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BASIC TRANSFORMATIONS OF SYMMETRIC R-SPACES

Dedicated to Professor Ichiro Satake on his sixtieth birthday

MASARU TAKEUCHI

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Introduction

There is a class of compact symmetric spaces, the so-called irreducible symmetric R-spaces. Each irreducible symmetric R-space M has a big transformation group G, called the group of basic transformations, greater than the group of isometries of M. And every compact symmetric space with a big transformation group is essentially a symmetric R-space (Nagano [6]).

For example, the sphere is an irreducible symmetric R-space, and G is the group of conformal transformations. Also the projective space $P_n(\boldsymbol{F})$ over $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}$ or real quaternions $\boldsymbol{H}$ and the Cayley projective plane are irreducible symmetric R-spaces, and G are the group of projective transformations of $P_n(\boldsymbol{F})$ and the connected simple Lie group of type EIV, respectively. Furthermore, an irreducible Hermitian symmetric space M of compact type is an irreducible symmetric R-space, and G is the group of holomorphic transformations and anti-holomorphic transformations of M.

In this paper, we want to characterize these big groups G in terms of Riemannian geometry of M, except for spheres.

A submanifold S of M is called a Helgason sphere if (1) S is a totally geodesic sphere with minimum radius; and (2) S has the maximum dimension among the submanifolds in (1). We define a distance $d(p, q)$ of $p, q\in M$, called the arithmetic distance, to be the minimum possible length of a chain of Helgason spheres connecting p and q. We prove the following theorem.

Theorem. (i) *Let M be a compact rank one symmetric space other than spheres. Then the group G of basic transformations of M is identical with the group of diffeomorphisms which carry each Helgason sphere to a Helgason sphere.*

(ii) *Let M be an irreducible symmetric R-space of rank greater than one. Then the group G of basic transformations of M is identical with the group of diffeomorphisms which preserve the arithmetic distance d on M.*

Our problem was originated by Chow. In Chow [1] he studied the transformations of certain homogeneous algebraic manifolds by purely algebraic methods. When the ground field is the complex number field, these manifolds are the irreducible compact Hermitian symmetric spaces M of classical type, and his result may be stated as follows. With respect to a family $\mathcal{H}$ of complex submanifolds of M, a distance d on M is defined in the above way. For example, for the complex Grassmann manifold M, $\mathcal{H}$ is the family of projective lines lying on M, where M is regarded as a complex submanifold of a complex projective space by the Plücker imbedding. It is verified that for each space M $\mathcal{H}$ is nothing but the family of Helgason spheres in our sense. He proved that then the group of holomorphic and anti-holomorphic transformations of M is identical with the group of isometric bijections of (M, d), except for complex projective spaces. So our Theorem (ii) may be thought of as a generalization of the theorem of Chow under differentiability.

Peterson [8] studied the arithmetic distance on irreducible compact symmetric spaces defined by means of Helgason spheres of dimension greater than one. His method is different from ours and to use the Radon duality in the sense of Nagano [7].

Our method is as follows. The spaces M in Theorem (i) are the projective spaces, and the Helgason spheres are the projective lines. Thus Theorem (i) follows from the fundamental theorem in projective geometry, along with a theorem of Springer [11]. For the proof of Theorem (ii), we make use of the characterization of G by Tanaka [17] as the automorphism group $\mathrm{Aut}(P)$ of a G_0-structure P of M, G_0 being a Lie subgroup of $GL(n, \boldsymbol{R})$, $n=\dim M$. We will prove that $\mathrm{Aut}(P)$ is equal to the isometric diffeomorphism group of (M, d).

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1. Symmetric $\boldsymbol{R}$-spaces

In this section we give the definition of symmetric R-spaces and recall some properties of them (cf. Takeuchi [13], [15], Tanaka [17]).

Let

$$(1.1) \qquad \mathcal{G}\colon \mathfrak{g} = \mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1\,, \quad [\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$$

be a graded Lie algebra over $\boldsymbol{R}$ with $\mathfrak{g}$ real simple and $\mathfrak{g}_{-1}\neq\{0\}$, and τ a Cartan involution of $\mathfrak{g}$ with

$$(1.2) \qquad \tau\mathfrak{g}_p = \mathfrak{g}_{-p} \qquad (-1\leq p\leq 1)\,.$$

Such a pair $(\mathcal{G}, \tau)$ is called a *compact simple symmetric graded Lie algebra over* $\boldsymbol{R}$. Two compact simple symmetric graded Lie algebras over $\boldsymbol{R}$ $(\mathcal{G}, \tau)$ and $(\mathcal{G}', \tau')$ are said to be *isomorphic* to each other, if there is a Lie isomorphism $\varphi\colon \mathfrak{g}\to\mathfrak{g}'$ with $\varphi\mathfrak{g}_p=\mathfrak{g}'_p$ $(-1\leq p\leq 1)$ and $\varphi\circ\tau=\tau'\circ\varphi$.

Let $(\mathcal{G}, \tau)$ be a compact simple symmetric graded Lie algebra over $\boldsymbol{R}$. Then, from (1.1) and the semi-simplicity of $\mathfrak{g}$, there is uniquely an element $E \in \mathfrak{g}_0$ with

$$\mathfrak{g}_p = \{X \in \mathfrak{g};\ [E, X] = pX\} \qquad (-1 \leq p \leq 1), \tag{1.3}$$

which is called the *characteristic element* of $\mathcal{G}$. From (1.2) one has

$$\tau E = -E. \tag{1.4}$$

Now we define several subgroups of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. Let $\bar{\mathfrak{g}} = \mathfrak{g}^C$ denote the complexified Lie algebra of $\mathfrak{g}$ and regard $\mathrm{Aut}(\mathfrak{g})$ as a subgroup of $\mathrm{Aut}(\bar{\mathfrak{g}})$. Denoting by $\mathrm{Inn}(\bar{\mathfrak{g}})$ the group of inner automorphisms of $\bar{\mathfrak{g}}$, we define

$$G' = \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Inn}(\bar{\mathfrak{g}}),$$

which is an open normal subgroup of $\mathrm{Aut}(\mathfrak{g})$. Let G_0 denote the group of automorphisms of the graded Lie algebra $\mathcal{G}$, that is,

$$G_0 = \{a \in \mathrm{Aut}(\mathfrak{g});\ a\mathfrak{g}_p = \mathfrak{g}_p\ (-1 \leq p \leq 1)\},$$

which is also described as

$$G_0 = \{a \in \mathrm{Aut}(\mathfrak{g});\ aE = E\},$$

in virtue of (1.3). Under the identification of $\mathfrak{g}$ with $\mathrm{Lie\,Aut}(\mathfrak{g})$, the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$, through the adjoint representation, we have $\mathrm{Lie}\, G_0 = \mathfrak{g}_0$. Note that G_0 leaves $\mathfrak{g}_{\pm 1}$ invariant. Next we define an open subgroup G of $\mathrm{Aut}(\mathfrak{g})$, thus $\mathrm{Lie}\, G = \mathfrak{g}$, by

$$G = G_0 G'.$$

Then G_0 is a closed subgroup of G. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \tag{1.5}$$

be the Cartan decomposition associated to τ. Note that from (1.4) one has $E \in \mathfrak{p}$. Since $\tau\mathfrak{g}_0 = \mathfrak{g}_0$, (1.5) induces a Cartan decomposition of $\mathfrak{g}_0$:

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0, \quad \text{where} \quad \mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

Extending τ to a conjugate linear automorphism $\bar{\tau}$ of $\bar{\mathfrak{g}}$, and denoting by $(\ ,\ )$ the Killing form of $\bar{\mathfrak{g}}$, we define a Hermitian inner product $\langle\ ,\ \rangle$ on $\bar{\mathfrak{g}}$ by

$$\langle X, Y \rangle = -(X, \bar{\tau} Y) \qquad \text{for} \quad X, Y \in \bar{\mathfrak{g}},$$

which is invariant under the group

$$\mathrm{Aut}(\bar{\mathfrak{g}}, \bar{\tau}) = \{a \in \mathrm{Aut}(\bar{\mathfrak{g}});\ a\bar{\tau} = \bar{\tau} a\}.$$

Its restriction $\langle\ ,\ \rangle|\mathfrak{g}\times\mathfrak{g}$ to $\mathfrak{g}\times\mathfrak{g}$ is an inner product on $\mathfrak{g}$ which is invariant under the group

$$\mathrm{Aut}(\mathfrak{g}, \tau) = \{a\in\mathrm{Aut}(\mathfrak{g});\ a\tau = \tau a\}.$$

We define compact subgroups K and K_0 of G with Lie $K=\mathfrak{k}$ and Lie $K_0=\mathfrak{k}_0$ by

$$K = G\cap\mathrm{Aut}(\mathfrak{g}, \tau),$$
$$K_0 = G_0\cap\mathrm{Aut}(\mathfrak{g}, \tau) = G_0\cap K.$$

Then we have polar decompositions

$$G = K\exp\mathfrak{p}, \tag{1.6}$$
$$G_0 = K_0\exp\mathfrak{p}_0. \tag{1.7}$$

The second one is the polar decomposition of the self-adjoint (with respect to $\langle\ ,\ \rangle|\mathfrak{g}\times\mathfrak{g}$) real algebraic group $G_0\subset GL(\mathfrak{g})$. Also the first one follows from the polar decomposition

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g}, \tau)\exp\mathfrak{p}$$

of the self-adjoint real algebraic group $\mathrm{Aut}(\mathfrak{g})\subset GL(\mathfrak{g})$. In particular, K is a maximal compact subgroup of G. Next we define a parabolic subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ by

$$\mathfrak{u} = \mathfrak{g}_0+\mathfrak{g}_1,$$

and a closed subgroup U of G with Lie $U=\mathfrak{u}$ by

$$U = G_0\exp\mathfrak{g}_1, \tag{1.8}$$

which is also described as

$$U = \{a\in\mathrm{Aut}(\mathfrak{g});\ a\mathfrak{u} = \mathfrak{u}\}. \tag{1.9}$$

The homogeneous space

$$M = G/U \tag{1.10}$$

is called the *R-space* associated to $\mathcal{G}$, which is known to be compact, connected and real projective algebraic. The group G acts on M effectively, so that it is identified with a subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ of M. We call G the *group of basic transformations* of M. Furthermore it is shown that

$$G = KU, \quad K\cap U = K_0, \tag{1.11}$$

and hence we have a natural identification

$$M = K/K_0. \tag{1.12}$$

Our homogeneous space $M=G/U$ may be also described as a homogeneous space of the open normal subgroup G' of G in the following way. We define $G_0'=G'\cap G_0$, $K'=G'\cap K$, $K_0'=G'\cap K_0$, $U'=G'\cap U$. Then G_0', K', K_0', U' are open normal subgroups of G_0, K, K_0, U respectively, and equalities (1.6)~(1.12) hold also for these groups, which will be denoted by the same numbers with primes. From these we have

$$(1.13) \qquad G=K_0G', \quad G_0=K_0G_0', \quad U=K_0U', \quad K=K_0K',$$

which implies that

$$(1.14) \qquad G/G' \simeq G_0/G_0' \simeq U/U' \simeq K/K' \simeq K_0/K_0'.$$

Note that our group G is given also by

$$G=G_0\,\mathrm{Inn}(\mathfrak{g}),$$

where $\mathrm{Inn}(\mathfrak{g})$ denotes the group of inner automorphisms of $\mathfrak{g}$, since one has $G'=G_0'\,\mathrm{Inn}(\mathfrak{g})$ (Takeuchi [12]).

Next we want to define a Riemannian metric on M. We define $\theta\in\mathrm{Aut}(\mathfrak{g})$ with $\theta^2=1$ by

$$\theta|\mathfrak{g}_0=1, \quad \theta|(\mathfrak{g}_1+\mathfrak{g}_{-1})=-1,$$

where 1 designates the identity map. From $\theta\tau=\tau\theta$ and $\theta E=E$, it follows that $\theta\in G_0\cap\mathrm{Aut}(\mathfrak{g},\tau)=K_0$ and hence $\theta G\theta^{-1}=G$. Thus an automorphism θ of G whose differential is $\theta\in\mathrm{Aut}(\mathfrak{g})$, is defined by

$$\theta(a)=\theta a\theta^{-1} \qquad \text{for } a\in G.$$

It has the properties that $\theta(K)=K$, $\theta(k)=k$ for any $k\in K_0$, and $\mathfrak{k}_0=\{X\in\mathfrak{k};\ \theta X=X\}$, whence (K, K_0, θ) is a compact symmetric pair. We define a K_0-invariant subspace $\mathfrak{m}$ of $\mathfrak{k}$ with $\mathfrak{k}=\mathfrak{k}_0+\mathfrak{m}$ by

$$\mathfrak{m}=\{X\in\mathfrak{k};\ \theta X=-X\}=\{X\in\mathfrak{g}_1+\mathfrak{g}_{-1};\ \tau X=X\},$$

which is identified with the tangent space T_oM to M at the origin $o=U\in M$. Since $\langle\ ,\ \rangle|\mathfrak{m}\times\mathfrak{m}$ is a K_0-invariant inner product, for any $c>0$ there is uniquely a K-invariant Riemannian metric g on $M=K/K_0$ such that $g_o=c\langle\ ,\ \rangle|\mathfrak{m}\times\mathfrak{m}$. The Riemannian manifold (M, g) is a compact symmetric space *with a cubic lattice* in the sense that a maximal torus A_M of (M, g) has an expression

$$A_M=\boldsymbol{R}^r/\Gamma_M, \quad r=\mathrm{rank}(M, g)$$

by a lattice Γ_M generated by an orthogonal basis of $\boldsymbol{R}^r$ of the same length. Furthermore (M, g) is *irreducible* in the sense that it is not a Riemannian product of compact symmetric spaces with cubic lattices. Conversely, any irreducible (in the above sense) compact symmetric space (M, g) with a cubic lattice is obtained in this way from a compact simple symmetric graded Lie algebra $(\mathcal{G}, \tau)$ over $\boldsymbol{R}$ (unique up to isomorphism) and a constant $c>0$ (Loos [5]). Our symmetric space (M, g) is called an *irreducible symmetric R-space.* In the following, for the simplicity we assume that $c=1$.

REMARK 1.1. From the definition of g, it is obvious that K is a subgroup of the group $\mathrm{I}(M, g)$ of all isometries of (M, g). Actually, K is equal to $\mathrm{I}(M, g)$ (cf. Corollary 6.9).

Take a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ with $E\in\mathfrak{a}$, and extend it to a Cartan subalgebra $\mathfrak{h}=\mathfrak{b}+\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{b}\subset\mathfrak{k}$. Then the complexification $\bar{\mathfrak{h}}=\mathfrak{h}^C$ is a Cartan subalgebra of $\bar{\mathfrak{g}}$. The real part $\mathfrak{h}_{\boldsymbol{R}}$ of $\bar{\mathfrak{h}}$ is given by $\mathfrak{h}_{\boldsymbol{R}}=\sqrt{-1}\mathfrak{b}+\mathfrak{a}$. We identify the root system $\bar{\Sigma}$ of $\bar{\mathfrak{g}}$ relative to $\bar{\mathfrak{h}}$ with a subset of $\mathfrak{h}_{\boldsymbol{R}}$ by means of the duality defined by (,), and set

$$\bar{\Sigma}_p = \{\alpha\in\bar{\Sigma};\ (\alpha, E) = p\} \qquad (-1\leq p\leq 1)\,.$$

Let

$$\bar{\mathfrak{g}} = \bar{\mathfrak{h}}+\sum_{\alpha\in\bar{\Sigma}}\bar{\mathfrak{g}}^{\alpha}$$

be the $\bar{\mathfrak{h}}$-root space decomposition of $\bar{\mathfrak{g}}$. Then $\bar{\Sigma}=\bar{\Sigma}_{-1}\cup\bar{\Sigma}_0\cup\bar{\Sigma}_1$, and the complexifications $\bar{\mathfrak{g}}_p=\mathfrak{g}_p^C$ $(-1\leq p\leq 1)$ are given by

$$\bar{\mathfrak{g}}_0 = \bar{\mathfrak{h}}+\sum_{\alpha\in\bar{\Sigma}_0}\bar{\mathfrak{g}}^{\alpha}\,,\quad \bar{\mathfrak{g}}_{\pm1} = \sum_{\alpha\in\bar{\Sigma}_{\pm1}}\bar{\mathfrak{g}}^{\alpha}\,.$$

Denoting by $\pi_{\mathfrak{a}}\colon \mathfrak{h}_{\boldsymbol{R}}\to\mathfrak{a}$ the orthogonal projection, we set $\Sigma=\pi_{\mathfrak{a}}(\bar{\Sigma})-\{0\}$, which is the root system of $\mathfrak{g}$ relative to $\mathfrak{a}$, and set

$$\Sigma_p = \{\gamma\in\Sigma;\ (\gamma, E) = p\} \qquad (-1\leq p\leq 1)\,.$$

Then Σ is an irreducible reduced root system in $\mathfrak{a}$, $\Sigma=\Sigma_{-1}\cup\Sigma_0\cup\Sigma_1$, and the $\mathfrak{a}$-root space decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{g}^0+\sum_{\gamma\in\Sigma}\mathfrak{g}^{\gamma}\,,$$

where

$$\mathfrak{g}^{\lambda} = \{X\in\mathfrak{g};\ [H, X] = (\lambda, H)X \text{ for each } H\in\mathfrak{a}\}\,.$$

Let $m(\gamma)=\dim\mathfrak{g}^{\gamma}$ denote the multiplicity of $\gamma\in\Sigma$. Furthermore one has

$$\mathfrak{g}_0 = \mathfrak{g}^0+\sum_{\gamma\in\Sigma_0}\mathfrak{g}^{\gamma}\,,\quad \mathfrak{g}_{\pm1} = \sum_{\gamma\in\Sigma_{\pm1}}\mathfrak{g}^{\gamma}\,,$$

and thus Σ_0 is the root system of $\mathfrak{g}_0$ relative to $\mathfrak{a}$. Let σ denote the complex conjugation of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{g}$. Choose a σ-order $>$ on $\mathfrak{h}_{\boldsymbol{R}}$ in the sense of

Satake [10] and denote by $\bar{\Sigma}^+$ (resp. $\bar{\Sigma}^-$) the set of positive (resp. negative) roots in $\bar{\Sigma}$ (with respect to $>$). Let $\bar{\Pi}\subset\bar{\Sigma}^+$ be the σ-fundamental system, $\bar{\Pi}^0=\{\alpha\in\bar{\Pi};\, \sigma\alpha=-\alpha\}$, and $\tilde{\sigma}$ the Satake involution of $\bar{\Pi}-\bar{\Pi}^0$. We define $\mathrm{Aut}(\bar{\Pi}, \sigma)$ to be the subgroup of the automorphism group $\mathrm{Aut}(\bar{\Pi})$ of $\bar{\Pi}$ consistng of all the $t\in\mathrm{Aut}(\bar{\Pi})$ such that $t\bar{\Pi}^0=\bar{\Pi}^0$ and $t\tilde{\sigma}=\tilde{\sigma}t$ on $\bar{\Pi}-\bar{\Pi}^0$. It is also identified as

$$(1.15)\qquad \mathrm{Aut}(\bar{\Pi}, \sigma) = \{t\in GL(\mathfrak{h}_{\boldsymbol{R}});\, t\bar{\Sigma}=\bar{\Sigma},\, t\bar{\Sigma}^+=\bar{\Sigma}^+,\, t\sigma=\sigma t\}.$$

It is known (Takeuchi [12]) that

$$(1.16)\qquad \mathrm{Aut}(\mathfrak{g})/G' \simeq \mathrm{Aut}(\bar{\Pi}, \sigma)$$

in a natural way. We set $\Sigma^\pm=\pi_{\mathfrak{a}}(\bar{\Sigma}^\pm)-\{0\}$ and $\Pi=\pi_{\mathfrak{a}}(\bar{\Pi})-\{0\}$, and thus $\Pi\subset\Sigma^+$ is a fundamental system of Σ. Now we choose a σ-order $>$ on $\mathfrak{h}_{\boldsymbol{R}}$ such that $(\alpha, E)\geq 0$ for any $\alpha\in\bar{\Sigma}^+$ once and for all. Then one has $\bar{\Sigma}_0=\bar{\Sigma}\cap\{\bar{\Pi}_0\}_{\boldsymbol{Z}}$ for $\bar{\Pi}_0=\bar{\Pi}\cap\bar{\Sigma}_0$, where $\{\bar{\Pi}_0\}_{\boldsymbol{Z}}$ denotes the subgroup of $\mathfrak{h}_{\boldsymbol{R}}$ generated by $\bar{\Pi}_0$, and hence $\bar{\Pi}_0$ is a σ-fundamental system of $\bar{\Sigma}_0$. Furthermore one has $\bar{\Sigma}_{\pm1}=\bar{\Sigma}^\pm-\bar{\Sigma}_0$. From these it follows that

(1.17) $\Sigma_0=\Sigma\cap\{\Pi_0\}_{\boldsymbol{Z}}$ for $\Pi_0=\Pi\cap\Sigma_0$,

(1.18) $\Sigma_{\pm1}=\Sigma^\pm-\Sigma_0$,

(1.19) (Π, Π_0) is an *irreducible symmetric pair* in the sense that Π is irreducible and $\Pi-\Pi_0$ consists of only one root γ_1, called the *distinguished root,* such that the highest root $\delta\in\Sigma^+$ has an expression

$$\delta=\gamma_1+\sum_{\gamma\in\Pi_0} m_\gamma\gamma\,, \qquad m_\gamma\in\boldsymbol{Z}\,.$$

We define $\mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma)$ to be the subgroup of $\mathrm{Aut}(\bar{\Pi}, \sigma)$ consisting of all the $t\in\mathrm{Aut}(\bar{\Pi}, \sigma)$ such that $t\bar{\Pi}_0=\bar{\Pi}_0$. Under the identification (1.15), it is also given by

$$(1.20)\qquad \mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma) = \{t\in\mathrm{Aut}(\bar{\Pi}, \sigma);\, tE=E\}.$$

Lemma 1.2. *The quotient group G_0/G_0' is isomorphic to* $\mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma)$ *in a natural way. Therefore also G/G' is isomorphic to* $\mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma)$.

Proof. In general, for subsets $A, B, \cdots$ of $\bar{\mathfrak{g}}$, we denote by $\mathrm{Aut}(\mathfrak{g}, A, B, \cdots)$ the group of all the $a\in\mathrm{Aut}(\mathfrak{g})$ such that $aA=A$, $aB=B$, $\cdots$.

Let $a\in G_0$ be arbitrary. Then, since $\bar{\Pi}_0$ is a σ-fundamental system of $\bar{\Sigma}_0$, there is $a'\in G_0'$ such that $b=a'a\in\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_{\boldsymbol{R}}, \mathfrak{a}, \bar{\Pi}_0)$. Since $b\bar{\Pi}_0=\bar{\Pi}_0$, b leaves $\bar{\Sigma}_0^+=\bar{\Sigma}_0\cap\bar{\Sigma}^+$ invariant. Furthermore, since $bE=E$, b leaves also $\bar{\Sigma}_1$ invariant. Therefore b leaves $\bar{\Sigma}^+=\bar{\Sigma}_0^+\cup\bar{\Sigma}_1$ invariant, whence $b\bar{\Pi}=\bar{\Pi}$. Thus we have proved that

$$G_0=G_0'\,\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_{\boldsymbol{R}}, \mathfrak{a}, \bar{\Pi}, \bar{\Pi}_0)\,.$$

On the other hand, by (1.16) the natural homomorphism $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_{\boldsymbol{R}}, \mathfrak{a}, \bar{\Pi}, \bar{\Pi}_0) \to \mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma)$ is surjective, and has the kernel

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_{\boldsymbol{R}}, \mathfrak{a}, \bar{\Pi}, \bar{\Pi}_0) \cap G' = \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}_{\boldsymbol{R}}, \mathfrak{a}, \bar{\Pi}, \bar{\Pi}_0) \cap G'_0 .$$

Thus we obtain the lemma. q.e.d.

Let

$$W = N_{K'}(\mathfrak{a})/Z_{K'}(\mathfrak{a}), \quad W_0 = N_{K'_0}(\mathfrak{a})/Z_{K'_0}(\mathfrak{a})$$

be the Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}_0$ respectively, which may be regarded as finite subgroups of $O(\mathfrak{a})$ through the adjoint action, $O(\mathfrak{a})$ being the orthogonal group on $\mathfrak{a}$ with respect to the inner product $\langle\ ,\ \rangle$. Here $N_*(\mathfrak{a})$ (resp. $Z_*(\mathfrak{a})$) denotes the normalizer (resp. centralizer) of $\mathfrak{a}$ in $*$. The groups W, W_0 are related by that $W_0 = \{s \in W;\ sE = E\}$. It is known that (Π, Π_0) is *of rank* r, $r = \mathrm{rank}(M, g)$, in the sense that the irreducible symmetric bounded domain corresponding to (Π, Π_0) has the rank r. Thus we can choose (cf. Takeuchi [14]) a maximal system

$$\Delta = \{\beta_1, \cdots, \beta_r\}, \qquad \text{with } \beta_1 = \delta ,$$

of strongly orthogonal roots in Σ_1 of the same length. Here, by the length of $X \in \mathfrak{g}$ we mean the norm $|X|$ of X with respect to $\langle\ ,\ \rangle$. Note that $\Delta \subset W\delta$. Let us fix a root $\beta \in \Delta$. We choose an element $X_\beta \in \mathfrak{g}^\beta$ with $|X_\beta|^2 = 2/(\beta, \beta)$, and set $X_{-\beta} = \tau X_\beta \in \mathfrak{g}^{-\beta}$, $A_\beta = \pi(X_\beta + X_{-\beta}) \in \mathfrak{m}$. Then one has $[X_\beta, X_{-\beta}] = -(2/(\beta, \beta))\beta$. We define a basis $\{X_+, X_-, H\}$ of $\mathfrak{sl}(2, \boldsymbol{R})$ and an element $A \in \mathfrak{sl}(2, \boldsymbol{R})$ by

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}.$$

Then the correspondence

$$\phi_\beta\colon X_+ \mapsto X_\beta,\ X_- \mapsto X_{-\beta},\ H \mapsto (2/(\beta, \beta))\beta$$

(thus $\phi_\beta\colon A \mapsto A_\beta$) defines an injective Lie homomorphism $\phi_\beta\colon \mathfrak{sl}(2, \boldsymbol{R}) \to \mathfrak{g}$. The extension $\phi_\beta\colon SL(2, \boldsymbol{R}) \to G'$ of ϕ_β sends the parabolic subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix};\ a \in \boldsymbol{R}^*,\ b \in \boldsymbol{R} \right\}$$

of $SL(2, \boldsymbol{R})$ into U', and hence it induces a ϕ_β-equivariant imbedding

$$\psi_\beta\colon P_1(\boldsymbol{R}) = SL(2, \boldsymbol{R})/P \to M = G'/U' .$$

Therefore

$$T_\beta = \{(\exp tA_\beta)o;\ 0 \leq t \leq 1\}$$

is a simply closed geodesic of (M, g) of the length $2\pi/|\delta|$. In fact,

$$\begin{aligned} |A_\beta|^2 &= \langle A_\beta, A_\beta\rangle = -(A_\beta, A_\beta) = -2\pi^2(X_\beta, X_{-\beta}) \\ &= 2\pi^2|X_\beta|^2 = 4\pi^2/(\beta, \beta) = 4\pi^2/|\delta|^2 . \end{aligned}$$

For each i $(1 \leq i \leq r)$, we set

$$\begin{aligned} &X_{\pm i} = X_{\pm\beta i}\,, \quad A_i = A_{\beta_i} = \pi(X_i + X_{-i})\,, \quad T_i = T_{\beta i}\,, \\ &\phi_i = \phi_{\beta i}\,, \quad \psi_i = \psi_{\beta i}\,. \end{aligned}$$

Then the $\boldsymbol{R}$-linear span

$$\mathfrak{a}_M = \{A_1, \cdots, A_r\}_{\boldsymbol{R}}$$

of $\{A_1, \cdots, A_r\}$ is a maximal abelian subalgebra in $\mathfrak{m}$, and hence

$$A_M = (\exp \mathfrak{a}_M)o \simeq \mathfrak{a}_M/\Gamma_M$$

is a maximal torus of (M, g), where

$$\Gamma_M = \{H \in \mathfrak{a}_M;\ (\exp H)o = o\}\,.$$

Furthermore, the Lie homomorphism $\phi = \phi_1 \times \cdots \times \phi_r\colon SL(2, \boldsymbol{R})^r \to G'$ sends P^r into U', and induces a ϕ-equivariant imbedding

$$\psi\colon P_1(\boldsymbol{R})^r \to M\,,$$

which gives rise to a diffeomorphism $\psi = \psi_1 \times \cdots \times \psi_r\colon P_1(\boldsymbol{R})^r \to A_M = T_1 \times \cdots \times T_r$. Therefore, the lattice Γ_M is given by

(1.21) $$\Gamma_M = \{A_1, \cdots, A_r\}_{\boldsymbol{Z}} \quad \text{with} \quad \langle A_i, A_j\rangle = \delta_{ij} 4\pi^2/|\delta|^2\,.$$

For $H \in \mathfrak{a}_M$ we use the linear coordinate $(x_1, \cdots, x_r)$ determined by

$$H = \sum_{i=1}^{r} x_i A_i\,.$$

The real part $(\mathfrak{a}_M)_{\boldsymbol{R}}$ of $(\mathfrak{a}_M)^C$ is given by $(\mathfrak{a}_M)_{\boldsymbol{R}} = \sqrt{-1}\mathfrak{a}_M$. For $\gamma \in (\mathfrak{a}_M)_{\boldsymbol{R}}$, let $|\gamma|$ denote the norm of γ with respect to (,), which is positive definite on $(\mathfrak{a}_M)_{\boldsymbol{R}}$. We define $h_i \in (\mathfrak{a}_M)_{\boldsymbol{R}}$ $(1 \leq i \leq r)$ by

(1.22) $$\sqrt{-1}(h_i, A_j) = \delta_{ij}\pi \qquad (1 \leq i, j \leq r)\,.$$

Note that $(h_i, h_j) = \delta_{ij}|\delta|^2/4$ $(1 \leq i, j \leq r)$ in virtue of (1.21). For $\gamma \in (\mathfrak{a}_M)_{\boldsymbol{R}}$ we define

$$(\mathfrak{k}^C)^\gamma = \{X \in \mathfrak{k}^C;\ [H, X] = (\gamma, H)X \text{ for each } H \in \mathfrak{a}_M\}\,,$$

and set

$$\Sigma_M = \{\gamma \in (\mathfrak{a}_M)_R - \{0\} ; (\mathfrak{k}^C)^\gamma \neq \{0\}\},$$

which is the root system of (M, g) relative to $\mathfrak{a}_M$. Then, irreducible symmetric R-spaces are divided into the following five classes.

(I) Hermitian type.

$\bar{\mathfrak{g}}$ is not simple; $\pi_1(M) = \{0\}$;
$\Sigma_M = \{\pm h_i \pm h_j \ (1 \leq i < j \leq r), \ \pm 2h_i \ (1 \leq i \leq r)\}$ or
$\{\pm h_i \pm h_j \ (1 \leq i < j \leq r), \ \pm h_i, \ \pm 2h_i \ (1 \leq i \leq r)\}$.

(II) Type $Sp(r)$.

$\bar{\mathfrak{g}}$ is simple; $\pi_1(M) = \{0\}$;
Σ_M is the same as in (I).

(III) Type $U(r)$.

$\bar{\mathfrak{g}}$ is simple; $\pi_1(M) = \boldsymbol{Z}$;
$\Sigma_M = \{\pm(h_i - h_j) \ (1 \leq i < j \leq r)\}$.

(IV) Type $SO(2r+1)$.

$\bar{\mathfrak{g}}$ is simple; $\pi_1(M) = \boldsymbol{Z}_2$;
$\Sigma_M = \{\pm h_i \pm h_j \ (1 \leq i < j \leq r), \ \pm h_i \ (1 \leq i \leq r)\}$.

(V) Type $SO(2r)$ $(r \geq 2)$.

$\bar{\mathfrak{g}}$ is simple; $\pi_1(M) = \boldsymbol{Z}_2$;
$\Sigma_M = \{\pm h_i \pm h_j \ (1 \leq i < j \leq r)\}$.

Let

$$W_M = N_{K_0'}(\mathfrak{a}_M)/Z_{K_0'}(\mathfrak{a}_M) \subset O(\mathfrak{a}_M)$$

be the Weyl group of (M, g). We denote by $\mathfrak{S}_r$ the subgroup of $O(\mathfrak{a}_M)$ consisting of transformations $(x_1, \cdots, x_r) \mapsto (x_{p(1)}, \cdots, x_{p(r)})$, p being a permutation of $\{1, \cdots, r\}$, and by $(\boldsymbol{Z}_2)^r$ the subgroup of $O(\mathfrak{a}_M)$ consisting of transformations $(x_1, \cdots, x_r) \mapsto (\varepsilon_1 x_1, \cdots, \varepsilon_r x_r)$, $\varepsilon_i = \pm 1$.

Lemma 1.3. $\mathfrak{S}_r \subset W_M \subset \mathfrak{S}_r \cdot (\boldsymbol{Z}_2)^r$.

Proof. Since W_M is generated by the reflections of $\mathfrak{a}_M$ with respect to $\sqrt{-1}\gamma$ for $\gamma \in \Sigma_M$, this follows by the above table of Σ_M. q.e.d.

2. Stratifications

In this section we define certain stratifications of M and $\mathfrak{g}_{-1}$ by means of the group orbits. We retain the notation in Section 1.

We define $B_l \in \mathfrak{a}_M$ $(0 \leq l \leq r)$ by

$$B_0 = 0, \quad B_l = (1/2)(A_1 + \cdots + A_l) \quad (1 \leq l \leq r),$$

and set

$$b_l = \exp B_l \in K', \quad \mathcal{V}_l = U' b_l o \subset M, \quad d_l = \dim \mathcal{V}_l \quad (0 \leq l \leq r).$$

Furthermore we define $s_l \in W$ $(0 \leq l \leq r)$ by

$$s_0 = 1, \quad s_l = s_{\beta_1} \cdots s_{\beta_l} \quad (1 \leq l \leq r),$$

where $s_\gamma \in W \subset O(\mathfrak{a})$ denotes the reflection with respect to $\gamma \in \Sigma$.

Lemma 2.1. 1) $\exp(1/2)A_i \in N_{K'}(\mathfrak{a})$ *and* $\exp(1/2)A_i | \mathfrak{a} = s_{\beta_i}$ $(1 \leq i \leq r)$. *Therefore* $b_l \in N_{K'}(\mathfrak{a})$ *and* $b_l | \mathfrak{a} = s_l$ $(0 \leq l \leq r)$.

2) $\{s_0, s_1, \cdots, s_r\}$ *is a set of complete representatives of the double coset space* $W_0 \backslash W / W_0$.

3) $M = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_r$ *(disjoint union).*

4) *If* $0 \leq l' \leq l \leq r$, $\mathcal{V}_{l'}$ *is contained in the closure* $\mathrm{Cl}\,\mathcal{V}_l$ *of* $\mathcal{V}_l$.

Proof. 1), 2) See Takeuchi [15].

3) We define a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ by

$$\mathfrak{n} = \sum_{\gamma \in \Sigma^+} \mathfrak{g}^\gamma,$$

and set

$$N = \exp \mathfrak{n}, \quad B' = \{a \in G';\, a\mathfrak{n} = \mathfrak{n}\}.$$

In the same way we define

$$\mathfrak{n}_0 = \sum_{\gamma \in \Sigma_0^+} \mathfrak{g}^\gamma, \quad \text{where} \quad \Sigma_0^+ = \Sigma_0 \cap \Sigma^+,$$

$$N_0 = \exp \mathfrak{n}_0, \quad B'_0 = \{a \in G'_0;\, a\mathfrak{n}_0 = \mathfrak{n}_0\}.$$

Then we have Bruhat decompositions

$$G' = \bigcup_{s \in W} NsB' = \bigcup_{s \in W} B'sB', \tag{2.1}$$

$$G'_0 = \bigcup_{t \in W_0} N_0 t B'_0 = \bigcup_{t \in W_0} B'_0 t B'_0. \tag{2.2}$$

Therefore, by (2.2) together with (1.8)$'$, we have

$$U' = G'_0 \exp \mathfrak{g}_1 = \bigcup_{t \in W_0} N_0 t B'_0 \exp \mathfrak{g}_1 = \bigcup_{t \in W_0} N_0 t B'. \tag{2.3}$$

Hence, for each $s \in W$ we obtain

$$U'sU' = \bigcup_{t,t' \in W_0} B'tN_0sN_0t'B' = \bigcup_{t,t' \in W_0} B'N_0tst'N_0B' = \bigcup_{t,t' \in W_0} B'tst'B' = \bigcup_{w \in W_0 s W_0} B'wB' . \tag{2.4}$$

Since

$$W = \bigcup_{l=0}^{r} W_0 s_l W_0 \text{ (disjoint union)}$$

by 2), together with the Bruhat decomposition (2.1), we get

$$G' = \bigcup_{l=0}^{r} U's_l U' \text{ (disjoint union)} .$$

This implies the assertion 3).

4) For each coset $[s]=sW_0 \in W/W_0$, choosing an element $k \in N_{K'}(\mathfrak{a})$ with $k|\mathfrak{a}=s$, we set $\mathcal{V}_{[s]}=Nko$. Then (Takeuchi [13])

$$M = \bigcup_{[s] \in W/W_0} \mathcal{V}_{[s]} \text{ (disjoint union)}$$

gives a cellular decomposition of M with the closure relations: $\mathcal{V}_{[s']} \subset \mathrm{Cl}\,\mathcal{V}_{[s]}$ if and only if

$$s'E - sE = \sum_{\gamma \in \Pi} m_\gamma \gamma \qquad \text{with some } m_\gamma \geq 0 . \tag{2.5}$$

Moreover, by (2.4) and (2.3) we have

$$U's_l U' = \bigcup_{s \in W_0 s_l W_0} B'sB' = \bigcup_{s \in W_0 s_l} B'sU' = \bigcup_{s \in W_0 s_l} NsU' ,$$

and hence

$$\mathcal{V}_l = \bigcup_{s \in W_0 s_l} \mathcal{V}_{[s]} . \tag{2.6}$$

Suppose that $0 \leq l' \leq l \leq r$. Then we have

$$s_{l'}E - s_l E = \sum_{l'+1 \leq i \leq l} \frac{2}{(\beta_i, \beta_i)} \beta_i .$$

Therefore, by (2.5) $\mathcal{V}_{[s_{l'}]} \subset \mathrm{Cl}\,\mathcal{V}_{[s_l]}$, and hence by (2.6) we get $\mathcal{V}_{l'} \subset \mathrm{Cl}\,\mathcal{V}_l$.

q.e.d.

We define subsets $D_l \subset \mathfrak{a}_M$ and $\mathcal{D}_l \subset A_M$ $(0 \leq l \leq r)$ by

$$D_l = \{(x_1, \cdots, x_r) \in \mathfrak{a}_M ;\ |x_i| \leq 1/2\ (1 \leq i \leq r),\ \#\{i;\ x_i \neq 0\} = l\} ,$$
$$\mathcal{D}_l = (\exp D_l)o ,$$

to get

$$A_M = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_r \text{ (disjoint union)} . \tag{2.7}$$

Lemma 2.2. 1) $\mathcal{V}_l = K_0' \mathcal{D}_l$ $(0 \leq l \leq r)$.
2) $\mathcal{V}_l = U b_l o = K_0 \mathcal{D}_l$ $(0 \leq l \leq r)$. *Therefore*

$$G = \overset{r}{\underset{l=0}{\dot{\bigcup}}} U s_l U \ (\textit{disjoint union}) .$$

Proof. If we denote by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the homogeneous coordinate of $P_1(\boldsymbol{R}) = SL(2, \boldsymbol{R})/P$, we have

$$\left(\exp \frac{A}{2}\right) P = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$(\exp tX_+)\left(\exp \frac{A}{2}\right) P = \begin{bmatrix} t \\ 1 \end{bmatrix} \qquad \text{for } t \in \boldsymbol{R} ,$$

$$(\exp xA)P = \begin{bmatrix} \cos \pi x \\ -\sin \pi x \end{bmatrix} = \begin{bmatrix} -\cot \pi x \\ 1 \end{bmatrix} \qquad \text{for } 0 < |x| \leq 1/2 .$$

Therefore, for each $x \in \boldsymbol{R}$ with $0 < |x| \leq 1/2$, there is $a \in P$ such that $a \exp(A/2)P = (\exp xA)P$. The ϕ-equivariance of ψ: $P_1(\boldsymbol{R})^r \to M$ implies that for any $p \in \mathcal{D}_l$ there is $a \in P^r$ such that $p = \phi(a) \exp (1/2)(A_{i_1} + \cdots + A_{i_l})o$, $1 \leq i_1 < \cdots < i_l \leq r$. By $\phi(P^r) \subset U'$ and Lemma 1.3 we have that $\mathcal{D}_l \subset U' b_l o = \mathcal{V}_l$, which implies $K_0' \mathcal{D}_l \subset \mathcal{V}_l$ since $K_0' \subset U'$. On the other hand, by (2.7) we get

$$M = K_0' A_M = \overset{r}{\underset{l=0}{\dot{\bigcup}}} K_0' \mathcal{D}_l .$$

Thus the assertion 1) holds.

2) Since $U = U' K_0$ by (1.13) and $K_0 = K_0' N_{K_0}(\mathfrak{a}_M)$, it suffices to show that $N_{K_0}(\mathfrak{a}_M) \mathcal{D}_l = \mathcal{D}_l$ $(0 \leq l \leq r)$. But, since $N_{K_0}(\mathfrak{a}_M)$ acts on $\mathfrak{a}_M$ as orthogonal transformations leaving Γ_M invariant, for any $k \in N_{K_0}(\mathfrak{a}_M)$ there is a permutation p of $\{1, \cdots, r\}$ such that $kA_i = \varepsilon_i A_{p(i)}$, $\varepsilon_i = \pm 1$ $(1 \leq i \leq r)$. Thus $kD_l = D_l$ and hence $k\mathcal{D}_l = \mathcal{D}_l$. q.e.d.

We define a K_0-equivariant linear isomorphism ϖ_{-1}: $\mathfrak{m} \to \mathfrak{g}_{-1}$ by

$$\varpi_{-1}(X) = (1/2)(X - [E, X]) \qquad \text{for } X \in \mathfrak{m} .$$

Since $\varpi_{-1}(A_i) = \pi X_{-i}$ $(1 \leq i \leq r)$, ϖ_{-1} induces a linear isomorphism from $\mathfrak{a}_M$ onto the subspace $\mathfrak{a}_{-1}$ of $\mathfrak{g}_{-1}$ defined by

$$\mathfrak{a}_{-1} = \{X_{-1}, \cdots, X_{-r}\}_{\boldsymbol{R}} .$$

By $\mathfrak{m} = K_0' \mathfrak{a}_M$, one has

$$\mathfrak{g}_{-1} = K_0' \mathfrak{a}_{-1} . \tag{2.8}$$

We define subsets E_l of $\mathfrak{a}_{-1}$ $(0 \leq l \leq r)$ by

$$E_l = \{\sum_{i=1}^{r} x_i X_{-i};\ \#\{i;\ x_i \neq 0\} = l\}\ .$$

Then $\mathfrak{a}_{-1}$ is a disjoint union of these E_l $(0 \leq l \leq r)$, and hence by (2.8)

$$\mathfrak{g}_{-1} = \bigcup_{l=0}^{r} K_0' E_l\ . \tag{2.9}$$

Now we identify $\mathfrak{g}_{-1}$ with a G_0-invariant open dense subset of M through the G_0-equivariant imbedding $\iota\colon \mathfrak{g}_{-1} \to M$ defined by $\iota(X) = (\exp X)o$ for $X \in \mathfrak{g}_{-1}$, and set

$$V_l = \mathcal{V}_l \cap \mathfrak{g}_{-1} \qquad (0 \leq l \leq r)\ .$$

Then each V_l $(0 \leq l \leq r)$ is non-empty (cf. Lemma 2.3), and so it is a G_0-invariant submanifold of $\mathfrak{g}_{-1}$ with $\dim V_l = d_l$. These V_l give a stratification

$$\mathcal{S}\colon \mathfrak{g}_{-1} = V_0 \cup V_1 \cup \cdots \cup V_r \tag{2.10}$$

of $\mathfrak{g}_{-1}$.

Lemma 2.3. 1) $V_l = K_0' E_l = K_0 E_l$ $(0 \leq l \leq r)$.

2) *For $p, q \geq 0$ with $p+q \leq r$, we define*

$$Y_{p,q} = X_{-1} + \cdots + X_{-p} - X_{-(p+1)} - \cdots - X_{-(p+q)} \in E_{p+q}\ .$$

Then

$$V_l = \bigcup_{p+q=l} G_0' Y_{p,q} \qquad (0 \leq l \leq r)\ .$$

3) *Each V_l $(0 \leq l \leq r)$ is a finite union of G_0'-orbits (resp. G_0-orbits) in $\mathfrak{g}_{-1}$.*

Proof. 1) Under the notation in the proof of Lemma 2.2, we have

$$(\exp xX_-)P = \begin{bmatrix} & 1 \\ -x & \end{bmatrix} \qquad \text{for } x \in \boldsymbol{R}\,,$$

$$(\exp tA)P = \begin{bmatrix} & 1 \\ -\tan \pi t & \end{bmatrix} \qquad \text{for } t \in \boldsymbol{R} \text{ with } |t| < 1/2\,.$$

Therefore, for each $X = \sum x_i X_{-i} \in E_l$ we have

$$\frac{1}{\pi} \sum_{i=1}^{r} (\mathrm{Tan}^{-1} x_i) A_i \in D_l\,,$$

$$(\exp X)o = \exp\left(\frac{1}{\pi} \sum_{i=1}^{r} (\mathrm{Tan}^{-1} x_i) A_i\right) o \in \mathcal{D}_l\,,$$

and hence $E_l \subset V_l$ by Lemma 2.2. This implies $K_0' E_l \subset V_l$. On the other hand, by (2.9) one has

$$\bigcup_{l=0}^{r} K_0' E_l = \bigcup_{l=0}^{r} V_l\,.$$

Thus we get $K'_0 E_l = V_l$, which also implies $K_0 E_l = V_l$ by the K_0-invariance of V_l.

2) Let $X \in V_l$ be arbitrary. By 1) there is $k \in K'_0$ with $kX \in E_l$. Furthermore, since $\mathfrak{S}_r \subset W_M$ by Lemma 1.3 and $\varpi_{-1}\colon \mathfrak{a}_M \to \mathfrak{a}_{-1}$ is K'_0-equivariant, there is $k' \in N_{K'_0}(\mathfrak{a}_{-1})$ such that

$$k'kX = \sum_{i=1}^{p} x_i X_{-i} - \sum_{j=1}^{q} x_{p+j} X_{-(p+j)} ,$$

with x_i, $x_{p+j} > 0$, $p+q=l$. On the other hand, the connected Lie subgroup A_Δ of G'_0 generated by

$$\mathfrak{a}_\Delta = \{\beta_1, \cdots, \beta_r\}_{\boldsymbol{R}}$$

leaves $\mathfrak{a}_{-1}$ invariant, and its matricial representation on $\mathfrak{a}_{-1}$ with respect to $\{X_{-1}, \cdots, X_{-r}\}$ is all the real diagonal matrices with positive entries. Thus there is $a \in A_\Delta$ such that $ak'kX = Y_{p,q}$. Since $ak'k \in G'_0$, we get the assertion 2).

3) This follows from 2) and the G_0-invariance of V_l. q.e.d.

REMARK 2.4. We define

$$Y_0 = 0,\ Y_l = X_{-1} + \cdots + X_{-l} \in E_l \quad (1 \leq l \leq r) .$$

Then, as is seen from the above proof, if W_M contains $(\boldsymbol{Z}_2)^r$, then $V_l = G'_0 Y_l = G_0 Y_l$ $(0 \leq l \leq r)$, that is, each V_l consists of a single G'_0-orbit (resp. a single G_0-orbit).

Lemma 2.5. *Each* V_l *or the closure* $\mathrm{Cl}V_l$ *in* $\mathfrak{g}_{-1}$ $(0 \leq l \leq r)$ *is invariant under the transformation* $X \mapsto tX$ *of* $\mathfrak{g}_{-1}$ *for any* $t > 0$.

Proof. This follows from the fact that $\mathrm{ad}E$, $E \in \mathfrak{g}_0$, acts on $\mathfrak{g}_{-1}$ as -1, together with the G_0-invariance of V_l. q.e.d.

Let $u_0\colon \mathfrak{g}_{-1} \to T_o M$ be the natural linear isomorphism induced by the differential of the projection $G \to M = G/U$. We identify as $GL(T_o M) = GL(\mathfrak{g}_{-1})$ through the isomorphism u_0. Let $\rho\colon U \to GL(\mathfrak{g}_{-1})$ be the linear isotropy representation of $M = G/U$. It is known (Tanaka [17]) that the restriction $\rho | G_0$ to G_0 is an injective Lie homomorphism. We identify G_0 with a Lie subgroup of $GL(\mathfrak{g}_{-1})$ through $\rho | G_0$. We define

$$GL(\mathfrak{g}_{-1}, \mathcal{S}) = \{a \in GL(\mathfrak{g}_{-1});\ aV_l = V_l\ (0 \leq l \leq r)\} ,$$

and call it the *group of automorphisms* of the stratification $\mathcal{S}$. Then, from the G_0-invariance of each V_l, one has $G_0 \subset GL(\mathfrak{g}_{-1}, \mathcal{S})$.

3. Complex symmetric $\boldsymbol{R}$-spaces

In this section we consider the symmetric R-spaces in complex category.

Let

$$\mathcal{G}\colon \mathfrak{g} = \mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1, \ [\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$$

be a graded Lie algebra over $\boldsymbol{C}$ with $\mathfrak{g}$ complex simple and $\mathfrak{g}_{-1}\neq\{0\}$, and τ a Cartan involution of $\mathfrak{g}$, regarded as a real semi-simple Lie algebra, with $\tau\mathfrak{g}_p=\mathfrak{g}_{-p}$ $(-1\leq p\leq 1)$. Such a pair $(\mathcal{G}, \tau)$ is called a *compact simple symmetric graded Lie algebra over* $\boldsymbol{C}$. The *characteristic element* $E\in\mathfrak{g}_0$ is defined in the same way as in Section 1. Let

$$G_0 = \{a\in \mathrm{Aut}(\mathfrak{g});\ a\mathfrak{g}_p = \mathfrak{g}_p\ (-1\leq p\leq 1)\}\,,$$
$$G' = \mathrm{Inn}(\mathfrak{g})\,,\quad G = G_0G'\,.$$

The various groups and Lie algebras, their subspaces are defined in the same way as in Section 1. Note that then G', G_0', U', K', K_0' are all connected. Various equalities hold also for these groups. In our case, the homogeneous space

$$M = G/U = K/K_0$$

is a simply connected complex projective algebraic manifold, and is called the *complex R-space* associated to $\mathcal{G}$. The group G may be regarded as a subgroup of the holomorphic automorphism group $\mathrm{Aut}(M)$ of M. In the same way as in Section 1 we define a K-invariant Hermitian metric g on M by making use of the $\mathrm{Aut}(\mathfrak{g}, \tau)$-invariant Hermitian inner product given by

$$\langle X, Y\rangle = -(X, \tau Y) \qquad \text{for} \quad X, Y\in\mathfrak{g}\,,$$

where (,) denotes the Killing form of $\mathfrak{g}$. The Hermitian manifold (M, g) is an irreducible (in the sense of de Rham) Hermitian symmetric space of compact type, and is called an *irreducible complex symmetric R-space*.

REMARK 3.1. Actually we have that $G=\mathrm{Aut}(M)$ and $K=\mathrm{Aut}(M)\cap \mathrm{I}(M, g)$. See the equality (1.9) in complex category and Remark 1.1.

In our case, the real subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ defined as in Section 1 is a Cartan subalgebra of the complex Lie algebra $\mathfrak{g}$, and $\mathfrak{a}$ is nothing but the real part of $\mathfrak{h}$. Let $\Sigma\subset\mathfrak{a}$ denote the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We introduce a linear order $>$ on $\mathfrak{a}$ such that $(\alpha, E)\geq 0$ for any positive root α in Σ. We define Σ_p $(-1\leq p\leq 1)$, Π, Π_0, W, W_0 and so on in the same way as in Section 1. Then (Π, Π_0) is also an irreducible symmetric pair of rank $r=\mathrm{rank}(M, g)$, and hence we can take a maximal system $\Delta=\{\beta_1, \cdots, \beta_r\}$ of strongly orthogonal roots in Σ_1 of the same length with $\beta_1=\delta$, the highest root in Σ. Thus we can define $X_{\pm i}\in\mathfrak{g}_{\pm 1}$, $A_i\in\mathfrak{m}$, $b_i\in K'$, $Y_i\in\mathfrak{g}_{-1}$ and so on. Making use of the A_i $(1\leq i\leq r)$ we can construct a maximal abelian subalgebra $\mathfrak{a}_M$ in $\mathfrak{m}$ with the coordinate

$(x_1, \cdots, x_r)$. Then (cf. Takeuchi [14]) the Weyl group $W_M \subset O(\mathfrak{a}_M)$ of (M, g) is given by

$$W_M = \mathfrak{S}_r \cdot (\boldsymbol{Z}_2)^r . \tag{3.1}$$

We define

$$\mathcal{V}_l = U'b_l o,\ V_l = \mathcal{V}_l \cap \mathfrak{g}_{-1},\ d_l = \dim_{\boldsymbol{C}} \mathcal{V}_l\ (0 \le l \le r),$$

regarding $\mathfrak{g}_{-1}$ as a G_0-invariant open dense subset of M through the natural imbedding $\iota\colon \mathfrak{g}_{-1} \to M$. Also G_0 is identified with a complex algebraic group in $GL(\mathfrak{g}_{-1})$, through the linear isotropy representation. We can prove the following lemma in the same way as Lemma 1.2.

Lemma 3.2. *If we define* $\mathrm{Aut}(\Pi, \Pi_0) = \{t \in \mathrm{Aut}(\Pi);\ t\Pi_0 = \Pi_0\}$, *we have*

$$G/G' \simeq G_0/G_0' \simeq \mathrm{Aut}(\Pi, \Pi_0) .$$

Lemma 3.3. 1) $M = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_r$ *(disjoint union), and therefore we get a stratification*

$$\mathcal{S}\colon \mathfrak{g}_{-1} = V_0 \cup V_1 \cup \cdots \cup V_r \ \textit{(disjoint union)} .$$

2) $\mathcal{V}_l = Ub_l o\ (0 \le l \le r)$.

3) $V_l = G_0 Y_l = G_0' Y_l\ (0 \le l \le r)$.

4) $\mathrm{Cl}\, \mathcal{V}_l$ *is an algebraic subvariety of* M, *and*

$$\mathrm{Cl}\, \mathcal{V}_l = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_l \qquad (0 \le l \le r) .$$

5) $\mathrm{Cl}\, V_l$ *is an affine algebraic subvariety of* $\mathfrak{g}_{-1}$, *and*

$$\mathrm{Cl}\, V_l = V_0 \cup V_1 \cup \cdots \cup V_l \qquad (0 \le l \le r) .$$

6) $0 = d_0 < d_1 < \cdots < d_r = \dim_{\boldsymbol{C}} M$.

Proof. The proof of the assertions 1), 2) is the same as in Section 2. The assertion 3) follows by the same argument as in Section 2 and the same reasoning as in Remark 2.4, recalling the equality (3.1).

4) In the same way as in Lemma 2.1, 4) we get

$$l' \le l \Rightarrow \mathcal{V}_{l'} \subset \mathrm{Cl}\, \mathcal{V}_l . \tag{3.2}$$

Since the complex linear algebraic group G' acts on M regularly and U' is an algebraic subgroup of G', by a well known fact in algebraic geometry (cf. for example, A. Borel: Linear Algebraic Groups, Benjamin, 1969), $\mathcal{V}_l = U'b_l o$ contains a Zariski open subset of the Zariski closure $\mathrm{Cl}^Z(\mathcal{V}_l)$ of $\mathcal{V}_l$. Thus we have $\mathrm{Cl}\, \mathcal{V}_l = \mathrm{Cl}^Z(\mathcal{V}_l)$. The Zariski connectedness of $\mathrm{Cl}\mathcal{V}_l$ follows from that of U'. Also from the above we have

(3.3) $$\mathcal{V}_{l'}\subset \mathrm{Cl}\,\mathcal{V}_l,\ l\neq l' \Rightarrow d_{l'}<d_l\,.$$

In particular, by (3.2) we get

(3.4) $$l'<l \Rightarrow d_{l'}<d_l\,.$$

Suppose that $\mathcal{V}_{l'}\subset \mathrm{Cl}\,\mathcal{V}_l$ for $l'>l$. Then, by (3.3) we would have $d_{l'}<d_l$, which is a contradiction to (3.4). Thus we have proved that $l'\leq l$ if and only if $\mathcal{V}_{l'}\subset \mathrm{Cl}\,\mathcal{V}_l$, which completes the proof of the assertion 4).

5) Since actually $\mathfrak{g}_{-1}$ is Zariski open in M, this follows from assertions 3), 4) and the Zariski connectedness of G_0'.

6) Follows from (3.4). q.e.d.

REMARK 3.4. The defining polynomials of the affine algebraic variety $\mathrm{Cl}V_l$ in $\mathfrak{g}_{-1}$ are given as follows. Taking a basis $\{e_1, \cdots, e_n\}$ of $\mathfrak{g}_{-1}$ we identify $\mathfrak{g}_{-1}$ with $\boldsymbol{C}^n$, and then $\mathfrak{gl}(\mathfrak{g}_{-1})$ with the space of $n\times n$ complex matrices. Take a basis $\{X_1, \cdots, X_N\}$ of $\mathfrak{g}_0\subset\mathfrak{gl}(\mathfrak{g}_{-1})$. Then the set $\{F^{(l)}_\alpha(z_1, \cdots, z_n)\}$ of all minor determinants of degree d_l+1 of the $n\times N$ matrix

$$(X_1 z, \cdots, X_N z), \quad \text{where} \quad z=\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

is a set of defining polynomials of $\mathrm{Cl}V_l$. This follows from assertions 3), 5), 6) in Lemma 3.3.

By Lemma 3.3, 5), the automorphism group of $\mathcal{S}$:

$$GL(\mathfrak{g}_{-1}, \mathcal{S}) = \{a\in GL(\mathfrak{g}_{-1});\ aV_l=V_l\ \ (0\leq l\leq r)\}$$

is also given by

(3.5) $$GL(\mathfrak{g}_{-1}, \mathcal{S}) = \{a\in GL(\mathfrak{g}_{-1});\ a(\mathrm{Cl}\,V_l) = \mathrm{Cl}\,V_l\ \ (0\leq l\leq r)\}\,.$$

Thus it is a complex algebraic group in $GL(\mathfrak{g}_{-1})$. Note that $G_0\subset GL(\mathfrak{g}_{-1}, \mathcal{S})$.

Theorem 3.5. $G_0=GL(\mathfrak{g}_{-1}, \mathcal{S})$.

Proof. First we claim that $\mathfrak{g}_0$ is a maximal subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$. Since $\mathfrak{g}_0$ contains the scalar endomorphisms $\boldsymbol{C}1$ of $\mathfrak{g}_{-1}$, it suffices to show that the semi-simple part $\mathfrak{s}_0$ of $\mathfrak{g}_0$ is a maximal subalgebra of $\mathfrak{sl}(\mathfrak{g}_{-1})$. Note here that $\mathfrak{s}_0$ acts irreducibly on $\mathfrak{g}_{-1}$ because (M, g) is an irreducible Hermitian symmetric space. Now our claim can be verified for each (M, g), by seeing the classification of irreducible maximal subalgebras of $\mathfrak{sl}(N, \boldsymbol{C})$ by Dynkin [2].

So $\mathrm{Lie}\,GL(\mathfrak{g}_{-1}, \mathcal{S})$ is equal to either $\mathfrak{g}_0$ or $\mathfrak{gl}(\mathfrak{g}_{-1})$. In the latter case, since in general the number of $GL(\mathfrak{g}_{-1}, \mathcal{S})$-orbits in $\mathfrak{g}_{-1}$ is $r+1$, we have $r=1$ and hence $\mathfrak{g}_0=\mathfrak{gl}(\mathfrak{g}_{-1})$. Thus we have always $\mathrm{Lie}\,GL(\mathfrak{g}_{-1}, \mathcal{S})=\mathfrak{g}_0$, whence the identity component of $GL(\mathfrak{g}_{-1}, \mathcal{S})$ is equal to G_0'. In the following we follow the argument in Gyoja [3]. Since $GL(\mathfrak{g}_{-1}, \mathcal{S})$ normalizes G_0' by the above, Ad induces a homomorphism $\varphi: GL(\mathfrak{g}_{-1}, \mathcal{S}) \to \mathrm{Aut}(\mathfrak{g}_0)$ with

$$\varphi^{-1}(\mathrm{Inn}(\mathfrak{g}_0)) = G_0' . \tag{3.6}$$

Here (3.6) follows from the Schur's lemma together with the fact that $\boldsymbol{C}^*1 \subset G_0'$. We will show

$$\varphi(GL(\mathfrak{g}_{-1}, \mathcal{S})) = \mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0)\, \mathrm{Inn}(\mathfrak{g}_0) . \tag{3.7}$$

Let $a \in GL(\mathfrak{g}_{-1}, \mathcal{S})$ be arbitrary. Since $\varphi(a)\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0$, there is $a' \in G_0'$ with $\varphi(a'a) \in \mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi_0)$. Set $b = a'a \in GL(\mathfrak{g}_{-1}, \mathcal{S})$. Denoting by $\rho: \mathfrak{g}_0 \hookrightarrow \mathfrak{gl}(\mathfrak{g}_{-1})$ the identity representation, we define a representation ρ^b of $\mathfrak{g}_0$ by

$$\rho^b(X) = \rho(\varphi(b)X) \qquad \text{for } X \in \mathfrak{g}_0 .$$

Then, since $\rho(\varphi(b)X) = b\rho(X)b^{-1}$, ρ^b is equivalent to ρ. But the highest weight of ρ is $-\alpha_1$, where α_1 is the distinguished root in Π, and hence $\varphi(b)\alpha_1 = \alpha_1$. Thus $\varphi(b) \in \mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0)$, and so $\varphi(a) = \varphi(a')^{-1}\varphi(b) \in \mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0)\cdot \mathrm{Inn}(\mathfrak{g}_0)$. For the proof of the inclusion $\mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0)\mathrm{Inn}(\mathfrak{g}_0) \subset \varphi(GL(\mathfrak{g}_{-1}, \mathcal{S}))$, it suffices to show $\mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0) \subset \varphi(GL(\mathfrak{g}_{-1}, \mathcal{S}))$. Let $a \in \mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0)$ be arbitrary. If we define a representation ρ^a of $\mathfrak{g}_0$ by

$$\rho^a(X) = \rho(aX) \qquad \text{for } X \in \mathfrak{g}_0 ,$$

then by $a\alpha_1 = \alpha_1$, ρ^a is equivalent to ρ. Hence there is $b \in GL(\mathfrak{g}_{-1})$ such that $\rho^a(X) = b\rho(X)b^{-1}$ for each $X \in \mathfrak{g}_0$, which implies $G_0' = bG_0'^{-1}b$. We claim that $b \in GL(\mathfrak{g}_{-1}, \mathcal{S})$. In fact, for each $X \in V_l$ $(0 \leq l \leq r)$, $G_0'bX = bG_0'X = bV_l$. Thus $\dim_{\boldsymbol{C}}(G_0'bX) = \dim_{\boldsymbol{C}} V_l$, and hence by Lemma 3.3, 6) $G_0'bX = V_l$. In particular, we have $bX \in V_l$, whence the claim. Since $\varphi(b) = a$, we are done.

Now (3.6), (3.7) imply that $GL(\mathfrak{g}_{-1}, \mathcal{S})/G_0' \simeq \varphi(GL(\mathfrak{g}_{-1}, \mathcal{S}))/\mathrm{Inn}(\mathfrak{g}_0) \simeq \mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0)/\mathrm{Aut}(\mathfrak{g}_0, \mathfrak{a}, \Pi, \Pi_0) \cap \mathrm{Inn}(\mathfrak{g}_0) \simeq \mathrm{Aut}(\Pi, \Pi_0)$, and so by Lemma 3.2 we get the assertion of the theorem. q.e.d.

Lemma 3.6. *We define*

$$GL_e(\mathfrak{g}_{-1}, \mathcal{S}) = \{a \in GL(\mathfrak{g}_{-1});\ a(\mathrm{Cl}\, V_{2k}) = \mathrm{Cl}\, V_{2k}\ (0 \leq k \leq [r/2])\} .$$

Then, $GL_e(\mathfrak{g}_{-1}, \mathcal{S}) = GL(\mathfrak{g}_{-1}, \mathcal{S})$ *if* $r \geq 3$.

Proof. By (3.5) we have $GL(\mathfrak{g}_{-1}, \mathcal{S}) \subset GL_e(\mathfrak{g}_{-1}, \mathcal{S})$. Since $\mathfrak{g}_0 \subset \mathrm{Lie}\, GL_e(\mathfrak{g}_{-1}, \mathcal{S})$ and $\dim_{\boldsymbol{C}} GL_e(\mathfrak{g}_{-1}, \mathcal{S}) < \dim_{\boldsymbol{C}} GL(\mathfrak{g}_{-1})$ in virtue of $r \geq 3$, we see by the same argument as in the proof of Theorem 3.5 that the identity component of $GL_e(\mathfrak{g}_{-1}, \mathcal{S})$ is equal to G_0'. Since $GL(\mathfrak{g}_{-1}, \mathcal{S})$ contains the normalizer of G_0'

in $GL(\mathfrak{g}_{-1})$ as we have seen in the proof of Theorem 3.5, we get $GL_e(\mathfrak{g}_{-1}, \mathcal{S})\subset GL(\mathfrak{g}_{-1}, \mathcal{S})$. q.e.d.

4. Automorphisms of stratification

We come back to an irreducible symmetric R-space (M, g) and use the same notation as in Sections 1 and 2. First we prove the following theorem, making use of the results in complex category.

Theorem 4.1. 1) $\mathrm{Cl}\,\mathcal{V}_l=\mathcal{V}_0\cup\mathcal{V}_1\cup\cdots\cup\mathcal{V}_l$ $(0\leq l\leq r)$.
2) $\mathrm{Cl}\,V_l=V_0\cup V_1\cup\cdots\cup V_l$ $(0\leq l\leq r)$.
3) $0=d_0<d_1<\cdots<d_r=\dim M$.

Suppose first that $\bar{\mathfrak{g}}$ is simple. Then the complexification $\bar{\mathcal{G}}$: $\bar{\mathfrak{g}}=\bar{\mathfrak{g}}_{-1}+\bar{\mathfrak{g}}_0+\bar{\mathfrak{g}}_1$ of $\mathcal{G}$, together with the conjugate linear extension $\bar{\tau}$ to $\bar{\mathfrak{g}}$ of τ, becomes a compact simple symmetric graded Lie algebra $(\bar{\mathcal{G}}, \bar{\tau})$ over $\boldsymbol{C}$. Various objects for $(\bar{\mathcal{G}}, \bar{\tau})$ considered in Section 3 are denoted by the same symbols with $-$. We define an involutive automorphism σ of $\mathrm{Aut}(\bar{\mathfrak{g}})$ as a real Lie group by

$$\sigma(a)=\sigma a\sigma^{-1} \qquad \text{for } a\in\mathrm{Aut}(\bar{\mathfrak{g}})\,.$$

For a σ-invariant subgroup H of $\mathrm{Aut}(\bar{\mathfrak{g}})$, H_σ will denote the group of all fixed points of σ in H. For example, we have

$$(\bar{G}')_\sigma=G'\,,\quad (\bar{G}_0)_\sigma=G_0\,,\quad (\bar{K}_0)_\sigma=K_0\,.$$

Lemma 4.2. $G=(\bar{G})_\sigma\,.$

Proof. It is obvious that $G=G_0G'=(\bar{G}_0)_\sigma(\bar{G}')_\sigma\subset(\bar{G})_\sigma$. For the proof of $(\bar{G})_\sigma\subset G$, we note first that by (1.9), (1.10)′ in complex category, $\bar{M}=\bar{G}/\bar{U}$ can be identified with the set of all complex parabolic subalgebras of $\bar{\mathfrak{g}}$ which are conjugate to the complexification $\bar{\mathfrak{u}}$ of $\mathfrak{u}$ under $\bar{G}$ or under $\mathrm{Inn}(\bar{\mathfrak{g}})$. Let $a\in(\bar{G})_\sigma$ be arbitrary. Then $a\in\mathrm{Aut}(\mathfrak{g})$ and so $a\mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$ whose complexification is conjugate to $\bar{\mathfrak{u}}$ under $\mathrm{Inn}(\bar{\mathfrak{g}})$ by the above remark. Since two parabolic subalgebras of $\mathfrak{g}$ are conjugate to each other under $\mathrm{Inn}(\mathfrak{g})$ if and only if their complexifications are conjugate to each other under $\mathrm{Inn}(\bar{\mathfrak{g}})$, there is $b\in G'$ such that $ba\mathfrak{u}=\mathfrak{u}$. It follows from (1.9) that $ba\in U$ and hence $a\in G'U=G$. q.e.d.

From the above lemma it follows that if $\bar{H}$ (resp. H) is one of the groups $\bar{G}$, $\bar{G}_0$, $\bar{U}$, $\bar{K}$, $\bar{K}_0$, $\bar{G}'$, $\bar{G}_0'$, $\bar{U}'$, $\bar{K}'$, $\bar{K}_0'$ (resp. $G, G_0, U, K, K_0, G', G_0', U', K', K_0'$), then $H=(\bar{H})_\sigma$. In particular, we have a natural G-equivariant imbedding

$$j\colon M=G/U\hookrightarrow\bar{M}=\bar{G}/\bar{U}\,.$$

Furthermore, an involutive diffeomorphism σ of $\bar{M}$ can be defined by

$$\sigma(ao) = \sigma(a)o \qquad \text{for } a \in \bar{G}.$$

For a σ-invariant subset N of $\bar{M}$, the set of all fixed points of σ in N will be denoted by N_σ. It is known (Takeuchi [13]) that σ is an involutive isometry of $(\bar{M}, \bar{g})$ such that

$$M = (\bar{M})_\sigma. \tag{4.1}$$

Furthermore, $\bar{\iota}$: $\bar{\mathfrak{g}}_{-1} \to \bar{M}$ is σ-equivariant and j: $(M, g) \to (\bar{M}, \bar{g})$ is isometric. As a maximal abelian subalgebra $\bar{\mathfrak{a}}$ of $\bar{\mathfrak{p}} = \mathfrak{p} + \sqrt{-1}\mathfrak{k}$, we take

$$\bar{\mathfrak{a}} = \mathfrak{h}_{\boldsymbol{R}} = \mathfrak{a} + \sqrt{-1}\mathfrak{b},$$

and use the σ-order $>$ on $\mathfrak{h}_{\boldsymbol{R}}$ for $\mathfrak{g}$ as an order on $\bar{\mathfrak{a}}$ for $\bar{\mathfrak{g}}$.

Lemma 4.3. *We can choose* $\bar{\Delta} = \{\bar{\beta}_1, \cdots, \bar{\beta}_{\bar{r}}\} \subset \bar{\Sigma}_1$ *and* $\bar{X}_i \in \bar{\mathfrak{g}}^{\bar{\beta}_i}$ $(1 \leq i \leq \bar{r})$ *for* $\bar{\mathfrak{g}}$ *in the following way.*

(a) *Class (II).*

$\bar{r} = 2r$; $\sigma(\bar{\beta}_{2i-1}) = \bar{\beta}_{2i}$, $\sigma\bar{X}_{2i-1} = \bar{X}_{2i}$ $(1 \leq i \leq r)$; *If we set* $\beta_i = \pi_{\mathfrak{a}}(\bar{\beta}_{2i})$ $(1 \leq i \leq r)$, *then* $\Delta = \{\beta_1, \cdots, \beta_r\} \subset \Sigma_1$ *is a system of orthogonal roots for* $\mathfrak{g}$.

(b) *Otherwise.*

$\bar{r} = r$; $\sigma(\bar{\beta}_i) = \bar{\beta}_i$, $\sigma\bar{X}_i = \bar{X}_i$ $(1 \leq i \leq r)$; *If we set* $\beta_i = \bar{\beta}_i$ $(1 \leq i \leq r)$, *then* $\Delta = \{\beta_1, \cdots, \beta_r\} \subset \Sigma_1$ *is a system of orthogonal roots for* $\mathfrak{g}$.

Proof. See Takeuchi [13]. However, for $\bar{\mathfrak{g}}$ of type E_6 or E_7, $\{\bar{\beta}_i\}$ in [13] should be replaced by the following, under the numbering of roots of $(\bar{\Pi}, \bar{\Pi}_0)$ in [13].

$$\begin{aligned} \bar{\mathfrak{g}} = E_6 \colon \bar{\beta}_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6, \\ \bar{\beta}_2 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5. \\ \bar{\mathfrak{g}} = E_7 \colon \bar{\beta}_1 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7, \\ \bar{\beta}_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_7, \\ \bar{\beta}_3 &= \alpha_1. \end{aligned}$$

q.e.d.

Lemma 4.4. (a) *Class (II).*

$$\mathcal{V}_l = (\bar{\mathcal{V}}_{2l})_\sigma,\ V_l = (\bar{V}_{2l})_\sigma,\ d_l = \bar{d}_{2l}\ (0 \leq l \leq r),$$
$$\bar{\mathcal{V}}_{2l-1} \cap M = \phi,\ \bar{V}_{2l-1} \cap \mathfrak{g}_{-1} = \phi\ (1 \leq l \leq r).$$

(b) *Otherwise.*

$$\mathcal{V}_l = (\bar{\mathcal{V}}_l)_\sigma,\ V_l = (\bar{V}_l)_\sigma,\ d_l = \bar{d}_l\ (0 \leq l \leq r).$$

Proof. (a) By Lemam 4.3, we can take as

$$X_i = \bar{X}_{2i-1} + \bar{X}_{2i} \qquad (1 \leq i \leq r).$$

Then $b_l=\bar{b}_{2l}$ and $Y_l=\bar{Y}_{2l}$ $(0\leq l\leq r)$. Now $\mathcal{V}_{2l}=\bar{U}\bar{b}_{2l}o$ is σ-invariant because of $\bar{b}_{2l}o=b_lo\in M$. Moreover $\mathcal{V}_l=Ub_lo\subset\bar{U}\bar{b}_{2l}o=\bar{\mathcal{V}}_{2l}$. Thus, by Lemma 2.1, 3) together with (4.1), we have that $\bar{\mathcal{V}}_{2l-1}\cap M=\phi$ and $\mathcal{V}_l=(\bar{\mathcal{V}}_{2l})_\sigma$. This implies also $d_l=\bar{d}_{2l}$. In the same way as above, by $Y_l=\bar{Y}_{2l}$ we obtain the σ-invariance of $\bar{V}_{2l}$, and

$$(\bar{V}_{2l})_\sigma=(\bar{\mathcal{V}}_{2l}\cap\bar{\mathfrak{g}}_{-1})_\sigma=(\bar{\mathcal{V}}_{2l})_\sigma\cap(\bar{\mathfrak{g}}_{-1})_\sigma$$
$$=\mathcal{V}_l\cap\mathfrak{g}_{-1}=V_l\,.$$

This implies also $\bar{V}_{2l-1}\cap\mathfrak{g}_{-1}=\phi$ by virtue of (2.10).

(b) This is proved in the same way as above, by taking as $X_i=\bar{X}_i$. q.e.d.

Now Theorem 4.1 for simple $\bar{\mathfrak{g}}$ follows from the above lemma and the assertions 4), 5), 6) in Lemma 3.3.

Suppose next that $\bar{\mathfrak{g}}$ is not simple. Then $\mathfrak{g}$ is the scalar restriction to $\boldsymbol{R}$ of a complex simple Lie algebra $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{g}}_p$ $(-1\leq p\leq 1)$ be the subspace of $\tilde{\mathfrak{g}}$ such that the scalar restriction to $\boldsymbol{R}$ of $\tilde{\mathfrak{g}}_p$ is $\mathfrak{g}_p$. Then the graded Lie algebra $\tilde{\mathcal{G}}'$: $\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{-1}+\tilde{\mathfrak{g}}_0+\tilde{\mathfrak{g}}_1$, together with $\tilde{\tau}=\tau$, becomes a compact simple symmetric graded Lie algebra $(\tilde{\mathcal{G}},\tilde{\tau})$ over $\boldsymbol{C}$. Various objects for $(\tilde{\mathcal{G}},\tilde{\tau})$ are denoted by the same symbols as in Section 3, but with $\sim$. Note that in particular we have

$$G'=\tilde{G}'\,, \tag{4.2}$$

under the identification $\mathrm{Aut}(\tilde{\mathfrak{g}})\subset\mathrm{Aut}(\mathfrak{g})$. In our case, $\mathfrak{k}$ is a compact real form of $\tilde{\mathfrak{g}}$ and $\mathfrak{p}=I\mathfrak{k}$, where I is the complex structure of $\mathfrak{g}$. Thus $\mathfrak{h}=\mathfrak{b}+\mathfrak{a}=I\mathfrak{a}+\mathfrak{a}$ is the scalar restriction to $\boldsymbol{R}$ of a Cartan subalgebra of $\tilde{\mathfrak{g}}$ whose real part is $\mathfrak{a}$. The real part $\mathfrak{h}_{\boldsymbol{R}}$ of $\bar{\mathfrak{h}}$ is given by

$$\mathfrak{h}_{\boldsymbol{R}}=\sqrt{-1}\,I\mathfrak{a}+\mathfrak{a}\,.$$

Denoting the $\boldsymbol{C}$-linear extension to $\bar{\mathfrak{g}}$ of I by the same I, we define

$$\mathfrak{g}^\pm=\{X\in\bar{\mathfrak{g}};\,IX=\pm\sqrt{-1}\,X\}\,,$$

to get a decomposition

$$\bar{\mathfrak{g}}=\mathfrak{g}^+\oplus\mathfrak{g}^- \tag{4.3}$$

with $\sigma\mathfrak{g}^\pm=\mathfrak{g}^\mp$. Then $\boldsymbol{R}$-linear isomorphisms $\varpi^\pm$: $\mathfrak{g}\to\mathfrak{g}^\pm$ are defined by

$$\varpi^\pm(X)=X^\pm=(1/2)(X\mp\sqrt{-1}\,IX)\qquad\text{for } X\in\mathfrak{g}\,.$$

If we set $\mathfrak{a}^\pm=\varpi^\pm(\mathfrak{a})$, we have

$$\mathfrak{h}_{\boldsymbol{R}}=\mathfrak{a}^++\mathfrak{a}^-\,.$$

We may identify $\Sigma \subset \mathfrak{a}$ with $\tilde{\Sigma}$. For $\alpha \in \Sigma$, $\alpha^{\pm} \in \mathfrak{a}^{\pm}$ is defined by

$$(\alpha^{\pm}, H^{\pm}) = (\alpha, H) \qquad \text{for each } H \in \mathfrak{a} .$$

Then we have

$$\begin{aligned} &\sigma\alpha^{\pm} = \alpha^{\mp} \qquad \text{for } \alpha \in \Sigma , \\ &\bar{\Sigma} = \{\alpha^+, \alpha^- ; \alpha \in \Sigma\}, \ \bar{\Sigma}^+ = \{\alpha^+, \alpha^- ; \alpha \in \Sigma^+\} , \\ &\bar{\Pi} = \Pi^+ \cup \Pi^-, \ \bar{\Pi}_0 = \Pi_0^+ \cup \Pi_0^- , \end{aligned}$$

where $\Pi^{\pm} = \{\alpha^{\pm}; \alpha \in \Pi\}$ and $\Pi_0^{\pm} = \{\alpha^{\pm}; \alpha \in \Pi_0\}$. In particular, σ (restricted to $\mathfrak{h}_{\boldsymbol{R}}$) belongs to $\mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma)$. We regard $\mathrm{Aut}(\Pi)$ as a normal subgroup of $\mathrm{Aut}(\bar{\Pi})$ by the correspondence $t \mapsto (t^+, t^-)$ for $t \in \mathrm{Aut}(\Pi)$, where $t^{\pm} \in \mathrm{Aut}(\Pi^{\pm})$ is defined by

$$t^{\pm}(\alpha^{\pm}) = (t\alpha)^{\pm} \qquad \text{for } \alpha \in \Pi .$$

Then we have a semi-direct decomposition

$$\mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma) = \boldsymbol{Z}_2 \cdot \mathrm{Aut}(\Pi, \Pi_0) , \qquad \text{where} \quad \boldsymbol{Z}_2 = \{1, \sigma\} . \tag{4.4}$$

An involutive automorphism k_0 of $\mathfrak{g}$ with $k_0 | \mathfrak{h}_{\boldsymbol{R}} = \sigma$ is constructed as follows. Choose $\kappa \in \mathrm{Aut}(\tilde{\mathfrak{g}}, \mathfrak{k}, \mathfrak{h}_{\boldsymbol{R}})$ with $\kappa | \mathfrak{h}_{\boldsymbol{R}} = -1$ and $\kappa^2 = 1$, and set

$$k_0 = \tau\kappa \in \mathrm{Aut}(\mathfrak{g}) .$$

Then it is verified that $k_0 \mathfrak{h}_{\boldsymbol{R}} = \mathfrak{h}_{\boldsymbol{R}}$ and $k_0 \alpha^{\pm} = \alpha^{\mp}$ for each $\alpha \in \Pi$, and hence $k_0 | \mathfrak{h}_{\boldsymbol{R}} = \sigma$. Actually k_0 belongs to K_0, because $k_0 E = E$ and $k_0 \tau = \tau k_0$. Recall that $G/G' \simeq \mathrm{Aut}(\bar{\Pi}, \bar{\Pi}_0, \sigma)$ by Lemma 1.2, $\tilde{G}/\tilde{G}' \simeq \mathrm{Aut}(\Pi, \Pi_0)$ by Lemma 3.2, and $G' = \tilde{G}'$ by (4.2). So by (4.4) we get a semi-direct decomposition

$$G = \boldsymbol{Z}_2 \cdot \tilde{G} \qquad \text{where} \quad \boldsymbol{Z}_2 = \{1, k_0\} .$$

This, together with (4.2), (1.14), implies semi-direct decompositions

$$G_0 = \boldsymbol{Z}_2 \cdot \tilde{G}_0, \ U = \boldsymbol{Z}_2 \cdot \tilde{U}, \ K = \boldsymbol{Z}_2 \cdot \tilde{K}_0, \ K_0 = \boldsymbol{Z}_2 \cdot \tilde{K}_0 , \tag{4.5}$$

and equalities

$$G_0' = \tilde{G}_0', \ U' = \tilde{U}', \ K' = \tilde{K}', \ K_0' = \tilde{K}_0' .$$

Thus we have a natural identification

$$M = G/U \simeq \tilde{M} = \tilde{G}/\tilde{U} ,$$

which is a homothety between (M, g) and $(\tilde{M}, \tilde{g})$. It is easy to see that under this identification we have

$$\mathcal{V}_l = \tilde{\mathcal{V}}_l, \ V_l = \tilde{V}_l \ (0 \leq l \leq r = \tilde{r}) , \tag{4.6}$$

and hence $\dim \mathcal{C}V_l = \dim V_l = 2\tilde{d}_l$ $(0 \leq l \leq r)$. Now Theorem 4.1 for non-simple $\bar{\mathfrak{g}}$ follows from the assertions 4), 5), 6) in Lemma 3.3.

REMARK 4.5. Each Cl V_l $(0 \leq l \leq r)$ is a real affine algebraic variety in $\mathfrak{g}_{-1}$. In case where $\bar{\mathfrak{g}}$ is not simple, this is obvious from Remark 3.4. In case where $\bar{\mathfrak{g}}$ is simple, Cl $V_l = (\text{Cl } \bar{V}_m) \cap \mathfrak{g}_{-1}$, $m = l$ or $2l$, by Lemma 4.4. In the construction of defining polynomials $\{F_\alpha^{(m)}\}$ of Cl $\bar{V}_m$ in Remark 3.4, we choose basis $\{e_1, \cdots, e_n\}$ and $\{X_1, \cdots, X_N\}$ of $\bar{\mathfrak{g}}_{-1}$ and $\bar{\mathfrak{g}}_0$ from $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$, respectively. Then $\{F_\alpha^{(m)}\}$ are real polynomials by which Cl V_l is defined in $\mathfrak{g}_{-1}$.

REMARK 4.6. If $\bar{\mathfrak{g}}$ is not simple, that is, if (M, g) is an irreducible Hermitian symmetric space of compact type, the group G is equal to the group of holomorphic transformations and anti-holomorphic transformations of M. This follows from Remark 3.1 and the decomposition $G = \boldsymbol{Z}_2 \cdot \tilde{G}$.

Theorem 4.7. *$GL(\mathfrak{g}_{-1}, \mathcal{S}) = G_0$ if $r \geq 2$.*

Proof. Note first that by Theorem 4.1 $GL(\mathfrak{g}_{-1}, \mathcal{S})$ is given also by

$$GL(\mathfrak{g}_{-1}, \mathcal{S}) = \{a \in GL(\mathfrak{g}_{-1});\ a(\text{Cl } V_l) = \text{Cl } V_l \ (0 \leq l \leq r)\}, \tag{4.7}$$

and hence $GL(\mathfrak{g}_{-1}, \mathcal{S})$ is a closed subgroup of $GL(\mathfrak{g}_{-1})$.

Suppose that $\bar{\mathfrak{g}}$ is simple. Then $G_0 = (\bar{G}_0)_\sigma$, as is seen in the proof of Theorem 4.1, and $GL(\bar{\mathfrak{g}}_{-1}, \bar{\mathcal{S}}) = \bar{G}_0$ by Theorem 3.5. Thus it suffices to show

$$GL(\mathfrak{g}_{-1}, \mathcal{S}) = GL(\bar{\mathfrak{g}}_{-1}, \bar{\mathcal{S}}) \cap GL(\mathfrak{g}_{-1}) \qquad \text{if} \quad r \geq 2, \tag{4.8}$$

$GL(\mathfrak{g}_{-1})$ being regarded as a subgroup of $GL(\bar{\mathfrak{g}}_{-1})$. In case where M is of class (II), by Lemma 4.4 we have $\text{Cl } V_l = (\text{Cl } \bar{V}_{2l}) \cap \mathfrak{g}_{-1}$ $(0 \leq l \leq r)$. Hence, by (3.5), (4.7) and Remark 4.5, we have

$$GL(\mathfrak{g}_{-1}, \mathcal{S}) = GL_e(\bar{\mathfrak{g}}_{-1}, \bar{\mathcal{S}}) \cap GL(\mathfrak{g}_{-1}),$$

under the notation in Lemma 3.6. Since $r \geq 2$, Lemma 3.6 implies (4.8). If M is not of class (II), we have $\text{Cl } V_l = (\text{Cl } \bar{V}_l) \cap \mathfrak{g}_{-1}$ $(0 \leq l \leq r)$, which implies (4.8) in the same way.

Suppose next that $\bar{\mathfrak{g}}$ is not simple. Then $G_0 = \boldsymbol{Z}_2 \cdot \tilde{G}_0$ with $\boldsymbol{Z}_2 = \{1, k_0\}$ by (4.5), and $GL(\tilde{\mathfrak{g}}_{-1}, \tilde{\mathcal{S}}) = \tilde{G}_0$ by Theorem 3.5. Thus it suffices to show

$$GL(\mathfrak{g}_{-1}, \mathcal{S}) = \boldsymbol{Z}_2 \cdot GL(\tilde{\mathfrak{g}}_{-1}, \tilde{\mathcal{S}}) \qquad \text{if} \quad r \geq 2. \tag{4.9}$$

First we show

$$\mathfrak{g}_0 = \text{Lie } GL(\mathfrak{g}_{-1}, \mathcal{S}) \qquad \text{if} \quad r \geq 2. \tag{4.10}$$

We set $\mathfrak{g}_s = \text{Lie } GL(\mathfrak{g}_{-1}, \mathcal{S})$. Let $I \in \text{End } \mathfrak{g}_{-1}$ denote the complex structure of $\mathfrak{g}_{-1}$ induced by that of $\mathfrak{g}$, and

(4.11) $$\tilde{\mathfrak{g}}_{-1}=\mathfrak{g}^{+}_{-1}+\mathfrak{g}^{-}_{-1}$$

be the decomposition by I which is the one (4.3) for $\tilde{\mathfrak{g}}_{-1}$. We define a complex structure I of End $\mathfrak{g}_{-1}$ by

$$I(X)=I\circ X \qquad \text{for } X\in \operatorname{End}\mathfrak{g}_{-1}.$$

Since each V_l is a complex submanifold of $\tilde{\mathfrak{g}}_{-1}$ by (4.6), $\mathfrak{g}_s$ is inavriant under I. Also $\mathfrak{g}_0$ is invariant under I, and $I|\mathfrak{g}_0$ is the complex structure of $\mathfrak{g}_0$ induced by that of $\mathfrak{g}$. Thus we have the following three decompositions by I.

$$\begin{aligned}
&\operatorname{End}\tilde{\mathfrak{g}}_{-1}=(\operatorname{End}\mathfrak{g}_{-1})^{C}=(\operatorname{End}\mathfrak{g}_{-1})^{+}+(\operatorname{End}\mathfrak{g}_{-1})^{-},\\
&\tilde{\mathfrak{g}}_s=\mathfrak{g}^{+}_s+\mathfrak{g}^{-}_s,\\
&\tilde{\mathfrak{g}}_0=\mathfrak{g}^{+}_0\oplus\mathfrak{g}^{-}_0,
\end{aligned}$$

where

$$\begin{aligned}
(\operatorname{End}\mathfrak{g}_{-1})^{\pm}&=\operatorname{Hom}(\mathfrak{g}^{+}_{-1},\mathfrak{g}^{\pm}_{-1})+\operatorname{Hom}(\mathfrak{g}^{-}_{-1},\mathfrak{g}^{\pm}_{-1}),\\
&\mathfrak{g}^{\pm}_0\subset\operatorname{Hom}(\mathfrak{g}^{\pm}_{-1},\mathfrak{g}^{\pm}_{-1}).
\end{aligned}$$

By $[\tilde{\mathfrak{g}}_0,\tilde{\mathfrak{g}}_s]\subset\tilde{\mathfrak{g}}_s$ and $[\tilde{\mathfrak{g}}_0, I]=\{0\}$, we get

(4.12) $$[\tilde{\mathfrak{g}}_0,\mathfrak{g}^{\pm}_s]\subset\mathfrak{g}^{\pm}_s.$$

Let us consider the adjoint action of $\tilde{\mathfrak{g}}_0$ on End $\tilde{\mathfrak{g}}_{-1}$. Since $\mathfrak{g}^{\pm}_0$ is included in $\operatorname{Hom}(\mathfrak{g}^{\pm}_{-1},\mathfrak{g}^{\pm}_{-1})$ irreducibly, $\tilde{\mathfrak{g}}_0$ leaves $\operatorname{Hom}(\mathfrak{g}^{-}_{-1},\mathfrak{g}^{+}_{-1})\subset\operatorname{End}\tilde{\mathfrak{g}}_{-1}$ invariant and acts on it irreducibly. We set

$$\begin{aligned}
\mathfrak{g}^{++}_s&=\mathfrak{g}^{+}_s\cap\operatorname{Hom}(\mathfrak{g}^{+}_{-1},\mathfrak{g}^{+}_{-1}),\\
\mathfrak{g}^{-+}_s&=\mathfrak{g}^{+}_s\cap\operatorname{Hom}(\mathfrak{g}^{-}_{-1},\mathfrak{g}^{+}_{-1}).
\end{aligned}$$

We will show

(4.13) $$\mathfrak{g}^{+}_s=\mathfrak{g}^{++}_s+\mathfrak{g}^{-+}_s,$$

(4.14) $\mathfrak{g}^{-+}_s\subset\operatorname{Hom}(\mathfrak{g}^{-}_{-1},\mathfrak{g}^{+}_{-1})$ is invariant under $\tilde{\mathfrak{g}}_0$.

We denote a general element $X\in\operatorname{End}\tilde{\mathfrak{g}}_{-1}$ by a matricial form

$$X=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\quad \begin{matrix}A\in\operatorname{Hom}(\mathfrak{g}^{+}_{-1},\mathfrak{g}^{+}_{-1}), & B\in\operatorname{Hom}(\mathfrak{g}^{-}_{-1},\mathfrak{g}^{+}_{-1}),\\ C\in\operatorname{Hom}(\mathfrak{g}^{+}_{-1},\mathfrak{g}^{-}_{-1}), & D\in\operatorname{Hom}(\mathfrak{g}^{-}_{-1},\mathfrak{g}^{-}_{-1}).\end{matrix}$$

Then, for

$$X_0=\begin{pmatrix}A_0 & 0\\ 0 & D_0\end{pmatrix}\in\tilde{\mathfrak{g}}_0,\quad A_0\in\mathfrak{g}^{+}_0,\quad D_0\in\mathfrak{g}^{-}_0,$$

$$X=\begin{pmatrix}A & B\\ 0 & 0\end{pmatrix}\in\mathfrak{g}^{+}_s,\quad A\in\operatorname{Hom}(\mathfrak{g}^{+}_{-1},\mathfrak{g}^{+}_{-1}),\quad B\in\operatorname{Hom}(\mathfrak{g}^{-}_{-1},\mathfrak{g}^{+}_{-1}),$$

we have

$$[X_0, X] = \begin{pmatrix} A_0A - AA_0 & A_0B - BD_0 \\ 0 & 0 \end{pmatrix}.$$

Since $\mathfrak{g}_0^{\pm}$ contains the scalar endomorphisms $\boldsymbol{C}1_{\mathfrak{g}_{-1}^{\pm}}$, we can take as $A_0 = (1/2)1$ and $D_0 = -(1/2)1$. Then, for any $X \in \mathfrak{g}_s^+$ we have

$$[X_0, X] = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

which belongs to $\mathfrak{g}_s^+$ by (4.12). Hence $B \in \mathfrak{g}_s^{-+}$ and so $A = X - B \in \mathfrak{g}_s^{++}$. Thus we get (4.13). Since the action of $\bar{\mathfrak{g}}_0$ on $\mathrm{Hom}(\mathfrak{g}_{-1}^-, \mathfrak{g}_{-1}^+)$ is given by

$$B \mapsto A_0B - BD_0 \qquad \text{for } (A_0, D_0) \in \mathfrak{g}_0^+ \oplus \mathfrak{g}_0^- = \bar{\mathfrak{g}}_0,$$

we get also (4.14).

Now by the $\bar{\mathfrak{g}}_0$-irrecibility of $\mathrm{Hom}(\mathfrak{g}_{-1}^-, \mathfrak{g}_{-1}^+)$ and (4.14) we have either (a) $\mathfrak{g}_s^{-+} = \{0\}$ or (b) $\mathfrak{g}_s^{-+} = \mathrm{Hom}(\mathfrak{g}_{-1}^-, \mathfrak{g}_{-1}^+)$. In case (a), by (4.13) $\mathfrak{g}_s^+$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1}^+)$ with $\mathfrak{g}_0^+ \subset \mathfrak{g}_s^+$. Since $r \geq 2$, by the same argument as in the proof of Theorem 3.5, we get $\mathfrak{g}_0^+ = \mathfrak{g}_s^+$, which implies $\mathfrak{g}_0 = \mathfrak{g}_s$. In case (b), noting that $\boldsymbol{C}1_{\mathfrak{g}_{-1}^+} \subset \mathfrak{g}_s^{++}$, we have $\mathfrak{g}_s^{++} = \mathfrak{gl}(\mathfrak{g}_{-1}^+)$. This, together with (b), implies $\mathfrak{g}_s^+ = (\mathrm{End}\,\mathfrak{g}_{-1})^+$. Therefore we have $\mathfrak{g}_s = \mathfrak{gl}(\mathfrak{g}_{-1})$, which is a contradiction to $r \geq 2$. Thus we have proved (4.10). In particular, we see that any element $a \in GL(\mathfrak{g}_{-1}, \mathcal{S})$ normalizes $\mathfrak{g}_0$.

Now let $a \in GL(\mathfrak{g}_{-1}, \mathcal{S})$ be arbitrary. Since the $\mathfrak{g}_0$-module $\bar{\mathfrak{g}}_{-1}$ has the decomposition (4.11) with inequivalent irreducible $\mathfrak{g}_0$-submodules $\mathfrak{g}_{-1}^{\pm}$, a, regarded as an element of $GL(\bar{\mathfrak{g}}_{-1})$, permutes $\mathfrak{g}_{-1}^+$ and $\mathfrak{g}_{-1}^-$. On the other hand, $k_0 \in G_0 \subset GL(\mathfrak{g}_{-1}, \mathcal{S})$ is anti-linear as a map of $\tilde{\mathfrak{g}}_{-1}$, and hence k_0, regarded as an element of $GL(\bar{\mathfrak{g}}_{-1})$, interchanges $\mathfrak{g}_{-1}^+$ and $\mathfrak{g}_{-1}^-$. Thus, either a or ak_0 leaves $\mathfrak{g}_{-1}^{\pm}$ invariant, that is, either a or ak_0 is $\boldsymbol{C}$-linear as a map of $\tilde{\mathfrak{g}}_{-1}$. Therefore, either a or ak_0 belongs to $GL(\tilde{\mathfrak{g}}_{-1}, \tilde{\mathcal{S}})$, because $\mathcal{S} = \tilde{\mathcal{S}}$ by (4.6). This proves (4.9). q.e.d.

REMARK 4.8. For classical $\mathcal{G}$ such that the symmetric domain corresponding to (Π, Π_0) is of tube type, Theorem 4.7 was proved by Tanaka [18].

5. Helgason spheres

In this section we define the notion of a Helgason sphere of an irreducible symmetric R-space (M, g), and prove that the group G permutes Helgason spheres of (M, g).

Under the notation in Section 1, we introduce a linear order $>$ on $(\mathfrak{a}_M)_{\boldsymbol{R}}$ in such a way that $h_1 > \cdots > h_r > 0$. Let Σ_M^+ denote the set of positive roots in Σ_M. For each $\gamma \in \Sigma_M$ we define

$$\mathfrak{k}_0^\gamma = \mathfrak{k}_0 \cap ((\mathfrak{k}^C)^\gamma + (\mathfrak{k}^C)^{-\gamma}),$$
$$\mathfrak{m}^\gamma = \mathfrak{m} \cap ((\mathfrak{k}^C)^\gamma + (\mathfrak{k}^C)^{-\gamma}), \quad m_M(\gamma) = \dim \mathfrak{m}^\gamma,$$

and set

$$\mathfrak{k}_0^0 = \mathfrak{k}_0 \cap (\mathfrak{k}^C)^0 = \{X \in \mathfrak{k}_0;\ [X, \mathfrak{a}_M] = \{0\}\},$$
$$\mathfrak{m}^0 = \mathfrak{m} \cap (\mathfrak{k}^C)^0 = \mathfrak{a}_M .$$

Then we have orthogonal decompositions

$$\mathfrak{k}_0 = \mathfrak{k}_0^0 + \sum_{\gamma \in \Sigma_M^+} \mathfrak{k}_0^\gamma,$$
$$\mathfrak{m} = \mathfrak{m}^0 + \sum_{\gamma \in \Sigma_M^+} \mathfrak{m}^\gamma .$$

For $\gamma \in \Sigma_M$, we set

$$A^\gamma = -\frac{2\pi\sqrt{-1}}{(\gamma, \gamma)} \gamma \in \mathfrak{a}_M .$$

Note that $|A^\gamma| = 2\pi/|\gamma|$. Let

$$\Sigma_M = (\Sigma_M)_1 \cup \cdots \cup (\Sigma_M)_s$$

be the decomposition of Σ_M into the sum of irreducible components $(\Sigma_M)_k$, and δ_k $(1 \le k \le s)$ the highest root in $(\Sigma_M)_k$. We choose δ_M of the largest length among these δ_k $(1 \le k \le s)$ once and for all.

Suppose that $\gamma \in \Sigma_M^+$ satisfy $2\gamma \notin \Sigma_M$. We define

$$\mathfrak{n}_\gamma = \boldsymbol{R}\sqrt{-1}\gamma + \mathfrak{m}^\gamma,$$
$$\mathfrak{t}_\gamma = [\mathfrak{n}_\gamma, \mathfrak{n}_\gamma] = \mathfrak{k}_0^\gamma + [\mathfrak{m}^\gamma, \mathfrak{m}^\gamma]$$
$$= \mathfrak{k}_0^\gamma + [\mathfrak{k}_0^\gamma, \mathfrak{k}_0^\gamma],$$
$$\mathfrak{s}_\gamma = \mathfrak{t}_\gamma + \mathfrak{n}_\gamma .$$

Then $\mathfrak{s}_\gamma$ is a subalgebra of $\mathfrak{k}$, and by virtue of $2\gamma \notin \Sigma_M$ one has $[[\mathfrak{n}_\gamma, \mathfrak{n}_\gamma], \mathfrak{n}_\gamma] \subset \mathfrak{n}_\gamma$. Therefore

$$N_\gamma = (\exp \mathfrak{n}_\gamma) o \subset M$$

is a totally geodesic submanifold of (M, g) with

$$\dim N_\gamma = m_M(\gamma) + 1 . \tag{5.1}$$

Lemma 5.1 (Helgason [4]). *Suppose that* $\dim M \ge 2$.

1) *The maximum of sectional curvatures of* (M, g) *is equal to* $|\delta_M|^2$.

2) N_γ *has constant sectional curvature* $|\gamma|^2$ *(with respect to the metric on* N_γ *induced by* g*), and therefore the symmetric pair* $(\mathfrak{s}_\gamma, \mathfrak{t}_\gamma)$ *is isomorphic to*

$$(\mathfrak{o}(m_M(\gamma)+2), \mathfrak{o}(m_M(\gamma)+1)) .$$

3) *If M is simply connected and Σ_M is irreducible, then*

(a) *N_{δ_M} is a sphere, and*

$$T = \{(\exp tA^{\delta_M})o;\ 0 \leq t \leq 1\}$$

is a simply closed geodesic in N_{δ_M} of length $2\pi/|\delta_M|$, and has the minimum length among all the closed geodesics of (M, g);

(b) *Any totally geodesic sphere in (M, g) with dimension ≥ 2 of constant curvature $|\delta_M|^2$ is conjugate to a submanifold of N_{δ_M} under the largest connected isometry group $I^0(M, g)$ of (M, g).*

REMARK 5.2. Acutally, Lemma 5.1 holds for any non-flat compact symmetric space (M, g).

Theorem 5.3. *Suppose that* $\dim M \geq 2$.

1) *Shortest closed geodesics of (M, g) are conjugate to each other under $I^0(M, g)$ (up to parametrization).*

2) *We have an inequality*

$$|A_1| \leq |A^{\delta_M}|,$$

which is equivalent to that $|\delta_M| \leq |\delta|$, and the equality holds if and only if M is simply connected.

3) *The length of a shortest closed geodesic is $2\pi/|\delta|$.*

Proof. Let $c(t)$ $(0 \leq t \leq 1)$ be any shortest simply closed geodesic of (M, g). Recalling the fact that any vector in $\mathfrak{m}$ can be transformed into $\mathfrak{a}_M$ by an element of the identity component of K_0, we see that $c(t)$ is conjugate to a geodesic

$$c_A(t) = (\exp tA)o \ \ (0 \leq t \leq 1), \ \text{with} \ A \in \Gamma_M,$$

under $I^0(M, g)$. Since A has the shortest length among $\Gamma_M - \{0\}$, we have $A = \pm A_i$ $(1 \leq i \leq r)$. Furthermore we may assume that $A = A_1$ or $-A_1$ since W_M contains $\mathfrak{S}_r$ by Lemma 1.3. But the corresponding closed geodesics c_{A_1} and c_{-A_1} are the same up to parametrization, and so we get the assertion 1). We prove the inequality in 2) for each of the five classes in Section 1 separately.

Classes (I) and (II). In these cases, Σ_M is irreducible and $\delta_M = 2h_1$. A^{δ_M} can be computed by (1.22) to get $A^{\delta_M} = A_1$. Thus $|A^{\delta_M}| = |A_1|$.

Class (III). By $\dim M \geq 2$, we have $r \geq 2$. Thus Σ_M is irreducible and $\delta_M = h_1 - h_r$. One has $A^{\delta_M} = A_1 - A_r$, and hence $|A^{\delta_M}| = \sqrt{2}\,|A_1| > |A_1|$.

Class (IV). Σ_M is irreducible, $\delta_M = h_1 + h_2$ $(r \geq 2)$ or h_1 $(r = 1)$. If $r \geq 2$, $A^{\delta_M} = A_1 + A_2$ and $|A^{\delta_M}| = \sqrt{2}\,|A_1| > |A_1|$. If $r = 1$, $A^{\delta_M} = 2A_1$ and $|A^{\delta_M}| = 2|A_1| > |A_1|$.

Class (V). Suppose first that $r \geq 3$. Then Σ_M is irreducible and $\delta_M = h_1 + h_2$.

One has $A^{\delta_M}=A_1+A_2$ and $|A^{\delta_M}|=\sqrt{2}|A_1|>|A_1|$. Suppose that $r=2$. In this case, Σ_M is not irreducible and decomposed as $\Sigma_M=(\Sigma_M)_1\cup(\Sigma_M)_2$ with

$$(\Sigma_M)_1=\{h_1+h_2,\ -(h_1+h_2)\}\ ,\quad \delta_1=h_1+h_2\,,$$
$$(\Sigma_M)_2=\{h_1-h_2,\ h_2-h_1\}\ ,\qquad \delta_2=h_1-h_2\,.$$

We take as $\delta_M=h_1+h_2$. Then $A^{\delta_M}=A_1+A_2$ and $|A^{\delta_M}|=\sqrt{2}|A_1|>|A_1|$.

Now we obtain the assertion 2) by comparing the above computations of $|A^{\delta_M}|$ with $\pi_1(M)$. The assertion 3) follows from that $|A_1|=2\pi/|\delta|$. q.e.d.

Seeing the above proof, we get the following

Corollary 5.4. *Any shortest closed geodesic of (M, g) through the origin o is conjugate to the geodesic*

$$T_1=\{(\exp tA_1)o;\ |t|\leq 1/2\}$$

under the group K_0' (up to parametrization).

A submanifold S of M is called a *Helgason sphere* of (M, g) if

(H1) S is a totally geodesic sphere with minimum radius; and

(H2) S has the maximum dimension among the submanifolds with the property (H1).

REMARK 5.5. A "Helgason sphere" in Nagano [7] or Peterson [8] is a submanifold S with (H1), (H2) and $\dim S\geq 2$.

Theorem 5.6. 1) *Helgason spheres of (M, g) are conjugate to each other under $I^0(M, g)$.*

2) *For any shortest closed geodesic c of (M,g) there is a Helgason sphere which includes c.*

3) *M is simply connected if and only if $2\leq$dimension of a Helgason sphere S. In this case, one has*

$$\dim S=m_M(\delta_M)+1\,.$$

4) *The radius of a Helgason sphere is $1/|\delta|$.*

Proof. We may assume that $\dim M\geq 2$. Suppose first that M is simply connected, that is, (M, g) is of class (I) or (II). Since Σ_M is irreducible in this case, by Lemma 5.1, 2), 3) (a) and (5.1), N_{δ_M} is a totally geodesic sphere of the radius $1/|\delta_M|$ with

$$\dim N_{\delta_M}=m_M(\delta_M)+1\,.$$

Let N be an arbitrary totally geodesic sphere in (M,g). If $\dim N\geq 2$, by Lemma 5.1, 1) the sectional curvature κ of N satisfies $\kappa\leq|\delta_M|^2$. Hence the radius of

$N=1/\sqrt{\kappa}\geq 1/|\delta_M|$. If $\dim N=1$, that is, N is a closed geodesic, then by Lemma 5.1, 3) (a) the length of $N\geq 2\pi/|\delta_M|$, and hence the radius of $N\geq 1/|\delta_M|$. Therefore N_{δ_M} satisfies the property (H1). It has also the property (H2) by virtue of Lemma 5.1, 3) (b). Thus N_{δ_M} is a Helgason sphere, and hence the assertion 1) follows from Lemma 5.1, 3) (b). The assertion 2) follows from Lemma 5.1, 3) (a) and Theorem 5.3, 1).

Suppose next that M is not simply connected. If N is a totally geodesic sphere with $\dim N\geq 2$, then the radius of $N\geq 1/|\delta_M|$, as is shown in the above. The radius of a shortest closed geodesic is $1/|\delta|$ by Theorem 5.3, 3). Thus, by Theorem 5.3, 2) the Helgason spheres are the shortest closed geodesics. Therefore the assertion 1) follows from Theorem 5.3, 1). The assertion 2) is trivial.

The assertions 3) and 4) are obvious from the above arguments and Theorem 5.3, 2). q.e.d.

We fix a root $\beta\in\Delta$ and define a subalgebra $\mathfrak{g}_\beta$ of $\mathfrak{g}$ by

$$\mathfrak{g}_\beta=[\mathfrak{g}^\beta,\mathfrak{g}^{-\beta}]+\mathfrak{g}^\beta+\mathfrak{g}^{-\beta}.$$

It has a Cartan decomposition

$$\mathfrak{g}_\beta=\mathfrak{k}_\beta+\mathfrak{p}_\beta,\quad\text{where}\quad \mathfrak{k}_\beta=\mathfrak{k}\cap\mathfrak{g}_\beta,\ \mathfrak{p}_\beta=\mathfrak{p}\cap\mathfrak{g}_\beta. \tag{5.2}$$

The symmetric pair dual to $(\mathfrak{g}_\beta, \mathfrak{k}_\beta)$ is in the same situation as $(\mathfrak{s}_\gamma, \mathfrak{t}_\gamma)$ in Lemma 5.1, by virtue of $2\beta\notin\Sigma$. Therefore, by Lemma 5.1, 2) and Remark 5.2 one has

$$(\mathfrak{g}_\beta,\mathfrak{k}_\beta)\simeq(\mathfrak{o}(1, m(\beta)+1),\mathfrak{o}(m(\beta)+1)). \tag{5.3}$$

Furthermore we have a decomposition

$$\mathfrak{k}_\beta=(\mathfrak{k}_\beta)_0+\mathfrak{m}_\beta,\quad\text{where}\quad(\mathfrak{k}_\beta)_0=\mathfrak{k}_\beta\cap\mathfrak{k}_0,\ \mathfrak{m}_\beta=\mathfrak{k}_\beta\cap\mathfrak{m}$$

with the property

$$[[\mathfrak{m}_\beta,\mathfrak{m}_\beta],\mathfrak{m}_\beta]\subset\mathfrak{m}_\beta. \tag{5.4}$$

Now let G_β be the connected Lie subgroup of G generated by $\mathfrak{g}_\beta$ and set $S_\beta=G_\beta o\subset M$. Then we have

$$S_\beta=(\exp\mathfrak{m}_\beta)o\simeq G_\beta/U_\beta,$$

where $U_\beta=U\cap G_\beta$ is a parabolic subgroup of G_β. Therefore, by (5.3) S_β is the symmetric R-space associated to $\mathfrak{o}(1, m(\beta)+1)$, and hence it is a sphere. Together with (5.4), it follows that S_β is a totally geodesic sphere in (M, g) with dimension $m(\beta)$.

Lemma 5.7. *S_β is a Helgason sphere.*

Proof. The closed geodesic T_β in Section 1 is contained in S_β and has the length $2\pi/|\delta|$, and hence the radius of S_β is $1/|\delta|$. Therefore, by Theorem 5.6, 4) S_β has the property (H1).

Suppose first that M is not simply connected. If $\dim S_\beta$ would be greater than one, then the radius of $S_\beta \geq 1/|\delta_M| > 1/|\delta|$ by Theorem 5.3, 2), which is a contradiction. Therefore $\dim S_\beta = 1$, whence S_β is a Helgason sphere.

Suppose next that M is simply connected. By Theorem 5.6, 2) it suffices to prove

$$m(\beta) = m_M(\delta_M)+1, \quad \text{where} \quad \delta_M = 2h_1 . \tag{5.5}$$

Denoting by $\varpi_\Delta \colon \mathfrak{a} \to \mathfrak{a}_\Delta$ the orthogonal projection, we have (cf. Takeuchi [14])

$$\#\{\gamma \in \Sigma;\ \varpi_\Delta(\gamma) = \beta_i\} = 1 \qquad (1 \leq i \leq r) . \tag{5.6}$$

Moreover (Takeuchi [16]) there is $c \in \mathrm{Inn}(\bar{\mathfrak{g}})$ such that

$$c\mathfrak{a}_\Delta = (\mathfrak{a}_M)_{\boldsymbol{R}}, \quad c\beta_i = 2h_i \ (1 \leq i \leq r) . \tag{5.7}$$

Since $\Delta \subset W\delta = W\beta_1$, we have $m(\beta) = m(\beta_1)$, and so

$$\begin{aligned} m(\beta) &= \dim_{\boldsymbol{C}}\{X \in \bar{\mathfrak{g}};\ [H, X] = (\beta_1, H)X \quad \text{for each } H \in \mathfrak{a}\} \\ &= \dim_{\boldsymbol{C}}\{X \in \bar{\mathfrak{g}};\ [H, X] = (\beta_1, H)X \quad \text{for each } H \in \mathfrak{a}_\Delta\} \ \text{by (5.6)} \\ &= \dim_{\boldsymbol{C}}\{X \in \bar{\mathfrak{g}};\ [H, X] = (2h_1, H)X \quad \text{for each } H \in \mathfrak{a}_M\} \ \text{by (5.7)} \\ &= \dim_{\boldsymbol{C}}(\mathfrak{k}^{\boldsymbol{C}})^{\delta_M} + \dim_{\boldsymbol{C}}(\mathfrak{p}^{\boldsymbol{C}})^{\delta_M} , \end{aligned}$$

where

$$(\mathfrak{p}^{\boldsymbol{C}})^{\delta_M} = \{X \in \mathfrak{p}^{\boldsymbol{C}};\ [H, X] = (\delta_M, H)X \quad \text{for each } H \in \mathfrak{a}_M\} .$$

Since $\dim_{\boldsymbol{C}}(\mathfrak{k}^{\boldsymbol{C}})^{\delta_M} = m_M(\delta_M)$ and $\dim_{\boldsymbol{C}}(\mathfrak{p}^{\boldsymbol{C}})^{\delta_M} = 1$ (Takeuchi [16]), we get (5.5).

q.e.d.

Corollary 5.8.

$$m(\beta) = \begin{cases} m_M(\delta_M)+1 & \text{if} \quad \pi_1(M) = \{0\} , \\ 1 & \text{if} \quad \pi_1(M) \neq \{0\} . \end{cases}$$

For $\lambda \in \boldsymbol{R}$ and $-1 \leq p \leq 1$, we define

$$\Sigma^{(\lambda)} = \{\gamma \in \Sigma;\ 2(\gamma, \delta)/(\delta, \delta) = \lambda\} , \quad \Sigma_p^{(\lambda)} = \Sigma^{(\lambda)} \cap \Sigma_p .$$

Then we have decompositions

$$\begin{aligned} &\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)} \cup \Sigma^{(-1)} \cup \Sigma^{(2)} \cup \Sigma^{(-2)} \ \text{with} \ \Sigma^{(\pm 2)} = \{\pm\delta\} , \\ &\Sigma_0 = \Sigma_0^{(0)} \cup \Sigma_0^{(1)} \cup \Sigma_0^{(-1)}, \quad \Sigma_1 = \Sigma_1^{(0)} \cup \Sigma_1^{(1)} \cap \Sigma^{(2)} . \end{aligned}$$

Furthermore we set

$$\Sigma_\delta = \{\gamma\in\Sigma;\ (\gamma, E_\delta)\geq 0\},\quad \text{where}\quad E_\delta = 2E-\frac{2}{(\delta, \delta)}\delta\,,$$

$$(\Sigma_\delta)_q = \{\gamma\in\Sigma_\delta;\ (\gamma, E_\delta) = q\}\quad \text{for } q\geq 0\,.$$

Then we have decompositions

$$\begin{aligned}\Sigma_\delta &= (\Sigma_\delta)_0\cup(\Sigma_\delta)_1\cup(\Sigma_\delta)_2\,,\\ (\Sigma_\delta)_0 &= \Sigma_0^{(0)}\cup\Sigma^{(2)}\cup\Sigma^{(-2)}\,,\\ (\Sigma_\delta)_1 &= \Sigma_0^{(-1)}\cup\Sigma_1^{(1)}\,,\\ (\Sigma_\delta)_2 &= \Sigma_1^{(0)}\,.\end{aligned}$$

We define a parabolic subalgebra $\mathfrak{l}_\delta$ of $\mathfrak{g}$ and several subalgebras and a subspace of $\mathfrak{l}_\delta$ by

$$\begin{aligned}\mathfrak{l}_\delta &= \mathfrak{g}^0+\sum_{\gamma\in\Sigma_\delta}\mathfrak{g}^\gamma\,,\\ \mathfrak{l}_0 &= \mathfrak{g}^0+\sum_{\gamma\in(\Sigma_\delta)_0}\mathfrak{g}^\gamma\supset\mathfrak{g}_\delta\,,\\ \mathfrak{l}_1 &= \sum_{\gamma\in(\Sigma_\delta)_1}\mathfrak{g}^\gamma\subset\mathfrak{u}\ \text{(a subspace)}\,,\\ \mathfrak{l}_2 &= \sum_{\gamma\in(\Sigma_\delta)_2}\mathfrak{g}^\gamma\subset\mathfrak{g}_1\,,\\ \mathfrak{z}_0 &= \{X\in\mathfrak{l}_0;\ [X, \mathfrak{g}_\delta] = \{0\}\}\subset\mathfrak{g}_0\,.\end{aligned}$$

Note that $\mathfrak{l}_0=\mathfrak{g}_\delta\oplus\mathfrak{z}_0$. The corresponding connected Lie subgroups of G are denoted by L_δ, L_0, L_2 and Z_0, and set $L_1=\exp\mathfrak{l}_1$. Then we have that $L_\delta=L_0L_1L_2$ since $\mathfrak{l}_1+\mathfrak{l}_2$ is a nilpotent ideal of $\mathfrak{l}_\delta=\mathfrak{l}_0+\mathfrak{l}_1+\mathfrak{l}_2$ with $[\mathfrak{l}_1, \mathfrak{l}_1]\subset\mathfrak{l}_2$, and that $L_0=G_\delta Z_0$.

Lemma 5.9. $L_\delta S_\delta=S_\delta$.

Proof. We have that $L_\delta=L_0L_1L_2=G_\delta Z_0L_1L_2\subset G_\delta U$ since $Z_0\subset G_0$, $L_1\subset U$ and $L_2\subset\exp\mathfrak{g}_1$. Thus, for each $l\in L_\delta$ and each $p=ao\in S_\delta$ $(a\in G_\delta)$, one has

$$lp = lao\in L_\delta o\subset G_\delta Uo = G_\delta o = S_\delta\,.\qquad \text{q.e.d.}$$

Theorem 5.10. *Any element of the group G of basic transformations permutes the Helgason spheres of (M, g).*

Proof. We claim first that for any $a\in G$ aS_δ is also a Helgason sphere. If we denote by G^0 and K^0 the identity components of G and K respectively, we have the polar decomposition $G^0=K^0\exp\mathfrak{p}$ and $G^0=K^0L_\delta$. The latter follows from the fact that the parabolic subalgebra $\mathfrak{l}_\delta$ contains an Iwasawa subalgebra. Thus, together with (1.6), we get $G=KL_\delta$. Hence the claim follows by Lemma 5.9.

Now let S be an arbitrary Helgason sphere, and $a\in G$ be arbitrary. By Theorem 5.6, 1) there is $k\in K$ such that $kS_\delta=S$. Therefore $aS=akS_\delta$ is a

Helgason sphere by the above claim. q.e.d.

EXAMPLE 5.11. Let $M \hookrightarrow P_N(\boldsymbol{C})$ be the canonical equivariant projective imbedding of an irreducible Hermitian symmetric space (M, g) of compact type, in the sense of Sakane-Takeuchi [9]. Then a submanifold S of M is a Helgason sphere of (M, g) if and only if S is a projective line in $P_N(\boldsymbol{C})$. In fact, S_β in Lemma 5.7 is a projective line in $P_N(\boldsymbol{C})$.

REMARK 5.12. It can be shown that a Helgason sphere generates the homotopy $\pi_{m(\delta)}(M)$ and the homology $H_{m(\delta)}(M, \boldsymbol{Z})$.

6. Arithmetic distance

In this section we define a discrete valued distance d on an irreducible symmetric R-space (M, g) in terms of Helgason spheres, and characterize the group G as the group of isometries of d.

Lemma 6.1. *For each $p \in \mathcal{V}_l$ $(1 \leq l \leq r)$ there is a chain of Helgason spheres of length l connecting o and p, that is, there are Helgason spheres $S_1, \cdots, S_l$ such that $o \in S_1$, $p \in S_l$ and $S_k \cap S_{k+1} \neq \phi$ $(1 \leq k \leq l-1)$.*

Proof. Since each element of K_0 permutes the Helgason spheres by $K_0 \subset \mathrm{I}(M, g)$, and $\mathcal{V}_l = K_0 \mathcal{D}_l$ by Lemma 2.2, we may assume that $p \in \mathcal{D}_l$. Furthermore, since $\mathfrak{S}_r \subset W_M$ by Lemma 1.3, we may assume that

$$p = (\exp H)o\,, \quad H = \sum_{i=1}^{l} x_i A_i\,, \quad 0 < |x_i| \leq 1/2 \ (1 \leq i \leq l)\,.$$

We set

$$p_0 = o, \quad p_k = (\exp \sum_{i=1}^{k} x_i A_i)o \ (1 \leq k \leq l-1), \ p_l = p\,,$$

$$c_k = \{\exp(\sum_{i=1}^{k-1} x_i A_i + t A_k)o;\ |t| \leq 1/2\} \qquad (1 \leq k \leq l)\,.$$

Then c_k is a shortest closed geodesic (of length $2\pi/|\delta|$) through p_{k-1} and p_k $(1 \leq k \leq l)$. By Theorem 5.6, 2) there are Helgason spheres S_k with $c_k \subset S_k$ $(1 \leq k \leq l)$. The chain $\{S_k\}$ is the required one. q.e.d.

By this lemma and the transitivity of G on M, it follows that any two points of M can be connected by a chain of Helgason spheres. So we may give the following definition.

We define a distance d on (M, g), called the *arithmetic distance*, as follows. For $p, q \in M$ with $p \neq q$, $d(p, q)$ is defined to be the minimum possible length of a chain of Helgason spheres connecting p and q; and $d(p, q) = 0$ if $p = q$. Let

$$\mathrm{I}(M, d) = \{\varphi \in \mathrm{Diff}(M);\ d(\varphi(p), \varphi(q)) = d(p, q) \text{ for any } p, q \in M\}$$

denote the isometric diffeomorphism group of (M, d). Note that $G\subset \mathrm{I}(M, d)$ in virtue of Theorem 5.10. We define

$$M_l=\{p\in M;\ d(o, p)=l\} \qquad (0\leq l\leq r).$$

Then Lemma 6.1 is restated as

$$\mathcal{V}_l\subset M_0\cup M_1\cup\cdots\cup M_l \qquad (0\leq l\leq r).$$

Lemma 6.2. *If we define*

$$s_l'=s_{\beta_2}\cdots s_{\beta_{l+1}}\in W \qquad \textit{for } 1\leq l\leq r-1,$$

then $s_l'(\Sigma_0^{(1)}\cup\Sigma^{(2)})\subset\Sigma_0\cup\Sigma_1$, *under the notation in Section* 5.

Proof. Note that for $\gamma\in\Sigma$, $s_l'\gamma\in\Sigma_0\cup\Sigma_1$ if and only if $(s_l'\gamma, E)\geq 0$, where

$$\begin{aligned}(s_l'\gamma, E)=(\gamma, s_l'E)&=\left(\gamma,\ E-\frac{2}{(\beta_2,\beta_2)}\beta_2-\cdots-\frac{2}{(\beta_{l+1},\beta_{l+1})}\beta_{l+1}\right)\\&=(\gamma, E)-\frac{2}{(\delta,\delta)}\sum_{i=2}^{l+1}(\varpi_\Delta(\gamma),\beta_i).\end{aligned}$$

If $\gamma\in\Sigma_0^{(1)}$, then $(\gamma, E)=0$ and $\varpi_\Delta(\gamma)=(1/2)(\beta_1-\beta_j)$ $(2\leq j\leq r)$ or $(1/2)\beta_1$ (cf. Takeuchi [14]), whence $(s_l'\gamma, E)=0$ or 1. If $\gamma\in\Sigma^{(2)}$, that is, $\gamma=\delta$, then $(\gamma, E)=1$ and $\varpi_\Delta(\gamma)=\beta_1$, whence $(s_l'\gamma, E)=1$. q.e.d.

Next we want to know the structure of $\mathcal{V}_1=U'b_1o$. Under the notation in Section 5, we define

$$\begin{aligned}&\mathfrak{g}_0^{(0)}=\mathfrak{g}^0+\sum_{\gamma\in\Sigma_0^{(0)}}\mathfrak{g}^\gamma,\quad \mathfrak{g}_0^{(\pm1)}=\sum_{\gamma\in\Sigma_0^{(\pm1)}}\mathfrak{g}^\gamma,\\&\mathfrak{g}_1^{(1)}=\sum_{\gamma\in\Sigma_1^{(1)}}\mathfrak{g}^\gamma,\quad \mathfrak{g}_1^{(0)}=\sum_{\gamma\in\Sigma_1^{(0)}}\mathfrak{g}^\gamma,\quad \mathfrak{g}^{(2)}=\mathfrak{g}^\delta,\end{aligned}$$

and define a parabolic subalgebra $\mathfrak{u}_0$ of $\mathfrak{g}_0$ by

$$\mathfrak{u}_0=\mathfrak{g}_0^{(0)}+\mathfrak{g}_0^{(-1)}.$$

Then we have

$$\mathfrak{g}_0=\mathfrak{u}_0+\mathfrak{g}_0^{(1)},\quad \mathfrak{g}_1=\mathfrak{g}^{(2)}+\mathfrak{g}_1^{(1)}+\mathfrak{g}_1^{(0)}.$$

Let $G_0^{(1)}$, $G_1^{(1)}$, $G_1^{(0)}$ and $G^{(2)}$ denote the connected Lie subgroups of G generated by $\mathfrak{g}_0^{(1)}$, $\mathfrak{g}_1^{(1)}$, $\mathfrak{g}_1^{(0)}$ and $\mathfrak{g}^{(2)}$ respectively, and let

$$U_0'=\{a\in G_0';\ a\mathfrak{u}_0=\mathfrak{u}_0\}.$$

Then Lie $U_0'=\mathfrak{u}_0$, $\exp\mathfrak{g}_1=G^{(2)}G_1^{(1)}G_1^{(0)}$ and $U'=G_0'G^{(2)}G_1^{(1)}G_1^{(0)}$. Since $b_1|\mathfrak{a}=s_1$ by Lemma 2.1, 1), we have

$$\mathfrak{u}\cap b_1\mathfrak{u} = \mathfrak{g}^0+\sum_{\gamma}\mathfrak{g}^{\gamma},$$

where γ runs through all the $\gamma\in\Sigma$ such that $(\gamma, E)\geq 0$ and $(\gamma, s_1E)\geq 0$. But, since

$$(\gamma, s_1E) = \left(\gamma, E-\frac{2}{(\delta, \delta)}\delta\right) = (\gamma, E)-\frac{2(\gamma, \delta)}{(\delta, \delta)},$$

we have

$$\mathfrak{u}\cap b_1\mathfrak{u} = \mathfrak{u}_0+\mathfrak{g}_1^{(1)}+\mathfrak{g}_1^{(0)}.$$

Now U_0' leaves invariant $\mathfrak{g}_1$ and $\mathfrak{g}_1^{(1)}+\mathfrak{g}_1^{(0)}$, and hence U_0' acts on $\mathfrak{g}^{(2)}\simeq \mathfrak{g}_1/(\mathfrak{g}_1^{(1)}+\mathfrak{g}_1^{(0)})$ linearly. Then in the same way as in Takeuchi [15] we can prove

$$U'\cap b_1U'b_1^{-1} = U_0'G_1^{(1)}G_1^{(0)},$$

by which the following lemma is derived.

Lemma 6.3. *The correspondence* $(a, X)\mapsto a \exp Xb_1o$ $(a\in G_0', X\in\mathfrak{g}^{(2)})$ *induces a bijection* $\Psi\colon G_0'\times_{U_0'}\mathfrak{g}^{(2)}\to\mathcal{V}_1=U'b_1o$.

Theorem 6.4. $M_l=\mathcal{V}_l$ $(0\leq l\leq r)$. *Therefore, the range of d is* $\{0, 1, \cdots, r\}$.

Proof. We prove this by the induction on l. If $l=0$, this is obvious. Let $p\in M_1$ be arbitrary. Then there is a Helgason sphere through o and p, and hence there is a shortest closed geodesic c through o and p. Therefore, by Corollary 5.4 there is $k\in K_0'$ such that $kc=T_1$, whence $kp\in\mathcal{D}_1$. Thus p belongs to $K_0'\mathcal{D}_1$, which is equal to $\mathcal{V}_1$ by Lemma 2.2, 1). Thus we get $M_1\subset\mathcal{V}_1$. Together with Lemma 6.1, we obtain $M_1=\mathcal{V}_1$.

Assume that $1\leq l\leq r-1$ and $M_i=\mathcal{V}_i$ holds for each i with $0\leq i\leq l$. We show first that

$$M_{l+1}\subset \mathrm{Cl}\,\mathcal{V}_{l+1}. \tag{6.1}$$

Let $p\in M_{l+1}$ be arbitrary. Then there is $q\in M_l$ such that $d(q, p)=1$. By the assumption, $q\in\mathcal{V}_l=U'b_lo$. Here, since $\mathfrak{S}_r\subset W_M$ by Lemma 1.3, $\mathcal{V}_l$ is also written as $\mathcal{V}_l=U'b_l'o$ with

$$b_l' = \exp(1/2)(A_2+\cdots+A_{l+1}).$$

Therefore there is $a\in U'$ such that $q=ab_l'o$. Since $b_l'^{-1}a^{-1}q=o$ with $b_l'^{-1}a^{-1}\in G\subset \mathrm{I}(M, d)$, we have $b_l'^{-1}a^{-1}p\in M_1$. By the fact: $M_1=\mathcal{V}_1$ just proved, there is $b\in U'$ such that $b_l'^{-1}a^{-1}p=bb_1o$. Thus $p=ab_l'bb_1o\in U'b_l'U'b_1o$. So it suffices to show

$$b_l'U'b_1o\subset \mathrm{Cl}\,\mathcal{V}_{l+1}, \tag{6.2}$$

because of the U'-invariance of $\mathrm{Cl}\,\mathcal{V}_{l+1}$. Let $\pi\colon G_0'\times_{U_0'}\mathfrak{g}^{(2)}\to R=G_0'b_1o\simeq G_0'/U_0'$

denote the vector bundle projection. Since $G_0^{(1)}b_1o$ is an open dense subset of R, by Lemma 6.3 $\Psi(\pi^{-1}(G_0^{(1)}b_1o))$ is a dense subset of $\mathcal{V}_1$. But $\Psi(\pi^{-1}(G_0^{(1)}b_1o)) = G_0^{(1)}G^{(2)}b_1o$, and so $b_l'G_0^{(1)}G^{(2)}b_1o$ is dense in $b_l'U'b_1o$. On the other hand, since $b_l'|\mathfrak{a}=s_l'$, by Lemma 6.2 we have

$$b_l'G_0^{(1)}G^{(2)}b_1o \subset U'b_l'b_1o = U'b_{l+1}o = \mathcal{V}_{l+1}.$$

Thus we get (6.2).

Now, by (6.1) and Theorem 4.1 we have $M_{l+1} \subset \mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{l+1}$. This, together with the assumption, implies $M_{l+1} \subset \mathcal{V}_{l+1}$. On the other hand, by Lemma 6.1 $\mathcal{V}_{l+1} \subset M_0 \cup M_1 \cup \cdots \cup M_{l+1}$. This, together with the assumption, implies $\mathcal{V}_{l+1} \subset M_{l+1}$. Thus we have proved $M_{l+1} = \mathcal{V}_{l+1}$. q.e.d.

Lemma 6.5. *If $\varphi \in \mathrm{I}(M, d)$ with $\varphi(o)=o$, then $(\varphi_*)_o \in GL(\mathfrak{g}_{-1}, \mathcal{S})$, under the identification $GL(T_oM) = GL(\mathfrak{g}_{-1})$ through the isomorphism u_0: $\mathfrak{g}_{-1} \to T_oM$ defined in Section 2.*

Proof. By Theorem 6.4 we have

$$\varphi\mathcal{V}_l = \mathcal{V}_l \qquad (0 \leq l \leq r). \tag{6.3}$$

Since $\mathfrak{g}_{-1}$ is an open subset of M with $o \in \mathfrak{g}_{-1}$, there is an open set $\mathcal{U}$ of $\mathfrak{g}_{-1}$ with $o \in \mathcal{U}$ such that $\varphi\mathcal{U} \subset \mathfrak{g}_{-1}$. Let $X \in \mathrm{Cl}\ V_l$ be arbitrary. If $t>0$ is sufficiently small, we have $tX \in \mathcal{U}$. Then, by Lemma 2.5 $tX \in \mathrm{Cl}\ V_l \cap \mathcal{U}$, and so by (6.3) $\varphi(tX) \in \mathrm{Cl}\ V_l$. Thus, by Lemma 2.5 again we have $(1/t)\varphi(tX) \in \mathrm{Cl}\ V_l$. Therefore

$$(\varphi_*)_o(X) = \lim_{t \downarrow 0} \frac{1}{t}\varphi(tX) \in \mathrm{Cl}\ V_l\,,$$

and hence $(\varphi_*)_o(\mathrm{Cl}\ V_l) = \mathrm{Cl}\ V_l$ $(0 \leq l \leq r)$. This implies $(\varphi_*)_o \in GL(\mathfrak{g}_{-1}, \mathcal{S})$ in virtue of (4.7). q.e.d.

Let $F(M)$ denote the bundle of frames of M, that is, the bundle of all linear isomorphisms from $\mathfrak{g}_{-1}$ to tangent spaces to M. We define a subbundle P of $F(M)$ by

$$P = \{a_*u_0;\ a \in G\},$$

which is a G_0-structure over M. Let

$$\mathrm{Aut}(P) = \{\varphi \in \mathrm{Diff}(M);\ \varphi_*P = P\}$$

denote the group of automorphisms of the G_0-structure P.

Lemma 6.6. (Tanaka [17]). *If M is neither $P_n(\boldsymbol{R})$ $(n \geq 1)$ nor $P_n(\boldsymbol{C})$ $(n \geq 1)$, one has*

$$G = \mathrm{Aut}(P).$$

Theorem 6.7. *Let (M, g) be an irreducible symmetric R-space with $r = \mathrm{rank}(M, g) \geq 2$. Then the group G of basic transformations of M is identical with the group $\mathrm{I}(M, d)$ of isometric diffeomorphisms of the arithmetic distance d.*

Proof. We have seen the inclusion $G \subset \mathrm{I}(M, d)$. For the inclusion $\mathrm{I}(M, d) \subset G$, it suffices to show $\mathrm{I}(M, d) \subset \mathrm{Aut}(P)$ in virtue of Lemma 6.6. To prove this we follow the argument in Tanaka [18].

Let $\psi \in \mathrm{I}(M, d)$ and $u \in P$ be arbitrary. Then there is $a \in G$ such that $a_* u_0 = u$. Moreover, by transitivity of G on M, there is $b \in G$ such that $b\psi a(o) = o$. Set $\varphi = b\psi a \in \mathrm{I}(M, d)$. Then $\varphi(o) = o$ and

$$\psi_* u = b_*^{-1} \varphi_* u_0 . \tag{6.4}$$

By Lemma 6.5, $(\varphi_*)_o$ belongs to $GL(\mathfrak{g}_{-1}, \mathcal{S})$, which is equal to G_0 by Theorem 4.7. Therefore $\varphi_* u_0 \in P$, and hence $\psi_* u \in P$ by (6.4). This shows that $\psi \in \mathrm{Aut}(P)$. Thus we have proved $\mathrm{I}(M, d) \subset \mathrm{Aut}(P)$. q.e.d.

Theorem 6.8. *Let (M, g) be an irreducible symmetric R-space with $r = 1$ other than spheres. Then the group G of basic transformations of M is identical with the group of diffeomorphisms of M which send each Helgason sphere to a Helgason sphere.*

Proof. Our (M, g) are the projective spaces $P_n(\boldsymbol{F})$ $(n \geq 2)$ over $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or real quaternions $\boldsymbol{H}$ and the Cayley projective plane $P_2(\boldsymbol{O})$. In these cases, Helgason spheres are projective lines. The groups G are the group of projective transformations of $P_n(\boldsymbol{F})$ and the connected simple Lie group of type EIV, respectively. Here, by a projective transformation of $P_n(\boldsymbol{F})$ we mean a diffeomorphism of $P_n(\boldsymbol{F})$ induced by a semi-linear automorphism φ of $\boldsymbol{F}^{n+1}$, that is, a bijection $\varphi: \boldsymbol{F}^{n+1} \to \boldsymbol{F}^{n+1}$ such that

$$\begin{aligned} \varphi(u+v) &= \varphi(u) + \varphi(v) && \text{for } u, v \in \boldsymbol{F}^{n+1}, \\ \varphi(u\lambda) &= \varphi(u)\sigma(\lambda) && \text{for } u \in \boldsymbol{F}^{n+1}, \ \lambda \in \boldsymbol{F}, \end{aligned}$$

σ being an automorphism of $\boldsymbol{F}$. So our theorem follows from the fundamental theorem in projective geometry (for $P_n(\boldsymbol{F})$) and a theorem of Springer [11] (for $P_2(\boldsymbol{O})$). q.e.d.

Corollary 6.9. *For an irreducible symmetric R-space (M, g), one has $K = \mathrm{I}(M, g)$.*

Proof. In case where (M, g) is an n-sphere $(n \geq 1)$, it is seen that $K = O(n+1)$. Thus we have $K = \mathrm{I}(M, g)$. Suppose that (M, g) is not a sphere. Since any element of $\mathrm{I}(M, g)$ carries each Helgason sphere to a Helgason sphere, by

Theorems 6.7 and 6.8 $I(M, g)$ is a subgroup of G. Recalling that $K \subset I(M, g)$ and K is a maximal compact subgroup of G, we get $K = I(M, g)$. q.e.d.

Appendix

Table of the dimension $\dim S$ of a Helgason sphere S and the quotient group G/G^0 of G modulo the identity component G^0 of G for irreducible symmetric R-spaces M

M	dim S	G/G^0
$SU(r+s)/S(U(r)\times U(s))$ $(1\le r\le s)$	2	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$ $r=s\ge 2$
		$\boldsymbol{Z}_2$ otherwise
$SO(2n)/U(n)$ $(n\ge 5)$	2	$\boldsymbol{Z}_2$
$Sp(r)/U(r)$ $(r\ge 2)$	2	$\boldsymbol{Z}_2$
$SO(n+2)/SO(2)\times SO(n)$ $(n\ge 5)$	2	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$ n even
		$\boldsymbol{Z}_2$ n odd
$E_6/T\cdot$Spin (10)	2	$\boldsymbol{Z}_2$
$E_7/T\cdot E_6$	2	$\boldsymbol{Z}_2$
$SO(r+s)/S(O(r)\times O(s))$ $(1\le r\le s)$	1	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$ $r=s\ge 2$
		$\boldsymbol{Z}_2$ $r=s=1$ or $r<s$, $r+s$ even
		$\{1\}$ otherwise
$Sp(r+s)/Sp(r)\times Sp(s)$ $(1\le r\le s)$	4	$\boldsymbol{Z}_2$ $r=s$
		$\{1\}$ $r<s$
$U(r)$ $(r\ge 3)$	1	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$
$SO(n+1)/SO(n)$ $(n\ge 5)$	n	$\boldsymbol{Z}_2$
$O(p)\times O(q)/(O(p-1)\times O(q-1))\cdot\boldsymbol{Z}_2$	1	$\boldsymbol{Z}_2\cdot(\boldsymbol{Z}_2+\boldsymbol{Z}_2)^{*)}$ $p=q$ even
$(2\le p\le q,\ (p,q)\ne(2,2),\ (3,3))$		$\boldsymbol{Z}_2+\boldsymbol{Z}_2$ $p<q$, p, q even, or $p=q$ odd
		$\boldsymbol{Z}_2$ otherwise
$Sp(r)$ $(r\ge 1)$	3	$\boldsymbol{Z}_2$
$U(r)/O(r)$ $(r\ge 3)$	1	$\boldsymbol{Z}_2$
$SO(n)$ $(n\ge 5)$	1	$\boldsymbol{Z}_2+\boldsymbol{Z}_2$ n even
		$\boldsymbol{Z}_2$ n odd
$U(2r)/Sp(r)$ $(r\ge 3)$	1	$\boldsymbol{Z}_2$
$Sp(4)/(Sp(2)\times Sp(2))\cdot\boldsymbol{Z}_2$	1	$\{1\}$
$F_4/$Spin(9)	8	$\{1\}$
$SU(8)/Sp(4)\cdot\boldsymbol{Z}_2$	1	$\boldsymbol{Z}_2$
$T\cdot E_6/F_4$	1	$\boldsymbol{Z}_2$

*) semi-direct product with $N=\boldsymbol{Z}_2+\boldsymbol{Z}_2$ normal; the generator of $\boldsymbol{Z}_2$ interchanges two $\boldsymbol{Z}_2$'s of N.

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Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka, Japan