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Osaka University
A REMARK ON SIMPLE SYMMETRIC SETS

Dedicated to Professor Yoshikazu Nakai on his 60th birthday

HIROSI NAGAO

(Received June 22, 1978)

Nobusawa [1] has shown that if \( A \) is a simple symmetric set then the group of displacements \( H(\Lambda) \) is almost simple. The purpose of this note is to prove the converse. For the completeness we shall restate the result of Nobusawa in a slightly extended way.

A symmetric set is a set \( A \) carrying a binary operation \( a \circ b \) which satisfies the following identical relations:

\[
\begin{align*}
(1) & \quad a \circ a = a. \\
(2) & \quad (x \circ a) \circ a = x. \\
(3) & \quad (x \circ y) \circ a = (x \circ a) \circ (y \circ a).
\end{align*}
\]

The mapping \( \iota(a): A \rightarrow A \) defined by \( x^{\iota(a)} = x \circ a \) is an automorphism of \( A \) by (3) and we have the following:

\[
\begin{align*}
(4) & \quad a^{\iota(a)} = a. \\
(5) & \quad \iota(a)^2 = 1. \\
(6) & \quad \text{For any automorphism } \sigma \text{ of } A \\
& \quad \sigma^{-1} \iota(a) \sigma = \iota(a^\sigma).
\end{align*}
\]

Particularly we have

\[
(7) \quad \iota(b)^{-1} \iota(a) \iota(b) = \iota(a^{\iota(b)}) = \iota(a \circ b).
\]

The group \( G(\Lambda) \) is the subgroup of \( \text{Aut} \ A \) (the automorphism group of \( A \)) generated by \( \iota(A) = \{ \iota(a) \mid a \in A \} \). The group \( H(\Lambda) \) is the subgroup of \( G(\Lambda) \) generated by \( \{ \iota(a) \circ \iota(b) \mid a, b \in A \} \) and is called the group of displacements. Then \( |G(\Lambda)|: H(\Lambda)| \leq 2 \) and \( H(\Lambda) = \langle \iota(e) \iota(a) \mid a \in A \rangle \) for a fixed element \( e \) of \( A \).

The set \( \iota(A) \) is a collection of conjugate classes of involutions in \( G(\Lambda) \), and is a symmetric set with the binary operation \( \iota(a) \circ \iota(b) = \iota(b)^{-1} \iota(a) \iota(b) \). The mapping \( \iota: A \rightarrow \iota(A) \) is an epimorphism, and if \( \iota \) is an isomorphism then \( A \) is called effective.

If \( G(\Lambda) \) acts transitively (or primitively) on \( A \), then we call \( A \) transitive (or primitive). Note that \( A \) is transitive if and only if \( H(\Lambda) \) is transitive on \( A \), and if \( A \) is transitive then \( \iota(A) \) is a conjugate class of involutions in \( G(\Lambda) \).
Let $f: A \to B$ be a homomorphism of a symmetric set $A$ to another symmetric set $B$. An inverse image $f^{-1}(b)$ for $b \in f(A)$ is called a coset of $f$. We call $f$ proper if $|f(A)| > 1$ and $f$ is not a monomorphism. Thus $f$ is proper if and only if there is a coset $C$ such that $A \cong C$ and $|C| > 1$.

A symmetric set $A$ is called simple if $|A| > 2$ and there is no proper homomorphism of $A$ to another symmetric set. If $A$ is simple then $A$ is not trivial, where $A$ is called trivial if $G(A) = 1$.

**Proposition 1.** Let $A$ be a symmetric set and $K$ a normal subgroup of $G(A)$. Denote the $K$-orbit containing $a \in A$ by $a$ and let $\bar{A} = \{a | a \in A\}$. Define a binary operation on $\bar{A}$ by $a \circ b = a \circ b$. Then this is well defined and $\bar{A}$ is a symmetric set. Further the mapping $f: A \to \bar{A}$ $(a \mapsto a)$ is an epimorphism.

**Proof.** Let $a' = a^\sigma$ and $b' = b^\rho$ for $\sigma, \rho \in K$. Then

$$a' \circ b' = a'^\sigma b'^\rho = (a \circ b)_{\sigma^{-1}}^\sigma = (a \circ b)^{i(\rho)}_{\sigma^{-1}}^\sigma,$$

where $i(b)_{\rho} (b_{\rho^{-1}})_{\sigma^{-1}} = i(b)^{-1} (i(b)^{-1})_{\rho^{-1}} i(b)(i(b)^{-1})_{\rho^{-1}} \in K$. Hence $a' \circ b' = a \circ b$ and the binary operation on $\bar{A}$ is well defined. The other parts are evident.

The epimorphism $f: A \to \bar{A}$ in the proposition above is called the canonical epimorphism.

**Proposition 2.** If a symmetric set $A$ is simple then it is transitive.

**Proof.** Suppose that $A$ is simple and intransitive. Let $a$ be the $G(A)$-orbit containing $a \in A$ and $\bar{A} = \{a | a \in A\}$. Then there is the canonical epimorphism $f: A \to \bar{A}$. Here $|\bar{A}| > 1$ by the intransitivity. Hence $f$ must be an isomorphism by simplicity and we have $a^{G(A)} = a$ for any $a \in A$. Thus $G(A) = 1$, which is a contradiction.

**Proposition 3.** If a symmetric set $A$ is primitive, then $A$ is simple.

**Proof.** Suppose $A$ is not simple. Then there is a proper epimorphism $f: A \to B$. Thus there exists a coset $C = f^{-1}(b)$ such that $A \cong C$ and $|C| > 1$. Clearly $C$ is a non-trivial set of imprimitivity of $G(A)$, and hence $A$ is imprimitive.

Now we have the following

**Theorem 1** (Nobusawa). If $A$ is a simple symmetric set, then $H(A)$ is a unique minimal normal subgroup of $G(A)$. Hence $H(A)$ is either a simple group or a direct product of two simple groups which are isomorphic.

**Proof.** Let $K \neq 1$ be a normal subgroup of $G(A)$, $\bar{A}$ the symmetric set con-
sisting of all $K$-orbits on $A$ and let $f: A \to \tilde{A}$ be the canonical epimorphism. Since $K \neq 1$ there is an element $a \in A$ such that $|a^K| > 1$, and hence $f$ is not an isomorphism. Then by the simplicity of $A$ we have $|\tilde{A}| = 1$, that is, $K$ is transitive on $A$. Thus for any $a, b \in A$ there is an element $\rho \in K$ such that $b = a^\rho$, and then $\iota(a)(b) = \iota(a)^{-1}\rho^{-1}\iota(a)\rho \in K$. Therefore $K \supseteq H(A)$ and this shows that $H(A)$ is a unique minimal normal subgroup of $G(A)$. The second half of the theorem follows from the fact $|G(A): H(A)| \leq 2$.

Remark. Nobusawa [1] states the theorem under the assumption that $A$ is primitive. We also remark the following

**Proposition 4.** If $A$ is simple then it is effective.

**Proof.** Consider the epimorphism $\iota: A \to \iota(A)$. If $|\iota(A)| = 1$, then for any $x, a \in A$, $x^{\iota(a)} = x^{\iota(\iota(a))} = x$ and hence $G(A) = 1$, which contradicts the simplicity of $A$. Thus $|\iota(A)| > 1$ and hence $\iota$ is an isomorphism.

From Proposition 2 and 4, a simple symmetric set $A$ may be regarded as a conjugate class of involutions in $G(A)$, where the group $G(A)$ has the property as in Theorem 1.

We have also the following

**Theorem 2.** Let $A$ be a simple symmetric set. If $A$ is imprimitive then $H(A)$ is a simple group.

**Proof.** Let $G = G(A)$ and $H = H(A)$. we may assume that $A$ is a set of involutions in $G$. Suppose $H$ is not simple. Then $H = K \times K^a$ with $a \in A$ and $K$ is simple. Let $L = \{kk^a | k \in K\}$. Then $L$ is a subgroup of $H$, $C_0(a) \cap H = L$ and $C_0(a) = L + La$. We claim that $C_0(a)$ is a maximal subgroup of $G$ and hence $A$ is primitive. Suppose that there is a subgroup $M$ of $G$ which contains $C_0(a)$ properly. Then $M = M \cap H \subseteq L$ and hence there is an element $kk^a$ in $M^*$ such that $k \neq k^a$. Then $(kk^a)(kk^a)^{-1} = (kk^a)^{-1} \in M^* \cap K^a$ and hence $M^* \cap K^a + 1$. Since the projection of $M^*$ on $K^a$ covers the whole of $K^a$, $M^* \cap K^a$ is a normal subgroup of $K^a$ and hence we have $K^a \leq M^*$. In the same way we have $K^a \leq M^*$. Thus $M$ contains $H$ and we have $M = G$, which shows the maximality of $C_0(a)$.

To prove the converse of Theorem 1, we need the following

**Proposition 5.** Let $f: A \to \tilde{A}$ be an epimorphism of a symmetric set $A$ to another symmetric set $\tilde{A}$ and denote $f(a)$ by $a$. Then there is an epimorphism $f^*: G(A) \to G(\tilde{A})$ such that $\iota(a) = \iota(f(a))$ and $\sigma = \sigma^a$ for $a \in A$ and $\sigma \in G(A)$, where $\iota(a)$ and $\sigma$ denote $f^*(\iota(a))$ and $f^*(\sigma)$ respectively.

**Proof.** For $x, a \in A$, $x^a = x^a$, i.e. $x^{\iota(a)} = x^{\iota(a)}$. Hence for $\sigma = \iota(a_1) \cdots \iota(a_r) \in G(A)$ $x^{\iota(a_1) \cdots \iota(a_r)} \in G(A)$ $x^{\iota(a_1) \cdots \iota(a_r)} = x^{\iota(a_1) \cdots \iota(a_r)}$. Particularly if $\sigma = 1$ then $\iota(a_1) \cdots \iota(a_r) = 1$, and hence the
mapping \( f^*: G(A) \to G(\bar{A}) \) defined by \( f^*(\iota(a_1) \cdots \iota(a_r)) = \iota(a_1) \cdots \iota(a_r) \) is well-defined. It is now easy to see that \( f^* \) is an epimorphism satisfying the conditions in the proposition.

The epimorphism \( f^* \) in the proposition is called the extension of \( f \).

Now we have the following

**Theorem 3.** Let \( G \) be a group, \( A \) a conjugate class of involutions in \( G \) and suppose \( G = \langle A \rangle \). If \( H = \langle ab \mid a, b \in A \rangle \) is a minimal normal subgroup of \( G \) and \( |A| > 2 \), then the symmetric set \( A \) with binary operation \( a \circ b = b^{-1}a^r ab \) is simple.

Proof. Suppose \( A \) is not simple and \( |A| > 2 \). Then there is a proper epimorphism \( f: A \to A \). Denote \( f(a) \) by \( a \). Then \( f \) can be extended to the group epimorphism \( f^*: G(A) \to G(\bar{A}) \). There is also a natural epimorphism \( g: G \to G(A) \) such that \( g(a_1 \cdots a_r) = \iota(a_1) \cdots \iota(a_r) \) for \( a_i \in A \) (\( i = 1, \ldots, r \)). Thus we have an epimorphism \( f** = f^* \circ g: G \to G(\bar{A}) \), and we denote \( f**(\sigma) \) by \( \sigma \). Then we have

\[
\sigma^x = x^\sigma
\]

for \( x \in A \) and \( \sigma \in G \).

Since \( f**(H) = H(\bar{A}) \), \( f** \) induces an epimorphism \( f**|_H: H \to H(\bar{A}) \), and \( \text{Ker}(f**|_H) \) is a normal subgroup of \( G \) which is contained in \( H \).

Now since \( f \) is not an isomorphism there are different elements \( a \) and \( b \) of \( A \) such that \( a = b \). Then \( f**(ab^{-1}) = \iota(a)\iota(b)^{-1} = 1 \) and hence we have \( 1 \neq ab^{-1} \in \text{Ker}(f**|_H) \). Thus \( \text{Ker}(f**|_H) \neq 1 \). Next we show that \( H \neq \text{Ker}(f**|_H) \). Since \( |\bar{A}| > 1 \) there are elements \( a, b \in A \) such that \( a \neq b \). Let \( b = a^\sigma \) with \( \sigma = c_1 \cdots c_r, c_i \in A \) (\( i = 1, \ldots, r \)). Then there is an \( i \) such that \( a^{c_i^{-1}c_i} = a \) and \( a^{c_i^{-1}c_{i+1}} = a \). Let \( d = a^{c_i^{-1}c_i}, c = c_{i+1} \). Then \( d, c \in A \) and \( \bar{d}^c = \bar{d} \). Since \( f**(dc) = \iota(\bar{d})\iota(c) \) takes \( \bar{d} \) to \( \bar{d}^{c_i} = \bar{d}^c \), \( dc \in H \) is not a minimal normal subgroup of \( G \).

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Reference