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A REMARK ON SIMPLE SYMMETRIC SETS

Dedicated to Professor Yoshikazu Nakai on his 60th birthday

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Nobusawa [1] has shown that if A is a simple symmetric set then the group of displacements $H(A)$ is almost simple. The purpose of this note is to prove the converse. For the completeness we shall restate the result of Nobusawa in a slightly extended way.

A symmetric set is a set A carrying a binary operation $a\circ b$ which satisfies the following identical relations:

(1) $a\circ a=a$.
(2) $(x\circ a)\circ a=x$.
(3) $(x\circ y)\circ a=(x\circ a)\circ(y\circ a)$.

The mapping $\iota(a)\colon A\rightarrow A$ defined by $x^{\iota(a)}=x\circ a$ is an automorphism of A by (3) and we have the following:

(4) $a^{\iota(a)}=a$.
(5) $\iota(a)^2=1$.
(6) For any automorphism σ of A

$$\sigma^{-1}\iota(a)\sigma=\iota(a^{\sigma})\,.$$

Particularly we have

(7) $\iota(b)^{-1}\iota(a)\iota(b)=\iota(a^{\iota(b)})=\iota(a\circ b)$.

The group $G(A)$ is the subgroup of Aut A (the automorphism group of A) generated by $\iota(A)=\{\iota(a)\,|\,a\in A\}$. The group $H(A)$ is the subgroup of $G(A)$ generated by $\{\iota(a)\iota(b)\,|\,a,b\in A\}$ and is called the group of displacements. Then $|G(A)\colon H(A)|\leqq 2$ and $H(A)=\langle\iota(e)\iota(a)\,|\,a\in A\rangle$ for a fixed element e of A.

The set $\iota(A)$ is a collection of conjugate classes of involutions in $G(A)$, and is a symmetric set with the binary operation $\iota(a)\circ\iota(b)=\iota(b)^{-1}\iota(a)\iota(b)$. The mapping $\iota\colon A\rightarrow\iota(A)$ is an epimorphism, and if ι is an isomorphism then A is called *effective*.

If $G(A)$ acts transitively (or primitively) on A, then we call A *transitive* (or *primitive*). Note that A is transitive if and only if $H(A)$ is transitive on A, and if A is transitive then $\iota(A)$ is a conjugate class of involutions in $G(A)$.

Let f: $A \rightarrow B$ be a homomorphism of a symmetric set A to another symmetric set B. An inverse image $f^{-1}(b)$ for $b \in f(A)$ is called a *coset* of f. We call f *proper* if $|f(A)|>1$ and f is not a monomorphism. Thus f is proper if and only if there is a coset C such that $A \supsetneqq C$ and $|C|>1$.

A symmetric set A is called *simple* if $|A|>2$ and there is no proper homomorphism of A to another symmetric set. If A is simple then A is not trivial, where A is called *trivial* if $G(A)=1$.

Proposition 1. *Let A be a symmetric set and K a normal subgroup of $G(A)$. Denote the K-orbit containing $a \in A$ by $\bar{a}$ and let $\bar{A}=\{\bar{a} \mid a \in A\}$. Define a binary operation on $\bar{A}$ by $\bar{a} \circ \bar{b}=\overline{a \circ b}$. Then this is well defined and $\bar{A}$ is a symmetric set. Further the mapping f: $A \rightarrow \bar{A}$ $(a \mapsto \bar{a})$ is an epimorphism.*

Proof. Let $a'=a^{\sigma}$ and $b'=b^{\rho}$ for σ, $\rho \in K$. Then

$$\begin{aligned} a' \circ b' &= a^{\sigma} \circ b^{\rho} = (a \circ b^{\rho\sigma^{-1}})^{\sigma} \\ &= (((a \circ b) \circ b) \circ b^{\rho\sigma^{-1}})^{\sigma} \\ &= (a \circ b)^{\iota(b)\iota(b^{\rho\sigma^{-1}})\sigma}, \end{aligned}$$

where $\iota(b)\iota(b^{\rho\sigma^{-1}})\sigma=\iota(b)^{-1}(\rho\sigma^{-1})^{-1}\iota(b)(\rho\sigma^{-1})\sigma \in K$. Hence $\overline{a' \circ b'}=\overline{a \circ b}$ and the binary operation on $\bar{A}$ is well defined. The other parts are evident.

The epimorphism f: $A \rightarrow \bar{A}$ in the proposition above is called the *canonical epimorphism*.

Proposition 2. *If a symmetric set A is simple then it is transitive.*

Proof. Suppose that A is simple and intransitive. Let $\bar{a}$ be the $G(A)$-orbit containing $a \in A$ and $\bar{A}=\{\bar{a} \mid a \in A\}$. Then there is the canonical epimorphism f: $A \rightarrow \bar{A}$. Here $|\bar{A}|>1$ by the intransitivity. Hence f must be an isomorphism by simplicity and we have $a^{G(A)}=a$ for any $a \in A$. Thus $G(A)=1$, which is a contradiction.

Proposition 3. *If a symmetric set A is primitive, then A is simple.*

Proof. Suppose A is not simple. Then there is a proper epimorphism f: $A \rightarrow B$. Thus there exists a coset $C=f^{-1}(b)$ such that $A \supsetneqq C$ and $|C|>1$. Clearly C is a non-trivial set of imprimitivity of $G(A)$, and hence A is imprimitive.

Now we have the following

Theorem 1 (Nobusawa). *If A is a simple symmetric set, then $H(A)$ is a unique minimal normal subgroup of $G(A)$. Hence $H(A)$ is either a simple group or a direct product of two simple groups which are isomorphic.*

Proof. Let $K \neq 1$ be a normal subgroup of $G(A)$, $\bar{A}$ the symmetric set consisting of all K-orbits on A and let f: $A \rightarrow \bar{A}$ be the canonical epimorphism. Since $K \neq 1$ there is an element $a \in A$ such that $|a^K| > 1$, and hence f is not an isomorphism. Then by the simplicity of A we have $|\bar{A}| = 1$, that is, K is transitive on A. Thus for any $a, b \in A$ there is an element $\rho \in K$ such that $b = a^\rho$, and then $\iota(a)\iota(b) = \iota(a)^{-1}\rho^{-1}\iota(a)\rho \in K$. Therefore $K \supseteq H(A)$ and this shows that $H(A)$ is a unique minimal normal subgroup of $G(A)$. The second half of the theorem follows from the fact $|G(A)$: $H(A)| \leqq 2$.

Remark. Nobusawa [1] states the theorem under the assumption that A is primitive.

We also remark the following

Proposition 4. *If A is simple then it is effective.*

Proof. Consider the epimorphism ι: $A \rightarrow \iota(A)$. If $|\iota(A)| = 1$, then for any x, $a \in A$ $x^{\iota(a)} = x^{\iota(x)} = x$ and hence $G(A) = 1$, which contradicts the simplicity of A. Thus $|\iota(A)| > 1$ and hence ι is an isomorphism.

From Proposition 2 and 4, a simple symmetric set A may be regarded as a conjugate class of involutions in $G(A)$, where the group $G(A)$ has the property property as in Theorem 1.

We have also the following

Theorem 2. *Let A be a simple symmetric set. If A is imprimitive then $H(A)$ is a simple group.*

Proof. Let $G = G(A)$ and $H = H(A)$. we may assume that A is a set of involutions in G. Suppose H is not simple. Then $H = K \times K^a$ with $a \in A$ and K is simple. Let $L = \{kk^a | k \in K\}$. Then L is a subgroup of H, $C_G(a) \cap H = L$ and $C_G(a) = L + La$. We claim that $C_G(a)$ is a maximal subgroup of G and hence A is primitive. Suppose that there is a subgroup M of G which contains $C_G(a)$ properly. Then $M^* = M \cap H \gneqq L$ and hence there is an element kk_1^a in M^* such that $k \neq k_1$. Then $(kk^a)(kk_1^a)^{-1} = (kk_1^{-1})^a \in M^* \cap K^a$ and hence $M^* \cap K^a \neq 1$. Since the projection of M^* on K^a covers the whole of K^a, $M^* \cap K^a$ is a normal subgroup of K^a and hence we have $K^a \leqq M^*$. In the same way we have $K \leqq M^*$. Thus M contains H and we have $M = G$, which shows the maximality of $C_G(a)$.

To prove the converse of Theorem 1, we need the following

Proposition 5. *Let f: $A \rightarrow \bar{A}$ be an epimorphism of a symmetric set A to another symmetric set $\bar{A}$ and denote $f(a)$ by $\bar{a}$. Then there is an epimorphism f^*: $G(A) \rightarrow G(\bar{A})$ such that $\overline{\iota(a)} = \iota(\bar{a})$ and $\overline{a^\sigma} = \bar{a}^{\bar{\sigma}}$ for $a \in A$ and $\sigma \in G(A)$, where $\overline{\iota(a)}$ and $\bar{\sigma}$ denote $f^*(\iota(a))$ and $f^*(\sigma)$ respectively.*

Proof. For x, $a \in A$, $\overline{x \circ a} = \bar{x} \circ \bar{a}$, *i.e.* $\overline{x^{\iota(a)}} = \bar{x}^{\iota(\bar{a})}$. Hence for $\sigma = \iota(a_1) \cdots \iota(a_r)$ $\in G(A)$ $\overline{x^\sigma} = \bar{x}^{\iota(\bar{a}_1) \cdots \iota(\bar{a}_r)}$. Particularly if $\sigma = 1$ then $\iota(\bar{a}_1) \cdots \iota(\bar{a}_r) = 1$, and hence the mapping $f^*: G(A) \to G(\bar{A})$ defined by $f^*(\iota(a_1)\cdots\iota(a_r)) = \iota(\bar{a}_1)\cdots\iota(\bar{a}_r)$ is well defined. It is now easy to see that f^* is an epimorphism satisfying the conditions in the proposition.

The epimorphism f^* in the proposition is called the *extension* of f.

Now we have the following

Theorem 3. *Let G be a group, A a conjugate class of involutions in G and suppose $G=\langle A\rangle$. If $H=\langle ab \mid a, b\in A\rangle$ is a minimal normal subgroup of G and $|A|>2$, then the symmetric set A with binary operation $a\circ b=b^{-1}ab$ is simple.*

Proof. Suppose A is not simple and $|A|>2$. Then there is a proper epimorphism $f: A\to\bar{A}$. Denote $f(a)$ by $\bar{a}$. Then f can be extended to the group epimorphism $f^*: G(A)\to G(\bar{A})$. There is also a natural epimorphism $g: G\to G(A)$ such that $g(a_1\cdots a_r)=\iota(a_1)\cdots\iota(a_r)$ for $a_i\in A$ $(i=1, \cdots, r)$. Thus we have an epimorphism $f^{**}=f^*\circ g: G\to G(\bar{A})$, and we denote $f^{**}(\sigma)$ by $\bar{\sigma}$. Then we have

$$\overline{x^{\sigma}} = \bar{x}^{\bar{\sigma}}$$

for $x\in A$ and $\sigma\in G$.

Since $f^{**}(H)=H(\bar{A})$, f^{**} induces an epimorphism $f^{**}|_H: H\to H(\bar{A})$, and $\mathrm{Ker}\,(f^{**}|_H)$ is a normal subgroup of G which is contained in H.

Now since f is not an isomorphism there are different elements a and b of A such that $\bar{a}=\bar{b}$. Then $f^{**}(ab^{-1})=\iota(\bar{a})\iota(\bar{b})^{-1}=1$ and hence we have $1\neq ab^{-1}\in\mathrm{Ker}\,(f^{**}|_H)$. Thus $\mathrm{Ker}(f^{**}|_H)\neq 1$. Next we show that $H\neq \mathrm{Ker}(f^{**}|_H)$. Since $|\bar{A}|>1$ there are elements $a, b\in A$ such that $\bar{a}\neq\bar{b}$. Let $b=a^{\sigma}$ with $\sigma=c_1\cdots c_r$, $c_i\in A$ $(i=1, \cdots, r)$. Then there is an i such that $\overline{a^{c_1\cdots c_i}}=\bar{a}$ and $\overline{a^{c_1\cdots c_{i+1}}}\neq\bar{a}$. Let $d=a^{c_1\cdots c_i}$, $c=c_{i+1}$. Then $d, c\in A$ and $\bar{d}^{\bar{c}}\neq\bar{d}$. Since $f^{**}(dc)=\iota(\bar{d})\iota(\bar{c})$ takes $\bar{d}$ to $\bar{d}^{\iota(\bar{c})}=\bar{d}^{\bar{c}}$, $dc\in H-\mathrm{Ker}(f^{**}|_H)$.

Thus $\mathrm{Ker}(f^{**}|_H)$ is a non-trivial normal subgroup of G which is properly contained in H, and H is not a minimal normal subgroup of G.

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Reference

[1] N. Nobusawa: *Simple symmetric sets and simple groups,* Osaka J. Math. **14** (1977), 411–415.