THE TAIL ESTIMATION OF THE QUADRATIC VARIATION OF A QUASI LEFT CONTINUOUS LOCAL MARTINGALE

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Abstract

We discuss some estimates of the tail distributions of the supremum and the quadratic variation of a local martingale. The assumption made so far in the literature on exponential moments involving a quasi left continuous local martingale is improved.

1. Introduction and main result

There have been a number of works on tail distributions of the supremum and the quadratic variation of a local martingale. On the other hand, in the paper [7] Kotani gave a necessary and sufficient condition for one-dimensional diffusion processes to be martingales. In Azéma, Gundy, and Yor [1], the uniform integrability of a continuous martingale in terms of tails of its supremum and quadratic variation was first characterized. The existence of the limits of the tails was considered by Galtchouk and Novikov [5] (for a discrete time martingale), Novikov [10], Elworthy, Li, and Yor [2], [3], Madan and Yor [9] (for a continuous local martingale), Liptser and Novikov [8], and Kaji [6] (for a càdlàg local martingale) by using the Tauberian theorem. In the statements on the quadratic variation of a local martingale, the existence of some exponential moments involving a local martingale is assumed, but Takaoka [11] relaxed its assumption for a continuous local martingale. In this paper we also do so for a càdlàg local martingale.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}, P)\) be a filtered probability space with usual conditions, where \(\mathbb{R}_+ = [0, \infty)\), and \(M = \{M_t\}_{t \in \mathbb{R}_+}\) is a càdlàg local martingale with \(M_0 = 0\) defined on it. We denote by \(\mu\) the random measure on \(\mathbb{R}_+ \times X\) such that for all \(t \in \mathbb{R}_+\) and Borel subsets \(U\) of \(X\)

\[\mu(\cdot, (0, t] \times U) = \sum_{0 < s \leq t} 1_U(\Delta M_s),\]

where \(X = \mathbb{R} - \{0\}\) and \(\Delta M_t = M_t - M_{t-}, t > 0\). That is, \(\mu\) is the counting measure of jumps of \(M\). Then we denote by \(\hat{\mu}\) its predictable compensator. If \(M\) is a locally square integrable martingale, then it is well-known that we can define a predictable

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quadratic variation process \( \langle M \rangle = \{ \langle M \rangle_t \}_{t \in \mathbb{R}} \), and an optional quadratic variation process \([M] = \{ [M]_t \}_{t \in \mathbb{R}} \), and the canonical decomposition

\[
M = M^c + M^d
\]

holds, where \( M^c \) is a continuous local martingale with \( M^c_0 = 0 \) and \( M^d \) is a stochastic integral process with respect to \( \mu - \tilde{\mu} \) defined as

\[
M^d_t = \int_{(0,t] \times X} x[\mu(\cdot, ds \, dx) - \tilde{\mu}(\cdot, ds \, dx)], \quad t \in \mathbb{R}_+.
\]

Moreover recall that

\[
\langle M^d \rangle_t = \int_{(0,t] \times X} x^2 \tilde{\mu}(\cdot, ds \, dx), \quad t \in \mathbb{R}_+.
\]

First, we recall the result by Liptser and Novikov [8].

**Theorem 1.1.** Assume that \( M \) is a locally square integrable martingale, \( \langle M \rangle_\infty = \lim_{t \to \infty} \langle M \rangle_t < \infty \) a.s., and \( \{M^*_\tau \}_{\tau \in T} \) is uniformly integrable, where \( T \) is the set of stopping times \( \tau \). Then

(i) \( 0 \leq E[M_\infty^*] \leq E[M^*_\infty] < \infty \).

Besides,

(ii) if \( \{\Delta M_\tau \}_{\tau \in T} \) is uniformly integrable, then

\[
\lim_{\lambda \to \infty} \lambda P \left( \sup_{t \in \mathbb{R}_+} (\Delta M^*_t) > \lambda \right) = E[M^*_\infty];
\]

(iii) if \( |\Delta M| \leq K \) and \( E[e^{e^M_M}] < \infty \) for some \( K > 0 \), and \( \epsilon > 0 \), then

\[
\lim_{\lambda \to \infty} \lambda P \left( \sqrt{\langle M \rangle_\infty} > \lambda \right) = \lim_{\lambda \to \infty} \lambda P \left( |\Delta M| > \lambda \right) = \sqrt{\frac{2}{\pi}} E[M^*_\infty].
\]

Here we notice that the uniform boundedness for jumps is assumed in the above result. But Kaji [6] gave the following improvement.

**Theorem 1.2.** Assume the existence of the random variable \( M^*_\infty \) such that

\[
\lim_{t \to \infty} M^*_t = M^*_\infty < \infty \quad \text{a.s. and that } \{M^*_\tau \}_{\tau \in T} \text{ is uniformly integrable. Then}
\]

(i) \( -\infty < -E[M^*_\infty] \leq E[M^*_\infty] \leq 0 \)

holds. Besides, if \( \{\Delta M^*_\tau \}_{\tau \in T} \) is uniformly integrable, then

(ii) \( \lim_{\lambda \to \infty} \lambda P(\sup_{t \in \mathbb{R}_+} M^*_t > \lambda) = -E[M^*_\infty]. \)
Theorem 1.3. Assume that $M$ is a locally square integrable martingale and that $\langle M \rangle_\infty < \infty$ a.s., $[M^-]_T$ is uniformly integrable, and there exists $\lambda_0 > 0$ such that

\[
E \left[ \exp \left\{ \lambda_0 M^-_\infty + \int_{R \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \tilde{\mu}(\cdot, ds \, dx) \right\} \right] < \infty
\]

for some $K > 0$, where $\phi_{\lambda}(x) = e^{-\lambda x} - 1 + \lambda x - (\lambda^2/2)x^2$. Then

(i) $\lim_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty].$

(ii) $\lim_{\lambda \to \infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty].$

As a remark, we note that the condition (1) refines the conditions "$|\Delta M| \leq K$ and $E[e^{\lambda_0 M^-}] < \infty$ for some $\lambda_0, K > 0$".

Finally, we introduce our main result:

Theorem 1.4. Assume that $M$ is a locally square integrable martingale and quasi left continuous, $\langle M \rangle_\infty < \infty$ a.s., $[M^-]_T$ is uniformly integrable.

(i) Assume moreover that there exists $\lambda_0 > 0$ such that

\[
E \left[ \int_{R \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \tilde{\mu}(\cdot, ds \, dx) \right] < \infty
\]

for some $K > 0$. Then

\[
\lim_{\lambda \to \infty} \lambda P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty].
\]

(ii) On the other hand, if we assume that there exists $\lambda_0 > 0$ such that

\[
E \left[ \left\{ \int_{R \times \{|x| > K\}} |\phi_{\lambda_0}(x)| \tilde{\mu}(\cdot, ds \, dx) \right\}^{2\alpha} \right] < \infty
\]

for some $K > 0, \alpha > 0$. Then

\[
\lim_{\lambda \to \infty} \lambda P(\sqrt{[M]_\infty} > \lambda) = -\sqrt{2/\pi} E[M_\infty].
\]

The proof of the above shall be divided in three steps. As a first step, we will relax the assumption involving the finiteness of some exponential moment of a local martingale in Theorem 1.3, but we assume its quasi left continuity:
Theorem 1.5. Assume that $M$ is a locally square integrable martingale and quasi left continuous, $\langle M \rangle_\infty < \infty$ a.s., $\{M^-_{i\in T}\}$ is uniformly integrable, and there exists $\lambda_0 > 0$ such that

$$E\left[ \exp \left\{ -\lambda_0 M_\infty + \int_{\mathbb{R}\times [|x|>K]} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) \right\} \right] < \infty$$

for some $K > 0$. Then

$$\lim_{\lambda \to \infty} \lambda \cdot P(\sqrt{\langle M \rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} \cdot E[M_\infty].$$

As a second step, in Subsection 3.2 we will describe the proof of (i) from Theorem 1.5 by Takaoka’s method [10]. Finally, we can obtain (ii) from (i). This proof is the same as in Subsection 6.4 of Kaji [6] and is omitted.

2. Proof of Theorem 1.5

2.1. Two lemmas. First, it is known that

$$\int_{\mathbb{R}^d\times X} |\phi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) < \infty \quad \text{a.s.}$$

and

$$\int_{\mathbb{R}^d\times X} |\psi_{\lambda_0}(x)| \hat{\mu}(\cdot, ds \, dx) < \infty \quad \text{a.s.,}$$

where $\psi_{\lambda}(x) = e^{-\lambda x} - 1 + \lambda x$. See Subsection 5.1 in Kaji [6].

Lemma 2.1.

$$E\left[ e^{-\lambda_0 M_\infty - (\lambda^2/2) \langle M^2 \rangle_\infty - \int_{\mathbb{R}^d\times X} \psi_{\lambda_0}(x) \hat{\mu}(\cdot, ds \, dx) } \right] = 1, \quad 0 < \forall \lambda < \lambda_0.$$

Proof. According to Lemma 5.2 of Kaji [6], the condition $E[e^{\lambda_0 M_\infty}] < \infty$ implies the desired conclusion. In fact, we can see

$$E[e^{\lambda_0 M_\infty}] \leq E[e^{-\lambda_0 M_\infty}] + 1,$$

where the right hand side is $< \infty$ by the assumption (3).

Lemma 2.2.

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \left( E\left[ e^{-\lambda_0 M_\infty - (\lambda^2/2) \langle M^2 \rangle_\infty - \int_{\mathbb{R}^d\times X} \psi_{\lambda_0}(x) \hat{\mu}(\cdot, ds \, dx) } \right] - E[e^{-(\lambda^2/2) \langle M^2 \rangle_\infty}] \right) = -E[M_\infty].$$
Proof. First, we will show
\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty} - (\lambda^2/2) (M^\prime \| \alpha) - \int_{R \times X} \psi(x) \hat{\mu}(\cdot, dx) } - e^{-\lambda (\lambda^2/2) (M^\prime \| \alpha) } \right\} = -M_{\infty} \quad \text{a.s.}
\]

Observe the equality
\[
\frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty} - (\lambda^2/2) (M^\prime \| \alpha) - \int_{R \times X} \psi(x) \hat{\mu}(\cdot, dx) } - e^{-\lambda (\lambda^2/2) (M^\prime \| \alpha) } \right\}
= \frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty} - (\lambda^2/2) (M^\prime \| \alpha) - \int_{R \times X} \psi(x) \hat{\mu}(\cdot, dx) } - e^{-\lambda M_{\infty} - (\lambda^2/2) (M^\prime \| \alpha) } \right\}
+ \frac{1}{\lambda} \left\{ e^{-\lambda M_{\infty} - (\lambda^2/2) (M^\prime \| \alpha) - \int_{R \times X} \psi(x) \hat{\mu}(\cdot, dx) } - 1 \right\}
+ \lambda \left\{ e^{-\lambda M_{\infty} - (\lambda^2/2) (M^\prime \| \alpha) } - 1 \right\},
\]
where the last “=” holds by the fact \( \langle M \rangle_{\infty} = \langle M^\prime \rangle_{\infty} + \int_{R \times X} x^2 \hat{\mu}(\cdot, ds) dx \). Since it is clear that
\[
\lim_{\lambda \to 0} \frac{e^{-\lambda M_{\infty} - 1}}{\lambda} = -M_{\infty} \quad \text{a.s.}
\]
holds, the second term of the right-hand side of the observation converges to \(-M_{\infty}\) a.s. Therefore, to get (6), it is sufficient to show that the first term of the right-hand side of the observation converges to 0 a.s. According to the dominated convergence theorem with respect to \( \hat{\mu}(\cdot, ds) dx \), Lemma 4.1 of Kaji [6], (4), and the fact \( \lim_{\lambda \to 0} \phi \lambda / \lambda = 0 \) imply
\[
\lim_{\lambda \to 0} \int_{R \times X} \left| \frac{\phi(x)}{\lambda} \hat{\mu}(\cdot, ds) dx \right| = 0 \quad \text{a.s.}
\]
On the other hand, by using the inequality
\[
\left| e^{vx} - 1 \right| \leq |x| e^{v|x|}, \quad v > 0,
\]
we have
\[
\left| \frac{1}{\lambda} \left\{ e^{-\int_{R \times X} \phi(x) \hat{\mu}(\cdot, dx) } - 1 \right\} \right|
\leq \int_{R \times X} \left| \frac{\phi(x)}{\lambda} \hat{\mu}(\cdot, ds) dx \right| \exp \left\{ \int_{R \times X} \phi(x) \hat{\mu}(\cdot, ds) dx \right\}
\leq \int_{R \times X} \left| \frac{\phi(x)}{\lambda} \hat{\mu}(\cdot, ds) dx \right| \exp \left\{ \int_{R \times X} \left| \phi(x) \hat{\mu}(\cdot, ds) dx \right| \right\}.
\]
where the last line holds, since $\lambda \rightarrow |\phi_k(x)|$ is increasing for each $x \in X$. By (7) and (8) the left-hand side of the last inequality converges to 0 a.s. as $\lambda \rightarrow 0$. Hence (6) holds.

Next, we show that for all $0 < \lambda < \lambda_0 \wedge 1/(2c_0K)$

$$e^{-\frac{x^2}{2}} \leq e^{-\lambda M_\infty} \int_{\mathbb{R} \times \{|x| > K\}} \phi_j(x) \hat{\mu}(\cdot, ds \, d\sigma) + \mu_j \hat{\mu}(\cdot, ds \, d\sigma)$$

$$+ M_\infty^+ + 1 + 2c_0K e^{-1},$$

where the positive constant $c_0$ is such that for all $|x| \leq \lambda_0 K$

$$e^{-x} - 1 + x - \frac{x^2}{2} \leq c_0 |x|^3.$$
We will estimate $I_2$. By using the inequality

$$\left| \frac{e^{\nu x} - 1}{\nu} \right| \leq e^{\nu x} 1_{[x \geq 0]} + x^{-1} 1_{[x < 0]}, \quad \nu > 0,$$

we have

$$I_2 \leq e^{-\lambda M_\infty} \int [I_{[\rho(x)/\lambda] 1 \leq K}] \phi_\lambda(x) \mu(\cdot, ds dx) 1 \left\{ M_\infty + \int [I_{[\rho(x)/\lambda] 1 > K}] \phi_\lambda(x) \mu(\cdot, ds dx) > 0 \right\}$$

$$+ \left( -M_\infty - \int [I_{[\rho(x)/\lambda] 2 > K}] \phi_\lambda(x) \mu(\cdot, ds dx) \right) 1 \left\{ M_\infty + \int [I_{[\rho(x)/\lambda] 2 > K}] \phi_\lambda(x) \mu(\cdot, ds dx) > 0 \right\}$$

$$\leq e^{-\lambda M_\infty} \int [I_{[\rho(x)/\lambda] 1 \leq K}] \phi_\lambda(x) \mu(\cdot, ds dx) + M_\infty + \int [I_{[\rho(x)/\lambda] 2 > K}] \phi_\lambda(x) \mu(\cdot, ds dx).$$

By Lemma 4.1 of Kaji [6], the right-hand side of the last inequality is

$$\leq e^{-\lambda M_\infty} \int [I_{[\rho(x)/\lambda] 1 \leq K}] \phi_\lambda(x) \mu(\cdot, ds dx) + M_\infty + \frac{1}{\lambda_0} \int [I_{[\rho(x)/\lambda] 2 > K}] \phi_\lambda(x) \mu(\cdot, ds dx).$$

We now estimate $I_3$. By using the inequality

$$\left| \frac{e^{\nu x} - 1}{\nu} \right| \leq e^{\nu x} 1_{[x \geq 0]} + x^{-1} 1_{[x < 0]}, \quad \nu > 0,$$

we have

$$I_3 \leq e^{-\lambda_0 (M_\infty + \int [I_{[\rho(x)/\lambda] 1 \leq K}] \phi_\lambda(x) \mu(\cdot, ds dx))}$$

$$\left\{ e^{-\lambda_0 \int [I_{[\rho(x)/\lambda] 2 > K}] \phi_\lambda(x) \mu(\cdot, ds dx)} + \int [I_{[\rho(x)/\lambda] 2 > K}] \phi_\lambda(x) \mu(\cdot, ds dx) \right\}.$$
where we can see $(\lambda^2/2)(M)_\infty e^{-\lambda^2/2}(M)_\infty \leq e^{-1}$ by using the inequality $xe^{-x} \leq e^{-1}$.

Hence, the above three estimations of $I_1$, $I_2$, and $I_3$ imply (9).

Finally, according to the dominated convergence theorem, (6), (9), $E[M_{\infty}^+] < \infty$, and the assumption (3) imply the desired conclusion.

2.2. A Tauberian theorem.

**Theorem 2.1** ([4]). Let $X$ be an $R_+^*$-valued random variable such that
\[ \lim_{\lambda \to 0} \frac{1}{\lambda} (1 - E[e^{-\lambda^2/2}X]) \text{ exists in } R, \]
then
\[ \sqrt{\frac{2}{\pi}} \lim_{\lambda \to 0} \frac{1}{\lambda} (1 - E[e^{-\lambda^2/2}X]) = \lim_{\lambda \to \infty} \lambda P(\sqrt{X} > \lambda). \]

2.3. Proof of Theorem 1.5. According to Lemmas 2.1 and 2.2, we have
\[ \lim_{\lambda \to 0} \frac{1}{\lambda} (1 - E[e^{-\lambda^2/2}(M)_{\infty}]) = -E[M_{\infty}] \]
holds. Then, by using the Tauberian theorem the last result implies
\[ \lim_{\lambda \to \infty} \lambda P(\sqrt{(M)_{\infty}} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_{\infty}]. \]

3. Proof of Theorem 1.4

3.1. The lemma.

**Lemma 3.1.** Let $\rho$ be a stopping time. Then it follows that for any $0 < a < 1$
\[ \lim sup \frac{1}{\lambda} \lambda P(\sqrt{(M)_{\infty}} > \lambda) \leq \frac{1}{a} \lim sup \frac{1}{\lambda} \lambda P(\sqrt{(M)_{\rho}} > \lambda) \]
\[ + \frac{C}{\sqrt{1-a^2}} \sup_{t<\tau} E[(M_{\rho+t} - M_{\rho})^{-}; \rho < \infty], \]
where $C$ is a positive constant which does not depend on $M$, $a$, and $\rho$.

Proof. Fix $0 < a < 1$. We have
\[ P(\langle M \rangle_{\infty} > \lambda^2) \leq P(\langle M \rangle_{\rho} \leq a^2 \lambda^2, \langle M \rangle_{\infty} > \lambda^2) + P(\langle M \rangle_{\rho} > a^2 \lambda^2), \]
and so
\[ \lim sup \frac{1}{\lambda} \lambda P(\langle M \rangle_{\infty} > \lambda^2) \leq \frac{1}{a} \lim sup \frac{1}{\lambda} \lambda P(\langle M \rangle_{\rho} > \lambda^2) \]
\[ + \sup_{\lambda} \lambda P(\langle M \rangle_{\rho} \leq a^2 \lambda^2, \langle M \rangle_{\infty} > \lambda^2). \]
On the other hand, define the process \( N_t = \{ N_t \}_{t \in \mathbb{R}_+} \) and the filtration \( \{ G_t \}_{t \in \mathbb{R}_+} \) as

\[
N_t = M_{\rho + t} - M_{\rho}, \quad G_t = \mathcal{F}_{\rho + t}, \quad \forall t \in \mathbb{R}_+.
\]

Then \( N \) is a local martingale with respect to \( \{ G_t \}_{t \in \mathbb{R}_+} \) and

\[
\langle N \rangle_\infty = \langle M \rangle_\infty - \langle M \rangle_\rho.
\]

holds. Also, observe

\[
\sup_{\lambda} \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) \leq \sup_{\lambda} \lambda P(\langle N \rangle_\infty > \lambda^2 - a^2 \lambda^2)
\]

\[
= \frac{1}{\sqrt{1 - a^2}} \sup_{\lambda} \lambda P(\langle N \rangle_\infty > \lambda^2).
\]

Then, by using the appendix the right-hand side of the last inequality is

\[
\leq \frac{C}{\sqrt{1 - a^2}} \sup_{\lambda} \lambda P \left( \sup_{t \in \mathbb{R}_+} |N_t| > \lambda \right),
\]

where \( C \) is a positive constant which does not depend on \( M, a, \) and \( \rho \). If we let \( \lambda > 0 \) and

\[
\tau_\lambda = \begin{cases} 
\inf \{ t \in \mathbb{R}_+ \mid |N_t| > \lambda \} & \text{if } \{ t \} \neq \emptyset \\
\infty & \text{if } \{ t \} = \emptyset,
\end{cases}
\]

then \( |N_{\tau_\lambda}| \geq \lambda \) on \( \{ \tau_\lambda < \infty \} = \{ \sup_{t \in \mathbb{R}_+} |N_t| > \lambda \}, \) and so

\[
\lambda P \left( \sup_{t \in \mathbb{R}_+} |N_t| > \lambda \right) \leq E[|N_{\tau_\lambda}|].
\]

Therefore by the last result we have

\[
(12) \leq \frac{C}{\sqrt{1 - a^2}} \sup_{\tau \in \mathcal{T}(N)} E[|N_{\tau}|]
\]

\[
\leq \frac{C}{\sqrt{1 - a^2}} \sup_{\tau \in \mathcal{T}(N)} 2E[N_{-\tau}^-]
\]

\[
= \frac{2C}{\sqrt{1 - a^2}} \sup_{\tau \in \mathcal{T}(N)} \{ E[(M_{\rho + \tau} - M_{\rho})^-; \rho = \infty] + E[(M_{\rho + \tau} - M_{\rho})^-; \rho < \infty] \}
\]

\[
\leq \frac{2C}{\sqrt{1 - a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\rho + \tau} - M_{\rho})^-; \rho < \infty],
\]

where \( \mathcal{T}(N) = \{ \tau : \text{stopping time} \mid [N_{\tau \wedge \lambda}]_{t \in \mathbb{R}_+} \text{ is uniformly integrable} \}. \) That is,

\[
\sup_{\lambda} \lambda P(\langle M \rangle_\rho \leq a^2 \lambda^2, \langle M \rangle_\infty > \lambda^2) \leq \frac{2C}{\sqrt{1 - a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\rho + \tau} - M_{\rho})^-; \rho < \infty].
\]
Hence, by the last inequality and (11) we get the desired conclusion. \hfill \Box

3.2. Proof of (i). For any $u > 0$, introduce the stopping time

$$
\tau_u = \begin{cases} 
\inf \{ t \in \mathbb{R}_+ \mid -\lambda_0 M_t + A_t > u \} & \text{if } \{ \} \neq \emptyset \\
\infty & \text{if } \{ \} = \emptyset,
\end{cases}
$$

where

$$
A_t = \int_{(0,t] \times \{ [k-1, k) \}} [\phi_{\lambda_0}(x)] \tilde{\mu}(\cdot, ds \, dx), \quad t \in \mathbb{R}_+.
$$

Fix $u > 0$. We consider the process $M^{(u)} = \{ M_t^{(u)} \}_{t \in \mathbb{R}_+}$ defined as $M_t^{(u)} = M_{\tau_u \wedge t}$, $t \in \mathbb{R}_+$. Then it follows from the assumptions with respect to $M$ that $M^{(u)}$ is also a quasi left continuous and locally square integrable martingale which satisfying $M_0^{(u)} = 0$, $\langle M^{(u)} \rangle_{\infty} = \langle M \rangle_{\infty} < \infty$ a.s., and the uniform integrability of $\{ (M_t^{(u)})^- \}_{t \in T}$. Moreover, if we pick the random measure $\mu^{(u)}$ on $\Omega \times \mathbb{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbb{R}_+$ and Borel subsets $U$ of $\mathbf{X}$

$$
\mu^{(u)}(\cdot, (0, t] \times U) = \sum_{0 < s \leq t} 1_U(\Delta M_s^{(u)})
$$

and its compensator $\hat{\mu}^{(u)}$, then it follows that for all $t \in \mathbb{R}_+$ and Borel subsets $U$ of $\mathbf{X}$

$$
\mu^{(u)}(\cdot, (0, t] \times U) = \sum_{0 < s \leq \tau_u \wedge t} 1_U(\Delta M_s) = \mu^{(u)}(\cdot, (0, \tau_u \wedge t] \times U) \quad \text{a.s.,}
$$

and so $\hat{\mu}^{(u)}$ is the random measure on $\Omega \times \mathbb{R}_+ \times \mathbf{X}$ such that for all $t \in \mathbb{R}_+$ and Borel subsets $U$ of $\mathbf{X}$

$$
\hat{\mu}^{(u)}(\cdot, (0, t] \times U) = \hat{\mu}(\cdot, (0, \tau_u \wedge t] \times U) \quad \text{a.s.,}
$$

and therefore we can have that

$$
E\left[ e^{-\lambda_0 M_ \infty} \mathbf{1}_{L \times [1, \infty)} \left[ \phi_{\lambda_0}(x) \tilde{\mu}^{(u)}(\cdot, ds \, dx) \right] \right]
$$

$$
= E\left[ e^{-\lambda_0 M_ \infty + A_{\infty}} \right]
$$

$$
= E\left[ e^{-\lambda_0 M_{\tau_u \wedge t}} \wedge \tau_u < \infty \right] + E\left[ e^{-\lambda_0 M_{\tau_u \wedge t}} \wedge \tau_u = \infty \right]
$$

$$
\leq E\left[ e^{u - \lambda_0 \Delta M_{\tau_u \wedge t}} \wedge \tau_u < \infty \right] + e^u P(\tau_u = \infty)
$$

$$
= E\left[ e^{u - \lambda_0 \cdot 0} \wedge \tau_u < \infty \right] + e^u P(\tau_u = \infty) \quad (= e^u),
$$

where the fourth line of the above holds by the definition of $\tau_u$ and the last line does by the quasi left continuity of $M$. By applying Theorem 1.5 to the process $M^{(u)}$, we have

$$
-\infty < E[M_ \infty^{(u)}] \leq 0, \quad \lim_{\lambda \to \infty} \lambda P(\sqrt{\langle M^{(u)} \rangle_\infty} > \lambda) = -\sqrt{\frac{2}{\pi}} E[M_ \infty^{(u)}],
$$

$$
S. KAJI
$$
that is, \(-\infty < E[M_{\tau_u}] \leq 0\) and

\[
\lim_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{M_{\tau_u}} > \lambda) = -\frac{2}{\pi} E[M_{\tau_u}].
\]

Now we show

\[
\liminf_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{M_{\tau_u}} > \lambda) \geq -\frac{2}{\pi} E[M_{\infty}].
\]

Indeed, the left-hand side of (13) is

\[
\leq \liminf_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{M_{\tau_u}} > \lambda)
\]

and the right-hand side of (13) is

\[
= -\frac{2}{\pi} E[M_{\infty}; \tau_u = \infty] - \frac{2}{\pi} E[M_{\tau_u}; \tau_u < \infty] \\
\geq -\frac{2}{\pi} E[M_{\infty}; \tau_u = \infty] + \frac{2}{\pi} E\left[ \frac{\mu}{\lambda_0} - \frac{1}{\lambda_0} A_{\tau_u}; \tau_u < \infty \right] \\
\geq -\frac{2}{\pi} E[M_{\infty}; \tau_u = \infty] + \frac{1}{\lambda_0} \sqrt{\frac{2}{\pi}} E[A_{\tau_u}; \tau_u < \infty],
\]

where the second line of the above holds by the definition of \(\tau_u\). Also, the right-hand side of the above converges to \(-\sqrt{2/\pi} E[M_{\infty}]\) as \(u \to \infty\), because by the dominated convergence theorem, the fact \(E[|M_{\infty}|] < \infty\) we have known and the assumption (2) imply

\[
\lim_{u \to \infty} E[M_{\infty}; \tau_u = \infty] = E[M_{\infty}], \quad \lim_{u \to \infty} E[A_{\tau_u}; \tau_u < \infty] = 0.
\]

Therefore we can get (14).

On the other hand, we will show

\[
\limsup_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{M_{\tau_u}} > \lambda) \leq -\frac{2}{\pi} E[M_{\infty}].
\]

According to Lemma 3.1, we have for all \(0 < a < 1\)

\[
\limsup_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{M_{\tau_u}} > \lambda^2) \leq \frac{1}{a} \liminf_{\lambda \to \infty} \lambda \mathbb{P}(\sqrt{M_{\tau_u}} > \lambda^2) \\
+ \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in T} E[(M_{\tau_u} + \tau) - M_{\tau_u}]; \tau_u < \infty],
\]
where $C$ is a positive constant which does not depend on $a$ and $u$. Fix $0 < a < 1$. By (13) the first term on the right-hand side of the last inequality is

$$\frac{1}{a} \left( -\sqrt{\frac{2}{\pi}} E[M_{\infty}^{(u)}] \right).$$

Therefore

$$\limsup_{\lambda \to \infty} \lambda P(\langle M \rangle_{\infty} > \lambda^2) \leq \frac{1}{a} \left( -\sqrt{\frac{2}{\pi}} E[M_{\infty}^{(u)}] \right) + \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[(M_{\tau+\tau} - M_{\tau})^{-}; \tau_{u} < \infty].$$

By the definition of $\tau_u$ the second term on the right-hand side of the last inequality is

$$\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E\left[ \left( M_{\tau+\tau} + \frac{u}{\lambda_0} - \frac{1}{\lambda_0} A_{\tau} \right)^{-}; \tau_{u} < \infty \right]$$

$$\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E\left[ M_{\tau+\tau}^{-} + \frac{1}{\lambda_0} A_{\tau}; \tau_{u} < \infty \right]$$

$$\leq \frac{C}{\sqrt{1-a^2}} \sup_{\tau \in \mathcal{T}} E[M_{\infty}^{-}; \tau_{u} < \infty] + \frac{C}{\sqrt{1-a^2}} \frac{1}{\lambda_0} E[A_{\infty}; \tau_{u} < \infty].$$

By the uniform integrability of $\{M_{\tau}^{-}\}_{\tau \in \mathcal{T}}$ the first term on the right-hand side of the last inequality converges to 0 a.s. as $u \to \infty$ and from the dominated convergence theorem the assumption (2) implies that the second term of it does so, too. Therefore

$$\limsup_{\lambda \to \infty} \lambda P(\langle M \rangle_{\infty} > \lambda^2) \leq \limsup_{u \to \infty} \frac{1}{a} \left( -\sqrt{\frac{2}{\pi}} E[M_{\infty}^{(u)}] \right).$$

Moreover, the right-hand side of the last inequality is

$$\leq (1/a)(-\sqrt{2/\pi} E[M_{\infty}])$$

since $\liminf_{u \to \infty} E[M_{\tau}^{+}] \geq E[M_{\infty}^{+}]$ holds by the Fatou lemma and since $\lim_{u \to \infty} E[M_{\tau}^{-}] = E[M_{\infty}^{-}]$ holds by the uniform integrability of $\{M_{\tau}^{-}\}_{\tau \in \mathcal{T}}$. Therefore we can get (15).

Hence (14) and (15) imply the desired conclusion.

4. Appendix

Proposition 4.1. Assume that $M$ is a quasi left continuous and locally square integrable martingale. Then

$$\sup_{\lambda} \lambda P(\sqrt{\langle M \rangle_{\infty}} > \lambda) \leq C \sup_{\lambda} \lambda P\left( \sup_{t \to \infty} |M_t| > \lambda \right),$$

where $C$ is a universal positive constant.
Proof. Pick any stopping times $\rho$ and $\tau$ with $\rho \leq \tau$. First, it is clear that we can get

\begin{equation}
E[(\sqrt{M}_{\tau^-} - \sqrt{M}_{\rho^-})^2] \leq E[(M)_{\tau} - (M)_{\rho}].
\end{equation}

In fact, $(M)_{t}$ is continuous, since $M$ is quasi left continuous, and the inequality $(\sqrt{a} - \sqrt{b})^2 \leq a - b$ for $0 \leq b \leq a$ holds. Introduce the local martingale $N_t = M_{(\rho+t)\wedge \tau} - M_{\rho}, t < \infty$, and then we can see $(M)_{\tau} - (M)_{\rho} = (N)_{\infty}$. Therefore, (16) and the last result imply

\begin{equation}
E[(\sqrt{M}_{\tau^-} - \sqrt{M}_{\rho^-})^2] \leq E[(N)_{\infty}]
\end{equation}

where the last line of the last inequality holds by the property of a local martingale.

By the definition of $N$ we have

\begin{align*}
E \left[ \left( \sup_{t<\infty} |N_t| \right)^2 \right] &= E \left[ \left( \sup_{t<\infty} |N_t| \right)^2 ; \rho < \tau \right] \\
&\leq 2E \left[ \left( \sup_{t<\infty} |M_{t\wedge \tau}| \right)^2 + M_{\rho}^2 ; \rho < \tau \right] \\
&= 2E \left[ \left( \sup_{t<\infty} |M_t| \right)^2 + M_{\rho}^2 ; \rho < \tau = \infty \right] \\
&\quad + 2E \left[ \left( \sup_{t<\infty} |M_{t\wedge \tau}| \right)^2 + M_{\rho}^2 ; \rho < \tau < \infty \right]
\end{align*}

\begin{equation}
\leq 4E \left[ \left( \sup_{t<\infty} |M_t| \right)^2 ; \rho < \tau = \infty \right] \\
+ 2E \left[ \left( \sup_{t<\infty} |M_t| \right)^2 + \left( \sup_{t<\infty} |M_t| \right)^2 ; \rho < \tau < \infty \right]
\end{equation}

where the eighth line of the last inequality holds by the quasi left continuity of $t \rightarrow$
\[ \sup_{\tau \leq t} |M_t| . \] Hence, (17) and (18) imply
\[ E[(\sqrt{\langle M \rangle_\tau} - \sqrt{\langle M \rangle_\rho})^2] = 4E \left[ \left( \sup_{t < \tau} |M_t| \right)^2 ; \rho < \tau \right] . \]

Then, according to Corollary 6 of Azéma, Gundy, and Yor [1], the above implies the desired conclusion.

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**References**


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