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## SPECTRA AND SYMMETRIC EIGENTENSORS OF THE LICHNEROWICZ LAPLACIAN ON $S^n$

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### Abstract

We compute the eigenvalues with multiplicities of the Lichnerowicz Laplacian acting on the space of symmetric covariant tensor fields on the Euclidian sphere  $S^n$ . The spaces of symmetric eigentensors are explicitly given.

### 1. Introduction

Let  $(M, g)$  be a Riemannian  $n$ -manifold. For any  $p \in \mathbb{N}$ , we shall denote by  $\Gamma(\otimes^p T^*M)$ ,  $\Omega^p(M)$  and  $\mathcal{S}^p M$  the space of covariant  $p$ -tensor fields on  $M$ , the space of differential  $p$ -forms on  $M$  and the space of symmetric covariant  $p$ -tensor fields on  $M$ , respectively. Note that  $\Gamma(\otimes^0 T^*M) = \Omega^0(M) = \mathcal{S}^0 M = C^\infty(M, \mathbb{R})$ ,  $\Omega(M) = \sum_{p=0}^n \Omega^p(M)$  and  $\mathcal{S}(M) = \sum_{p \geq 0} \mathcal{S}^p(M)$ .

Let  $D$  be the Levi-Civita connection associated to  $g$ ; its curvature tensor field  $R$  is given by

$$R(X, Y)Z = D_{[X, Y]}Z - (D_X D_Y Z - D_Y D_X Z),$$

and the Ricci endomorphism field  $r: TM \rightarrow TM$  is given by

$$g(r(X), Y) = \sum_{i=1}^n g(R(X, E_i)Y, E_i),$$

where  $(E_1, \dots, E_n)$  is any local orthonormal frame.

For any  $p \in \mathbb{N}$ , the connection  $D$  induces a differential operator  $D: \Gamma(\otimes^p T^*M) \rightarrow \Gamma(\otimes^{p+1} T^*M)$  given by

$$\begin{aligned} DT(X, Y_1, \dots, Y_p) &= D_X T(Y_1, \dots, Y_p) \\ &= X.T(Y_1, \dots, Y_p) - \sum_{j=1}^p T(Y_1, \dots, D_X Y_j, \dots, Y_p). \end{aligned}$$

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Its formal adjoint  $D^* : \Gamma(\otimes^{p+1} T^*M) \rightarrow \Gamma(\otimes^p T^*M)$  is given by

$$D^*T(Y_1, \dots, Y_p) = - \sum_{j=1}^n D_{E_j} T(E_j, Y_1, \dots, Y_p),$$

where  $(E_1, \dots, E_n)$  is any local orthonormal frame.

Recall that, for any differential  $p$ -form  $\alpha$ , we have

$$(1) \quad d\alpha(X_1, \dots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} D_{X_j} \alpha(X_1, \dots, \hat{X}_j, \dots, X_{p+1}).$$

We denote by  $\delta$  the restriction of  $D^*$  to  $\Omega(M) \oplus \mathcal{S}(M)$  and we define  $\delta^* : \mathcal{S}^p(M) \rightarrow \mathcal{S}^{p+1}(M)$  by

$$\delta^*T(X_1, \dots, X_{p+1}) = \sum_{j=1}^{p+1} D_{X_j} T(X_1, \dots, \hat{X}_j, \dots, X_{p+1}).$$

Recall that the operator trace  $\text{Tr} : \mathcal{S}^p(M) \rightarrow \mathcal{S}^{p-2}(M)$  is given by

$$\text{Tr} T(X_1, \dots, X_{p-2}) = \sum_{j=1}^n T(E_j, E_j, X_1, \dots, X_{p-2}),$$

where  $(E_1, \dots, E_n)$  is any local orthonormal frame.

The Lichnerowicz Laplacian is the second order differential operator

$$\Delta_M : \Gamma\left(\otimes^p T^*M\right) \rightarrow \Gamma\left(\otimes^p T^*M\right)$$

given by

$$\Delta_M(T) = D^*D(T) + R(T),$$

where  $R(T)$  is the curvature operator given by

$$\begin{aligned} R(T)(Y_1, \dots, Y_p) &= \sum_{j=1}^p T(Y_1, \dots, r(Y_j), \dots, Y_p) \\ &\quad - \sum_{i < j} \sum_{l=1}^n \{T(Y_1, \dots, E_l, \dots, R(Y_i, E_l)Y_j, \dots, Y_p) \\ &\quad \quad + T(Y_1, \dots, R(Y_j, E_l)Y_i, \dots, E_l, \dots, Y_p)\}, \end{aligned}$$

where  $(E_1, \dots, E_n)$  is any local orthonormal frame and, in

$$T(Y_1, \dots, E_l, \dots, R(Y_i, E_l)Y_j, \dots, Y_p),$$

$E_l$  takes the place of  $Y_i$  and  $R(Y_i, E_l)Y_j$  takes the place of  $Y_j$ .

This differential operator, introduced by Lichnerowicz in [15] pp.26, is self-adjoint, elliptic and respects the symmetries of tensor fields. In particular,  $\Delta_M$  leaves invariant  $S(M)$  and the restriction of  $\Delta_M$  to  $\Omega(M)$  coincides with the Hodge-de Rham Laplacian, i.e., for any differential  $p$ -form  $\alpha$ ,

$$(2) \quad \Delta_M \alpha = (d\delta + \delta d)(\alpha).$$

We have shown in [6] that, for any symmetric covariant tensor field  $T$ ,

$$(3) \quad \Delta_M(T) = (\delta \circ \delta^* - \delta^* \delta)(T) + 2R(T).$$

Note that if  $T \in S(M)$  and  $g^l$  denotes the symmetric product of  $l$  copies of the Riemannian metric  $g$ , we have

$$(4) \quad (\text{Tr} \circ \Delta_M)T = (\Delta_M \circ \text{Tr})T,$$

$$(5) \quad \Delta_M(T \odot g^l) = (\Delta_M T) \odot g^l,$$

where  $\odot$  is the symmetric product.

The Lichnerowicz Laplacian acting on symmetric covariant tensor fields is of fundamental importance in mathematical physics (see for instance [9], [20] and [22]). Note also that the Lichnerowicz Laplacian acting on symmetric covariant 2-tensor fields appears in many problems in Riemannian geometry (see [3], [5], [19], . . .).

On a compact Riemannian manifold, the Lichnerowicz Laplacian  $\Delta_M$  has discrete eigenvalues with finite multiplicities. For a given compact Riemannian manifold, it may be an interesting problem to determine explicitly the eigenvalues and the eigentensors of  $\Delta_M$  on  $M$ .

Let us enumerate the cases where the spectra of  $\Delta_M$  was computed:

1.  $\Delta_M$  acting on  $C^\infty(M, \mathbb{C})$ :  $M$  is either flat torus or Klein bottles [4],  $M$  is a Hopf manifolds [1];
2.  $\Delta_M$  acting on  $\Omega(M)$ :  $M = S^n$  or  $P^n(\mathbb{C})$  [10] and [11],  $M = \mathbb{C}aP^2$  or  $G_2/SO(4)$  [16] and [18],  $M = SO(n+1)/SO(2) \times SO(n)$  or  $M = Sp(n+1)/Sp(1) \times Sp(n)$  [21];
3.  $\Delta_M$  acting on  $S^2(M)$  and  $M$  is the complex projective space  $P^2(\mathbb{C})$  [22];
4.  $\Delta_M$  acting on  $S^2(M)$  and  $M$  is either  $S^n$  or  $P^n(\mathbb{C})$  [6] and [7];
5. Brian and Richard Millman give in [2] a theoretical method for computing the spectra of Lichnerowicz Laplacian acting on  $\Omega(G)$  where  $G$  is a compact semisimple Lie group endowed with the biinvariant metric induced from the negative of the Killing form;
6. Some partial results where given in [12]–[14].

In this paper, we compute the eigenvalues and we determine the spaces of eigentensors of  $\Delta_M$  acting on  $S(M)$  in the case where  $M$  is the Euclidian sphere  $S^n$ .

Let us describe our method briefly. We consider the  $(n + 1)$ -Euclidian space  $\mathbb{R}^{n+1}$  with its canonical coordinates  $(x_1, \dots, x_{n+1})$ . For any  $k, p \in \mathbb{N}$ , we denote by  $S^p H_k^\delta$  the space of symmetric covariant  $p$ -tensor fields  $T$  on  $\mathbb{R}^{n+1}$  satisfying:

1.  $T = \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n+1} T_{i_1, \dots, i_p} dx_{i_1} \odot \dots \odot dx_{i_p}$  where  $T_{i_1, \dots, i_p}$  are homogeneous polynomials of degree  $k$ ;
2.  $\delta(T) = \Delta_{\mathbb{R}^{n+1}}(T) = 0$ .

The  $n$ -dimensional sphere  $S^n$  is the space of unitary vectors in  $\mathbb{R}^{n+1}$  and the Euclidian metric on  $\mathbb{R}^{n+1}$  induces a Riemannian metric on  $S^n$ . We denote by  $i : S^n \hookrightarrow \mathbb{R}^{n+1}$  the canonical inclusion.

For any tensor field  $T \in \Gamma(\otimes^p T^* \mathbb{R}^{n+1})$ , we compute  $i^*(\Delta_{\mathbb{R}^{n+1}} T) - \Delta_{S^n}(i^* T)$  and get a formula (see Theorem 2.1). Inspired by this formula and having in mind the fact that  $i^* : \sum_{k \geq 0} S^p H_k^\delta \rightarrow S^p S^n$  is injective and its image is dense in  $S^p S^n$  (see [10]), we give, for any  $k$ , a direct sum decomposition of  $S^p H_k^\delta$  composed by eigenspaces of  $\Delta_{S^n}$ . Thus we obtain the eigenvalues and the spaces of eigentensors with its multiplicities of  $\Delta_{S^n}$  acting on  $S(S^n)$  (see Section 4).

Note that the eigenvalues and the eigenspaces of  $\Delta_{S^n}$  acting on  $\Omega(S^n)$  was computed in [10] by using the representation theory. In [11], I. Iwasaki and K. Katase recover the result by a method using the restriction of harmonic tensor fields and a result in [8]. The formula obtained in Theorem 2.1 combined with the methods developed in [10] and [11] permit to present those results in a more precise form (see Section 3).

### 2. A relation between $\Delta_{\mathbb{R}^{n+1}}$ and $\Delta_{S^n}$

We consider the Euclidian space  $\mathbb{R}^{n+1}$  endowed with its canonical coordinates  $(x_1, \dots, x_{n+1})$  and its canonical Euclidian flat Riemannian metric  $\langle \cdot, \cdot \rangle$ . We denote by  $D$  be the Levi-Civita covariant derivative associated to  $\langle \cdot, \cdot \rangle$ . We consider the radial vector field given by

$$\vec{r} = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}.$$

For any  $p$ -tensor field  $T \in \Gamma(\otimes^p T^* \mathbb{R}^{n+1})$  and for any  $1 \leq i < j \leq p$ , we denote by  $i_{\vec{r}, j} T$  the  $(p - 1)$ -tensor field given by

$$i_{\vec{r}, j} T(X_1, \dots, X_{p-1}) = T(X_1, \dots, X_{j-1}, \vec{r}, X_j, \dots, X_{p-1}),$$

and by  $\text{Tr}_{i,j} T$  the  $(p - 2)$ -tensor field given by

$$\begin{aligned} & \text{Tr}_{i,j} T(X_1, \dots, X_{p-2}) \\ &= \sum_{l=1}^{n+1} T(X_1, \dots, X_{i-1}, E_l, X_i, \dots, X_{j-2}, E_l, X_{j-1}, \dots, X_{p-2}), \end{aligned}$$

where  $(E_1, \dots, E_{n+1})$  is any orthonormal basis of  $\mathbb{R}^{n+1}$ . Note that  $\text{Tr}_{i,j} T = 0$  if  $T$  is a differential form and  $\text{Tr}_{i,j} T = \text{Tr} T$  if  $T$  is symmetric.

For any permutation  $\sigma$  of  $\{1, \dots, p\}$ , we denote by  $T^\sigma$  the  $p$ -tensor field

$$T^\sigma(X_1, \dots, X_p) = T(X_{\sigma(1)}, \dots, X_{\sigma(p)}).$$

For  $1 \leq i < j \leq p$ , the transposition of  $(i, j)$  is the permutation  $\sigma_{i,j}$  of  $\{1, \dots, p\}$  such that  $\sigma_{i,j}(i) = j$ ,  $\sigma_{i,j}(j) = i$  and  $\sigma_{i,j}(k) = k$  for  $k \neq i, j$ . Let  $\mathcal{T}$  denote the set of the transpositions of  $\{1, \dots, p\}$ .

The sphere  $i: S^n \hookrightarrow \mathbb{R}^{n+1}$  is endowed with the Euclidian metric.

**Theorem 2.1.** *Let  $T$  be a covariant  $p$ -tensor field on  $\mathbb{R}^{n+1}$ . Then,*

$$\begin{aligned} & i^*(\Delta_{\mathbb{R}^{n+1}} T) \\ &= \Delta_{S^n} i^* T + i^* \left( p(1 - p)T + (2p - n + 1)L_{\vec{r}} T - L_{\vec{r}} \circ L_{\vec{r}} T - 2 \sum_{\sigma \in \mathcal{T}} T^\sigma + O(T) \right), \end{aligned}$$

where  $O(T)$  is given by

$$\begin{aligned} O(T)(X_1, \dots, X_p) &= 2 \sum_{i < j} \langle X_i, X_j \rangle \text{Tr}_{i,j}(X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p) \\ &\quad - 2 \sum_{j=1}^p D_{X_j}(i_{\vec{r},j} T)(X_1, \dots, \hat{X}_j, \dots, X_p), \end{aligned}$$

where the symbol  $\hat{\phantom{x}}$  means that the term is omitted.

**Proof.** The proof is a massive computation in a local orthonormal frame using the properties of the Riemannian embedding of the sphere in the Euclidian space.

We choose a local orthonormal frame of  $\mathbb{R}^{n+1}$  of the form  $(E_1, \dots, E_n, N)$  such that  $E_i$  is tangent to  $S^n$  for  $1 \leq i \leq n$  and  $N = (1/r)\vec{r}$  where  $r = \sqrt{x_1^2 + \dots + x_{n+1}^2}$ .

For any vector field  $X$  on  $\mathbb{R}^{n+1}$ , we have

$$(6) \quad D_X N = \frac{1}{r}(X - \langle X, N \rangle N),$$

$$(7) \quad D_N X = [N, X] + \frac{1}{r}(X - \langle X, N \rangle N).$$

Let  $\nabla$  be the Levi-Civita connexion of the Riemannian metric on  $S^n$ . We have, for any vector fields  $X, Y$  tangent to  $S^n$ ,

$$(8) \quad D_X Y = \nabla_X Y - \langle X, Y \rangle N.$$

Let  $T$  be a covariant  $p$ -tensor field on  $\mathbb{R}^{n+1}$  and  $(X_1, \dots, X_p)$  a family of vector fields on  $\mathbb{R}^{n+1}$  which are tangent to  $S^n$ . A direct calculation using the definition of the Lichnerowicz Laplacian gives

$$\begin{aligned} \Delta_{\mathbb{R}^{n+1}}(T)(X_1, \dots, X_p) &= D^* D(T)(X_1, \dots, X_p) \\ &= \sum_{i=1}^n \left( -E_i E_i \cdot T(X_1, \dots, X_p) + 2 \sum_{j=1}^p E_i \cdot T(X_1, \dots, D_{E_i} X_j, \dots, X_p) \right. \\ &\quad + D_{E_i} E_i \cdot T(X_1, \dots, X_p) - \sum_{j=1}^p T(X_1, \dots, D_{D_{E_i} E_i} X_j, \dots, X_p) \\ &\quad - \sum_{j=1}^p T(X_1, \dots, D_{E_i} D_{E_i} X_j, \dots, X_p) \\ &\quad \left. - 2 \sum_{l < j} T(X_1, \dots, D_{E_i} X_l, \dots, D_{E_i} X_j, \dots, X_p) \right) \\ &\quad - N \cdot N \cdot T(X_1, \dots, X_p) + 2 \sum_{j=1}^p N \cdot T(X_1, \dots, D_N X_j, \dots, X_p) \\ &\quad + D_N N \cdot T(X_1, \dots, X_p) - \sum_{j=1}^p T(X_1, \dots, D_{D_N N} X_j, \dots, X_p) \\ &\quad - \sum_{j=1}^p T(X_1, \dots, D_N D_N X_j, \dots, X_p) - 2 \sum_{l < j} T(X_1, \dots, D_N X_l, \dots, D_N X_j, \dots, X_p). \end{aligned}$$

(6)–(8) make it obvious that

$$(9) \quad \begin{aligned} D_{D_{E_i} E_i} X_j &= \nabla_{\nabla_{E_i} E_i} X_j - \langle \nabla_{E_i} E_i, X_j \rangle N - [N, X_j] \\ &\quad - \frac{1}{r} (X_j - \langle X_j, N \rangle N), \end{aligned}$$

$$(10) \quad \begin{aligned} D_{E_i} D_{E_i} X_j &= \nabla_{E_i} \nabla_{E_i} X_j - (\langle E_i, \nabla_{E_i} X_j \rangle + E_i \cdot \langle E_i, X_j \rangle) N \\ &\quad - \frac{1}{r} \langle E_i, X_j \rangle E_i, \end{aligned}$$

$$(11) \quad \begin{aligned} D_N D_N X &= [N, [N, X]] + \frac{2}{r} [N, X] + \left( \frac{1}{r^2} - \frac{1}{r} \right) (X - \langle X, N \rangle N) \\ &\quad - \frac{2}{r} N \cdot \langle X, N \rangle N. \end{aligned}$$

By (8)–(10), we get easily, in restriction to  $S^n$ ,

$$\begin{aligned}
& \sum_{i=1}^n \left( 2 \sum_{j=1}^p E_i \cdot T(X_1, \dots, D_{E_i} X_j, \dots, X_p) + D_{E_i} E_i \cdot T(X_1, \dots, X_p) \right. \\
& \quad \left. - \sum_{j=1}^p T(X_1, \dots, D_{D_{E_i} E_i} X_j, \dots, X_p) - \sum_{j=1}^p T(X_1, \dots, D_{E_i} D_{E_i} X_j, \dots, X_p) \right) \\
&= \sum_{i=1}^n \left( 2 \sum_{j=1}^p E_i \cdot T(X_1, \dots, \nabla_{E_i} X_j, \dots, X_p) + \nabla_{E_i} E_i \cdot T(X_1, \dots, X_p) \right. \\
& \quad \left. - \sum_{j=1}^p T(X_1, \dots, \nabla_{\nabla_{E_i} E_i} X_j, \dots, X_p) - \sum_{j=1}^p T(X_1, \dots, \nabla_{E_i} \nabla_{E_i} X_j, \dots, X_p) \right) \\
& \quad - 2 \sum_{j=1}^p X_j \cdot T(X_1, \dots, \overbrace{N}^j, \dots, X_p) + p(n+1)T(X_1, \dots, X_p) - nL_N T(X_1, \dots, X_p).
\end{aligned}$$

On other hand, also by using (8), we have

$$\begin{aligned}
& \sum_{l < j} \sum_{i=1}^n T(X_1, \dots, D_{E_i} X_l, \dots, D_{E_i} X_j, \dots, X_p) \\
&= \sum_{l < j} \sum_{i=1}^n T(X_1, \dots, D_{E_i} X_l, \dots, \nabla_{E_i} X_j, \dots, X_p) - \sum_{l < j} T(X_1, \dots, D_{X_j} X_l, \dots, \overbrace{N}^j, \dots, X_p) \\
&= \sum_{l < j} \sum_{i=1}^n T(X_1, \dots, \nabla_{E_i} X_l, \dots, \nabla_{E_i} X_j, \dots, X_p) - \sum_{l < j} T(X_1, \dots, \overbrace{N}^l, \dots, \nabla_{X_l} X_j, \dots, X_p) \\
& \quad - \sum_{l < j} T(X_1, \dots, D_{X_j} X_l, \dots, \overbrace{N}^j, \dots, X_p) \\
&= \sum_{l < j} \sum_{i=1}^n T(X_1, \dots, \nabla_{E_i} X_l, \dots, \nabla_{E_i} X_j, \dots, X_p) - \sum_{l < j} T(X_1, \dots, D_{X_j} X_l, \dots, \overbrace{N}^j, \dots, X_p) \\
& \quad - \sum_{l < j} T(X_1, \dots, \overbrace{N}^l, \dots, D_{X_l} X_j, \dots, X_p) \\
& \quad - \sum_{l < j} \langle X_l, X_j \rangle T(X_1, \dots, \overbrace{N}^l, \dots, \overbrace{N}^j, \dots, X_p).
\end{aligned}$$



So we get, in restriction to  $S^n$ , since  $D_N N = 0$

$$\begin{aligned} & \Delta_{\mathbb{R}^{n+1}}(X_1, \dots, X_p) - \nabla^* \nabla T(X_1, \dots, X_p) \\ &= p(n+1)T(X_1, \dots, X_p) - nL_N T(X_1, \dots, X_p) - 2 \sum_{j=1}^p D_{X_j}(i_{N,j} T)(X_1, \dots, \hat{X}_j, \dots, X_p) \\ &+ 2 \sum_{l < j} \langle X_l, X_j \rangle T(X_1, \dots, \overbrace{N}^l, \dots, \overbrace{N}^j, \dots, X_p) - N.N.T(X_1, \dots, X_p) \\ &+ 2 \sum_{j=1}^p N.T(X_1, \dots, D_N X_j, \dots, X_p) - \sum_{j=1}^p T(X_1, \dots, D_N D_N X_j, \dots, X_p) \\ &- 2 \sum_{i < j} T(X_1, \dots, D_N X_i, \dots, D_N X_j, \dots, X_p). \end{aligned}$$

Remark that, in restriction to  $S^n$ , the following equality holds

$$\sum_{j=1}^p D_{X_j}(i_{N,j} T)(X_1, \dots, \hat{X}_j, \dots, X_p) = \sum_{j=1}^p D_{X_j}(i_{\tilde{r},j} T)(X_1, \dots, \hat{X}_j, \dots, X_p).$$

Now by using (7) and (11) and by taking the restriction to  $S^n$ , we have

$$\begin{aligned} & 2 \sum_{j=1}^p N.T(X_1, \dots, D_N X_j, \dots, X_p) \\ &= 2 \sum_{j=1}^p N.T(X_1, \dots, [N, X_j], \dots, X_p) + 2 \sum_{j=1}^p N \left( \frac{1}{r} \right) T(X_1, \dots, X_j, \dots, X_p) \\ &- 2 \sum_{j=1}^p N(\langle X_j, N \rangle) T(X_1, \dots, \overbrace{N}^j, \dots, X_p) + 2 \sum_{j=1}^p N.T(X_1, \dots, X_j, \dots, X_p) \\ &= 2 \sum_{j=1}^p N.T(X_1, \dots, [N, X_j], \dots, X_p) - 2pT(X_1, \dots, X_p) + 2pN.T(X_1, \dots, X_j, \dots, X_p) \\ &- 2 \sum_{j=1}^p N(\langle X_j, N \rangle) T(X_1, \dots, \overbrace{N}^j, \dots, X_p). \\ & \sum_{j=1}^p T(X_1, \dots, D_N D_N X_j, \dots, X_p) \\ &= \sum_{j=1}^p T(X_1, \dots, [N, [N, X_j], \dots, X_p) - 2 \sum_{j=1}^p N(\langle X_j, N \rangle) T(X_1, \dots, \overbrace{N}^j, \dots, X_p). \end{aligned}$$

$$\begin{aligned} & \sum_{i < j} T(X_1, \dots, D_N X_i, \dots, D_N X_j, \dots, X_p) \\ &= \sum_{i < j} T(X_1, \dots, [N, X_i], \dots, [N, X_j], \dots, X_p) + \frac{p(p-1)}{2} T(X_1, \dots, X_p) \\ & \quad + \sum_{i < j} T(X_1, \dots, X_i, \dots, [N, X_j], \dots, X_p) + \sum_{i < j} T(X_1, \dots, [N, X_i], \dots, X_j, \dots, X_p). \end{aligned}$$

So we get, in restriction to  $S^n$

$$\begin{aligned} & -N \cdot N \cdot T(X_1, \dots, X_p) + 2 \sum_{j=1}^p N \cdot T(X_1, \dots, D_N X_j, \dots, X_p) \\ & - \sum_{j=1}^p T(X_1, \dots, D_N D_N X_j, \dots, X_p) - 2 \sum_{i < j} T(X_1, \dots, D_N X_i, \dots, D_N X_j, \dots, X_p) \\ & = -L_N \circ L_N T(X_1, \dots, X_p) + 2p L_N T(X_1, \dots, X_p) - p(1+p) T(X_1, \dots, X_p). \end{aligned}$$

The curvature of  $S^n$  is given by

$$R(X, Y)Z = \langle X, Y \rangle Z - \langle Y, Z \rangle X$$

and

$$r(X) = (n-1)X.$$

Hence, a direct computation gives that the curvature operator is given by

$$\begin{aligned} R(T)(X_1, \dots, X_p) &= p(n-1)T(X_1, \dots, X_p) + 2 \sum_{\sigma \in \mathcal{T}} T^\sigma(X_1, \dots, X_p) \\ & \quad - 2 \sum_{i < j} \sum_{l=1}^n \langle X_i, X_j \rangle T(X_1, \dots, E_l, \dots, E_l, \dots, X_p). \end{aligned}$$

Finally, we get

$$\begin{aligned} i^*(\Delta_{\mathbb{R}^{n+1}} T) &= \Delta_{S^n} i^* T \\ & \quad + i^* \left( p(1-p)T + (2p-n)L_N T - L_N \circ L_N T - 2 \sum_{\sigma \in \mathcal{T}} T^\sigma + O(T) \right), \end{aligned}$$

One can conclude the proof by remarking that

$$i^*(L_N T) = i^*(L_{\bar{r}} T)$$

and

$$i^*(L_N \circ L_N T) = -i^*(L_{\bar{r}} T) + i^*(L_{\bar{r}} \circ L_{\bar{r}} T). \quad \square$$

**Corollary 2.1.** *Let  $\alpha$  be a differential  $p$ -form on  $\mathbb{R}^{n+1}$ . Then*

$$i^*(\Delta_{\mathbb{R}^{n+1}}\alpha) = \Delta_{S^n}i^*\alpha + i^*((2p - n + 1)L_{\bar{r}}\alpha - L_{\bar{r}} \circ L_{\bar{r}}\alpha - 2di_{\bar{r}}\alpha)$$

**Corollary 2.2.** *Let  $T$  be a symmetric  $p$ -tensor field on  $\mathbb{R}^{n+1}$ . Then*

$$i^*(\Delta_{\mathbb{R}^{n+1}}T) = \Delta_{S^n}i^*T + i^*(2p(1 - p)T + (2p - n + 1)L_{\bar{r}}T - L_{\bar{r}} \circ L_{\bar{r}}T - 2\delta^*(i_{\bar{r}}T) + 2 \operatorname{Tr}(T) \odot \langle \cdot, \cdot \rangle),$$

where  $\odot$  is the symmetric product.

### 3. Eigenvalues and eigenforms of $\Delta_{S^n}$ acting on $\Omega(S^n)$

In this section, we will use Corollary 2.1 and the results developed in [10] to deduce the eigenvalues and the spaces of eigenforms of  $\Delta_{S^n}$  acting on  $\Omega^*(S^n)$ . We recover the results of [10] and [11] in a more precise form.

Let  $\bigwedge^p H_k$  be the space of all coclosed harmonic homogeneous  $p$ -forms of degree  $k$  on  $\mathbb{R}^{n+1}$ . A differential form  $\alpha$  belongs to  $\bigwedge^p H_k$  if  $\delta(\alpha) = 0$  and  $\alpha$  can be written

$$\alpha = \sum_{1 \leq i_1 < \dots < i_p \leq n+1} \alpha_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p},$$

where  $\alpha_{i_1 \dots i_p}$  are harmonic polynomial functions on  $\mathbb{R}^{n+1}$  of degree  $k$ . For any  $\alpha \in \bigwedge^p H_k$ , we have

$$(12) \quad L_{\bar{r}}\alpha = di_{\bar{r}}\alpha + i_{\bar{r}} d\alpha = (k + p)\alpha.$$

We have (see [10])

$$i^*: \sum_{k \geq 0} \bigwedge^p H_k \rightarrow \Omega^p(S^n)$$

is injective and its image is dense.

For any  $\alpha \in \bigwedge^p H_k$ , we put

$$(13) \quad \omega(\alpha) = \alpha - \frac{1}{p + k} di_{\bar{r}}\alpha.$$

**Lemma 3.1.** *We get a linear map  $\omega: \bigwedge^p H_k \rightarrow \bigwedge^p H_k$  which is a projector, i.e.,  $\omega \circ \omega = \omega$ . Moreover,*

$$\text{Ker } \omega = d\left(\bigwedge^{p-1} H_{k+1}\right), \quad \text{Im } \omega = \bigwedge^p H_k \cap \text{Ker } i_{\bar{r}},$$

and hence

$$\bigwedge^p H_k = \bigwedge^p H_k \cap \text{Ker } i_{\bar{r}} \oplus d\left(\bigwedge^{p-1} H_{k+1}\right).$$

The following lemma is an immediate consequence of Corollary 2.1 and (12).

**Lemma 3.2.** 1. *For any  $\alpha \in \bigwedge^p H_k \cap \text{Ker } i_{\bar{r}}$ , we have*

$$\Delta_{S^n} i^* \alpha = (k + p)(k + n - p - 1) i^* \alpha.$$

2. *For any  $\alpha \in d(\bigwedge^{p-1} H_{k+1})$ , we have*

$$\Delta_{S^n} i^* \alpha = (k + p)(k + n - p + 1) i^* \alpha.$$

REMARK 3.1. We have

$$(k + p)(k + n - p - 1) = (k' + p)(k' + n - p + 1) \Leftrightarrow k = k' + 1$$

and

$$n = 2p.$$

The following table gives explicitly the spectra of  $\Delta_{S^n}$  and the spaces of eigenforms with its multiplicities. The multiplicity was computed in [11].

Table I.

$p$	The eigenvalues	The space of eigenforms	Multiplicity
$p = 0$	$k(k + n - 1), k \in \mathbb{N}$	$\bigwedge^0 H_k$	$\frac{(n+k-2)!(n+2k-1)}{k!(n-1)!}$
$1 \leq p \leq n,$ $n \neq 2p$	$(k + p)(k + n - p - 1),$ $k \in \mathbb{N}^*$	$\omega(\bigwedge^p H_k)$	$\frac{(n+k-1)!(n+2k-1)}{p!(k-1)!(n-p-1)!(n+k-p-1)(k+p)}$
	$(k + p)(k + n - p + 1),$ $k \in \mathbb{N}$	$d(\bigwedge^{p-1} H_{k+1})$	$\frac{(n+k)!(n+2k+1)}{(p-1)!k!(n-p)!(n+k-p+1)(k+p)}$
$1 \leq p \leq n,$ $n = 2p$	$(k + p)(k + p + 1),$ $k \in \mathbb{N}$	$\omega(\bigwedge^p H_{k+1}) \oplus d(\bigwedge^{p-1} H_{k+1})$	$\frac{2(2p+k)!(2p+2k+1)}{p!(p-1)!k!(k+p+1)(k+p)}$

**4. Eigenvalues and eigentensors of  $\Delta_{S^n}$  acting on  $\mathcal{S}(S^n)$**

This section is devoted to the determination of the eigenvalues and the spaces of eigentensors of  $\Delta_{S^n}$  acting on  $\mathcal{S}(S^n)$ .

Let  $\mathcal{S}^p P_k$  be the space of  $T \in \mathcal{S}^p(\mathbb{R}^{n+1})$  of the form

$$T = \sum_{1 \leq i_1 \leq \dots \leq i_p \leq n+1} T_{i_1 \dots i_p} dx_{i_1} \odot \dots \odot dx_{i_p},$$

where  $T_{i_1 \dots i_p}$  are homogeneous polynomials of degree  $k$ . We put

$$\mathcal{S}^p H_k^\delta = \mathcal{S}^p P_k \cap \text{Ker } \Delta_{\mathbb{R}^{n+1}} \cap \text{Ker } \delta$$

and

$$\mathcal{S}^p H_k^{\delta 0} = \mathcal{S}^p H_k^\delta \cap \text{Ker } \text{Tr}.$$

In a similar manner as in [10] Lemma 6.4 and Corollary 6.6, we have

$$(14) \quad \mathcal{S}^p P_k = \mathcal{S}^p H_k^\delta \oplus (r^2 \mathcal{S}^p P_{k-2} + dr^2 \odot \mathcal{S}^{p-1} P_{k-1}),$$

and

$$i^* : \sum_{k \geq 0} \mathcal{S}^p H_k^\delta \rightarrow \mathcal{S}^p S^n$$

is injective and its image is dense in  $\mathcal{S}^p S^n$ .

Now, for any  $k \geq 0$ , we proceed to give a direct sum decomposition of  $\mathcal{S}^p H_k^\delta$  consisting of eigenspaces of  $\Delta_{S^n}$  and, hence, we determine completely the eigenvalues of  $\Delta_{S^n}$  acting on  $\mathcal{S}^p(S^n)$ . This will be done in several steps.

At first, we have the following direct sum decomposition:

$$(15) \quad \mathcal{S}^p H_k^\delta = \mathcal{S}^p H_k^{\delta 0} \oplus \bigoplus_{l=1}^{[p/2]} \mathcal{S}^{p-2l} H_k^{\delta 0} \odot \langle \ , \ \rangle^l,$$

where  $\langle \ , \ \rangle^l$  is the symmetric product of  $l$  copies of  $\langle \ , \ \rangle$ .

The task is now to decompose  $\mathcal{S}^p H_k^{\delta 0}$  as a sum of eigenspaces of  $\Delta_{S^n}$  and get, according to (5), all the eigenvalues. This decomposition needs some preparation.

**Lemma 4.1.** *Let  $T \in \mathcal{S}^p P_k$  and  $h \in \mathbb{N}^*$ . Then we have the following formulas:*

1.  $\delta^*(i_{\bar{r}} T) - i_{\bar{r}} \delta^*(T) = (p - k)T;$
2.  $\delta^{*(h)}(i_{\bar{r}} T) - i_{\bar{r}} \delta^{*(h)}(T) = h(p - k + h - 1)\delta^{*(h-1)}(T);$
3.  $\delta^*(i_{\bar{r}^h} T) - i_{\bar{r}^h} \delta^*(T) = h(p - k - h + 1)i_{\bar{r}^{h-1}} T,$

where  $i_{\bar{r}}^h = \overbrace{i_{\bar{r}} \circ \dots \circ i_{\bar{r}}}^h$  and  $\delta^{*(h)} = \overbrace{\delta^* \circ \dots \circ \delta^*}^h$ .

Proof. The first formula is easily verified and the others follow by induction on  $h$ .  $\square$

Note that the spaces  $S^p H_k^{\delta 0}$  are invariant by  $\delta^*$  and  $i_{\bar{r}}$ ; this is a consequence of the following formulas which one can check easily. For any symmetric tensor field  $T$  on  $\mathbb{R}^{n+1}$ , we have

$$(16) \quad \Delta_{\mathbb{R}^{n+1}}(i_{\bar{r}}T) = i_{\bar{r}}\Delta_{\mathbb{R}^{n+1}}(T) + 2\delta T,$$

$$(17) \quad \delta(i_{\bar{r}}T) = i_{\bar{r}}\delta(T) - \text{Tr}(T),$$

$$(18) \quad \text{Tr}(\delta^*(T)) = -2\delta(T) + \delta^*(\text{Tr}(T)),$$

$$(19) \quad \text{Tr}(i_{\bar{r}}T) = i_{\bar{r}}\text{Tr}(T).$$

Now the desired decomposition of  $S^p H_k^{\delta 0}$  is based on the following algebraic lemma.

**Lemma 4.2.** *Let  $V$  be a finite dimensional vectorial space,  $\phi$  and  $\psi$  are two endomorphisms of  $V$  and  $(A_k^p)_{k,p \in \mathbb{N} \cup \{-1\}}$  a family of vectorial subspaces of  $V$  such that:*

1. *for any  $p, k \in \mathbb{N}$ ,  $A_{-1}^p = A_k^{-1} = 0$ ;*
2. *for any  $p, k \in \mathbb{N}$ ,  $\phi(A_k^p) \subset A_{k-1}^{p+1}$  and  $\psi(A_k^p) \subset A_{k+1}^{p-1}$ ;*
3. *for any  $p, k \in \mathbb{N}$  and for any  $a \in A_k^p$ ,*

$$\phi \circ \psi(a) - \psi \circ \phi(a) = (p - k)a.$$

Then:

- (i) *for any  $k < p$ ,  $\psi: A_k^p \rightarrow A_{k+1}^{p-1}$  is injective;*
- (ii) *for  $k \leq p$ , we have*

$$A_k^p = (A_k^p \cap \text{Ker } \phi) \oplus \psi(A_{k-1}^{p+1})$$

and

$$A_k^p = \bigoplus_{l=0}^k \psi^l(A_{k-l}^{p+l} \cap \text{Ker } \phi).$$

Proof. Note that one can deduce easily, by induction, that for any  $l \in \mathbb{N}^*$  and for any  $a \in A_k^p$

$$(20) \quad \phi^l \circ \psi(a) - \psi \circ \phi^l(a) = l(p - k + l - 1)\phi^{l-1}(a),$$

$$(21) \quad \psi^l \circ \phi(a) - \phi \circ \psi^l(a) = l(k - p + l - 1)\psi^{l-1}(a).$$

(i) Let  $a \in A_k^p$  such that  $\psi(a) = 0$ . From (20) and since  $p - k > 0$ , for any  $l \geq 0$ , if  $\phi^l(a) = 0$  then  $\phi^{l-1}(a) = 0$ . Now, since  $\phi^l(a) \in A_{k-l}^{p+l}$  and since  $A_{-1}^{p+l} = 0$ , we have, for any  $l \geq k + 1$ ,  $\phi^l(a) = 0$  which implies, by induction, that  $a = 0$  and hence  $\psi : A_k^p \rightarrow A_{k+1}^{p-1}$  is injective.

(ii) Suppose that  $k \leq p$ . We define  $P_k^p : A_k^p \rightarrow A_k^p$  as follows

$$\begin{cases} P_k^p(a) = \sum_{s=0}^k \alpha_s \psi^s \circ \phi^s(a) \\ \alpha_0 = 1 \text{ and } \alpha_s - (s+1)(k-p-s-2)\alpha_{s+1} = 0 \text{ for } 1 \leq s \leq k-1. \end{cases}$$

$P_k^p$  satisfies

$$P_k^p \circ P_k^p = P_k^p, \quad \text{Ker } P_k^p = \psi(A_{k-1}^{p+1})$$

and

$$\text{Im } P_k^p = A_k^p \cap \text{Ker } \phi.$$

Indeed, let  $a \in A_{k-1}^{p+1}$ . We have

$$\begin{aligned} P_k^p(\psi(a)) &= \sum_{s=0}^k \alpha_s \psi^s \circ \phi^s(\psi(a)) \\ &\stackrel{(20)}{=} \sum_{s=0}^k \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=0}^k s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &\stackrel{\phi^k(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=1}^k s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &= \sum_{s=0}^{k-1} (\alpha_s + (s+1)(p-k+s+2)\alpha_{s+1}) \psi^{s+1} \circ \phi^s(a) \\ &= 0. \end{aligned}$$

Conversely, since  $P_k^p(a) = a + \sum_{s=1}^k \alpha_s \psi^s \circ \phi^s(a)$ , we deduce that  $P_k^p(a) = 0$  implies that  $a \in \psi(A_{k-1}^{p+1})$ , so we have shown that  $\text{Ker } P_k^p = \psi(A_{k-1}^{p+1})$ . The relation  $P_k^p \circ P_k^p = P_k^p$  is a consequence of the definition of  $P_k^p$  and  $P_k^p \circ \psi = 0$ .

Note that  $\phi(a) = 0$  implies that  $P_k^p(a) = a$  and hence  $A_k^p \cap \text{Ker } \phi \subset \text{Im } P_k^p$ . Conversely, let  $a \in A_k^p$ , we have

$$\begin{aligned}
 \phi \circ P_k^p(a) &= \sum_{s=0}^k \alpha_s \phi \circ \psi^s \circ \phi^s(a) \\
 &\stackrel{(21)}{=} \sum_{s=0}^k \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=0}^k \alpha_s s(k-p-s-1) \psi^{s-1} \circ \phi^s(a) \\
 &\stackrel{\phi^{k+1}(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=1}^k \alpha_s s(k-p-s-1) \psi^{s-1} \circ \phi^s(a) \\
 &= \sum_{s=0}^{k-1} (\alpha_s - (s+1)(k-p-s-2)\alpha_{s+1}) \psi^s \circ \phi^{s+1}(a) \\
 &= 0.
 \end{aligned}$$

We conclude that  $P_k^p$  is a projector,  $\text{Ker } P_k^p = \psi(A_{k-1}^{p+1})$  and  $A_k^p \cap \text{Ker } \phi = \text{Im } P_k^p$  and we deduce immediately that  $A_k^p = \psi(A_{k-1}^{p+1}) \oplus A_k^p \cap \text{Ker } \phi$ . The same decomposition holds for  $A_{k-1}^{p+1}$  and, since  $\psi: A_{k-1}^{p+1} \rightarrow A_k^p$  is injective, we get

$$A_k^p = \psi \circ \psi(A_{k-2}^{p+2}) \oplus \psi(A_{k-1}^{p+1} \cap \text{Ker } \phi) \oplus A_k^p \cap \text{Ker } \phi.$$

We proceed by induction and we get the desired decomposition.  $\square$

According to Lemma 4.1, the hypothesis of Lemma 4.2 are satisfied by the spaces  $S^p H_k^{\delta 0}$  and the operators  $\delta^*$  and  $i_{\bar{r}}$ . So we get, in a first time,

$$(22) \quad S^p H_k^{\delta 0} = S^p H_k^{\delta 0} \cap \text{Ker } \delta^* \oplus i_{\bar{r}}(S^{p+1} H_{k-1}^{\delta 0}), \quad \text{if } k \leq p,$$

$$(23) \quad S^p H_k^{\delta 0} = S^p H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}} \oplus \delta^*(S^{p-1} H_{k+1}^{\delta 0}), \quad \text{if } k \geq p,$$

and, in a second time, the desired decomposition of  $S^p H_k^{\delta 0}$ .

**Lemma 4.3.** *We have:*

1. *If  $k \leq p$*

$$S^p H_k^{\delta 0} = \bigoplus_{l=0}^k i_{\bar{r}}^l (S^{p+l} H_{k-l}^{\delta 0} \cap \text{Ker } \delta^*);$$

2. *If  $k \geq p$*

$$S^p H_k^{\delta 0} = \bigoplus_{l=0}^p \delta^{*l} (S^{p-l} H_{k+l}^{\delta 0} \cap \text{Ker } i_{\bar{r}});$$



3. If  $k = p$ , for any  $0 \leq l \leq p$ ,

$$\mathcal{S}^p H_p^{\delta 0} = \bigoplus_{l=0}^p i_{\bar{l}}(\mathcal{S}^{p+l} H_{p-l}^{\delta 0} \cap \text{Ker } \delta^*) = \bigoplus_{l=0}^p \delta^{*l}(\mathcal{S}^{p-l} H_{p+l}^{\delta 0} \cap \text{Ker } i_{\bar{l}}).$$

Now, we use Corollary 2.2 to show that the decompositions of  $\mathcal{S}^p H_k^{\delta 0}$  given in Lemma 4.3 are composed by eigenspaces of  $\Delta_{S^n}$ .

**Theorem 4.1.** *We have:*

1. If  $k \leq p$ , for any  $0 \leq q \leq k$  and any  $T \in i_{\bar{l}(k-q)}(\mathcal{S}^{p+k-q} H_q^{\delta 0} \cap \text{Ker } \delta^*)$ ,

$$\Delta_{S^n} i^* T = ((k+p)(n+p+k-2q-1) + 2q(q-1)) i^* T;$$

2. If  $k \geq p$ , for any  $0 \leq q \leq p$  and for any  $T \in \delta^{*(p-q)}(\mathcal{S}^q H_{k+p-q}^{\delta 0} \cap \text{Ker } i_{\bar{l}})$ ,

$$\Delta_{S^n} i^* T = ((k+p)(n+p+k-2q-1) + 2q(q-1)) i^* T.$$

*Proof.* 1. Let  $T = i_{\bar{l}(k-q)}(T_0)$  with  $T_0 \in \mathcal{S}^{p+k-q} H_q^{\delta 0} \cap \text{Ker } \delta^*$ . We have from Corollary 2.2

$$\begin{aligned} \Delta_{S^n} i^* T &= i^*(2p(p-1)T + (n-2p-1)L_{\bar{l}}T + L_{\bar{l}} \circ L_{\bar{l}}T \\ &\quad + 2\delta^*(i_{\bar{l}}T) - 2\text{Tr}(T) \odot \langle \cdot, \cdot \rangle). \end{aligned}$$

We have

$$\text{Tr } T = 0, \quad L_{\bar{l}}T = (k+p)T$$

and

$$L_{\bar{l}} \circ L_{\bar{l}}T = (k+p)^2 T.$$

Moreover, by using Lemma 4.1, we have

$$\begin{aligned} 2\delta^*(i_{\bar{l}}T) &= 2\delta^*(i_{\bar{l}(k-q+1)}T_0) \\ &\stackrel{\delta^*(T_0)=0}{=} 2(k-q+1)(p+k-q-q-k+q-1+1)i_{\bar{l}(k-q)}T_0 \\ &= 2(k-q+1)(p-q)T. \end{aligned}$$

Hence

$$\Delta_{S^n} i^* T = (2p(p-1) + (n-2p-1)(k+p) + (k+p)^2 + 2(p-q)(k-q+1)) i^* T.$$

One can deduce the desired relation by remarking that

$$2p(p-1) + 2(p-q)(k-q+1) = 2(k+p)(p-q) + 2q(q-1).$$

2. This follows by the same calculation as 1. □

From the fact that

$$i^*: \sum_{k \geq 0} \mathcal{S}^p H_k^\delta \rightarrow \mathcal{S}^p S^n$$

is injective and its image is dense in  $\mathcal{S}^p S^n$ , from (15), and from Lemma 4.3 and Theorem 4.1, note that we have actually proved that the eigenvalues of  $\Delta_{\mathcal{S}^p}$  acting on  $\mathcal{S}^p S^n$  belongs to

$$\left\{ (k + p - 2l)(n + p + k - 2l - 2q - 1) + 2q(q - 1), \right. \\ \left. k \in \mathbb{N}, 0 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor, 0 \leq q \leq \min(k, p - 2l) \right\}.$$

Our next goal is to sharpen this result by computing  $\dim \mathcal{S}^p H_k^{\delta 0} \cap \text{Ker } \delta^*$  if  $k \leq p$  and  $\dim \mathcal{S}^p H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}}$  if  $k \geq p$ .

**Lemma 4.4.** *We have the following formulas:*

1.  $\dim \mathcal{S}^p H_k^\delta = \dim \mathcal{S}^p P_k - \dim \mathcal{S}^p P_{k-2} - \dim \mathcal{S}^{p-1} P_{k-1} + \dim \mathcal{S}^{p-1} P_{k-3}$ ,
  2.  $\dim \mathcal{S}^p H_k^{\delta 0} = \dim \mathcal{S}^p H_k^\delta - \dim \mathcal{S}^{p-2} H_k^\delta$ ,
  3.  $\dim(\mathcal{S}^p H_k^{\delta 0} \cap \text{Ker } \delta^*) = \dim \mathcal{S}^p H_k^{\delta 0} - \dim \mathcal{S}^{p+1} H_{k-1}^{\delta 0}$  ( $k \leq p$ ),
  4.  $\dim(\mathcal{S}^p H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}}) = \dim \mathcal{S}^p H_k^{\delta 0} - \dim \mathcal{S}^{p-1} H_{k+1}^{\delta 0}$  ( $k \geq p$ ).
- Note that we use the convention that  $\mathcal{S}^p P_k = \mathcal{S}^p H_k^\delta = \mathcal{S}^p H_k^{\delta 0} = 0$  if  $k < 0$  or  $p < 0$ .*

*Proof.* 1. The formula is a consequence of (14), the relation

$$(r^2 \mathcal{S}^p P_{k-2}) \cap (dr^2 \odot \mathcal{S}^{p-1} P_{k-1}) = r^2 (dr^2 \odot \mathcal{S}^{p-1} P_{k-3})$$

and the fact that  $dr^2 \odot \cdot : \mathcal{S}^p P_k \rightarrow \mathcal{S}^{p+1} P_{k+1}$  is injective.

2. The formula is a consequence of (15).
3. The formula is a consequence of (22) and Lemma 4.2.
4. The formula is a consequence of (23) and Lemma 4.2. □

A straightforward calculation using Lemma 4.4 and the formula

$$\dim \mathcal{S}^p P_k = \frac{(n + p)! (n + k)!}{n! p! n! k!}$$

gives  $\dim \mathcal{S}^p H_k^{\delta 0} \cap \text{Ker } \delta^*$  if  $k \leq p$  and  $\dim \mathcal{S}^p H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}}$  if  $k \geq p$ . We summarize the results on the following table.

Table II.

Space	Dimension	Conditions on $k$ and $p$
$S^0 H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}}$	$\frac{(n+k-2)! (n+2k-1)}{k! (n-1)!}$	$k \geq 0$
$S^p H_0^{\delta 0} \cap \text{Ker } \delta^*$	$\frac{(n+p-2)! (n+2p-1)}{p! (n-1)!}$	$p \geq 0$
$S^1 H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}}$	$\frac{(n+k-3)! k(n+2k-1)(n+k-1)}{(n-2)! (k+1)!}$	$k \geq 1$
$S^p H_1^{\delta 0} \cap \text{Ker } \delta^*$	$\frac{(n+p-3)! p(n+2p-1)(n+p-1)}{(n-2)! (p+1)!}$	$p \geq 1$
$S^p H_k^{\delta 0} \cap \text{Ker } \delta^*$	$\frac{(n+k-4)! (n+p-3)! (n+p+k-2)}{k! (p+1)! (n-1)! (n-2)!} \times$ $(n-2)(n+2k-3)(n+2p-1)(p-k+1)$	$2 \leq k \leq p$
$S^p H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}}$	$\frac{(n+k-3)! (n+p-4)! (n+p+k-2)}{(k+1)! p! (n-1)! (n-2)!} \times$ $(n-2)(n+2k-1)(n+2p-3)(k-p+1)$	$k \geq p \geq 2$

REMARK 4.1. Note that, for  $n = 2$ , we have

$$\begin{aligned} \dim(S^p H_k^{\delta 0} \cap \text{Ker } \delta^*) &= 0 \quad \text{for } 2 \leq k \leq p, \\ \dim(S^p H_k^{\delta 0} \cap \text{Ker } i_{\bar{r}}) &= 0 \quad \text{for } k \geq p \geq 2. \end{aligned}$$

For simplicity we introduce the following notations.

$$\begin{aligned} S_0 &= \left\{ (k, l, q) \in \mathbb{N}^3, 0 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor, 0 \leq k \leq p - 2l, 0 \leq q \leq k \right\}, \\ S_1 &= \left\{ (k, l, q) \in \mathbb{N}^3, 0 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor, k > p - 2l, 0 \leq q \leq p - 2l \right\}, \\ V_{q,l}^k &= i_{\bar{r}^{k-q}} (S^{p-2l+k-q} H_q^{\delta 0} \cap \text{Ker } \delta^*) \odot \langle \cdot, \cdot \rangle^l \quad \text{for } (k, l, q) \in S_0, \\ W_{q,l}^k &= \delta^{*(p-2l-q)} (S^q H_{p-2l+k-q}^{\delta 0} \cap \text{Ker } i_{\bar{r}}) \odot \langle \cdot, \cdot \rangle^l \quad \text{for } (k, l, q) \in S_1. \end{aligned}$$

Let us summarize all the results above.

**Theorem 4.2.** 1. For  $n = 2$ , we have:

(a) The set of the eigenvalues of  $\Delta_{S^2}$  acting on  $S^p S^2$  is

$$\left\{ (k+p-2l)(p+k-2l+1), \quad k \in \mathbb{N}, 0 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor \right\};$$

(b) *The eigenspace associated to the eigenvalue  $\lambda(k, l) = (k + p - 2l)(k + p - 2l + 1)$  is given by*

$$V_{\lambda(k,l)} = \begin{cases} \bigoplus_{a=0}^{\min(l, \lfloor k/2 \rfloor)} (V_{0,l-a}^{k-2a} \oplus V_{1,l-a}^{k+1-2a}) & \text{if } 0 \leq k \leq p - 2l, \\ \bigoplus_{a=0}^{\min(l, \lfloor k/2 \rfloor)} (W_{0,l-a}^{k-2a} \oplus W_{1,l-a}^{k+1-2a}) & \text{if } k > p - 2l; \end{cases}$$

(c) *The multiplicity of  $\lambda(k, l)$  is given by*

$$m(\lambda(k, l)) = 2 \left( \min \left( l, \left\lfloor \frac{k}{2} \right\rfloor \right) + 1 \right) (1 + 2p + 2k - 4l).$$

2. *For  $n \geq 3$ , we have:*

(a) *The set of the eigenvalues of  $\Delta_{S^n}$  acting on  $S^p S^n$  is*

$$\left\{ (k + p - 2l)(n + p + k - 2l - 2q - 1) + 2q(q - 1), \right. \\ \left. k \in \mathbb{N}, 0 \leq l \leq \left\lfloor \frac{p}{2} \right\rfloor, 0 \leq q \leq \min(k, p - 2l) \right\};$$

(b) *The space*

$$\mathcal{P} = \sum_{k \geq 0} S^p H_k^\delta = \left( \bigoplus_{(k,l,q) \in S_0} V_{q,l}^k \right) \oplus \left( \bigoplus_{(k,l,q) \in S_1} W_{q,l}^k \right)$$

*is dense in  $S^p S^n$  and, for any  $(k, q, l) \in S_0$  (resp.  $(k, q, l) \in S_1$ ),  $V_{q,l}^k$  (resp.  $W_{q,l}^k$ ) is a subspace of the eigenspace associated to the eigenvalue  $(k + p - 2l)(n + p + k - 2l - 2q - 1) + 2q(q - 1)$ ;*

(c) *The dimensions of  $V_{q,l}^k$  and  $W_{q,l}^k$  are given in Table II since*

$$\dim V_{q,l}^k = \dim(S^{p-2l+k-q} H_q^{\delta_0} \cap \text{Ker } \delta^*) \quad \text{for } (k, l, q) \in S_0, \\ \dim W_{q,l}^k = \dim(S^q H_{p-2l+k-q}^{\delta_0} \cap \text{Ker } i_{\bar{r}}) \quad \text{for } (k, l, q) \in S_1.$$

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