is $< \infty$ by the assumption (3). □

Lemma 2.2.

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\bigl(E\bigl[e^{-\lambda M_\infty - (\lambda^2/2)\langle M^c\rangle_\infty - \int_{\mathbf{R}_+\times\mathbf{X}} \psi_\lambda(x)\hat{\mu}(\cdot, ds\,dx)}\bigr] - E[e^{-(\lambda^2/2)\langle M\rangle_\infty}]\bigr) = -E[M_\infty].$$

Proof. First, we will show

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\left\{e^{-\lambda M_\infty-(\lambda^2/2)\langle M^c\rangle_\infty-\int_{\mathbf{R}_+\times\mathbf{X}}\psi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-e^{-(\lambda^2/2)\langle M\rangle_\infty}\right\}=-M_\infty\quad\text{a.s.}\tag{6}$$

Observe the equality

$$\begin{aligned}&\frac{1}{\lambda}\left\{e^{-\lambda M_\infty-(\lambda^2/2)\langle M^c\rangle_\infty-\int_{\mathbf{R}_+\times\mathbf{X}}\psi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-e^{-(\lambda^2/2)\langle M\rangle_\infty}\right\}\\&=\frac{1}{\lambda}\left\{e^{-\lambda M_\infty-(\lambda^2/2)\langle M^c\rangle_\infty-\int_{\mathbf{R}_+\times\mathbf{X}}\psi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-e^{-\lambda M_\infty-(\lambda^2/2)\langle M\rangle_\infty}\right\}\\&\quad+\frac{1}{\lambda}\left\{e^{-\lambda M_\infty-(\lambda^2/2)\langle M\rangle_\infty}-e^{-(\lambda^2/2)\langle M\rangle_\infty}\right\}\\&=e^{-\lambda M_\infty-(\lambda^2/2)\langle M\rangle_\infty}\cdot\frac{1}{\lambda}\left\{e^{-\int_{\mathbf{R}_+\times\mathbf{X}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-1\right\}\\&\quad+e^{-(\lambda^2/2)\langle M\rangle_\infty}\cdot\frac{1}{\lambda}\left\{e^{-\lambda M_\infty}-1\right\},\end{aligned}$$

where the last "=" holds by the fact $\langle M\rangle_\infty=\langle M^c\rangle_\infty+\int_{\mathbf{R}_+\times\mathbf{X}}x^2\hat{\mu}(\cdot,ds\,dx)$. Since it is clear that

$$\lim_{\lambda\to 0}\frac{e^{-\lambda M_\infty}-1}{\lambda}=-M_\infty\quad\text{a.s.}$$

holds, the second term of the right-hand side of the observation converges to $-M_\infty$ a.s. Therefore, to get (6), it is sufficient to show that the first term of the right-hand side of the observation converges to 0 a.s. According to the dominated convergence theorem with respect to $\hat{\mu}(\cdot,ds\,dx)$, Lemma 4.1 of Kaji [6], (4), and the fact $\lim_{\lambda\to 0}\phi_\lambda/\lambda=0$ imply

$$\lim_{\lambda\to 0}\int_{\mathbf{R}_+\times\mathbf{X}}\left|\frac{\phi_\lambda(x)}{\lambda}\right|\hat{\mu}(\cdot,ds\,dx)=0\quad\text{a.s.}\tag{7}$$

On the other hand, by using the inequality

$$\left|\frac{e^{\nu x}-1}{\nu}\right|\le|x|e^{\nu|x|},\quad\nu>0,$$

we have

$$\begin{aligned}&\left|\frac{1}{\lambda}\left\{e^{-\int_{\mathbf{R}_+\times\mathbf{X}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-1\right\}\right|\\&\le\left|\int_{\mathbf{R}_+\times\mathbf{X}}\frac{\phi_\lambda(x)}{\lambda}\hat{\mu}(\cdot,ds\,dx)\right|\exp\left\{\left|\int_{\mathbf{R}_+\times\mathbf{X}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)\right|\right\}\\&\le\int_{\mathbf{R}_+\times\mathbf{X}}\left|\frac{\phi_\lambda(x)}{\lambda}\right|\hat{\mu}(\cdot,ds\,dx)\exp\left\{\int_{\mathbf{R}_+\times\mathbf{X}}|\phi_{\lambda_0}(x)|\hat{\mu}(\cdot,ds\,dx)\right\},\end{aligned}\tag{8}$$

where the last line holds, since $\lambda \to |\phi_\lambda(x)|$ is increasing for each $x \in \mathbf{X}$. By (7) and (8) the left-hand side of the last inequality converges to 0 a.s. as $\lambda \to 0$. Hence (6) holds.

Next, we show that for all $0 < \lambda < \lambda_0 \wedge 1/(2c_0K)$

$$
\begin{aligned}
&\left|\frac{1}{\lambda}\left\{e^{-\lambda M_\infty-(\lambda^2/2)\langle M^c\rangle_\infty-\int_{\mathbf{R}_+\times\mathbf{X}}\psi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-e^{-(\lambda^2/2)\langle M\rangle_\infty}\right\}\right| \\
&\le e^{-\lambda_0 M_\infty+\int_{\mathbf{R}_+\times\{|x|>K\}}|\phi_{\lambda_0}(x)|\hat{\mu}(\cdot,ds\,dx)}+\int_{\mathbf{R}_+\times\{|x|>K\}}\left|\frac{\phi_{\lambda_0}(x)}{\lambda_0}\right|\hat{\mu}(\cdot\,,ds\,dx) \\
&\quad+M_\infty^+ + 1 + 2c_0Ke^{-1},
\end{aligned}
\tag{9}
$$

where the positive constant c_0 is such that for all $|x| \le \lambda_0 K$

$$
\left|e^{-x}-1+x-\frac{x^2}{2}\right| \le c_0|x|^3. \tag{10}
$$

Fix $0 < \lambda < \lambda_0 \wedge 1/(2c_0K)$. Observe the inequality

$$
\begin{aligned}
&\left|\frac{1}{\lambda}\left\{e^{-\lambda M_\infty-(\lambda^2/2)\langle M^c\rangle_\infty-\int_{\mathbf{R}_+\times\mathbf{X}}\psi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-e^{-(\lambda^2/2)\langle M\rangle_\infty}\right\}\right| \\
&=\left|\frac{1}{\lambda}\left\{e^{-\lambda M_\infty-(\lambda^2/2)\langle M\rangle_\infty+(\lambda^2/2)\int_{\mathbf{R}_+\times\mathbf{X}}x^2\hat{\mu}(\cdot,ds\,dx)-\int_{\mathbf{R}_+\times\mathbf{X}}\psi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-e^{-(\lambda^2/2)\langle M\rangle_\infty}\right\}\right| \\
&=e^{-(\lambda^2/2)\langle M\rangle_\infty}\cdot\frac{1}{\lambda}\left|e^{-\lambda M_\infty-\int_{\mathbf{R}_+\times\mathbf{X}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-1\right| \\
&=e^{-(\lambda^2/2)\langle M\rangle_\infty}\cdot\frac{1}{\lambda}\left|e^{-\lambda M_\infty-\int_{\mathbf{R}_+\times\{0<|x|\le K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)-\int_{\mathbf{R}_+\times\{|x|>K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}\right. \\
&\qquad\left.-e^{-\int_{\mathbf{R}_+\times\{0<|x|\le K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}+e^{-\int_{\mathbf{R}_+\times\{0<|x|\le K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-1\right| \\
&\le e^{-(\lambda^2/2)\langle M\rangle_\infty-\int_{\mathbf{R}_+\times\{0<|x|\le K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}\times\frac{1}{\lambda}\left|e^{-\lambda M_\infty-\int_{\mathbf{R}_+\times\{|x|>K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-1\right| \\
&\quad+e^{-(\lambda^2/2)\langle M\rangle_\infty}\cdot\frac{1}{\lambda}\left|e^{-\int_{\mathbf{R}_+\times\{0<|x|\le K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)}-1\right| \\
&=I_1\times I_2+I_3.
\end{aligned}
$$

We will estimate I_1. By (10) we obtain

$$
\begin{aligned}
I_1 &\le e^{-(\lambda^2/2)\langle M\rangle_\infty+\int_{\mathbf{R}_+\times\{0<|x|\le K\}}|\phi_\lambda(x)|\hat{\mu}(\cdot,ds\,dx)} \\
&\le e^{-(\lambda^2/2)\langle M\rangle_\infty+c_0K\lambda^3\int_{\mathbf{R}_+\times\{0<|x|\le K\}}x^2\hat{\mu}(\cdot,ds\,dx)} \\
&\le e^{-(\lambda^2/2)\langle M\rangle_\infty+c_0K\lambda^3\langle M\rangle_\infty} \\
&\le 1.
\end{aligned}
$$

We will estimate I_2. By using the inequality

$$\left|\frac{e^{\nu x}-1}{\nu}\right| \le e^{\nu x}1_{\{x\ge 0\}} + x^{-}1_{\{x<0\}}, \quad \nu > 0,$$

we have

$$\begin{aligned}
I_2 &\le e^{-\lambda M_\infty - \int_{\mathbf{R}_+\times\{|x|>K\}} \phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)} 1_{\{M_\infty + \int_{\mathbf{R}_+\times\{|x|>K\}}(\phi_\lambda(x)/\lambda)\hat{\mu}(\cdot,ds\,dx)\le 0\}} \\
&\quad + \left(-M_\infty - \int_{\mathbf{R}_+\times\{|x|>K\}} \frac{\phi_\lambda(x)}{\lambda}\hat{\mu}(\cdot,ds\,dx)\right)^{-} 1_{\{M_\infty + \int_{\mathbf{R}_+\times\{|x|>K\}}(\phi_\lambda(x)/\lambda)\hat{\mu}(\cdot,ds\,dx)>0\}} \\
&\le e^{\lambda_0(-M_\infty - \int_{\mathbf{R}_+\times\{|x|>K\}}(\phi_\lambda(x)/\lambda)\hat{\mu}(\cdot,ds\,dx))} 1_{\{M_\infty + \int_{\mathbf{R}_+\times\{|x|>K\}}(\phi_\lambda(x)/\lambda)\hat{\mu}(\cdot,ds\,dx)\le 0\}} \\
&\quad + \left(M_\infty + \int_{\mathbf{R}_+\times\{|x|>K\}} \frac{\phi_\lambda(x)}{\lambda}\hat{\mu}(\cdot,ds\,dx)\right) 1_{\{M_\infty + \int_{\mathbf{R}_+\times\{|x|>K\}}(\phi_\lambda(x)/\lambda)\hat{\mu}(\cdot,ds\,dx)>0\}} \\
&\le e^{-\lambda_0 M_\infty + \lambda_0\int_{\mathbf{R}_+\times\{|x|>K\}}|\phi_\lambda(x)/\lambda|\hat{\mu}(\cdot,ds\,dx)} + M_\infty^{+} + \int_{\mathbf{R}_+\times\{|x|>K\}} \left|\frac{\phi_\lambda(x)}{\lambda}\right| \hat{\mu}(\cdot,ds\,dx).
\end{aligned}$$

By Lemma 4.1 of Kaji [6], the right-hand side of the last inequality is

$$\le e^{-\lambda_0 M_\infty + \int_{\mathbf{R}_+\times\{|x|>K\}}|\phi_{\lambda_0}(x)|\hat{\mu}(\cdot,ds\,dx)} + M_\infty^{+} + \frac{1}{\lambda_0}\int_{\mathbf{R}_+\times\{|x|>K\}} |\phi_{\lambda_0}(x)|\hat{\mu}(\cdot,ds\,dx).$$

We now estimate I_3. By using the inequality

$$\left|\frac{e^{\nu x}-1}{\nu}\right| \le e^{\nu x}1_{\{x\ge 0\}} + x^{-}1_{\{x<0\}}, \quad \nu > 0,$$

we have

$$\begin{aligned}
I_3 &\le e^{-(\lambda^2/2)\langle M\rangle_\infty}\Bigg\{ e^{-\int_{\mathbf{R}_+\times\{0<|x|\le K\}}\phi_\lambda(x)\hat{\mu}(\cdot,ds\,dx)} 1_{\{\int_{\mathbf{R}_+\times\{0<|x|\le K\}}(\phi_\lambda(x)/\lambda)\hat{\mu}(\cdot,ds\,dx)\le 0\}} \\
&\qquad + \left(-\int_{\mathbf{R}_+\times\{0<|x|\le K\}} \frac{\phi_\lambda(x)}{\lambda}\hat{\mu}(\cdot,ds\,dx)\right)^{-} 1_{\{\int_{\mathbf{R}_+\times\{0<|x|\le K\}}(\phi_\lambda(x)/\lambda)\hat{\mu}(\cdot,ds\,dx)>0\}}\Bigg\} \\
&\le e^{-(\lambda^2/2)\langle M\rangle_\infty}\left\{ e^{\int_{\mathbf{R}_+\times\{0<|x|\le K\}}|\phi_\lambda(x)|\hat{\mu}(\cdot,ds\,dx)} + \int_{\mathbf{R}_+\times\{0<|x|\le K\}} \left|\frac{\phi_\lambda(x)}{\lambda}\right| \hat{\mu}(\cdot,ds\,dx)\right\}.
\end{aligned}$$

Moreover, by (10) the right-hand side of the last inequality is

$$\begin{aligned}
&\le e^{-(\lambda^2/2)\langle M\rangle_\infty}\left\{ e^{c_0K\lambda^3\int_{\mathbf{R}_+\times\{0<|x|\le K\}}x^2\hat{\mu}(\cdot,ds\,dx)} + c_0K\lambda^2\int_{\mathbf{R}_+\times\{0<|x|\le K\}} x^2\hat{\mu}(\cdot,ds\,dx)\right\} \\
&\le e^{-(\lambda^2/2)\langle M\rangle_\infty}\left\{ e^{c_0K\lambda^3\langle M\rangle_\infty} + c_0K\lambda^2\langle M\rangle_\infty\right\} \\
&\le e^{(\lambda^2/2)(-1+2c_0K\lambda)\langle M\rangle_\infty} + 2c_0K\cdot\frac{\lambda^2}{2}\langle M\rangle_\infty e^{-(\lambda^2/2)\langle M\rangle_\infty} \\
&\le 1 + 2c_0Ke^{-1},
\end{aligned}$$

where we can see $(\lambda^2/2)\langle M\rangle_\infty e^{-(\lambda^2/2)\langle M\rangle_\infty} \le e^{-1}$ by using the inequality $xe^{-x} \le e^{-1}$. Hence, the above three estimations of I_1, I_2, and I_3 imply (9).

Finally, according to the dominated convergence theorem, (6), (9), $E[M_\infty^+] < \infty$, and the assumption (3) imply the desired conclusion. □

2.2. A Tauberian theorem.

Theorem 2.1 ([4]). *Let X be an $\mathbf{R}_+$-valued random variable such that $\lim_{\lambda\to 0}(1/\lambda)(1 - E[e^{-(\lambda^2/2)X}])$ exists in $\mathbf{R}$, then*

$$\sqrt{\frac{2}{\pi}} \lim_{\lambda\to 0} \frac{1}{\lambda}(1 - E[e^{-(\lambda^2/2)X}]) = \lim_{\lambda\to\infty} \lambda P(\sqrt{X} > \lambda).$$

2.3. Proof of Theorem 1.5. According to Lemmas 2.1 and 2.2, we have

$$\lim_{\lambda\to 0} \frac{1}{\lambda}(1 - E[e^{-(\lambda^2/2)\langle M\rangle_\infty}]) = -E[M_\infty]$$

holds. Then, by using the Tauberian theorem the last result implies

$$\lim_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty].$$

3. Proof of Theorem 1.4

3.1. The lemma.

Lemma 3.1. *Let ρ be a stopping time. Then it follows that for any $0 < a < 1$*

$$\begin{aligned}\limsup_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) \le \frac{1}{a} &\limsup_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\rho} > \lambda) \\ &+ \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty],\end{aligned}$$

where C is a positive constant which does not depend on M, a, and ρ.

Proof. Fix $0 < a < 1$. We have

$$P(\langle M\rangle_\infty > \lambda^2) \le P(\langle M\rangle_\rho \le a^2\lambda^2, \langle M\rangle_\infty > \lambda^2) + P(\langle M\rangle_\rho > a^2\lambda^2),$$

and so

$$\begin{aligned}\limsup_{\lambda\to\infty} \lambda P(\langle M\rangle_\infty > \lambda^2) \le \frac{1}{a} &\limsup_{\lambda\to\infty} \lambda P(\langle M\rangle_\rho > \lambda^2) \\ &+ \sup_{\lambda} \lambda P(\langle M\rangle_\rho \le a^2\lambda^2, \langle M\rangle_\infty > \lambda^2).\end{aligned} \tag{11}$$

On the other hand, define the process $N = \{N_t\}_{t\in\mathbf{R}_+}$ and the filtration $\{\mathcal{G}_t\}_{t\in\mathbf{R}_+}$ as

$$N_t = M_{\rho+t} - M_\rho, \quad \mathcal{G}_t = \mathcal{F}_{\rho+t}, \quad \forall t \in \mathbf{R}_+.$$

Then N is a local martingale with respect to $\{\mathcal{G}_t\}_{t\in\mathbf{R}_+}$ and

$$\langle N\rangle_\infty = \langle M\rangle_\infty - \langle M\rangle_\rho$$

holds. Also, observe

$$\begin{aligned}\sup_\lambda \lambda P(\langle M\rangle_\rho \le a^2\lambda^2, \langle M\rangle_\infty > \lambda^2) &\le \sup_\lambda \lambda P(\langle N\rangle_\infty > \lambda^2 - a^2\lambda^2)\\ &= \frac{1}{\sqrt{1-a^2}} \sup_\lambda \lambda P(\langle N\rangle_\infty > \lambda^2).\end{aligned}$$

Then, by using the appendix the right-hand side of the last inequality is

$$\le \frac{C}{\sqrt{1-a^2}} \sup_\lambda \lambda P\left(\sup_{t\in\mathbf{R}_+} |N_t| > \lambda\right), \tag{12}$$

where C is a positive constant which does not depend on M, a, and ρ. If we let $\lambda > 0$ and

$$\tau_\lambda = \begin{cases} \inf\{t \in \mathbf{R}_+ \mid |N_t| > \lambda\} & \text{if} \quad \{\} \ne \emptyset \\ \infty & \text{if} \quad \{\} = \emptyset, \end{cases}$$

then $|N_{\tau_\lambda}| \ge \lambda$ on $\{\tau_\lambda < \infty\} = \{\sup_{t\in\mathbf{R}_+} |N_t| > \lambda\}$, and so

$$\lambda P\left(\sup_{t\in\mathbf{R}_+} |N_t| > \lambda\right) \le E[|N_{\tau_\lambda}|].$$

Therefore by the last result we have

$$\begin{aligned}(12) &\le \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}(N)} E[|N_\tau|]\\ &\le \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}(N)} 2E[N_\tau^-]\\ &= \frac{2C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}(N)} \{E[(M_{\rho+\tau} - M_\rho)^-; \rho = \infty] + E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty]\}\\ &\le \frac{2C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty],\end{aligned}$$

where $\mathcal{T}(N) = \{\tau : \text{stopping time} \mid \{N_{\tau\wedge t}\}_{t\in\mathbf{R}_+} \text{ is uniformly integrable}\}$. That is,

$$\sup_\lambda \lambda P(\langle M\rangle_\rho \le a^2\lambda^2, \langle M\rangle_\infty > \lambda^2) \le \frac{2C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty].$$

Hence, by the last inequality and (11) we get the desired conclusion. □

3.2. Proof of (i). For any $u > 0$, introduce the stopping time

$$\tau_u = \begin{cases} \inf\{t \in \mathbf{R}_+ \mid -\lambda_0 M_t + A_t > u\} & \text{if} \quad \{\} \neq \emptyset \\ \infty & \text{if} \quad \{\} = \emptyset, \end{cases}$$

where

$$A_t = \int_{(0,t]\times\{|x|>K\}} |\phi_{\lambda_0}(x)|\hat{\mu}(\,\cdot\,, ds\,dx), \quad t \in \mathbf{R}_+.$$

Fix $u > 0$. We consider the process $M^{(u)} = \{M_t^{(u)}\}_{t\in\mathbf{R}_+}$ defined as $M_t^{(u)} = M_{\tau_u \wedge t}$, $t \in \mathbf{R}_+$. Then it follows from the assumptions with respect to M that $M^{(u)}$ is also a quasi left continuous and locally square integrable martingale which satisfing $M_0^{(u)} = 0$, $\langle M^{(u)}\rangle_\infty$ $(= \langle M\rangle_{\tau_u}) \leq \langle M\rangle_\infty < \infty$ a.s., and the uniform integrability of $\{(M_\tau^{(u)})^-\}_{\tau\in\mathcal{T}}$. Moreover, if we pick the random measure $\mu^{(u)}$ on $\Omega \times \mathbf{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbf{R}_+$ and Borel subsets U of $\mathbf{X}$

$$\mu^{(u)}(\,\cdot\,, (0, t] \times U) = \sum_{0<s\leq t} 1_U(\Delta M_s^{(u)})$$

and its compensator $\hat{\mu}^{(u)}$, then it follows that for all $t \in \mathbf{R}_+$ and Borel subsets U of $\mathbf{X}$

$$\mu^{(u)}(\,\cdot\,, (0, t] \times U) = \sum_{0<s\leq\tau_u\wedge t} 1_U(\Delta M_s) = \mu(\,\cdot\,, (0, \tau_u \wedge t] \times U) \quad \text{a.s.,}$$

and so $\hat{\mu}^{(u)}$ is the random measure on $\Omega \times \mathbf{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbf{R}_+$ and Borel subsets U of $\mathbf{X}$

$$\hat{\mu}^{(u)}(\,\cdot\,, (0, t] \times U) = \hat{\mu}(\,\cdot\,, (0, \tau_u \wedge t] \times U) \quad \text{a.s.,}$$

and therefore we can have that

$$\begin{aligned} &E\big[e^{-\lambda_0 M_\infty^{(u)} + \int_{\mathbf{R}_+\times\{|x|>K\}} |\phi_{\lambda_0}(x)|\hat{\mu}^{(u)}(\cdot,ds\,dx)}\big] \\ &= E[e^{-\lambda_0 M_{\tau_u} + A_{\tau_u}}] \\ &= E[e^{-\lambda_0 M_{\tau_u} + A_{\tau_u}}; \tau_u < \infty] + E[e^{-\lambda_0 M_{\tau_u} + A_{\tau_u}}; \tau_u = \infty] \\ &\leq E[e^{u-\lambda_0 \Delta M_{\tau_u}}; \tau_u < \infty] + e^u P(\tau_u = \infty) \\ &= E[e^{u-\lambda_0\times 0}; \tau_u < \infty] + e^u P(\tau_u = \infty) \quad (= e^u), \end{aligned}$$

where the fourth line of the above holds by the definition of τ_u and the last line does by the quasi left continuity of M. By applying Theorem 1.5 to the process $M^{(u)}$, we have

$$-\infty < E[M_\infty^{(u)}] \leq 0, \quad \lim_{\lambda\to\infty} \lambda P(\sqrt{\langle M^{(u)}\rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty^{(u)}],$$

that is, $-\infty < E[M_{\tau_u}] \le 0$ and

$$\lim_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_{\tau_u}} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_{\tau_u}]. \tag{13}$$

Now we show

$$\liminf_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) \ge -\sqrt{\frac{2}{\pi}} E[M_\infty]. \tag{14}$$

Indeed, the left-hand side of (13) is

$$\le \liminf_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda)$$

and the right-hand side of (13) is

$$\begin{aligned} &= -\sqrt{\frac{2}{\pi}} E[M_\infty; \tau_u = \infty] - \sqrt{\frac{2}{\pi}} E[M_{\tau_u}; \tau_u < \infty] \\ &\ge -\sqrt{\frac{2}{\pi}} E[M_\infty; \tau_u = \infty] + \sqrt{\frac{2}{\pi}} E\left[\frac{u}{\lambda_0} - \frac{1}{\lambda_0} A_{\tau_u}; \tau_u < \infty\right] \\ &\ge -\sqrt{\frac{2}{\pi}} E[M_\infty; \tau_u = \infty] - \frac{1}{\lambda_0}\sqrt{\frac{2}{\pi}} E[A_{\tau_u}; \tau_u < \infty], \end{aligned}$$

where the second line of the above holds by the definition of τ_u. Also, the right-hand side of the above converges to $-\sqrt{2/\pi}\, E[M_\infty]$ as $u \to \infty$, because by the dominated convergence theorem, the fact $E[|M_\infty|] < \infty$ we have known and the assumption (2) imply

$$\lim_{u\to\infty} E[M_\infty; \tau_u = \infty] = E[M_\infty], \qquad \lim_{u\to\infty} E[A_{\tau_u}; \tau_u < \infty] = 0.$$

Therefore we can get (14).

On the other hand, we will show

$$\limsup_{\lambda\to\infty} \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) \le -\sqrt{\frac{2}{\pi}} E[M_\infty]. \tag{15}$$

According to Lemma 3.1, we have for all $0 < a < 1$

$$\begin{aligned} \limsup_{\lambda\to\infty} \lambda P(\langle M\rangle_\infty > \lambda^2) \le \frac{1}{a} &\liminf_{\lambda\to\infty} \lambda P(\langle M\rangle_{\tau_u} > \lambda^2) \\ &+ \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E[(M_{\tau_u+\tau} - M_{\tau_u})^-; \tau_u < \infty], \end{aligned}$$

where C is a positive constant which does not depend on a and u. Fix $0 < a < 1$. By (13) the first term on the right-hand side of the last inequality is

$$= \frac{1}{a}\left(-\sqrt{\frac{2}{\pi}}E[M_\infty^{(u)}]\right).$$

Therefore

$$\limsup_{\lambda\to\infty} \lambda P(\langle M\rangle_\infty > \lambda^2) \le \frac{1}{a}\left(-\sqrt{\frac{2}{\pi}}E[M_\infty^{(u)}]\right) + \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E[(M_{\tau_u+\tau} - M_{\tau_u})^-; \tau_u < \infty].$$

By the definition of τ_u the second term on the right-hand side of the last inequality is

$$\begin{aligned}
&\le \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E\left[\left(M_{\tau_u+\tau} + \frac{u}{\lambda_0} - \frac{1}{\lambda_0}A_{\tau_u}\right)^-; \tau_u < \infty\right] \\
&\le \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E\left[M_{\tau_u+\tau}^- + \frac{1}{\lambda_0}A_{\tau_u}; \tau_u < \infty\right] \\
&\le \frac{C}{\sqrt{1-a^2}} \sup_{\tau\in\mathcal{T}} E[M_\tau^-; \tau_u < \infty] + \frac{C}{\sqrt{1-a^2}}\frac{1}{\lambda_0}E[A_\infty; \tau_u < \infty].
\end{aligned}$$

By the uniform integrability of $\{M_\tau^-\}_{\tau\in\mathcal{T}}$ the first term on the right-hand side of the last inequality converges to 0 a.s. as $u \to \infty$ and from the dominated convergence theorem the assumption (2) implies that the second term of it does so, too. Therefore

$$\limsup_{\lambda\to\infty} \lambda P(\langle M\rangle_\infty > \lambda^2) \le \limsup_{u\to\infty} \frac{1}{a}\left(-\sqrt{\frac{2}{\pi}}E[M_\infty^{(u)}]\right).$$

Moreover, the right-hand side of the last inequality is $\le (1/a)(-\sqrt{2/\pi}E[M_\infty])$ since $\liminf_{u\to\infty} E[M_{\tau_u}^+] \ge E[M_\infty^+]$ holds by the Fatou lemma and since $\lim_{u\to\infty} E[M_{\tau_u}^-] = E[M_\infty^-]$ holds by the uniform integrability of $\{M_\tau^-\}_{\tau\in\mathcal{T}}$. Therefore we can get (15).

Hence (14) and (15) imply the desired conclusion.

4. Appendix

Proposition 4.1. *Assume that M is a quasi left continuous and locally square integrable martingale. Then*

$$\sup_\lambda \lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) \le C \sup_\lambda \lambda P\left(\sup_{t<\infty} |M_t| > \lambda\right),$$

where C is a universal positive constant.

Proof. Pick any stopping times ρ and τ with $\rho \le \tau$. First, it is clear that we can get

$$E[(\sqrt{\langle M\rangle_{\tau-}} - \sqrt{\langle M\rangle_{\rho-}})^2] \le E[\langle M\rangle_\tau - \langle M\rangle_\rho]. \tag{16}$$

In fact, $\langle M\rangle_t$ is continuous, since M is quasi left continuous, and the inequality $(\sqrt{a} - \sqrt{b})^2 \le a - b$ for $0 \le b \le a$ holds. Introduce the local martingale $N_t = M_{(\rho+t)\wedge\tau} - M_\rho$, $t < \infty$, and then we can see $\langle M\rangle_\tau - \langle M\rangle_\rho = \langle N\rangle_\infty$. Therefore, (16) and the last result imply

$$\begin{aligned} E[(\sqrt{\langle M\rangle_{\tau-}} - \sqrt{\langle M\rangle_{\rho-}})^2] &\le E[\langle N\rangle_\infty] \\ &\le E\left[\left(\sup_{t<\infty} |N_t|\right)^2\right], \end{aligned} \tag{17}$$

where the last line of the last inequality holds by the property of a local martingale. By the definision of N we have

$$\begin{aligned} E\left[\left(\sup_{t<\infty}|N_t|\right)^2\right] &= E\left[\left(\sup_{t<\infty}|N_t|\right)^2; \rho < \tau\right] \\ &\le 2E\left[\left(\sup_{t<\infty}|M_{t\wedge\tau}|\right)^2 + M_\rho{}^2; \rho < \tau\right] \\ &= 2E\left[\left(\sup_{t<\infty}|M_t|\right)^2 + M_\rho{}^2; \rho < \tau = \infty\right] \\ &\quad + 2E\left[\left(\sup_{t<\infty}|M_{t\wedge\tau}|\right)^2 + M_\rho{}^2; \rho < \tau < \infty\right] \\ &\le 4E\left[\left(\sup_{t<\infty}|M_t|\right)^2; \rho < \tau = \infty\right] \\ &\quad + 2E\left[\left(\sup_{t\le\tau}|M_t|\right)^2 + \left(\sup_{t<\tau}|M_t|\right)^2; \rho < \tau < \infty\right] \\ &= 4E\left[\left(\sup_{t<\infty}|M_t|\right)^2; \rho < \tau = \infty\right] \\ &\quad + 4E\left[\left(\sup_{t<\tau}|M_t|\right)^2; \rho < \tau < \infty\right] \\ &= 4E\left[\left(\sup_{t<\tau}|M_t|\right)^2; \rho < \tau\right], \end{aligned} \tag{18}$$

where the eighth line of the last inequality holds by the quasi left continuity of $t \to$

$\sup_{s \le t} |M_s|$. Hence, (17) and (18) imply

$$E[(\sqrt{\langle M\rangle_{\tau_-}} - \sqrt{\langle M\rangle_{\rho_-}})^2] = 4E\left[\left(\sup_{t<\tau} |M_t|\right)^2; \rho < \tau\right].$$

Then, according to Corollary 6 of Azéma, Gundy, and Yor [1], the above implies the desired conclusion. □

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