Boucetta, M.
Osaka J. Math.
46 (2009), 235–254

SPECTRA AND SYMMETRIC EIGENTENSORS OF THE LICHNEROWICZ LAPLACIAN ON S^n

MOHAMED BOUCETTA

(Received May 8, 2007, revised December 7, 2007)

Abstract

We compute the eigenvalues with multiplicities of the Lichnerowicz Laplacian acting on the space of symmetric covariant tensor fields on the Euclidian sphere S^n. The spaces of symmetric eigentensors are explicitly given.

1. Introduction

Let (M, g) be a Riemannian n-manifold. For any $p \in \mathbb{N}$, we shall denote by $\Gamma\left(\bigotimes^p T^*M\right)$, $\Omega^p(M)$ and $\mathcal{S}^p M$ the space of covariant p-tensor fields on M, the space of differential p-forms on M and the space of symmetric covariant p-tensor fields on M, respectively. Note that $\Gamma\left(\bigotimes^0 T^*M\right) = \Omega^0(M) = \mathcal{S}^0 M = C^\infty(M, \mathbb{R})$, $\Omega(M) = \sum_{p=0}^n \Omega^p(M)$ and $\mathcal{S}(M) = \sum_{p\geq 0} \mathcal{S}^p(M)$.

Let D be the Levi-Civita connection associated to g; its curvature tensor field R is given by

$$R(X, Y)Z = D_{[X,Y]}Z - (D_X D_Y Z - D_Y D_X Z),$$

and the Ricci endomorphism field $r\colon TM \to TM$ is given by

$$g(r(X), Y) = \sum_{i=1}^n g(R(X, E_i)Y, E_i),$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame.

For any $p \in \mathbb{N}$, the connection D induces a differential operator $D\colon \Gamma\left(\bigotimes^p T^*M\right) \to \Gamma\left(\bigotimes^{p+1} T^*M\right)$ given by

$$\begin{aligned} DT(X, Y_1, \dots, Y_p) &= D_X T(Y_1, \dots, Y_p)\\ &= X.T(Y_1, \dots, Y_p) - \sum_{j=1}^p T(Y_1, \dots, D_X Y_j, \dots, Y_p). \end{aligned}$$

2000 Mathematics Subject Classification. 53B21, 53B50, 58C40.

Recherche menée dans le cadre du Programme Thématique d'Appui à la Recherche Scientifique PROTARS III.

Its formal adjoint $D^*\colon \Gamma\left(\bigotimes^{p+1} T^*M\right) \to \Gamma\left(\bigotimes^{p} T^*M\right)$ is given by

$$D^*T(Y_1, \dots, Y_p) = -\sum_{j=1}^{n} D_{E_i} T(E_i, Y_1, \dots, Y_p),$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame.

Recall that, for any differential p-form α, we have

$$(1)\qquad d\alpha(X_1, \dots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} D_{X_j} \alpha(X_1, \dots, \hat{X}_j, \dots, X_{p+1}).$$

We denote by δ the restriction of D^* to $\Omega(M)\oplus\mathcal{S}(M)$ and we define $\delta^*\colon \mathcal{S}^p(M) \to \mathcal{S}^{p+1}(M)$ by

$$\delta^*T(X_1, \dots, X_{p+1}) = \sum_{j=1}^{p+1} D_{X_j} T(X_1, \dots, \hat{X}_j, \dots, X_{p+1}).$$

Recall that the operator trace $\mathrm{Tr}\colon \mathcal{S}^p(M) \to \mathcal{S}^{p-2}(M)$ is given by

$$\mathrm{Tr}\, T(X_1, \dots, X_{p-2}) = \sum_{j=1}^{n} T(E_j, E_j, X_1, \dots, X_{p-2}),$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame.

The Lichnerowicz Laplacian is the second order differential operator

$$\Delta_M\colon \Gamma\left(\bigotimes^{p} T^*M\right) \to \Gamma\left(\bigotimes^{p} T^*M\right)$$

given by

$$\Delta_M(T) = D^*D(T) + R(T),$$

where $R(T)$ is the curvature operator given by

$$\begin{aligned} R(T)(Y_1, \dots, Y_p) = \sum_{j=1}^{p} & T(Y_1, \dots, r(Y_j), \dots, Y_p) \\ & - \sum_{i<j} \sum_{l=1}^{n} \{T(Y_1, \dots, E_l, \dots, R(Y_i, E_l)Y_j, \dots, Y_p) \\ & \qquad + T(Y_1, \dots, R(Y_j, E_l)Y_i, \dots, E_l, \dots, Y_p)\}, \end{aligned}$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame and, in

$$T(Y_1, \dots, E_l, \dots, R(Y_i, E_l)Y_j, \dots, Y_p),$$

E_l takes the place of Y_i and $R(Y_i, E_l)Y_j$ takes the place of Y_j.

This differential operator, introduced by Lichnerowicz in [15] pp. 26, is self-adjoint, elliptic and respects the symmetries of tensor fields. In particular, Δ_M leaves invariant $\mathcal{S}(M)$ and the restriction of Δ_M to $\Omega(M)$ coincides with the Hodge-de Rham Laplacian, i.e., for any differential p-form α,

$$\Delta_M \alpha = (d\delta + \delta d)(\alpha). \tag{2}$$

We have shown in [6] that, for any symmetric covariant tensor field T,

$$\Delta_M(T) = (\delta \circ \delta^* - \delta^* \delta)(T) + 2R(T). \tag{3}$$

Note that if $T \in \mathcal{S}(M)$ and g^l denotes the symmetric product of l copies of the Riemannian metric g, we have

$$(\mathrm{Tr} \circ \Delta_M)T = (\Delta_M \circ \mathrm{Tr})T, \tag{4}$$

$$\Delta_M(T \odot g^l) = (\Delta_M T) \odot g^l, \tag{5}$$

where $\odot$ is the symmetric product.

The Lichnerowicz Laplacian acting on symmetric covariant tensor fields is of fundamental importance in mathematical physics (see for instance [9], [20] and [22]). Note also that the Lichnerowicz Laplacian acting on symmetric covariant 2-tensor fields appears in many problems in Riemannian geometry (see [3], [5], [19], ...).

On a compact Riemannian manifold, the Lichnerowicz Laplacian Δ_M has discrete eigenvalues with finite multiplicities. For a given compact Riemannian manifold, it may be an interesting problem to determine explicitly the eigenvalues and the eigentensors of Δ_M on M.

Let us enumerate the cases where the spectra of Δ_M was computed:

1. Δ_M acting on $C^\infty(M, \mathbb{C})$: M is either flat torus or Klein bottles [4], M is a Hopf manifolds [1];
2. Δ_M acting on $\Omega(M)$: $M = S^n$ or $P^n(\mathbb{C})$ [10] and [11], $M = \mathbb{C}aP^2$ or $G_2/SO(4)$ [16] and [18], $M = SO(n+1)/SO(2) \times SO(n)$ or $M = Sp(n+1)/Sp(1) \times Sp(n)$ [21];
3. Δ_M acting on $\mathcal{S}^2(M)$ and M is the complex projective space $P^2(\mathbb{C})$ [22];
4. Δ_M acting on $\mathcal{S}^2(M)$ and M is either S^n or $P^n(\mathbb{C})$ [6] and [7];
5. Brian and Richard Millman give in [2] a theoretical method for computing the spectra of Lichnerowicz Laplacian acting on $\Omega(G)$ where G is a compact semisimple Lie group endowed with the biinvariant metric induced from the negative of the Killing form;
6. Some partial results where given in [12]–[14].

In this paper, we compute the eigenvalues and we determine the spaces of eigentensors of Δ_M acting on $\mathcal{S}(M)$ in the case where M is the Euclidian sphere S^n.

Let us describe our method briefly. We consider the $(n+1)$-Euclidian space $\mathbb{R}^{n+1}$ with its canonical coordinates $(x_1, \dots, x_{n+1})$. For any k, $p \in \mathbb{N}$, we denote by $\mathcal{S}^p H_k^\delta$ the space of symmetric covariant p-tensor fields T on $\mathbb{R}^{n+1}$ satisfying:

1. $T = \sum_{1 \le i_1 \le \cdots \le i_p \le n+1} T_{i_1,\dots,i_p}\, dx_{i_1} \odot \cdots \odot dx_{i_p}$ where $T_{i_1,\dots,i_p}$ are homogeneous polynomials of degree k;
2. $\delta(T) = \Delta_{\mathbb{R}^{n+1}}(T) = 0$.

The n-dimensional sphere S^n is the space of unitary vectors in $\mathbb{R}^{n+1}$ and the Euclidian metric on $\mathbb{R}^{n+1}$ induces a Riemannian metric on S^n. We denote by $i \colon S^n \hookrightarrow \mathbb{R}^{n+1}$ the canonical inclusion.

For any tensor field $T \in \Gamma\left(\bigotimes^p T^*\mathbb{R}^{n+1}\right)$, we compute $i^*(\Delta_{\mathbb{R}^{n+1}} T) - \Delta_{S^n}(i^* T)$ and get a formula (see Theorem 2.1). Inspired by this formula and having in mind the fact that $i^* \colon \sum_{k \ge 0} \mathcal{S}^p H_k^\delta \to \mathcal{S}^p S^n$ is injective and its image is dense in $\mathcal{S}^p S^n$ (see [10]), we give, for any k, a direct sum decomposition of $\mathcal{S}^p H_k^\delta$ composed by eigenspaces of Δ_{S^n}. Thus we obtain the eigenvalues and the spaces of eigentensors with its multiplicities of Δ_{S^n} acting on $\mathcal{S}(S^n)$ (see Section 4).

Note that the eigenvalues and the eigenspaces of Δ_{S^n} acting on $\Omega(S^n)$ was computed in [10] by using the representation theory. In [11], I. Iwasaki and K. Katase recover the result by a method using the restriction of harmonic tensor fields and a result in [8]. The formula obtained in Theorem 2.1 combined with the methods developed in [10] and [11] permit to present those results in a more precise form (see Section 3).

2. A relation between $\Delta_{\mathbb{R}^{n+1}}$ and Δ_{S^n}

We consider the Euclidian space $\mathbb{R}^{n+1}$ endowed with its canonical coordinates $(x_1, \dots, x_{n+1})$ and its canonical Euclidian flat Riemannian metric $\langle\ ,\ \rangle$. We denote by D be the Levi-Civita covariant derivative associated to $\langle\ ,\ \rangle$. We consider the radial vector field given by

$$\vec{r} = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}.$$

For any p-tensor field $T \in \Gamma\left(\bigotimes^p T^*\mathbb{R}^{n+1}\right)$ and for any $1 \le i < j \le p$, we denote by $i_{\vec{r},j} T$ the $(p-1)$-tensor field given by

$$i_{\vec{r},j} T(X_1, \dots, X_{p-1}) = T(X_1, \dots, X_{j-1}, \vec{r}, X_j, \dots, X_{p-1}),$$

and by $\mathrm{Tr}_{i,j}\, T$ the $(p-2)$-tensor field given by

$$\begin{aligned}&\mathrm{Tr}_{i,j}\, T(X_1, \dots, X_{p-2})\\ &= \sum_{l=1}^{n+1} T(X_1, \dots, X_{i-1}, E_l, X_i, \dots, X_{j-2}, E_l, X_{j-1}, \dots, X_{p-2}),\end{aligned}$$

where $(E_1, \dots, E_{n+1})$ is any orthonormal basis of $\mathbb{R}^{n+1}$. Note that $\mathrm{Tr}_{i,j}\, T = 0$ if T is a differential form and $\mathrm{Tr}_{i,j}\, T = \mathrm{Tr}\, T$ if T is symmetric.

For any permutation σ of $\{1, \dots, p\}$, we denote by T^σ the p-tensor field

$$T^\sigma(X_1, \dots, X_p) = T(X_{\sigma(1)}, \dots, X_{\sigma(p)}).$$

For $1 \le i < j \le p$, the transposition of (i, j) is the permutation $\sigma_{i,j}$ of $\{1, \dots, p\}$ such that $\sigma_{i,j}(i) = j$, $\sigma_{i,j}(j) = i$ and $\sigma_{i,j}(k) = k$ for $k \ne i, j$. Let $\mathcal{T}$ denote the set of the transpositions of $\{1, \dots, p\}$.

The sphere $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ is endowed with the Euclidian metric.

Theorem 2.1. *Let T be a covariant p-tensor field on $\mathbb{R}^{n+1}$. Then,*

$$\begin{aligned}&i^*(\Delta_{\mathbb{R}^{n+1}} T)\\ &= \Delta_{S^n} i^* T + i^*\Bigl(p(1-p)T + (2p-n+1)L_{\vec{r}}T - L_{\vec{r}} \circ L_{\vec{r}} T - 2\sum_{\sigma\in\mathcal{T}} T^\sigma + O(T)\Bigr),\end{aligned}$$

where $O(T)$ is given by

$$\begin{aligned}O(T)(X_1, \dots, X_p) &= 2\sum_{i<j} \langle X_i, X_j\rangle\, \mathrm{Tr}_{i,j}(X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p)\\ &\quad - 2\sum_{j=1}^{p} D_{X_j}(i_{\vec{r},j} T)(X_1, \dots, \hat{X}_j, \dots, X_p),\end{aligned}$$

where the symbol $\hat{}$ means that the term is omitted.

Proof. The proof is a massive computation in a local orthonormal frame using the properties of the Riemannian embedding of the sphere in the Euclidian space.

We choose a local orthonormal frame of $\mathbb{R}^{n+1}$ of the form $(E_1, \dots, E_n, N)$ such that E_i is tangent to S^n for $1 \le i \le n$ and $N = (1/r)\vec{r}$ where $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$.

For any vector field X on $\mathbb{R}^{n+1}$, we have

$$D_X N = \frac{1}{r}(X - \langle X, N\rangle N), \tag{6}$$

$$D_N X = [N, X] + \frac{1}{r}(X - \langle X, N\rangle N). \tag{7}$$

Let ∇ be the Levi-Civita connexion of the Riemannian metric on S^n. We have, for any vector fields X, Y tangent to S^n,

$$D_X Y = \nabla_X Y - \langle X, Y\rangle N. \tag{8}$$

Let T be a covariant p-tensor field on $\mathbb{R}^{n+1}$ and $(X_1, \dots, X_p)$ a family of vector fields on $\mathbb{R}^{n+1}$ which are tangent to S^n. A direct calculation using the definition of the Lichnerowicz Laplacian gives

$$\begin{aligned}
&\Delta_{\mathbb{R}^{n+1}}(T)(X_1, \dots, X_p) = D^* D(T)(X_1, \dots, X_p)\\
&= \sum_{i=1}^{n}\Bigg(-E_i E_i.T(X_1, \dots, X_p) + 2\sum_{j=1}^{p} E_i.T(X_1, \dots, D_{E_i}X_j, \dots, X_p)\\
&\qquad + D_{E_i}E_i.T(X_1, \dots, X_p) - \sum_{j=1}^{p} T(X_1, \dots, D_{D_{E_i}E_i}X_j, \dots, X_p)\\
&\qquad - \sum_{j=1}^{p} T(X_1, \dots, D_{E_i}D_{E_i}X_j, \dots, X_p)\\
&\qquad - 2\sum_{l<j} T(X_1, \dots, D_{E_i}X_l, \dots, D_{E_i}X_j, \dots, X_p)\Bigg)\\
&\quad - N.N.T(X_1, \dots, X_p) + 2\sum_{j=1}^{p} N.T(X_1, \dots, D_N X_j, \dots, X_p)\\
&\quad + D_N N.T(X_1, \dots, X_p) - \sum_{j=1}^{p} T(X_1, \dots, D_{D_N N}X_j, \dots, X_p)\\
&\quad - \sum_{j=1}^{p} T(X_1, \dots, D_N D_N X_j, \dots, X_p) - 2\sum_{l<j} T(X_1, \dots, D_N X_l, \dots, D_N X_j, \dots, X_p).
\end{aligned}$$

(6)–(8) make it obvious that

$$\begin{aligned}
D_{D_{E_i}E_i}X_j &= \nabla_{\nabla_{E_i}E_i}X_j - \langle \nabla_{E_i}E_i, X_j\rangle N - [N, X_j]\\
&\qquad - \frac{1}{r}(X_j - \langle X_j, N\rangle N),
\end{aligned}\tag{9}$$

$$\begin{aligned}
D_{E_i}D_{E_i}X_j &= \nabla_{E_i}\nabla_{E_i}X_j - (\langle E_i, \nabla_{E_i}X_j\rangle + E_i.\langle E_i, X_j\rangle)N\\
&\qquad - \frac{1}{r}\langle E_i, X_j\rangle E_i,
\end{aligned}\tag{10}$$

$$\begin{aligned}
D_N D_N X &= [N, [N, X]] + \frac{2}{r}[N, X] + \left(\frac{1}{r^2} - \frac{1}{r}\right)(X - \langle X, N\rangle N)\\
&\qquad - \frac{2}{r}N.\langle X, N\rangle N.
\end{aligned}\tag{11}$$

By (8)–(10), we get easily, in restriction to S^n,

$$\sum_{i=1}^{n}\left(2\sum_{j=1}^{p}E_i.T(X_1,\dots,D_{E_i}X_j,\dots,X_p)+D_{E_i}E_i.T(X_1,\dots,X_p)\right.$$
$$\left.-\sum_{j=1}^{p}T(X_1,\dots,D_{D_{E_i}E_i}X_j,\dots,X_p)-\sum_{j=1}^{p}T(X_1,\dots,D_{E_i}D_{E_i}X_j,\dots,X_p)\right)$$
$$=\sum_{i=1}^{n}\left(2\sum_{j=1}^{p}E_i.T(X_1,\dots,\nabla_{E_i}X_j,\dots,X_p)+\nabla_{E_i}E_i.T(X_1,\dots,X_p)\right.$$
$$\left.-\sum_{j=1}^{p}T(X_1,\dots,\nabla_{\nabla_{E_i}E_i}X_j,\dots,X_p)-\sum_{j=1}^{p}T(X_1,\dots,\nabla_{E_i}\nabla_{E_i}X_j,\dots,X_p)\right)$$
$$-2\sum_{j=1}^{p}X_j.T(X_1,\dots,\overbrace{N}^{j},\dots,X_p)+p(n+1)T(X_1,\dots,X_p)-nL_NT(X_1,\dots,X_p).$$

On other hand, also by using (8), we have

$$\sum_{l<j}\sum_{i=1}^{n}T(X_1,\dots,D_{E_i}X_l,\dots,D_{E_i}X_j,\dots,X_p)$$
$$=\sum_{l<j}\sum_{i=1}^{n}T(X_1,\dots,D_{E_i}X_l,\dots,\nabla_{E_i}X_j,\dots,X_p)-\sum_{l<j}T(X_1,\dots,D_{X_j}X_l,\dots,\overbrace{N}^{j},\dots,X_p)$$
$$=\sum_{l<j}\sum_{i=1}^{n}T(X_1,\dots,\nabla_{E_i}X_l,\dots,\nabla_{E_i}X_j,\dots,X_p)-\sum_{l<j}T(X_1,\dots,\overbrace{N}^{l},\dots,\nabla_{X_l}X_j,\dots,X_p)$$
$$-\sum_{l<j}T(X_1,\dots,D_{X_j}X_l,\dots,\overbrace{N}^{j},\dots,X_p)$$
$$=\sum_{l<j}\sum_{i=1}^{n}T(X_1,\dots,\nabla_{E_i}X_l,\dots,\nabla_{E_i}X_j,\dots,X_p)-\sum_{l<j}T(X_1,\dots,D_{X_j}X_l,\dots,\overbrace{N}^{j},\dots,X_p)$$
$$-\sum_{l<j}T(X_1,\dots,\overbrace{N}^{l},\dots,D_{X_l}X_j,\dots,X_p)$$
$$-\sum_{l<j}\langle X_l,X_j\rangle T(X_1,\dots,\overbrace{N}^{l},\dots,\overbrace{N}^{j},\dots,X_p).$$

So we get, in restriction to S^n, since $D_N N = 0$

$$\begin{aligned}
&\Delta_{\mathbb{R}^{n+1}}(X_1,\dots,X_p) - \nabla^*\nabla T(X_1,\dots,X_p)\\
&= p(n+1)T(X_1,\dots,X_p) - nL_N T(X_1,\dots,X_p) - 2\sum_{j=1}^{p} D_{X_j}(i_{N,j}T)(X_1,\dots,\hat{X}_j,\dots,X_p)\\
&\quad + 2\sum_{l<j}\langle X_l, X_j\rangle T(X_1,\dots,\overbrace{N}^{l},\dots,\overbrace{N}^{j},\dots,X_p) - N.N.T(X_1,\dots,X_p)\\
&\quad + 2\sum_{j=1}^{p} N.T(X_1,\dots,D_N X_j,\dots,X_p) - \sum_{j=1}^{p} T(X_1,\dots,D_N D_N X_j,\dots,X_p)\\
&\quad - 2\sum_{i<j} T(X_1,\dots,D_N X_i,\dots,D_N X_j,\dots,X_p).
\end{aligned}$$

Remark that, in restriction to S^n, the following equality holds

$$\sum_{j=1}^{p} D_{X_j}(i_{N,j}T)(X_1,\dots,\hat{X}_j,\dots,X_p) = \sum_{j=1}^{p} D_{X_j}(i_{\vec{r},j}T)(X_1,\dots,\hat{X}_j,\dots,X_p).$$

Now by using (7) and (11) and by taking the restriction to S^n, we have

$$\begin{aligned}
&2\sum_{j=1}^{p} N.T(X_1,\dots,D_N X_j,\dots,X_p)\\
&= 2\sum_{j=1}^{p} N.T(X_1,\dots,[N,X_j],\dots,X_p) + 2\sum_{j=1}^{p} N\left(\frac{1}{r}\right)T(X_1,\dots,X_j,\dots,X_p)\\
&\quad - 2\sum_{j=1}^{p} N(\langle X_j, N\rangle)T(X_1,\dots,\overbrace{N}^{j},\dots,X_p) + 2\sum_{j=1}^{p} N.T(X_1,\dots,X_j,\dots,X_p)\\
&= 2\sum_{j=1}^{p} N.T(X_1,\dots,[N,X_j],\dots,X_p) - 2pT(X_1,\dots,X_p) + 2pN.T(X_1,\dots,X_j,\dots,X_p)\\
&\quad - 2\sum_{j=1}^{p} N(\langle X_j, N\rangle)T(X_1,\dots,\overbrace{N}^{j},\dots,X_p).
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{p} T(X_1,\dots,D_N D_N X_j,\dots,X_p)\\
&= \sum_{j=1}^{p} T(X_1,\dots,[N,[N,X_j],\dots,X_p) - 2\sum_{j=1}^{p} N(\langle X_j, N\rangle)T(X_1,\dots,\overbrace{N}^{j},\dots,X_p).
\end{aligned}$$

$$\begin{aligned}
&\sum_{i<j} T(X_1, \ldots, D_N X_i, \ldots, D_N X_j, \ldots, X_p) \\
&= \sum_{i<j} T(X_1, \ldots, [N, X_i], \ldots, [N, X_j], \ldots, X_p) + \frac{p(p-1)}{2} T(X_1, \ldots, X_p) \\
&\quad + \sum_{i<j} T(X_1, \ldots, X_i, \ldots, [N, X_j], \ldots, X_p) + \sum_{i<j} T(X_1, \ldots, [N, X_i], \ldots, X_j, \ldots, X_p).
\end{aligned}$$

So we get, in restriction to S^n

$$\begin{aligned}
&-N.N.T(X_1, \ldots, X_p) + 2\sum_{j=1}^{p} N.T(X_1, \ldots, D_N X_j, \ldots, X_p) \\
&\quad - \sum_{j=1}^{p} T(X_1, \ldots, D_N D_N X_j, \ldots, X_p) - 2\sum_{i<j} T(X_1, \ldots, D_N X_i, \ldots, D_N X_j, \ldots, X_p) \\
&\quad = -L_N \circ L_N T(X_1, \ldots, X_p) + 2pL_N T(X_1, \ldots, X_p) - p(1+p)T(X_1, \ldots, X_p).
\end{aligned}$$

The curvature of S^n is given by

$$R(X, Y)Z = \langle X, Y\rangle Z - \langle Y, Z\rangle X$$

and

$$r(X) = (n-1)X.$$

Hence, a direct computation gives that the curvature operator is given by

$$\begin{aligned}
R(T)(X_1, \ldots, X_p) &= p(n-1)T(X_1, \ldots, X_p) + 2\sum_{\sigma\in\mathcal{T}} T^{\sigma}(X_1, \ldots, X_p) \\
&\quad - 2\sum_{i<j}\sum_{l=1}^{n} \langle X_i, X_j\rangle T(X_1, \ldots, E_l, \ldots, E_l, \ldots, X_p).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&i^*(\Delta_{\mathbb{R}^{n+1}} T) = \Delta_{S^n} i^* T \\
&\qquad + i^*\left(p(1-p)T + (2p-n)L_N T - L_N \circ L_N T - 2\sum_{\sigma\in\mathcal{T}} T^{\sigma} + O(T)\right),
\end{aligned}$$

One can conclude the proof by remarking that

$$i^*(L_N T) = i^*(L_{\vec{r}} T)$$

and

$$i^*(L_N \circ L_N T) = -i^*(L_{\vec{r}} T) + i^*(L_{\vec{r}} \circ L_{\vec{r}} T). \qquad \square$$

Corollary 2.1. *Let α be a differential p-form on $\mathbb{R}^{n+1}$. Then*

$$i^*(\Delta_{\mathbb{R}^{n+1}}\alpha) = \Delta_{S^n} i^*\alpha + i^*((2p-n+1)L_{\vec{r}}\alpha - L_{\vec{r}} \circ L_{\vec{r}}\alpha - 2di_{\vec{r}}\alpha)$$

Corollary 2.2. *Let T be a symmetric p-tensor field on $\mathbb{R}^{n+1}$. Then*

$$\begin{aligned} i^*(\Delta_{\mathbb{R}^{n+1}}T) = \Delta_{S^n} i^*T + i^*(2p(1-p)T + (2p-n+1)L_{\vec{r}}T - L_{\vec{r}} \circ L_{\vec{r}}T \\ - 2\delta^*(i_{\vec{r}}T) + 2\,\mathrm{Tr}(T) \odot \langle\ ,\ \rangle), \end{aligned}$$

where $\odot$ is the symmetric product.

3. Eigenvalues and eigenforms of Δ_{S^n} acting on $\Omega(S^n)$

In this section, we will use Corollary 2.1 and the results developed in [10] to deduce the eigenvalues and the spaces of eigenforms of Δ_{S^n} acting on $\Omega^*(S^n)$. We recover the results of [10] and [11] in a more precise form.

Let $\bigwedge^p H_k$ be the space of all coclosed harmonic homogeneous p-forms of degree k on $\mathbb{R}^{n+1}$. A differential form α belongs to $\bigwedge^p H_k$ if $\delta(\alpha) = 0$ and α can be written

$$\alpha = \sum_{1 \le i_1 < \cdots < i_p \le n+1} \alpha_{i_1 \cdots i_p}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $\alpha_{i_1 \cdots i_p}$ are harmonic polynomial functions on $\mathbb{R}^{n+1}$ of degree k. For any $\alpha \in \bigwedge^p H_k$, we have

$$L_{\vec{r}}\alpha = di_{\vec{r}}\alpha + i_{\vec{r}}\, d\alpha = (k+p)\alpha. \tag{12}$$

We have (see [10])

$$i^*\colon \sum_{k \ge 0} \bigwedge^p H_k \to \Omega^p(S^n)$$

is injective and its image is dense.

For any $\alpha \in \bigwedge^p H_k$, we put

$$\omega(\alpha) = \alpha - \frac{1}{p+k}\, di_{\vec{r}}\alpha. \tag{13}$$

Lemma 3.1. *We get a linear map* ω: $\bigwedge^p H_k \to \bigwedge^p H_k$ *which is a projector, i.e.,* $\omega \circ \omega = \omega$. *Moreover,*

$$\operatorname{Ker} \omega = d\left(\bigwedge^{p-1} H_{k+1}\right), \quad \operatorname{Im} \omega = \bigwedge^{p} H_k \cap \operatorname{Ker} i_{\vec{r}},$$

and hence

$$\bigwedge^{p} H_k = \bigwedge^{p} H_k \cap \operatorname{Ker} i_{\vec{r}} \oplus d\left(\bigwedge^{p-1} H_{k+1}\right).$$

The following lemma is an immediate consequence of Corollary 2.1 and (12).

Lemma 3.2. 1. *For any* $\alpha \in \bigwedge^p H_k \cap \operatorname{Ker} i_{\vec{r}}$, *we have*

$$\Delta_{S^n} i^* \alpha = (k+p)(k+n-p-1) i^* \alpha.$$

2. *For any* $\alpha \in d\big(\bigwedge^{p-1} H_{k+1}\big)$, *we have*

$$\Delta_{S^n} i^* \alpha = (k+p)(k+n-p+1) i^* \alpha.$$

REMARK 3.1. We have

$$(k+p)(k+n-p-1) = (k'+p)(k'+n-p+1) \Leftrightarrow k = k'+1$$

and

$$n = 2p.$$

The following table gives explicitly the spectra of Δ_{S^n} and the spaces of eigenforms with its multiplicities. The multiplicity was computed in [11].

Table I.

p	The eigenvalues	The space of eigenforms	Multiplicity
$p = 0$	$k(k+n-1)$, $k \in \mathbb{N}$	$\bigwedge^0 H_k$	$\frac{(n+k-2)!\,(n+2k-1)}{k!\,(n-1)!}$
$1 \le p \le n$, $n \ne 2p$	$(k+p)(k+n-p-1)$, $k \in \mathbb{N}^*$	$\omega\big(\bigwedge^p H_k\big)$	$\frac{(n+k-1)!\,(n+2k-1)}{p!\,(k-1)!\,(n-p-1)!\,(n+k-p-1)(k+p)}$
	$(k+p)(k+n-p+1)$, $k \in \mathbb{N}$	$d\big(\bigwedge^{p-1} H_{k+1}\big)$	$\frac{(n+k)!\,(n+2k+1)}{(p-1)!\,k!\,(n-p)!\,(n+k-p+1)(k+p)}$
$1 \le p \le n$, $n = 2p$	$(k+p)(k+p+1)$, $k \in \mathbb{N}$	$\omega\big(\bigwedge^p H_{k+1}\big) \oplus d\big(\bigwedge^{p-1} H_{k+1}\big)$	$\frac{2(2p+k)!\,(2p+2k+1)}{p!\,(p-1)!\,k!\,(k+p+1)(k+p)}$

4. Eigenvalues and eigentensors of Δ_{S^n} acting on $\mathcal{S}(S^n)$

This section is devoted to the determination of the eigenvalues and the spaces of eigentensors of Δ_{S^n} acting on $\mathcal{S}(S^n)$.

Let $\mathcal{S}^pP_k$ be the space of $T \in \mathcal{S}^p(\mathbb{R}^{n+1})$ of the form

$$T = \sum_{1\leq i_1\leq\cdots\leq i_p\leq n+1} T_{i_1\cdots i_p}\, dx_{i_1} \odot \cdots \odot dx_{i_p},$$

where $T_{i_1\cdots i_p}$ are homogeneous polynomials of degree k. We put

$$\mathcal{S}^pH_k^\delta = \mathcal{S}^pP_k \cap \operatorname{Ker}\Delta_{\mathbb{R}^{n+1}} \cap \operatorname{Ker}\delta$$

and

$$\mathcal{S}^pH_k^{\delta 0} = \mathcal{S}^pH_k^\delta \cap \operatorname{Ker}\operatorname{Tr}.$$

In a similar manner as in [10] Lemma 6.4 and Corollary 6.6, we have

$$\mathcal{S}^pP_k = \mathcal{S}^pH_k^\delta \oplus (r^2\mathcal{S}^pP_{k-2} + dr^2 \odot \mathcal{S}^{p-1}P_{k-1}), \tag{14}$$

and

$$i^*\colon \sum_{k\geq 0}\mathcal{S}^pH_k^\delta \to \mathcal{S}^pS^n$$

is injective and its image is dense in $\mathcal{S}^pS^n$.

Now, for any $k \geq 0$, we proceed to give a direct sum decomposition of $\mathcal{S}^pH_k^\delta$ consisting of eigenspaces of Δ_{S^n} and, hence, we determine completely the eigenvalues of Δ_{S^n} acting on $\mathcal{S}^p(S^n)$. This will be done in several steps.

At first, we have the following direct sum decomposition:

$$\mathcal{S}^pH_k^\delta = \mathcal{S}^pH_k^{\delta 0} \oplus \bigoplus_{l=1}^{[p/2]} \mathcal{S}^{p-2l}H_k^{\delta 0} \odot \langle\ ,\ \rangle^l, \tag{15}$$

where $\langle\ ,\ \rangle^l$ is the symmetric product of l copies of $\langle\ ,\ \rangle$.

The task is now to decompose $\mathcal{S}^pH_k^{\delta 0}$ as a sum of eigenspaces of Δ_{S^n} and get, according to (5), all the eigenvalues. This decomposition needs some preparation.

Lemma 4.1. *Let $T \in \mathcal{S}^pP_k$ and $h \in \mathbb{N}^*$. Then we have the following formulas*:

1. $\delta^*(i_{\vec r}T) - i_{\vec r}\delta^*(T) = (p-k)T$;
2. $\delta^{*(h)}(i_{\vec r}T) - i_{\vec r}\delta^{*(h)}(T) = h(p-k+h-1)\delta^{*(h-1)}(T)$;
3. $\delta^*(i_{\vec r^h}T) - i_{\vec r^h}\delta^*(T) = h(p-k-h+1)i_{\vec r^{h-1}}T$,

where $i_{\vec{r}^h} = \overbrace{i_{\vec{r}} \circ \cdots \circ i_{\vec{r}}}^{h}$ *and* $\delta^{*(h)} = \overbrace{\delta^* \circ \cdots \circ \delta^*}^{h}$.

Proof. The first formula is easily verified and the others follow by induction on h. □

Note that the spaces $\mathcal{S}^p H_k^{\delta 0}$ are invariant by δ^* and $i_{\vec{r}}$; this is a consequence of the following formulas which one can check easily. For any symmetric tensor field T on $\mathbb{R}^{n+1}$, we have

$$\Delta_{\mathbb{R}^{n+1}}(i_{\vec{r}}T) = i_{\vec{r}}\Delta_{\mathbb{R}^{n+1}}(T) + 2\delta T, \tag{16}$$

$$\delta(i_{\vec{r}}T) = i_{\vec{r}}\delta(T) - \mathrm{Tr}(T), \tag{17}$$

$$\mathrm{Tr}(\delta^*(T)) = -2\delta(T) + \delta^*(\mathrm{Tr}(T)), \tag{18}$$

$$\mathrm{Tr}(i_{\vec{r}}T) = i_{\vec{r}}\,\mathrm{Tr}(T). \tag{19}$$

Now the desired decomposition of $\mathcal{S}^p H_k^{\delta 0}$ is based on the following algebraic lemma.

Lemma 4.2. *Let V be a finite dimensional vectorial space, ϕ and ψ are two endomorphisms of V and $(A_k^p)_{k,p\in\mathbb{N}\cup\{-1\}}$ a family of vectorial subspaces of V such that*:

1. *for any $p, k \in \mathbb{N}$, $A_{-1}^p = A_k^{-1} = 0$;*
2. *for any $p, k \in \mathbb{N}$, $\phi(A_k^p) \subset A_{k-1}^{p+1}$ and $\psi(A_k^p) \subset A_{k+1}^{p-1}$;*
3. *for any $p, k \in \mathbb{N}$ and for any $a \in A_k^p$,*

$$\phi \circ \psi(a) - \psi \circ \phi(a) = (p-k)a.$$

Then:

(i) *for any $k < p$, $\psi : A_k^p \to A_{k+1}^{p-1}$ is injective*;

(ii) *for $k \leq p$, we have*

$$A_k^p = (A_k^p \cap \mathrm{Ker}\,\phi) \oplus \psi\left(A_{k-1}^{p+1}\right)$$

and

$$A_k^p = \bigoplus_{l=0}^{k} \psi^l\left(A_{k-l}^{p+l} \cap \mathrm{Ker}\,\phi\right).$$

Proof. Note that one can deduce easily, by induction, that for any $l \in \mathbb{N}^*$ and for any $a \in A_k^p$

$$\phi^l \circ \psi(a) - \psi \circ \phi^l(a) = l(p-k+l-1)\phi^{l-1}(a), \tag{20}$$

$$\psi^l \circ \phi(a) - \phi \circ \psi^l(a) = l(k-p+l-1)\psi^{l-1}(a). \tag{21}$$

(i) Let $a \in A_k^p$ such that $\psi(a) = 0$. From (20) and since $p - k > 0$, for any $l \geq 0$, if $\phi^l(a) = 0$ then $\phi^{l-1}(a) = 0$. Now, since $\phi^l(a) \in A_{k-l}^{p+l}$ and since $A_{-1}^{p+l} = 0$, we have, for any $l \geq k+1$, $\phi^l(a) = 0$ which implies, by induction, that $a = 0$ and hence $\psi : A_k^p \to A_{k+1}^{p-1}$ is injective.

(ii) Suppose that $k \leq p$. We define $P_k^p : A_k^p \to A_k^p$ as follows

$$\begin{cases} P_k^p(a) = \displaystyle\sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^s(a) \\ \alpha_0 = 1 \text{ and } \alpha_s - (s+1)(k-p-s-2)\alpha_{s+1} = 0 \quad \text{for} \quad 1 \leq s \leq k-1. \end{cases}$$

P_k^p satisfies

$$P_k^p \circ P_k^p = P_k^p, \quad \operatorname{Ker} P_k^p = \psi\left(A_{k-1}^{p+1}\right)$$

and

$$\operatorname{Im} P_k^p = A_k^p \cap \operatorname{Ker} \phi.$$

Indeed, let $a \in A_{k-1}^{p+1}$. We have

$$\begin{aligned} P_k^p(\psi(a)) &= \sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^s(\psi(a)) \\ &\overset{(20)}{=} \sum_{s=0}^{k} \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=0}^{k} s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &\overset{\phi^k(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=1}^{k} s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &= \sum_{s=0}^{k-1} (\alpha_s + (s+1)(p-k+s+2)\alpha_{s+1})\psi^{s+1} \circ \phi^s(a) \\ &= 0. \end{aligned}$$

Conversely, since $P_k^p(a) = a + \sum_{s=1}^{k} \alpha_s \psi^s \circ \phi^s(a)$, we deduce that $P_k^p(a) = 0$ implies that $a \in \psi(A_{k-1}^{p+1})$, so we have shown that $\operatorname{Ker} P_k^p = \psi(A_{k-1}^{p+1})$. The relation $P_k^p \circ P_k^p = P_k^p$ is a consequence of the definition of P_k^p and $P_k^p \circ \psi = 0$.

Note that $\phi(a) = 0$ implies that $P_k^p(a) = a$ and hence $A_k^p \cap \operatorname{Ker}\phi \subset \operatorname{Im} P_k^p$. Conversely, let $a \in A_k^p$, we have

$$\begin{aligned}
\phi \circ P_k^p(a) &= \sum_{s=0}^{k} \alpha_s \phi \circ \psi^s \circ \phi^s(a) \\
&\overset{(21)}{=} \sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=0}^{k} \alpha_s s(k-p-s-1)\psi^{s-1} \circ \phi^s(a) \\
&\overset{\phi^{k+1}(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=1}^{k} \alpha_s s(k-p-s-1)\psi^{s-1} \circ \phi^s(a) \\
&= \sum_{s=0}^{k-1} (\alpha_s - (s+1)(k-p-s-2)\alpha_{s+1})\psi^s \circ \phi^{s+1}(a) \\
&= 0.
\end{aligned}$$

We conclude that P_k^p is a projector, $\operatorname{Ker} P_k^p = \psi\left(A_{k-1}^{p+1}\right)$ and $A_k^p \cap \operatorname{Ker}\phi = \operatorname{Im} P_k^p$ and we deduce immediately that $A_k^p = \psi\left(A_{k-1}^{p+1}\right) \oplus A_k^p \cap \operatorname{Ker}\phi$. The same decomposition holds for A_{k-1}^{p+1} and, since $\psi\colon A_{k-1}^{p+1} \to A_k^p$ is injective, we get

$$A_k^p = \psi \circ \psi\left(A_{k-2}^{p+2}\right) \oplus \psi\left(A_{k-1}^{p+1} \cap \operatorname{Ker}\phi\right) \oplus A_k^p \cap \operatorname{Ker}\phi.$$

We proceed by induction and we get the desired decomposition. □

According to Lemma 4.1, the hypothesis of Lemma 4.2 are satisfied by the spaces $\mathcal{S}^p H_k^{\delta 0}$ and the operators δ^* and $i_{\vec{r}}$. So we get, in a first time,

$$\mathcal{S}^p H_k^{\delta 0} = \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker}\delta^* \oplus i_{\vec{r}}\left(\mathcal{S}^{p+1} H_{k-1}^{\delta 0}\right), \quad \text{if} \quad k \leq p, \tag{22}$$

$$\mathcal{S}^p H_k^{\delta 0} = \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}} \oplus \delta^*\left(\mathcal{S}^{p-1} H_{k+1}^{\delta 0}\right), \quad \text{if} \quad k \geq p, \tag{23}$$

and, in a second time, the desired decomposition of $\mathcal{S}^p H_k^{\delta 0}$.

Lemma 4.3. *We have*:

1. *If* $k \leq p$

$$\mathcal{S}^p H_k^{\delta 0} = \bigoplus_{l=0}^{k} i_{\vec{r}^l}\left(\mathcal{S}^{p+l} H_{k-l}^{\delta 0} \cap \operatorname{Ker}\delta^*\right);$$

2. *If* $k \geq p$

$$\mathcal{S}^p H_k^{\delta 0} = \bigoplus_{l=0}^{p} \delta^{*l}\left(\mathcal{S}^{p-l} H_{k+l}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}\right);$$

3. *If $k = p$, for any $0 \le l \le p$,*

$$\mathcal{S}^p H_p^{\delta 0} = \bigoplus_{l=0}^{p} i_{\vec{r}^l}\left(\mathcal{S}^{p+l} H_{p-l}^{\delta 0} \cap \operatorname{Ker} \delta^*\right) = \bigoplus_{l=0}^{p} \delta^{*l}\left(\mathcal{S}^{p-l} H_{p+l}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}\right).$$

Now, we use Corollary 2.2 to show that the decompositions of $\mathcal{S}^p H_k^{\delta 0}$ given in Lemma 4.3 are composed by eigenspaces of Δ_{S^n}.

Theorem 4.1. *We have*:

1. *If $k \le p$, for any $0 \le q \le k$ and any $T \in i_{\vec{r}^{(k-q)}}(\mathcal{S}^{p+k-q} H_q^{\delta 0} \cap \operatorname{Ker} \delta^*)$,*

$$\Delta_{S^n} i^* T = ((k+p)(n+p+k-2q-1) + 2q(q-1)) i^* T;$$

2. *If $k \ge p$, for any $0 \le q \le p$ and for any $T \in \delta^{*(p-q)}(\mathcal{S}^q H_{k+p-q}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}})$,*

$$\Delta_{S^n} i^* T = ((k+p)(n+p+k-2q-1) + 2q(q-1)) i^* T.$$

Proof. 1. Let $T = i_{\vec{r}^{(k-q)}}(T_0)$ with $T_0 \in \mathcal{S}^{p+k-q} H_q^{\delta 0} \cap Ker\, \delta^*$. We have from Corollary 2.2

$$\begin{aligned}\Delta_{S^n} i^* T = i^*(&2p(p-1)T + (n-2p-1)L_{\vec{r}} T + L_{\vec{r}} \circ L_{\vec{r}} T \\ &+ 2\delta^*(i_{\vec{r}} T) - 2\operatorname{Tr}(T) \odot \langle\ ,\ \rangle).\end{aligned}$$

We have

$$\operatorname{Tr} T = 0, \quad L_{\vec{r}} T = (k+p) T$$

and

$$L_{\vec{r}} \circ L_{\vec{r}} T = (k+p)^2 T.$$

Moreover, by using Lemma 4.1, we have

$$\begin{aligned}2\delta^*(i_{\vec{r}} T) &= 2\delta^*(i_{\vec{r}^{(k-q+1)}} T_0) \\ &\overset{\delta^*(T_0)=0}{=} 2(k-q+1)(p+k-q-q-k+q-1+1) i_{\vec{r}^{(k-q)}} T_0 \\ &= 2(k-q+1)(p-q) T.\end{aligned}$$

Hence

$$\Delta_{S^n} i^* T = (2p(p-1) + (n-2p-1)(k+p) + (k+p)^2 + 2(p-q)(k-q+1)) i^* T.$$

One can deduce the desired relation by remarking that

$$2p(p-1) + 2(p-q)(k-q+1) = 2(k+p)(p-q) + 2q(q-1).$$

2. This follows by the same calculation as 1. □

From the fact that

$$i^*: \sum_{k\geq 0} \mathcal{S}^p H_k^\delta \to \mathcal{S}^p S^n$$

is injective and its image is dense in $\mathcal{S}^p S^n$, from (15), and from Lemma 4.3 and Theorem 4.1, note that we have actually proved that the eigenvalues of Δ_{S^n} acting on $\mathcal{S}^p S^n$ belongs to

$$\Big\{(k+p-2l)(n+p+k-2l-2q-1)+2q(q-1),$$
$$k \in \mathbb{N},\ 0 \leq l \leq \left[\frac{p}{2}\right],\ 0 \leq q \leq \min(k,\ p-2l)\Big\}.$$

Our next goal is to sharpen this result by computing $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*$ if $k \leq p$ and $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$ if $k \geq p$.

Lemma 4.4. *We have the following formulas*:

1. $\dim \mathcal{S}^p H_k^\delta = \dim \mathcal{S}^p P_k - \dim \mathcal{S}^p P_{k-2} - \dim \mathcal{S}^{p-1} P_{k-1} + \dim \mathcal{S}^{p-1} P_{k-3}$,
2. $\dim \mathcal{S}^p H_k^{\delta 0} = \dim \mathcal{S}^p H_k^\delta - \dim \mathcal{S}^{p-2} H_k^\delta$,
3. $\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*) = \dim \mathcal{S}^p H_k^{\delta 0} - \dim \mathcal{S}^{p+1} H_{k-1}^{\delta 0}$ $(k \leq p)$,
4. $\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}) = \dim \mathcal{S}^p H_k^{\delta 0} - \dim \mathcal{S}^{p-1} H_{k+1}^{\delta 0}$ $(k \geq p)$.

Note that we use the convention that $\mathcal{S}^p P_k = \mathcal{S}^p H_k^\delta = \mathcal{S}^p H_k^{\delta 0} = 0$ *if* $k < 0$ *or* $p < 0$.

Proof. 1. The formula is a consequence of (14), the relation

$$(r^2 \mathcal{S}^p P_{k-2}) \cap (dr^2 \odot \mathcal{S}^{p-1} P_{k-1}) = r^2 (dr^2 \odot \mathcal{S}^{p-1} P_{k-3})$$

and the fact that $dr^2 \odot .: \mathcal{S}^p P_k \to \mathcal{S}^{p+1} P_{k+1}$ is injective.

2. The formula is a consequence of (15).
3. The formula is a consequence of (22) and Lemma 4.2.
4. The formula is a consequence of (23) and Lemma 4.2. □

A straightforward calculation using Lemma 4.4 and the formula

$$\dim \mathcal{S}^p P_k = \frac{(n+p)!}{n!\,p!} \frac{(n+k)!}{n!\,k!}$$

gives $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*$ if $k \leq p$ and $\dim \mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$ if $k \geq p$. We summarize the results on the following table.

Table II.

Space	Dimension	Conditions on k and p
$\mathcal{S}^0 H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$	$\frac{(n+k-2)!\,(n+2k-1)}{k!\,(n-1)!}$	$k \geq 0$
$\mathcal{S}^p H_0^{\delta 0} \cap \operatorname{Ker} \delta^*$	$\frac{(n+p-2)!\,(n+2p-1)}{p!\,(n-1)!}$	$p \geq 0$
$\mathcal{S}^1 H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$	$\frac{(n+k-3)!\,k(n+2k-1)(n+k-1)}{(n-2)!\,(k+1)!}$	$k \geq 1$
$\mathcal{S}^p H_1^{\delta 0} \cap \operatorname{Ker} \delta^*$	$\frac{(n+p-3)!\,p(n+2p-1)(n+p-1)}{(n-2)!\,(p+1)!}$	$p \geq 1$
$\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*$	$\frac{(n+k-4)!\,(n+p-3)!\,(n+p+k-2)}{k!\,(p+1)!\,(n-1)!\,(n-2)!} \times (n-2)(n+2k-3)(n+2p-1)(p-k+1)$	$2 \leq k \leq p$
$\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}$	$\frac{(n+k-3)!\,(n+p-4)!\,(n+p+k-2)}{(k+1)!\,p!\,(n-1)!\,(n-2)!} \times (n-2)(n+2k-1)(n+2p-3)(k-p+1)$	$k \geq p \geq 2$

REMARK 4.1. Note that, for $n = 2$, we have

$$\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} \delta^*) = 0 \quad \text{for} \quad 2 \leq k \leq p,$$
$$\dim(\mathcal{S}^p H_k^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}) = 0 \quad \text{for} \quad k \geq p \geq 2.$$

For simplicity we introduce the following notations.

$$S_0 = \left\{(k, l, q) \in \mathbb{N}^3,\ 0 \leq l \leq \left[\frac{p}{2}\right],\ 0 \leq k \leq p - 2l,\ 0 \leq q \leq k\right\},$$
$$S_1 = \left\{(k, l, q) \in \mathbb{N}^3,\ 0 \leq l \leq \left[\frac{p}{2}\right],\ k > p - 2l,\ 0 \leq q \leq p - 2l\right\},$$
$$V_{q,l}^k = i_{\vec{r}^{k-q}}\left(\mathcal{S}^{p-2l+k-q} H_q^{\delta 0} \cap \operatorname{Ker} \delta^*\right) \odot \langle\ ,\ \rangle^l \quad \text{for} \quad (k, l, q) \in S_0,$$
$$W_{q,l}^k = \delta^{*(p-2l-q)}\left(\mathcal{S}^q H_{p-2l+k-q}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}\right) \odot \langle\ ,\ \rangle^l \quad \text{for} \quad (k, l, q) \in S_1.$$

Let us summarize all the results above.

Theorem 4.2. 1. *For $n = 2$, we have*:
(a) *The set of the eigenvalues of Δ_{S^2} acting on $\mathcal{S}^p S^2$ is*

$$\left\{(k + p - 2l)(p + k - 2l + 1), \quad k \in \mathbb{N},\ 0 \leq l \leq \left[\frac{p}{2}\right]\right\};$$

(b) *The eigenspace associated to the eigenvalue* $\lambda(k,l) = (k+p-2l)(k+p-2l+1)$ *is given by*

$$V_{\lambda(k,l)} = \begin{cases} \bigoplus_{a=0}^{\min(l,[k/2])} \left(V_{0,l-a}^{k-2a} \oplus V_{1,l-a}^{k+1-2a}\right) & \text{if} \quad 0 \le k \le p-2l, \\ \bigoplus_{a=0}^{\min(l,[k/2])} \left(W_{0,l-a}^{k-2a} \oplus W_{1,l-a}^{k+1-2a}\right) & \text{if} \quad k > p-2l; \end{cases}$$

(c) *The multiplicity of* $\lambda(k,l)$ *is given by*

$$m(\lambda(k,l)) = 2\left(\min\left(l, \left[\frac{k}{2}\right]\right) + 1\right)(1 + 2p + 2k - 4l).$$

2. *For* $n \ge 3$, *we have*:
(a) *The set of the eigenvalues of* Δ_{S^n} *acting on* $\mathcal{S}^p S^n$ *is*

$$\left\{(k+p-2l)(n+p+k-2l-2q-1) + 2q(q-1),\right.$$
$$\left. k \in \mathbb{N},\ 0 \le l \le \left[\frac{p}{2}\right],\ 0 \le q \le \min(k,\ p-2l)\right\};$$

(b) *The space*

$$\mathcal{P} = \sum_{k \ge 0} \mathcal{S}^p H_k^\delta = \left(\bigoplus_{(k,l,q)\in S_0} V_{q,l}^k\right) \oplus \left(\bigoplus_{(k,l,q)\in S_1} W_{q,l}^k\right)$$

is dense in $\mathcal{S}^p S^n$ *and, for any* $(k,q,l) \in S_0$ (*resp.* $(k,q,l) \in S_1$), $V_{q,l}^k$ (*resp.* $W_{q,l}^k$) *is a subspace of the eigenspace associated to the eigenvalue* $(k+p-2l)(n+p+k-2l-2q-1) + 2q(q-1)$;
(c) *The dimensions of* $V_{q,l}^k$ *and* $W_{q,l}^k$ *are given in Table II since*

$$\dim V_{q,l}^k = \dim(\mathcal{S}^{p-2l+k-q} H_q^{\delta 0} \cap \operatorname{Ker} \delta^*) \quad \text{for} \quad (k,l,q) \in S_0,$$
$$\dim W_{q,l}^k = \dim(\mathcal{S}^q H_{p-2l+k-q}^{\delta 0} \cap \operatorname{Ker} i_{\vec{r}}) \quad \text{for} \quad (k,l,q) \in S_1.$$

References

[1] E. Bedford and T. Suwa: *Eigenvalues of Hopf manifolds*, Proc. Amer. Math. Soc. **60** (1976), 259–264.

[2] B.L. Beers and R.S. Millman: *The spectra of the Laplace-Beltrami operator on compact, semisimple Lie groups,* Amer. J. Math. **99** (1977), 801–807.

[3] M. Berger and D. Ebin: *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Differential Geometry **3** (1969), 379–392.

[4] M. Berger, P. Gauduchon and E. Mazet: Le spectre d'une variété riemannienne, Lecture Notes in Math. **194**, Springer, Berlin, 1971.

[5] A.L. Besse: Einstein manifolds, Springer, Berlin, 1987.

[6] M. Boucetta: *Spectre des laplaciens de Lichnerowicz sur les sphères et les projectifs réels,* Publ. Mat. **43** (1999), 451–483.

[7] M. Boucetta: *Spectre du Laplacien de Lichnerowicz sur les projectifs complexes,* C.R. Acad. Sci. Paris Sér. I Math. **333** (2001), 571–576.

[8] S. Gallot and D. Meyer: *Opérateur de courbure et laplacien des formes différentielles d'une variété riemannienne*, J. Math. Pures Appl. (9) **54** (1975), 259–284.

[9] G.W. Gibbons and M.J. Perry: *Quantizing gravitational instantons*, Nuclear Phys. B **146** (1978), 90–108.

[10] A. Ikeda and Y. Taniguchi: *Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbf{C})$*, Osaka J. Math. **15** (1978), 515–546.

[11] I. Iwasaki and K. Katase: *On the spectra of Laplace operator on $\bigwedge^*(S^n)$*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), 141–145.

[12] E. Kaneda: *The spectra of 1-forms on simply connected compact irreducible Riemannian symmetric spaces*, J. Math. Kyoto Univ. **23** (1983), 369–395 and **24** (1984) 141–162.

[13] A. Lévy-Bruhl-Laperrière: *Spectre du de Rham-Hodge sur les formes de degré 1 des sphères de R^n $(n \geq 6)$*, Bull. Sci. Math. (2) **99** (1975), 213–240.

[14] A. Lévy-Bruhl-Laperrière: *Spectre du de Rham Hodge sur l'espace projectif complexe,* C.R. Acad. Sci. Paris Sér. A **284** (1977), 1265–1267.

[15] A. Lichnerowicz: *Propagateurs et commutateurs en relativité générale,* Inst. Hautes Études Sci. Publ. Math. **10** (1961), 56.

[16] K. Mashimo: *Spectra of Laplacian on $G_2/SO(4)$*, Bull. Fac. Gen. Ed. Tokyo Univ. of Agr. and Tech. **26** (1989), 85–92.

[17] K. Mashimo: *On branching theorem of the pair $(G_2, SU(3))$*, Nihonkai Math. J. **8** (1997), 101–107.

[18] K. Mashimo: *Spectra of the Laplacian on the Cayley projective plane*, Tsukuba J. Math. **21** (1997), 367–396.

[19] R. Michel: *Problèmes d'analyse géométrique liés à la conjecture de Blaschke*, Bull. Soc. Math. France **101** (1973), 17–69.

[20] K. Pilch and A.N. Schellekens: *Formulas for the eigenvalues of the Laplacian on tensor harmonics on symmetric coset spaces*, J. Math. Phys. **25** (1984), 3455–3459.

[21] C. Tsukamoto: *Spectra of Laplace-Beltrami operators on $\mathrm{SO}(n+2)/\mathrm{SO}(2) \times \mathrm{SO}(n)$ and $\mathrm{Sp}(n+1)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$*, Osaka J. Math. **18** (1981), 407–426.

[22] N.P. Warner: *The spectra of operators on $\mathbb{C}P^n$*, Proc. Roy. Soc. London Ser. A **383** (1982), 217–230.

Universite Cadi Ayyad
Faculté des Sciences et Techniques
BP 549 Marrakech
Morocco
e-mail: boucetta@fstg-marrakech.ac.ma