

Title	The tail estimation of the quadratic variation of a quasi left continuous local martingale
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Citation	Osaka Journal of Mathematics. 44(4) P.893-P.907
Issue Date	2007-12
Text Version	publisher
URL	https://doi.org/10.18910/4214
DOI	10.18910/4214
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THE TAIL ESTIMATION OF THE QUADRATIC VARIATION OF A QUASI LEFT CONTINUOUS LOCAL MARTINGALE

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(Received February 24, 2006, revised November 29, 2006)

Abstract

We discuss some estimates of the tail distributions of the supremum and the quadratic variation of a local martingale. The assumption made so far in the literature on exponential moments involving a quasi left continuous local martingale is improved.

1. Introduction and main result

There have been a number of works on tail distributions of the supremum and the quadratic variation of a local martingale. On the other hand, in the paper [7] Kotani gave a necessary and sufficient condition for one-dimensional diffusion processes to be martingales. In Azéma, Gundy, and Yor [1], the uniform integrability of a continuous martingale in terms of tails of its supremum and quadratic variation was first characterized. The existence of the limits of the tails was considered by Galtchouk and Novikov [5] (for a discrete time martingale), Novikov [10], Elworthy, Li, and Yor [2], [3], Madan and Yor [9] (for a continuous local martingale), Liptser and Novikov [8], and Kaji [6] (for a càdlàg local martingale) by using the Tauberian theorem. In the statements on the quadratic variation of a local martingale, the existence of some exponential moments involving a local martingale is assumed, but Takaoka [11] relaxed its assumption for a continuous local martingale. In this paper we also do so for a càdlàg local martingale.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbf{R}_+}, P)$ be a filtered probability space with usual conditions, where $\mathbf{R}_+ = [0, \infty)$, and $M = \{M_t\}_{t \in \mathbf{R}_+}$ is a càdlàg local martingale with $M_0 = 0$ defined on it. We denote by μ the random measure on $\mathbf{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbf{R}_+$ and Borel subsets U of \mathbf{X}

$$\mu(\cdot, (0, t] \times U) = \sum_{0 < s \leq t} 1_U(\Delta M_s),$$

where $\mathbf{X} = \mathbf{R} - \{0\}$ and $\Delta M_t = M_t - M_{t-}$, $t > 0$. That is, μ is the counting measure of jumps of M . Then we denote by $\hat{\mu}$ its predictable compensator. If M is a locally square integrable martingale, then it is well-known that we can define a predictable

quadratic variation process $\langle M \rangle = \{\langle M \rangle_t\}_{t \in \mathbf{R}_+}$ and an optional quadratic variation process $[M] = \{[M]_t\}_{t \in \mathbf{R}_+}$ and the canonical decomposition

$$M = M^c + M^d$$

holds, where M^c is a continuous local martingale with $M_0^c = 0$ and M^d is a stochastic integral process with respect to $\mu - \hat{\mu}$ defined as

$$M_t^d = \int_{(0,t] \times \mathbf{X}} x \{\mu(\cdot, ds dx) - \hat{\mu}(\cdot, ds dx)\}, \quad t \in \mathbf{R}_+.$$

Moreover recall that

$$\langle M^d \rangle_t = \int_{(0,t] \times \mathbf{X}} x^2 \hat{\mu}(\cdot, ds dx), \quad t \in \mathbf{R}_+.$$

First, we recall the result by Liptser and Novikov [8].

Theorem 1.1. *Assume that M is a locally square integrable martingale, $\langle M \rangle_\infty = \lim_{t \rightarrow \infty} \langle M \rangle_t < \infty$ a.s., and $\{M_\tau^+\}_{\tau \in \mathcal{T}}$ is uniformly integrable, where \mathcal{T} is the set of stopping times τ . Then*

(i) $0 \leq E[M_\infty] \leq E[M_\infty^+] < \infty$.

Besides,

(ii) if $\{\Delta M_\tau\}_{\tau \in \mathcal{T}}$ is uniformly integrable, then

$$\lim_{\lambda \rightarrow \infty} \lambda P\left(\sup_{t \in \mathbf{R}_+} (M_t^-) > \lambda\right) = E[M_\infty];$$

(iii) if $|\Delta M| \leq K$ and $E[e^{\epsilon M_\infty}] < \infty$ for some $K > 0$, and $\epsilon > 0$, then

$$\lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = \lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = \sqrt{\frac{2}{\pi}} E[M_\infty].$$

Here we notice that the uniform boundedness for jumps is assumed in the above result. But Kajii [6] gave the following improvement.

Theorem 1.2. *Assume the existence of the random variable M_∞ such that $\lim_{t \rightarrow \infty} M_t = M_\infty < \infty$ a.s. and that $\{M_\tau^-\}_{\tau \in \mathcal{T}}$ is uniformly integrable. Then*

(i) $-\infty < -E[M_\infty^-] \leq E[M_\infty] \leq 0$

holds. Besides, if $\{\Delta M_\tau\}_{\tau \in \mathcal{T}}$ is uniformly integrable, then

(ii) $\lim_{\lambda \rightarrow \infty} \lambda P(\sup_{t \in \mathbf{R}_+} M_t > \lambda) = -E[M_\infty]$.

Theorem 1.3. *Assume that M is a locally square integrable martingale and that $\langle M \rangle_\infty < \infty$ a.s., $\{M_\tau^-\}_{\tau \in \mathcal{T}}$ is uniformly integrable, and there exists $\lambda_0 > 0$ such that*

$$(1) \quad E \left[\exp \left\{ \lambda_0 M_\infty^- + \int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx) \right\} \right] < \infty$$

for some $K > 0$, where $\phi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x - (\lambda^2/2)x^2$. Then

- (i) $\lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty]$,
- (ii) $\lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty]$.

As a remark, we note that the condition (1) refines the conditions “ $|\Delta M| \leq K$ and $E[e^{\lambda_0 M_\infty^-}] < \infty$ for some $\lambda_0, K > 0$ ”.

Finally, we introduce our main result:

Theorem 1.4. *Assume that M is a locally square integrable martingale and quasi left continuous, $\langle M \rangle_\infty < \infty$ a.s., $\{M_\tau^-\}_{\tau \in \mathcal{T}}$ is uniformly integrable.*

- (i) *Assume moreover that there exists $\lambda_0 > 0$ such that*

$$(2) \quad E \left[\int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx) \right] < \infty$$

for some $K > 0$. Then

$$\lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty].$$

- (ii) *On the other hand, if we assume that there exists $\lambda_0 > 0$ such that*

$$E \left[\left[\int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx) \right]^{2+\alpha} \right] < \infty$$

for some $K > 0, \alpha > 0$. Then

$$\lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty].$$

The proof of the above shall be divided in three steps. As a first step, we will relax the assumption involving the finiteness of some exponential moment of a local martingale in Theorem 1.3, but we assume its quasi left continuity:

Theorem 1.5. *Assume that M is a locally square integrable martingale and quasi left continuous, $\langle M \rangle_\infty < \infty$ a.s., $\{M_\tau^-\}_{\tau \in \mathcal{T}}$ is uniformly integrable, and there exists $\lambda_0 > 0$ such that*

$$(3) \quad E \left[\exp \left\{ -\lambda_0 M_\infty + \int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx) \right\} \right] < \infty$$

for some $K > 0$. Then

$$\lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty].$$

As a second step, in Subsection 3.2 we will describe the proof of (i) from Theorem 1.5 by Takaoka’s method [10]. Finally, we can obtain (ii) from (i). This proof is the same as in Subsection 6.4 of Kaji [6] and is omitted.

2. Proof of Theorem 1.5

2.1. Two lemmas. First, it is known that

$$(4) \quad \int_{\mathbf{R}_+ \times \mathbf{X}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx) < \infty \quad \text{a.s.}$$

and

$$(5) \quad \int_{\mathbf{R}_+ \times \mathbf{X}} |\psi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx) < \infty \quad \text{a.s.,}$$

where $\psi_\lambda(x) = e^{-\lambda x} - 1 + \lambda x$. See Subsection 5.1 in Kaji [6].

Lemma 2.1.

$$E \left[e^{-\lambda M_\infty - (\lambda^2/2) \langle M^c \rangle_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} \right] = 1, \quad 0 < \forall \lambda < \lambda_0.$$

Proof. According to Lemma 5.2 of Kaji [6], the condition $E[e^{\lambda_0 M_\infty^-}] < \infty$ implies the desired conclusion. In fact, we can see

$$E[e^{\lambda_0 M_\infty^-}] \leq E[e^{-\lambda_0 M_\infty}] + 1,$$

where the right hand side is $< \infty$ by the assumption (3). □

Lemma 2.2.

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left(E \left[e^{-\lambda M_\infty - (\lambda^2/2) \langle M^c \rangle_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} \right] - E[e^{-(\lambda^2/2) \langle M \rangle_\infty}] \right) = -E[M_\infty].$$

Proof. First, we will show

$$(6) \quad \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty - (\lambda^2/2)\langle M^c \rangle_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - e^{-(\lambda^2/2)\langle M \rangle_\infty} \right\} = -M_\infty \quad \text{a.s.}$$

Observe the equality

$$\begin{aligned} & \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty - (\lambda^2/2)\langle M^c \rangle_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - e^{-(\lambda^2/2)\langle M \rangle_\infty} \right\} \\ &= \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty - (\lambda^2/2)\langle M^c \rangle_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - e^{-\lambda M_\infty - (\lambda^2/2)\langle M \rangle_\infty} \right\} \\ & \quad + \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty - (\lambda^2/2)\langle M \rangle_\infty} - e^{-(\lambda^2/2)\langle M \rangle_\infty} \right\} \\ &= e^{-\lambda M_\infty - (\lambda^2/2)\langle M \rangle_\infty} \cdot \frac{1}{\lambda} \left\{ e^{-\int_{\mathbf{R}_+ \times \mathbf{X}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - 1 \right\} \\ & \quad + e^{-(\lambda^2/2)\langle M \rangle_\infty} \cdot \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty} - 1 \right\}, \end{aligned}$$

where the last “=” holds by the fact $\langle M \rangle_\infty = \langle M^c \rangle_\infty + \int_{\mathbf{R}_+ \times \mathbf{X}} x^2 \hat{\mu}(\cdot, ds dx)$. Since it is clear that

$$\lim_{\lambda \rightarrow 0} \frac{e^{-\lambda M_\infty} - 1}{\lambda} = -M_\infty \quad \text{a.s.}$$

holds, the second term of the right-hand side of the observation converges to $-M_\infty$ a.s. Therefore, to get (6), it is sufficient to show that the first term of the right-hand side of the observation converges to 0 a.s. According to the dominated convergence theorem with respect to $\hat{\mu}(\cdot, ds dx)$, Lemma 4.1 of Kaji [6], (4), and the fact $\lim_{\lambda \rightarrow 0} \phi_\lambda/\lambda = 0$ imply

$$(7) \quad \lim_{\lambda \rightarrow 0} \int_{\mathbf{R}_+ \times \mathbf{X}} \left| \frac{\phi_\lambda(x)}{\lambda} \right| \hat{\mu}(\cdot, ds dx) = 0 \quad \text{a.s.}$$

On the other hand, by using the inequality

$$\left| \frac{e^{vx} - 1}{v} \right| \leq |x|e^{v|x|}, \quad v > 0,$$

we have

$$\begin{aligned} (8) \quad & \left| \frac{1}{\lambda} \left\{ e^{-\int_{\mathbf{R}_+ \times \mathbf{X}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - 1 \right\} \right| \\ & \leq \left| \int_{\mathbf{R}_+ \times \mathbf{X}} \frac{\phi_\lambda(x)}{\lambda} \hat{\mu}(\cdot, ds dx) \right| \exp \left\{ \left| \int_{\mathbf{R}_+ \times \mathbf{X}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx) \right| \right\} \\ & \leq \int_{\mathbf{R}_+ \times \mathbf{X}} \left| \frac{\phi_\lambda(x)}{\lambda} \right| \hat{\mu}(\cdot, ds dx) \exp \left\{ \int_{\mathbf{R}_+ \times \mathbf{X}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx) \right\}, \end{aligned}$$

where the last line holds, since $\lambda \rightarrow |\phi_\lambda(x)|$ is increasing for each $x \in \mathbf{X}$. By (7) and (8) the left-hand side of the last inequality converges to 0 a.s. as $\lambda \rightarrow 0$. Hence (6) holds.

Next, we show that for all $0 < \lambda < \lambda_0 \wedge 1/(2c_0K)$

$$(9) \quad \left| \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty - (\lambda^2/2)(M^c)_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - e^{-(\lambda^2/2)(M)_\infty} \right\} \right| \\ \leq e^{-\lambda_0 M_\infty + \int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx)} + \int_{\mathbf{R}_+ \times \{|x| > K\}} \left| \frac{\phi_{\lambda_0}(x)}{\lambda_0} \right| \hat{\mu}(\cdot, ds dx) \\ + M_\infty^+ + 1 + 2c_0K e^{-1},$$

where the positive constant c_0 is such that for all $|x| \leq \lambda_0 K$

$$(10) \quad \left| e^{-x} - 1 + x - \frac{x^2}{2} \right| \leq c_0 |x|^3.$$

Fix $0 < \lambda < \lambda_0 \wedge 1/(2c_0K)$. Observe the inequality

$$\left| \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty - (\lambda^2/2)(M^c)_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - e^{-(\lambda^2/2)(M)_\infty} \right\} \right| \\ = \left| \frac{1}{\lambda} \left\{ e^{-\lambda M_\infty - (\lambda^2/2)(M)_\infty + (\lambda^2/2) \int_{\mathbf{R}_+ \times \mathbf{X}} x^2 \hat{\mu}(\cdot, ds dx) - \int_{\mathbf{R}_+ \times \mathbf{X}} \psi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - e^{-(\lambda^2/2)(M)_\infty} \right\} \right| \\ = e^{-(\lambda^2/2)(M)_\infty} \cdot \frac{1}{\lambda} \left| e^{-\lambda M_\infty - \int_{\mathbf{R}_+ \times \mathbf{X}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - 1 \right| \\ = e^{-(\lambda^2/2)(M)_\infty} \cdot \frac{1}{\lambda} \left| e^{-\lambda M_\infty - \int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx) - \int_{\mathbf{R}_+ \times \{|x| > K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} \right. \\ \quad \left. - e^{-\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} + e^{-\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - 1 \right| \\ \leq e^{-(\lambda^2/2)(M)_\infty - \int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} \times \frac{1}{\lambda} \left| e^{-\lambda M_\infty - \int_{\mathbf{R}_+ \times \{|x| > K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - 1 \right| \\ + e^{-(\lambda^2/2)(M)_\infty} \cdot \frac{1}{\lambda} \left| e^{-\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} - 1 \right| \\ = I_1 \times I_2 + I_3.$$

We will estimate I_1 . By (10) we obtain

$$I_1 \leq e^{-(\lambda^2/2)(M)_\infty + \int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} |\phi_\lambda(x)| \hat{\mu}(\cdot, ds dx)} \\ \leq e^{-(\lambda^2/2)(M)_\infty + c_0 K \lambda^3 \int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} x^2 \hat{\mu}(\cdot, ds dx)} \\ \leq e^{-(\lambda^2/2)(M)_\infty + c_0 K \lambda^3 (M)_\infty} \\ \leq 1.$$

We will estimate I_2 . By using the inequality

$$\left| \frac{e^{\nu x} - 1}{\nu} \right| \leq e^{\nu x} 1_{\{x \geq 0\}} + x^{-1} 1_{\{x < 0\}}, \quad \nu > 0,$$

we have

$$\begin{aligned} I_2 &\leq e^{-\lambda M_\infty - \int_{\mathbf{R}_+ \times \{|x| > K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} 1_{\{M_\infty + \int_{\mathbf{R}_+ \times \{|x| > K\}} (\phi_\lambda(x)/\lambda) \hat{\mu}(\cdot, ds dx) \leq 0\}} \\ &\quad + \left(-M_\infty - \int_{\mathbf{R}_+ \times \{|x| > K\}} \frac{\phi_\lambda(x)}{\lambda} \hat{\mu}(\cdot, ds dx) \right)^- 1_{\{M_\infty + \int_{\mathbf{R}_+ \times \{|x| > K\}} (\phi_\lambda(x)/\lambda) \hat{\mu}(\cdot, ds dx) > 0\}} \\ &\leq e^{\lambda_0(-M_\infty - \int_{\mathbf{R}_+ \times \{|x| > K\}} (\phi_\lambda(x)/\lambda) \hat{\mu}(\cdot, ds dx))} 1_{\{M_\infty + \int_{\mathbf{R}_+ \times \{|x| > K\}} (\phi_\lambda(x)/\lambda) \hat{\mu}(\cdot, ds dx) \leq 0\}} \\ &\quad + \left(M_\infty + \int_{\mathbf{R}_+ \times \{|x| > K\}} \frac{\phi_\lambda(x)}{\lambda} \hat{\mu}(\cdot, ds dx) \right) 1_{\{M_\infty + \int_{\mathbf{R}_+ \times \{|x| > K\}} (\phi_\lambda(x)/\lambda) \hat{\mu}(\cdot, ds dx) > 0\}} \\ &\leq e^{-\lambda_0 M_\infty + \lambda_0 \int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx)} + M_\infty^+ + \int_{\mathbf{R}_+ \times \{|x| > K\}} \left| \frac{\phi_\lambda(x)}{\lambda} \right| \hat{\mu}(\cdot, ds dx). \end{aligned}$$

By Lemma 4.1 of Kaji [6], the right-hand side of the last inequality is

$$\leq e^{-\lambda_0 M_\infty + \lambda_0 \int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx)} + M_\infty^+ + \frac{1}{\lambda_0} \int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx).$$

We now estimate I_3 . By using the inequality

$$\left| \frac{e^{\nu x} - 1}{\nu} \right| \leq e^{\nu x} 1_{\{x \geq 0\}} + x^{-1} 1_{\{x < 0\}}, \quad \nu > 0,$$

we have

$$\begin{aligned} I_3 &\leq e^{-(\lambda^2/2)\langle M \rangle_\infty} \left\{ e^{-\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \phi_\lambda(x) \hat{\mu}(\cdot, ds dx)} 1_{\{\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} (\phi_\lambda(x)/\lambda) \hat{\mu}(\cdot, ds dx) \leq 0\}} \right. \\ &\quad \left. + \left(-\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \frac{\phi_\lambda(x)}{\lambda} \hat{\mu}(\cdot, ds dx) \right)^- 1_{\{\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} (\phi_\lambda(x)/\lambda) \hat{\mu}(\cdot, ds dx) > 0\}} \right\} \\ &\leq e^{-(\lambda^2/2)\langle M \rangle_\infty} \left\{ e^{\int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} |\phi_\lambda(x)| \hat{\mu}(\cdot, ds dx)} + \int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} \left| \frac{\phi_\lambda(x)}{\lambda} \right| \hat{\mu}(\cdot, ds dx) \right\}. \end{aligned}$$

Moreover, by (10) the right-hand side of the last inequality is

$$\begin{aligned} &\leq e^{-(\lambda^2/2)\langle M \rangle_\infty} \left\{ e^{c_0 K \lambda^3 \int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} x^2 \hat{\mu}(\cdot, ds dx)} + c_0 K \lambda^2 \int_{\mathbf{R}_+ \times \{0 < |x| \leq K\}} x^2 \hat{\mu}(\cdot, ds dx) \right\} \\ &\leq e^{-(\lambda^2/2)\langle M \rangle_\infty} \left\{ e^{c_0 K \lambda^3 \langle M \rangle_\infty} + c_0 K \lambda^2 \langle M \rangle_\infty \right\} \\ &\leq e^{(\lambda^2/2)(-1+2c_0 K \lambda)\langle M \rangle_\infty} + 2c_0 K \cdot \frac{\lambda^2}{2} \langle M \rangle_\infty e^{-(\lambda^2/2)\langle M \rangle_\infty} \\ &\leq 1 + 2c_0 K e^{-1}, \end{aligned}$$

where we can see $(\lambda^2/2)\langle M \rangle_\infty e^{-(\lambda^2/2)\langle M \rangle_\infty} \leq e^{-1}$ by using the inequality $xe^{-x} \leq e^{-1}$. Hence, the above three estimations of I_1 , I_2 , and I_3 imply (9).

Finally, according to the dominated convergence theorem, (6), (9), $E[M_\infty^+] < \infty$, and the assumption (3) imply the desired conclusion. \square

2.2. A Tauberian theorem.

Theorem 2.1 ([4]). *Let X be an \mathbf{R}_+ -valued random variable such that $\lim_{\lambda \rightarrow 0} (1/\lambda)(1 - E[e^{-(\lambda^2/2)X}])$ exists in \mathbf{R} , then*

$$\sqrt{\frac{2}{\pi}} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (1 - E[e^{-(\lambda^2/2)X}]) = \lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{X} > \lambda).$$

2.3. Proof of Theorem 1.5. According to Lemmas 2.1 and 2.2, we have

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (1 - E[e^{-(\lambda^2/2)\langle M \rangle_\infty}]) = -E[M_\infty]$$

holds. Then, by using the Tauberian theorem the last result implies

$$\lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty].$$

3. Proof of Theorem 1.4

3.1. The lemma.

Lemma 3.1. *Let ρ be a stopping time. Then it follows that for any $0 < a < 1$*

$$\begin{aligned} \limsup_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) &\leq \frac{1}{a} \limsup_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_\rho} > \lambda) \\ &\quad + \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty], \end{aligned}$$

where C is a positive constant which does not depend on M , a , and ρ .

Proof. Fix $0 < a < 1$. We have

$$P(\langle M \rangle_\infty > \lambda^2) \leq P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) + P(\langle M \rangle_\rho > a^2 \lambda^2),$$

and so

$$\begin{aligned} (11) \quad \limsup_{\lambda \rightarrow \infty} \lambda P(\langle M \rangle_\infty > \lambda^2) &\leq \frac{1}{a} \limsup_{\lambda \rightarrow \infty} \lambda P(\langle M \rangle_\rho > \lambda^2) \\ &\quad + \sup_{\lambda} \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2). \end{aligned}$$

On the other hand, define the process $N = \{N_t\}_{t \in \mathbf{R}_+}$ and the filtration $\{\mathcal{G}_t\}_{t \in \mathbf{R}_+}$ as

$$N_t = M_{\rho+t} - M_\rho, \quad \mathcal{G}_t = \mathcal{F}_{\rho+t}, \quad \forall t \in \mathbf{R}_+.$$

Then N is a local martingale with respect to $\{\mathcal{G}_t\}_{t \in \mathbf{R}_+}$ and

$$\langle N \rangle_\infty = \langle M \rangle_\infty - \langle M \rangle_\rho$$

holds. Also, observe

$$\begin{aligned} \sup_\lambda \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) &\leq \sup_\lambda \lambda P(\langle N \rangle_\infty > \lambda^2 - a^2 \lambda^2) \\ &= \frac{1}{\sqrt{1-a^2}} \sup_\lambda \lambda P(\langle N \rangle_\infty > \lambda^2). \end{aligned}$$

Then, by using the appendix the right-hand side of the last inequality is

$$(12) \leq \frac{C}{\sqrt{1-a^2}} \sup_\lambda \lambda P\left(\sup_{t \in \mathbf{R}_+} |N_t| > \lambda\right),$$

where C is a positive constant which does not depend on M , a , and ρ . If we let $\lambda > 0$ and

$$\tau_\lambda = \begin{cases} \inf\{t \in \mathbf{R}_+ \mid |N_t| > \lambda\} & \text{if } \{\} \neq \emptyset \\ \infty & \text{if } \{\} = \emptyset, \end{cases}$$

then $|N_{\tau_\lambda}| \geq \lambda$ on $\{\tau_\lambda < \infty\} = \{\sup_{t \in \mathbf{R}_+} |N_t| > \lambda\}$, and so

$$\lambda P\left(\sup_{t \in \mathbf{R}_+} |N_t| > \lambda\right) \leq E[|N_{\tau_\lambda}|].$$

Therefore by the last result we have

$$\begin{aligned} (12) &\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}(N)} E[|N_\tau|] \\ &\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}(N)} 2E[N_\tau^-] \\ &= \frac{2C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}(N)} \{E[(M_{\rho+\tau} - M_\rho)^-; \rho = \infty] + E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty]\} \\ &\leq \frac{2C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty], \end{aligned}$$

where $\mathcal{T}(N) = \{\tau : \text{stopping time} \mid \{N_{\tau \wedge t}\}_{t \in \mathbf{R}_+} \text{ is uniformly integrable}\}$. That is,

$$\sup_\lambda \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) \leq \frac{2C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\rho+\tau} - M_\rho)^-; \rho < \infty].$$

Hence, by the last inequality and (11) we get the desired conclusion. □

3.2. Proof of (i). For any $u > 0$, introduce the stopping time

$$\tau_u = \begin{cases} \inf\{t \in \mathbf{R}_+ \mid -\lambda_0 M_t + A_t > u\} & \text{if } \{\} \neq \emptyset \\ \infty & \text{if } \{\} = \emptyset, \end{cases}$$

where

$$A_t = \int_{(0,t] \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds dx), \quad t \in \mathbf{R}_+.$$

Fix $u > 0$. We consider the process $M^{(u)} = \{M_t^{(u)}\}_{t \in \mathbf{R}_+}$ defined as $M_t^{(u)} = M_{\tau_u \wedge t}$, $t \in \mathbf{R}_+$. Then it follows from the assumptions with respect to M that $M^{(u)}$ is also a quasi left continuous and locally square integrable martingale which satisfying $M_0^{(u)} = 0$, $\langle M^{(u)} \rangle_\infty (= \langle M \rangle_{\tau_u}) \leq \langle M \rangle_\infty < \infty$ a.s., and the uniform integrability of $\{(M_t^{(u)})^-\}_{t \in \mathcal{T}}$. Moreover, if we pick the random measure $\mu^{(u)}$ on $\Omega \times \mathbf{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbf{R}_+$ and Borel subsets U of \mathbf{X}

$$\mu^{(u)}(\cdot, (0, t] \times U) = \sum_{0 < s \leq t} 1_U(\Delta M_s^{(u)})$$

and its compensator $\hat{\mu}^{(u)}$, then it follows that for all $t \in \mathbf{R}_+$ and Borel subsets U of \mathbf{X}

$$\mu^{(u)}(\cdot, (0, t] \times U) = \sum_{0 < s \leq \tau_u \wedge t} 1_U(\Delta M_s) = \mu(\cdot, (0, \tau_u \wedge t] \times U) \quad \text{a.s.},$$

and so $\hat{\mu}^{(u)}$ is the random measure on $\Omega \times \mathbf{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbf{R}_+$ and Borel subsets U of \mathbf{X}

$$\hat{\mu}^{(u)}(\cdot, (0, t] \times U) = \hat{\mu}(\cdot, (0, \tau_u \wedge t] \times U) \quad \text{a.s.},$$

and therefore we can have that

$$\begin{aligned} & E\left[e^{-\lambda_0 M_\infty^{(u)} + \int_{\mathbf{R}_+ \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \hat{\mu}^{(u)}(\cdot, ds dx)}\right] \\ &= E\left[e^{-\lambda_0 M_{\tau_u} + A_{\tau_u}}\right] \\ &= E\left[e^{-\lambda_0 M_{\tau_u} + A_{\tau_u}}; \tau_u < \infty\right] + E\left[e^{-\lambda_0 M_{\tau_u} + A_{\tau_u}}; \tau_u = \infty\right] \\ &\leq E\left[e^{u - \lambda_0 \Delta M_{\tau_u}}; \tau_u < \infty\right] + e^u P(\tau_u = \infty) \\ &= E\left[e^{u - \lambda_0 \times 0}; \tau_u < \infty\right] + e^u P(\tau_u = \infty) \quad (= e^u), \end{aligned}$$

where the fourth line of the above holds by the definition of τ_u and the last line does by the quasi left continuity of M . By applying Theorem 1.5 to the process $M^{(u)}$, we have

$$-\infty < E[M_\infty^{(u)}] \leq 0, \quad \lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M^{(u)} \rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_\infty^{(u)}],$$

that is, $-\infty < E[M_{\tau_u}] \leq 0$ and

$$(13) \quad \lim_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_{\tau_u}} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_{\tau_u}].$$

Now we show

$$(14) \quad \liminf_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_{\infty}} > \lambda) \geq -\sqrt{\frac{2}{\pi}} E[M_{\infty}].$$

Indeed, the left-hand side of (13) is

$$\leq \liminf_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_{\infty}} > \lambda)$$

and the right-hand side of (13) is

$$\begin{aligned} &= -\sqrt{\frac{2}{\pi}} E[M_{\infty}; \tau_u = \infty] - \sqrt{\frac{2}{\pi}} E[M_{\tau_u}; \tau_u < \infty] \\ &\geq -\sqrt{\frac{2}{\pi}} E[M_{\infty}; \tau_u = \infty] + \sqrt{\frac{2}{\pi}} E\left[\frac{u}{\lambda_0} - \frac{1}{\lambda_0} A_{\tau_u}; \tau_u < \infty\right] \\ &\geq -\sqrt{\frac{2}{\pi}} E[M_{\infty}; \tau_u = \infty] - \frac{1}{\lambda_0} \sqrt{\frac{2}{\pi}} E[A_{\tau_u}; \tau_u < \infty], \end{aligned}$$

where the second line of the above holds by the definition of τ_u . Also, the right-hand side of the above converges to $-\sqrt{2/\pi} E[M_{\infty}]$ as $u \rightarrow \infty$, because by the dominated convergence theorem, the fact $E[|M_{\infty}|] < \infty$ we have known and the assumption (2) imply

$$\lim_{u \rightarrow \infty} E[M_{\infty}; \tau_u = \infty] = E[M_{\infty}], \quad \lim_{u \rightarrow \infty} E[A_{\tau_u}; \tau_u < \infty] = 0.$$

Therefore we can get (14).

On the other hand, we will show

$$(15) \quad \limsup_{\lambda \rightarrow \infty} \lambda P(\sqrt{\langle M \rangle_{\infty}} > \lambda) \leq -\sqrt{\frac{2}{\pi}} E[M_{\infty}].$$

According to Lemma 3.1, we have for all $0 < a < 1$

$$\begin{aligned} \limsup_{\lambda \rightarrow \infty} \lambda P(\langle M \rangle_{\infty} > \lambda^2) &\leq \frac{1}{a} \liminf_{\lambda \rightarrow \infty} \lambda P(\langle M \rangle_{\tau_u} > \lambda^2) \\ &\quad + \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in T} E[(M_{\tau_u+\tau} - M_{\tau_u})^-; \tau_u < \infty], \end{aligned}$$

where C is a positive constant which does not depend on a and u . Fix $0 < a < 1$. By (13) the first term on the right-hand side of the last inequality is

$$= \frac{1}{a} \left(-\sqrt{\frac{2}{\pi}} E[M_\infty^{(u)}] \right).$$

Therefore

$$\begin{aligned} \limsup_{\lambda \rightarrow \infty} \lambda P(\langle M \rangle_\infty > \lambda^2) &\leq \frac{1}{a} \left(-\sqrt{\frac{2}{\pi}} E[M_\infty^{(u)}] \right) \\ &\quad + \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\tau_u+\tau} - M_{\tau_u})^-; \tau_u < \infty]. \end{aligned}$$

By the definition of τ_u the second term on the right-hand side of the last inequality is

$$\begin{aligned} &\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E \left[\left(M_{\tau_u+\tau} + \frac{u}{\lambda_0} - \frac{1}{\lambda_0} A_{\tau_u} \right)^-; \tau_u < \infty \right] \\ &\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E \left[M_{\tau_u+\tau}^- + \frac{1}{\lambda_0} A_{\tau_u}; \tau_u < \infty \right] \\ &\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[M_\tau^-; \tau_u < \infty] + \frac{C}{\sqrt{1-a^2}} \frac{1}{\lambda_0} E[A_\infty; \tau_u < \infty]. \end{aligned}$$

By the uniform integrability of $\{M_\tau^-\}_{\tau \in \mathcal{T}}$ the first term on the right-hand side of the last inequality converges to 0 a.s. as $u \rightarrow \infty$ and from the dominated convergence theorem the assumption (2) implies that the second term of it does so, too. Therefore

$$\limsup_{\lambda \rightarrow \infty} \lambda P(\langle M \rangle_\infty > \lambda^2) \leq \limsup_{u \rightarrow \infty} \frac{1}{a} \left(-\sqrt{\frac{2}{\pi}} E[M_\infty^{(u)}] \right).$$

Moreover, the right-hand side of the last inequality is $\leq (1/a)(-\sqrt{2/\pi} E[M_\infty])$ since $\liminf_{u \rightarrow \infty} E[M_{\tau_u}^+] \geq E[M_\infty^+]$ holds by the Fatou lemma and since $\lim_{u \rightarrow \infty} E[M_{\tau_u}^-] = E[M_\infty^-]$ holds by the uniform integrability of $\{M_\tau^-\}_{\tau \in \mathcal{T}}$. Therefore we can get (15).

Hence (14) and (15) imply the desired conclusion.

4. Appendix

Proposition 4.1. *Assume that M is a quasi left continuous and locally square integrable martingale. Then*

$$\sup_{\lambda} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) \leq C \sup_{\lambda} \lambda P\left(\sup_{t < \infty} |M_t| > \lambda\right),$$

where C is a universal positive constant.

Proof. Pick any stopping times ρ and τ with $\rho \leq \tau$. First, it is clear that we can get

$$(16) \quad E[(\sqrt{\langle M \rangle_{\tau-}} - \sqrt{\langle M \rangle_{\rho-}})^2] \leq E[\langle M \rangle_{\tau} - \langle M \rangle_{\rho}].$$

In fact, $\langle M \rangle_t$ is continuous, since M is quasi left continuous, and the inequality $(\sqrt{a} - \sqrt{b})^2 \leq a - b$ for $0 \leq b \leq a$ holds. Introduce the local martingale $N_t = M_{(\rho+t) \wedge \tau} - M_{\rho}$, $t < \infty$, and then we can see $\langle M \rangle_{\tau} - \langle M \rangle_{\rho} = \langle N \rangle_{\infty}$. Therefore, (16) and the last result imply

$$(17) \quad \begin{aligned} E[(\sqrt{\langle M \rangle_{\tau-}} - \sqrt{\langle M \rangle_{\rho-}})^2] &\leq E[\langle N \rangle_{\infty}] \\ &\leq E\left[\left(\sup_{t < \infty} |N_t|\right)^2\right], \end{aligned}$$

where the last line of the last inequality holds by the property of a local martingale. By the definition of N we have

$$(18) \quad \begin{aligned} E\left[\left(\sup_{t < \infty} |N_t|\right)^2\right] &= E\left[\left(\sup_{t < \infty} |N_t|\right)^2; \rho < \tau\right] \\ &\leq 2E\left[\left(\sup_{t < \infty} |M_{t \wedge \tau}|\right)^2 + M_{\rho}^2; \rho < \tau\right] \\ &= 2E\left[\left(\sup_{t < \infty} |M_t|\right)^2 + M_{\rho}^2; \rho < \tau = \infty\right] \\ &\quad + 2E\left[\left(\sup_{t < \infty} |M_{t \wedge \tau}|\right)^2 + M_{\rho}^2; \rho < \tau < \infty\right] \\ &\leq 4E\left[\left(\sup_{t < \infty} |M_t|\right)^2; \rho < \tau = \infty\right] \\ &\quad + 2E\left[\left(\sup_{t \leq \tau} |M_t|\right)^2 + \left(\sup_{t < \tau} |M_t|\right)^2; \rho < \tau < \infty\right] \\ &= 4E\left[\left(\sup_{t < \infty} |M_t|\right)^2; \rho < \tau = \infty\right] \\ &\quad + 4E\left[\left(\sup_{t < \tau} |M_t|\right)^2; \rho < \tau < \infty\right] \\ &= 4E\left[\left(\sup_{t < \tau} |M_t|\right)^2; \rho < \tau\right], \end{aligned}$$

where the eighth line of the last inequality holds by the quasi left continuity of $t \rightarrow$

$\sup_{s \leq t} |M_s|$. Hence, (17) and (18) imply

$$E[(\sqrt{\langle M \rangle_{\tau_-}} - \sqrt{\langle M \rangle_{\rho_-}})^2] = 4E \left[\left(\sup_{t < \tau} |M_t| \right)^2 ; \rho < \tau \right].$$

Then, according to Corollary 6 of Azéma, Gundy, and Yor [1], the above implies the desired conclusion. \square

ACKNOWLEDGMENT. When Theorem 3.1 and 3.2 of Kaji [6] were obtained, Professor S. Kotani pointed out the possibility of their extension as in the statement of Theorem 1.4. The author is grateful to Professor S. Kotani for giving his conjecture and to the referee for useful advices.

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