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ON COMPACT HOMOGENEOUS AFFINE MANIFOLDS

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Introduction

If a differentiable manifold M is provided with an affine connection whose torsion and curvature vanish identically, we call M an affine manifold. The study of affine manifolds has been the subject of a number of recent publications including the papers by Auslander, Charlap, Koszul, Kamber and Tondeur, and Wolf. A general reference for the study of affine manifolds is [3], [4] or [6]. The subject of this paper is to study homogeneous affine manifolds.

If, for an affine manifold M, **aut** (M) denotes the Lie algebra of all infinitesimal affine transformations, then **aut** (M) has an associative algebra structure satisfying 1) $X\cdot Y-Y\cdot X=[X, Y]$ and 2) the isotropy subalgebra $\mathbf{aut}(M)_p=\{X\in\mathbf{aut}(M)\,|\,X_p=0\}$ at $p\in M$ is a left ideal of $\mathbf{aut}(M)$ (Theorem 1.2). Our study is essentially based upon these properties of **aut** (M). A pair $(\mathfrak{g}, \mathfrak{a})$ of a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is called an $\mathcal{A}$-pair if $\mathfrak{g}$ has an associative algebra structure satisfying the above 1) and 2) for the subalgebra $\mathfrak{a}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and A a closed subgroup of G with Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Then if $(\mathfrak{g}, \mathfrak{a})$ is an $\mathcal{A}$-pair, then the homogeneous space G/A has a unique G-invariant flat affine connection ∇ satisfying $\nabla_{X^*}Y^*=(Y\cdot X)^*$ where X^* denotes the vector field on G/A induced by the action of $\exp tX$ (Theorem 2.2). We call such a homogeneous affine manifold an $\mathcal{A}$-space. Then a compact homogeneous affine manifold turns out to be an $\mathcal{A}$-space (Theorem 2.4).

To each $\mathcal{A}$-pair $(\mathfrak{g}, \mathfrak{a})$, we can associate in a canonical way a pair (G, A) of Lie groups such that the Lie algebras of G and A are $\mathfrak{g}$ and $\mathfrak{a}$ respectively, and A is a closed subgroup of G (§4). Then for such a pair (G, A) of a Lie group and a closed subgroup, the $\mathcal{A}$-space G/A is embedded equivariantly into an affine space as a domain, which is called an $\mathcal{A}$-domain (Theorem 4.5).

The F-Stiefel manifold $V_r(F^n)$, consisting of all r linear frames in F^n ($F=R$, C or H), is naturally imbedded into the affine space F^{nr} as a domain,

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and moreover $V_r(F^n)$ turns out to be an $\mathcal{A}$-domain (Theorem 5.1). By using a well known theorem, the so called Wedderburn Theorem on associative algebras, we determine $\mathcal{A}$-domains; an $\mathcal{A}$-domain is affinely diffeomorphic to a direct product of an affine space and Stiefel manifolds (Theorem 5.5).

The following theorem shows the importance of Theorem 5.5 in the study of compact homogeneous affine manifolds: if M is a compact homogeneous affine manifold which is a quotient space of a domain D in an affine space, then the domain D is an $\mathcal{A}$-domain (Theorem 6.1). By applying this theorem, we consider the case in which M is convex or complete (Theorem 6.3 and 6.4).

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1. Affine manifolds

In what follows, by differentiable we always mean differentiable of class C^∞. All manifolds and affine connections are assumed differentiable. For general notations and definitions we refer to [4].

An affine connection is said to be *flat* if the torsion and curvature tensors vanish identically. A manifold provided with a flat affine connection is called an *affine manifold.* An affine transformation of an affine manifold is called an *automorphism.* **Aut** (M) denotes the Lie group of all automorphisms of an affine manifold M. M is said to be *homogeneous* if **Aut** (M) acts on M transitively.

Let N be a totally geodesic submanifold of an affine manifold M. Then N is an affine manifold and the inclusion of N into M is an affine mapping. N is called an *affine submanifold* of M.

Let V be a finite dimensional vector space over $\boldsymbol{R}$. In the canonical way, V is considered as an affine manifold, which is called an *affine space.* In general, an n dimensional affine space will be denoted by $\boldsymbol{A}^n$. Let U be a vector subspace of V and $x \in V$. Then the subset $x+U=\{x+y \in V \mid y \in U\}$ is an affine submanifold of the affine space V, which is called the *affine subspace* through x associated to U. We can easily see that *if M is a connected affine submanifold of $\boldsymbol{A}^n$ then there exists a unique affine subspace S of $\boldsymbol{A}^n$ such that M is an open subset of S.*

A connected open subset in $\boldsymbol{A}^n$ is called a domain. Let D be a *domain* of $\boldsymbol{A}^n$ and Γ a discrete group acting on D freely and properly discontinuously as a group of affine automorphisms of D. Then the quotient space $\Gamma\backslash D$ is an affine manifold and the projection is affine. When an affine manifold M is of the form $\Gamma\backslash D$, M is said to be *regular.* Moreover if D is a convex domain in $\boldsymbol{A}^n$, then M is said to be *convex.* It is well known that an affine manifold M is *complete* (i.e., the flat affine connection is complete) if and only if M is a regular affine manifold $\Gamma\backslash \boldsymbol{A}^n$. That is to say, a simply connected complete affine manifold is affinely diffeomorphic to $\boldsymbol{A}^n$.

In general we have the following:

Proposition 1.1 *Let M be an n dimensional simply connected affine manifold. Then there exists an affine immersion φ of M into $\boldsymbol{A}^n$ with the following universal property; if ψ is an affine mapping of M into $\boldsymbol{A}^m$ then there exists a unique affine mapping g of $\boldsymbol{A}^n$ into $\boldsymbol{A}^m$ such that $\psi = g\circ\varphi$.*

Proof. Let V be the vector space of all parallel differential forms of degree 1 on M. Since M is simply connected, $\dim V = n$. Take a point $p_0 \in M$ as a reference point and define a mapping φ of M into the dual space V^* of V by

$$\langle \varphi(p), \omega \rangle = \int_{p_0}^{p} \omega$$

for $p \in M$ and $\omega \in V$. Since M is simply connected, the integral does not depend on the choice of a path from p_0 to p and hence $\varphi(p)$ is well defined. Then φ is an affine immersion of M into the affine space V^* and satisfies the universal property. Q. E. D.

Let M be an affine manifold with flat affine connection ∇. *An infinitesimal automorphism of M* is, by definition, a vector field whose local one parameter group of local transformations consists of affine mappings. The set of all infinitesimal automorphisms of M forms a finite dimensional Lie subalgebra $\mathbf{aut}\,(M)$ of $\mathfrak{X}(M)$.

Theorem 1.2. *Let M be an affine manifold with flat affine connection ∇. Then*

1) *If X and $Y \in \mathbf{aut}\,(M)$, then $\nabla_X Y \in \mathbf{aut}\,(M)$.*
2) *If we define a multiplication $X\cdot Y$ in $\mathbf{aut}\,(M)$ by setting*

$$X\cdot Y = -\nabla_Y X\,,$$

then $\mathbf{aut}\,(M)$ forms an associative algebra over $\boldsymbol{R}$ such that

$$[X, Y] = X\cdot Y - Y\cdot X\,.$$

3) *Let $p \in M$ and $\mathfrak{a} = \{X \in \mathbf{aut}\,(M) \mid X_p = 0\}$. Then $\mathfrak{a}$ is a left ideal of the associative algebra $\mathbf{aut}\,(M)$.*

In order to make the computation easier, we introduce a tensor field A_X associated to a vector field X as follows: for $X \in \mathfrak{X}(M)$,

$$A_X = L_X - \nabla_X$$

where L_X denotes the Lie derivative by X. We have the following formulae on an affine manifold M.

1°) $A_X Y = -\nabla_Y X$ for $X, Y \in \mathfrak{X}(M)$.
2°) $X \in \mathbf{aut}(M)$ if and only if $\nabla(A_X)=0$.
3°) $A_{A_X Y}=A_X A_Y$ for $X \in \mathbf{aut}(M)$, $Y \in \mathfrak{X}(M)$.

Proof of Theorem 1.2. To prove 1), let $X, Y \in \mathbf{aut}(M)$. By 2°) it suffices to show $(\nabla_U(A_{\nabla_X Y}))V=0$ for any $U, V \in \mathfrak{X}(M)$. In fact $(\nabla_U(A_{\nabla_X Y}))V=\nabla_U(A_{\nabla_X Y}V)-A_{\nabla_X Y}(\nabla_U V)=-\nabla_U(A_{A_Y X}V)+A_{A_Y X}(\nabla_U V)=-\nabla_U(A_Y A_X V)+A_Y A_X(\nabla_U V)=-((\nabla_U A_Y)A_X V+A_Y(\nabla_U A_X)V+A_Y A_X \nabla_U V)+A_Y A_X(\nabla_U V)=0$ since A_X and A_Y are parallel by 2°). This proves 1). The second assertion of 2) follows from the triviality of the torsion of ∇. To complete the proof of 2), it is sufficient to prove that $(X\cdot Y)\cdot Z=X\cdot(Y\cdot Z)$ for X, Y and $Z \in \mathbf{aut}(M)$. In fact, $(X\cdot Y)\cdot Z=(-\nabla_Y X)\cdot Z=\nabla_Z(\nabla_Y X)=-\nabla_Z(A_X Y)=-A_X\nabla_Z Y=-\nabla_{Y\cdot Z}X=X\cdot(Y\cdot Z)$. This proves 2). Let $X \in \mathfrak{a}$ and $Y \in \mathbf{aut}(M)$. Then $Y\cdot X=-\nabla_X Y$. Since $X_p=0$, $(Y\cdot X)_p=0$, and hence $Y\cdot X \in \mathfrak{a}$. This proves 3). Q. E. D.

Let $\mathfrak{g}$ be a Lie algebra over the field $\boldsymbol{R}$. $\mathfrak{g}$ is called *an $\mathcal{A}$-Lie algebra* if $\mathfrak{g}$ is also an associative algebra over $\boldsymbol{R}$ such that for X and $Y \in \mathfrak{g}$

$$X\cdot Y-Y\cdot X=[X, Y]$$

where $X\cdot Y$ denotes the associative algebra multiplication. If $\mathfrak{a}$ is a left ideal of the underlying associative algebra of $\mathfrak{g}$, the pair $(\mathfrak{g}, \mathfrak{a})$ is called *an $\mathcal{A}$-pair of algebras*.

One can show easily that the underlying Lie algebra of an $\mathcal{A}$-Lie algebra is not semi-simple.

2. ($\mathcal{A}$)-Lie groups and ($\mathcal{A}$)-spaces.

Let G be a Lie group. The Lie algebra $\mathfrak{g}$ of G is, by definition, the Lie algebra of all *left invariant* vector fields on G. For $a \in G$, R_a and L_a denote the right and left translations of G by a, respectively; $R_a(g)=ga$, $L_a(g)=ag$. Let H be a closed subgroup of G. The action of G on the homogeneous space G/H is denoted by T; for $a \in G$ and $gH \in G/H$, $T_a(gH)=(ag)H$. This action induces an *anti*-Lie homomorphism of $\mathfrak{g}$ into $\mathfrak{X}(G/H)$ as follows: for any $X \in \mathfrak{g}$, let $a_t=\exp tX \in G$. The one parameter group $\{T_{a_t}\}$ of transformations of G/H defines a vector field X^* on G/H. Then the mapping, which assigns X^* to each X, is an anti-Lie homomorphism of $\mathfrak{g}$ into $\mathfrak{X}(G/H)$. The image of $\mathfrak{g}$ by this mapping will be denoted by $\mathfrak{g}'$. A vector field X^* in $\mathfrak{g}'$ is called the *induced vector field* of $X \in \mathfrak{g}$.

A Lie group G is called *an $\mathcal{A}$-Lie group with algebra* $\mathfrak{g}$ if the Lie algebra $\mathfrak{g}$ of G is an $\mathcal{A}$-Lie algebra over $\boldsymbol{R}$. A pair (G, A) of an $\mathcal{A}$-Lie group G and its subgroup A is called *an $\mathcal{A}$-pair of groups with algebras* $(\mathfrak{g}, \mathfrak{a})$ if the pair $(\mathfrak{g}, \mathfrak{a})$ of

Lie algebras of G and A is an $\mathcal{A}$-pair of algebras and if A is a closed subgroup of G.

Lemma 2.1. *Let G be a connected $\mathcal{A}$-Lie group with algebra $\mathfrak{g}$. Then for $a \in G$ and X, $Y \in \mathfrak{g}$,*

$$Ad(a)(X \cdot Y) = (Ad(a)X) \cdot (Ad(a)Y) .$$

That is, $Ad(a)$ is an automorphism of the associative algebra $\mathfrak{g}$.

This follows from the following formula; for any X, Y and $Z \in \mathfrak{g}$,

$$[Z, X \cdot Y] = [Z, X] \cdot Y + X \cdot [Z, Y] .$$

An affine connection on a homogeneous space G/H is said to be *G-invariant* if the transformation T_a of G/H is an affine mapping for any $a \in G$.

Theorem 2.2. *Let (G, A) be an $\mathcal{A}$-pair of groups with algebras $(\mathfrak{g}, \mathfrak{a})$. Then there exists a unique G-invariant flat affine connection ∇ on G/A such that*

$$\nabla_{X^*} Y^* = (Y \cdot X)^* \quad \text{for } X,\ Y \in \mathfrak{g} ,$$

where $Y \cdot X$ denotes the multiplication of the associative algebra $\mathfrak{g}$.

The invariant flat affine connection on G/A in Theorem 2.2 is called *the canonical flat affine connection.* The homogeneous space G/A provided with the canonical flat affine connection is called *an $\mathcal{A}$-space.*

Proof of Theorem 2.2. We shall construct an affine connection ∇ on G/A step by step.

1°) Let $p \in G/A$, $u \in T_p(G/A)$ and $Y \in \mathfrak{g}$. $\nabla_u Y^* \in T_p(G/A)$ is defined by

$$\nabla_u Y^* = (Y \cdot X)^*_p$$

where $X \in \mathfrak{g}$ and $X^*_p = u$. We show that this is well defined. It suffices to show that if $X \in \mathfrak{g}$ and $X^*_p = 0$, then $(Y \cdot X)^*_p = 0$ for any $Y \in \mathfrak{g}$. For $g \in G$ and $Z \in \mathfrak{g}$, $T_{g*}(Z^*) = (Ad(g)Z)^*$ on G/A. Let 0 denote the class of A in G/A and $p = T_a(0)$. Suppose $X \in \mathfrak{g}$ and $X^*_p = 0$. Then $(Ad(a^{-1})X)^*_0 = 0$ and hence $Ad\, a^{-1}(X) \in \mathfrak{a}$. Let $Y \in \mathfrak{g}$. $(Y \cdot X)^*_p = T_{a*}((Ad(a^{-1})(Y \cdot X))^*_0) = T_{a*}((Ad(a^{-1})Y) \cdot (Ad(a^{-1})X))^*_0$ by Lemma 2.1. Since $Ad(a^{-1})X \in \mathfrak{a}$ and $\mathfrak{a}$ is a left ideal of $\mathfrak{g}$, $(Ad(a^{-1})Y) \cdot (Ad(a^{-1})X) \in \mathfrak{a}$ and hence $(Y \cdot X)^*_p = 0$. Therefore the definition is consistent.

2°) Let $p \in G/A$, $u \in T_p(G/A)$ and $Z \in \mathfrak{X}(G/A)$. $\nabla_u Z \in T_p(G/A)$ is defined as follows: obviously there exist $Y_1 \cdots$, $Y_r \in \mathfrak{g}$ and smooth functions $f_1, \cdots, f_r$ defined around p such that $Z = \sum_i f^i Y^*_i$ around p. Then let $\nabla_u Z$ be defined by

$$\nabla_u Z = \sum_i (uf^i) Y^*_{ip} + \sum_i f^i(p) \nabla_u Y^*_i ,$$

where the second term of the right hand side has been defined in 1°). We show that this is well defined. It suffices to show that if $\sum_i f^i Y^*_i = 0$ around p, then $\sum_i (uf^i) Y^*_{ip} + \sum_i f^i(p) \nabla_u Y^*_i = 0$. Take $X \in \mathfrak{g}$ such that $X^*_p = u$. Then $\sum_i (uf^i) Y^*_{ip} + \sum_i f^i(p) \nabla_u Y^*_i = \sum_i (X^*_p f^i) Y^*_{ip} + \sum_i f^i(p)(Y_i \cdot X)^*_p = [X^*, \sum_i f^i Y^*_i]_p - \sum_i f^i(p)[X^*, Y^*_i]_p + \sum_i f^i(p)(Y_i X)^*_p$. Since $\sum_i f^i Y^*_i = 0$ around p, the first term vanishes. The rest is equal to $\sum_i f^i(p)(X \cdot Y_i)^*_p$ since $[X^*, Y^*_i]_p = -[X, Y_i]^*_p = (Y_i X)^*_p - (X \cdot Y_i)^*_p$. $\sum_i f^i(p)(X \cdot Y_i)^*_p = (X \cdot (\sum_i f^i(p) Y_i))^*_p$. On the other hand $\sum_i f^i Y^*_i = 0$ around p, in particular $0 = (\sum_i f^i(p) Y_i)^*_p = (\sum_i f^i(p) Y_i)^*_p$. By a method similar to that of 1°), $(X \cdot (\sum_i f^i(p) Y_i))^*_p = 0$.

3°) Let X and $Y \in \mathfrak{X}(G/A)$. $\nabla_X Y \in \mathfrak{X}(G/A)$ is defined by

$$(\nabla_X Y)_p = \nabla_{X_p} Y$$

where the right hand side has been defined in 2°). The differentiability of $\nabla_X Y$ is clear.

4°) Obviously ∇ satisfies the condition to be an affine connection on G/A.

By the definition of ∇, we get $\nabla_{X^*} Y^* = (Y \cdot X)^*$ on G/A for X, $Y \in \mathfrak{g}$. To show that ∇ is G-invariant, it is sufficient to prove that $T_{a^*}(\nabla_{X^*} Y^*) = \nabla_{T_a{}^*X^*} T_{a^*} Y^*$ for any $a \in G$ and X, $Y \in \mathfrak{g}$. In fact $T_{a^*}(\nabla_{X^*} Y^*) = T_{a^*}(Y \cdot X)^* = (Ad(a) Y \cdot X)^* = ((Ad(a) Y) \cdot (Ad(a) X))^* = \nabla_{T_a{}^*X^*} T_{a^*} Y^*$ since $T_{a^*} Z^* = (Ad(a) Z)^*$ for $a \in G$, $Z \in \mathfrak{g}$. ∇ is flat. In fact, for any X, Y and $Z \in \mathfrak{g}$, we have $\nabla_{X^*} Y^* - \nabla_{Y^*} X^* = [X^*, Y^*]$ and $[\nabla_{X^*}, \nabla_{Y^*}] Z^* = \nabla_{[X^*, Y^*]} Z^*$. Therefore we have proved the existence of such a flat affine connection on G/A. The uniqueness is trivial.

Q. E. D.

Suppose M is a homogeneous space G/H where G acts almost effectively (i.e., $\{g \in G \mid T_g = \text{the identity}\}$ is a discrete subgroup of G). Then $\mathfrak{g}$ is anti-Lie isomorphic to the Lie subalgebra $\mathfrak{g}'$ of vector fields on M induced by $\mathfrak{g}$. If ∇ is a G-invariant flat affine connection on M, then clearly $\mathfrak{g}'$ is a *Lie* subalgebra of **aut** (M). We recall that **aut** (M) is an $\mathcal{A}$-Lie algebra.

Proposition 2.3. *Let M be a homogeneous space G/H and ∇ a G-invariant flat affine connection on M where G acts on M almost effectively. If $\mathfrak{g}'$ is an associative subalgebra of* **aut** (M), *then (G, H) is an $\mathcal{A}$-pair of groups and ∇ is the canonical flat affine connection on the $\mathcal{A}$-space G/H.*

Proof. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. We define a multiplication $Y \cdot X$ on $\mathfrak{g}$ as follows; $\nabla_{X^*} Y^* = (Y \cdot X)^*$ for $X, Y \in \mathfrak{g}$. Since $\mathfrak{g}'$ is an associative subalgebra of **aut** (M) and the mapping of $\mathfrak{g}$ into $\mathfrak{g}'$ is bijective, $\nabla_{X^*} Y^* \in \mathfrak{g}'$, and hence there exists a unique element Z in $\mathfrak{g}$ such that $\nabla_{X^*} Y^* = Z^*$. Thus this multiplication $Y \cdot X$ is well defined and $\mathfrak{g}$ forms an associative algebra such that $[X, Y] = X \cdot Y - Y \cdot X$. Obviously $\mathfrak{h}$ is a left ideal of the associative algebra $\mathfrak{g}$. Thus (G, H) is an $\mathcal{A}$-pair of groups with algebras $(\mathfrak{g}, \mathfrak{h})$. The last assertion follows from the definition of the canonical flat affine connection on the $\mathcal{A}$-space G/H. Q. E. D.

In the case where M is *compact*, we have the following as a corollary of Proposition 2.3.

Theorem 2.4. *If M is a compact homogeneous affine manifold, then M is affinely diffeomorphic to an $\mathcal{A}$-space with the canonical flat affine connection.*

Let G be an $\mathcal{A}$-Lie group with algebra $\mathfrak{g}$. Naturally $G = G/(e)$ is an $\mathcal{A}$-space. As an $\mathcal{A}$-space, G has the canonical flat affine connection ∇. In this case, for $X \in \mathfrak{g}$, X^* is the *right* invariant vector field on G such that $X^*_e = X_e$.

Proposition 2.5. *Let ∇ be the canonical flat affine connection on an $\mathcal{A}$-Lie group G. Then*

1) $\nabla_{X^*} Y^* = (Y \cdot X)^*$ *and* $\nabla_X Y = X \cdot Y$ *for* $X, \in \mathfrak{g}$.
2) ∇ *is two sided invariant by* G.

Proof. The first assertion in 1) is the definition of ∇. Let $X, Y \in \mathfrak{g}$. $(\nabla_X Y)_e = \nabla_{X_e} Y = \nabla_{X^*_e} Y = \nabla_{Y_e} X^* + [X^*, Y]_e$ since the torsion vanishes. Obviously the second term vanishes. Then $(\nabla_X Y)_e = \nabla_{Y_e} X^* = (\nabla_{Y^*} X^*)_e = (X \cdot Y)^*_e = (X \cdot Y)_e$. Since ∇ is left invariant, for $a \in G$, $(\nabla_X Y)_a = L_{a*}(\nabla_{L^{-1}_{a*} X} L^{-1}_{a*} Y)_e = L_{a*}(\nabla_X Y)_e = L_{a*}(X \cdot Y)_e = (X \cdot Y)_a$. This proves 1). Take a in G. $R_{a*}(\nabla_{X^*} Y^*) = R_{a*}(Y \cdot X)^* = (Y \cdot X)^* = \nabla_{X^*} Y^* = \nabla_{R_{a*} X^*} R_{a*} Y^*$ for $X. Y \in \mathfrak{g}$. This shows that ∇ is invariant by R_a for a$\in G$. Therefore ∇ is two sided invariant by G.

Q. E. D.

As a consequence of Proposition 2.3 and Proposition 2.5, we have following characterization of $\mathcal{A}$-Lie groups.

Theorem 2.6. *A Lie group G is an $\mathcal{A}$-Lie group if and only if G has a two sided invariant flat affine connection.*

Proof. Suppose G has a two sided invariant flat affine connection ∇. Let $X, Y \in \mathfrak{g}$. Since ∇ is left invariant, X^* and Y^* are in **aut** (G) and hence $\nabla_{X^*} Y^*$ is in **aut** (G).$\nabla_{X^*} Y^*$ is a right invariant vector field since ∇ is right invariant. Therefore $\mathfrak{g}' = \{X^* \in \textbf{aut}\,(G) \mid X \in \mathfrak{g}\}$ is an associative subalgebra. By Propo-

sition 2.3, G is an $\mathcal{A}$-Lie group. The converse is Proposition 2.5.
Q. E. D.

Let (G, A) be an $\mathcal{A}$-pair of groups with algebras $(\mathfrak{g}, \mathfrak{a})$. ∇^G and $\nabla^{G/A}$ denote the canonical flat affine connections on G and G/A respectively.

Proposition 2.7. *The projection of G onto G/A is an affine mapping with respect to ∇^G and $\nabla^{G/A}$.*

Proof. Each $Z \in \mathfrak{g}$ induces a right invariant vector field on G and a vector field on G/A. As before, they will be denoted by the same letter Z^*. Then $p_*(Z^*)=Z^*$. Let $X, Y \in \mathfrak{g}$. $p_*(\nabla^G_{X^*} Y^*)=p_*(Y \cdot X)^*=(Y \cdot X)^*=\nabla^{G/A}_{X^*} Y^* =\nabla^{G/A}_{p_* X^*} p_* Y^*$. Thus p is an affine mapping. Q. E. D.

We state the following proposition without proof. The proof is straightforward.

Proposition 2.8. *Let (G, A) and (H, B) be $\mathcal{A}$-pairs of groups with algebras $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{h}, \mathfrak{b})$ respectively. Then*

1) *If φ is a homomorphism of G into H whose differential is an associative algebra homomorphism, then φ is an affine mapping. If, moreover, $\varphi(A) \subset B$, then the mapping of G/A into H/B induced by φ is also affine. In particular, a Lie subgroup of H whose algebra is an associative subalgebra of $\mathfrak{h}$ is an affine submanifold of H.*

2) *Let N be a closed normal subgroup of G which is contained in A. If the Lie subalgebra $\mathfrak{n}$ of N is a two sided ideal of $\mathfrak{g}$, then $(G/N, A/N)$ is an $\mathcal{A}$-pair of groups with $(\mathfrak{g}/\mathfrak{n}, \mathfrak{a}/\mathfrak{n})$, and $(G/N)/(A/N)$ and G/A are affinely diffeomorphic.*

3) *$(G \times H, A \times B)$ is an $\mathcal{A}$-pair of groups with algebras $(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{a} \oplus \mathfrak{b})$, and furthermore $G \times H/A \times B$ and $G/A \times H/B$ are affinely diffeomorphic.*

By using the structure of an affine manifold on an $\mathcal{A}$-Lie group, we shall prove the following theorem, which gives a sufficient condition for a subgroup of an $\mathcal{A}$-Lie group to be closed.

Theorem 2.9. *Let H be a connected Lie subgroup of an $\mathcal{A}$-Lie group G whose Lie algebra $\mathfrak{h}$ is an associative subalgebra of $\mathfrak{g}$. If there exists an affine immersion of G into the n dimensional affine space ($n=\dim G$), then H is a closed subgroup of G.*

Proof. Let φ be an affine immersion of G into the n dimensional affine space $\boldsymbol{A}^n$. Since H is a connected affine submanifold of G and $\varphi(G)$ is an open subset of $\boldsymbol{A}^n$, $\varphi(H)$ is a connected affine submanifold of $\boldsymbol{A}^n$. Thus, there exists an affine subspace S of $\boldsymbol{A}^n$ such that $\varphi(H)$ is an open subset of S. Since locally φ is a diffeomorphism, H is locally closed in G and hence H is a closed subgroup of G. Q.E.D.

Since any simply connected affine manifold can be affinely immersed into the affine space with the same dimension by Proposition 1.1, we have the following as a corollary to Theorem 2.9.

Corollary. *Let G be a simply connected $\mathcal{A}$-Lie group with algebra $\mathfrak{g}$. Then any connected Lie subgroup of G whose Lie algebra is an associative subalgebra of $\mathfrak{g}$ is a closed subgroup of G.*

3. The structure of $\mathcal{A}$-Lie algebras and $\mathcal{A}$-pairs of algebras

We assume that an associative algebra is always *finite dimensional throughout this section.* A general reference of associative algebras is [2] or [5].

Let $\mathfrak{g}$ be an associative algebra over $\boldsymbol{R}$. For each positive integer k, $\mathfrak{g}^k$ is defined by

$$\mathfrak{g}^k = \{\sum x_1 \cdots x_k\text{: finite sum} \mid x_i \in \mathfrak{g}\}\,.$$

We have $\mathfrak{g}=\mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots$. $\mathfrak{g}$ is said to be *nilpotent* if $\mathfrak{g}^k=(0)$ for some k. Any associative algebra has the maximal nilpotent two sided ideal, which will be called the *radical*. When the radical is zero, the associative algebra is said to be *semi-simple*. If $\mathfrak{r}$ is the radical of $\mathfrak{g}$, then $\mathfrak{g}/\mathfrak{r}$ is semisimple. An associative algebra is said to be *simple* if it has no non-trivial two sided ideal. When K is a ring, $\mathfrak{gl}(n; K)$ denotes the set of all $n \times n$ matrices with coefficients in K. If K is a division algebra over $\boldsymbol{R}$, then $\mathfrak{gl}(n; K)$ is a simple associative algebra over $\boldsymbol{R}$. We denote by $\boldsymbol{C}$ and $\boldsymbol{H}$ the fields of complex numbers and quaternions respectively. Then if K is a divison algebra over $\boldsymbol{R}$, then K is $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. The following is known as the Wedderburn Theorem ([2], [5]).

Theorem 3.1.

1) *Any simple associative algebra over $\boldsymbol{R}$ is isomorphic to one of $\mathfrak{gl}(n; \boldsymbol{R})$, $\mathfrak{gl}(n; \boldsymbol{C})$ and $\mathfrak{gl}(n; \boldsymbol{H})$ for some integer n.*

2) *A semi-simple associative algebra is isomorphic to a direct sum of simple associative algebras.*

Let $\mathfrak{g}$ be an associative algebra over $\boldsymbol{R}$. Let $\tilde{\mathfrak{g}}$ be the semi-direct sum $\boldsymbol{R}\cdot e + \mathfrak{g}$ where $\boldsymbol{R}\cdot e$ is a one dimensional vector space with the base (e) and the multiplication in $\tilde{\mathfrak{g}}$ is given by

$$(\alpha e, X)\cdot(\beta e, Y) = (\alpha\beta e, \alpha Y + \beta X + X \cdot Y)$$

for $\alpha, \beta \in \boldsymbol{R}$ and $X, Y \in \mathfrak{g}$. Then $\tilde{\mathfrak{g}}$ is an associative algebra with the identity $(e, 0)$ and $\mathfrak{g}$ is a two sided ideal of $\tilde{\mathfrak{g}}$. $\tilde{\mathfrak{g}}$ is called the *trivial extension of* $\mathfrak{g}$ *by adding the identity e.*

Let $\mathfrak{g}$ be a *semi-simple* associative algebra over $\boldsymbol{R}$. By Wedderburn's theorem, $\mathfrak{g}$ is isomorphic to a direct sum $\sum_i \mathfrak{g}_i$ where $\mathfrak{g}_i = \mathfrak{gl}(n_i; \boldsymbol{F}_i)$ and $\boldsymbol{F}_i = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. Let $\mathfrak{a}$ be a left ideal of $\mathfrak{g}$.Then $\mathfrak{a}_i = \mathfrak{a} \cap \mathfrak{g}_i$ is a left ideal of $\mathfrak{g}_i$, and $\mathfrak{a} = \sum \mathfrak{a}_i$. Let $\mathfrak{a}_r(n; \boldsymbol{F})$ denote the subspace of $\mathfrak{gl}(n; \boldsymbol{F})$ of all elements (x^i_j) such that $x^i_j = 0$, $(1 \leqq i \leqq n,\ 1 \leqq j \leqq r)$; *i.e.*,

$$\mathfrak{a}_r(n; \boldsymbol{F}) = \left(\overbrace{0}^{r} \middle| * \right) \Big\} n \,.$$

$\mathfrak{a}_r(n; \boldsymbol{F})$ is a left ideal of $\mathfrak{gl}(n; \boldsymbol{F})$.

Lemma 3.2. *If $\mathfrak{a}$ is a left ideal of $\mathfrak{gl}(n; \boldsymbol{F})$, then there exists $g \in GL(n; \boldsymbol{F})$ such that*

$$Ad(g)\mathfrak{a} = \mathfrak{a}_r(n; \boldsymbol{F}) \qquad \textit{for some } r.$$

The proof of this lemma is well-known. Since $Ad(g)$ is an automorphism of the associative algebra by Lemma 2.1, without loss of generality we may always assume that $\mathfrak{a} = \mathfrak{a}_r(n, \boldsymbol{F})$ if $\mathfrak{a}$ is a left ideal of $\mathfrak{gl}(n; \boldsymbol{F})$.

An $\mathcal{A}$-pair $(\mathfrak{g}, \mathfrak{a})$ of algebras is said to be *effective* if $\mathfrak{a}$ contains no non-trivial two sided ideal of the associative algebra $\mathfrak{g}$. This condition is equivalent to saying that $\mathfrak{a}$ contains no non-trivial ideal of the Lie algebra $\mathfrak{g}$.

4. Canonical $\mathcal{A}$-Lie groups and $\mathcal{A}$-domains

Let $\mathfrak{h}$ be an $\mathcal{A}$-Lie algebra over $\boldsymbol{R}$ whose underlying associative algebra, contains *the identity*. We denote by $\mathfrak{h}^*$ the group of all *invertible* elements of $\mathfrak{h}$. Obviously $\mathfrak{h}^*$ is an open subset of $\mathfrak{h}$ and a Lie group with respect to the relative topology. The Lie algebra of $\mathfrak{h}^*$ can be identified with the underlying Lie algebra of $\mathfrak{h}$ in a natural way since $\mathfrak{h}^*$ is an open subset of $\mathfrak{h}$. Therefore $\mathfrak{h}^*$ is an $\mathcal{A}$-Lie group with algebra $\mathfrak{h}$. Moreover, $\mathfrak{h}^*$ can be considered as a real algebraic group in a natural way. The Lie group $\mathfrak{h}^*$ acts *affinely* on the *affine space* $\mathfrak{h}$ on the left side through the multiplication of the underlying associative algebra of $\mathfrak{h}$. Obviously the inclusion of $\mathfrak{h}^*$ into $\mathfrak{h}$ is compatible with the actions of $\mathfrak{h}^*$ on $\mathfrak{h}^*$ itself and on $\mathfrak{h}$.

Proposiiton 4.1. *The inclusion mapping of an $\mathcal{A}$-Lie group $\mathfrak{h}^*$, provided with the canonical flat affine connection, into the affine space $\mathfrak{h}$ is an affine mapping.*

Proof. Let $\{X_1, \cdots, X_n\}$ be a base for the vector space $\mathfrak{h}$, and C^i_{jk} the structure constant with respect to $\{X_i\}$;

$$X_i X_j = \sum_k C^k_{ij} X_k \qquad (1 \leqq i, j \leqq n) \,.$$

$\{u^1, \cdots, u^n\}$ denotes the coordinate of the affine space $\mathfrak{h}$ defined by

$$X=\sum_i u^i(X)X_i \qquad \text{for } X\in\mathfrak{h}.$$

Let ∇ be the affine connection on the affine space $\mathfrak{h}$ and ∇^* the canonical flat affine connection on the $\mathcal{A}$-Lie group $\mathfrak{h}^*$. To prove the proposition, it is sufficient to show that

$$(\nabla_X Y)_g=(\nabla_X^* Y)_g$$

for any $g\in\mathfrak{h}^*$ and X, $Y\in\mathfrak{h}$ where $\mathfrak{h}$ is considered as the Lie algebra of the Lie group $\mathfrak{h}^*$. By the definition,

$$\begin{aligned} X_g &= \sum_j u^j(g\cdot X)(\partial/\partial u^j)_g \\ &= \sum_{i\cdot j\cdot k} u^i(g)u^k(X)C^j_{ik}(\partial/\partial u^j)_g \end{aligned}$$

where $g\cdot X$ deontes the multiplication of g and X in $\mathfrak{h}$. Thus,

$$\begin{aligned} (\nabla_X Y)_g &= \sum_{i\cdot j\cdot k\cdot l} \nabla_{u^i(g\cdot X)(\partial/\partial u^i)_g} u^j u^k(Y)C^l_{jk}(\partial/\partial u^l) \\ &= \sum_{i\cdot j\cdot k\cdot l} u^i(g\cdot X)\delta^j_i u^k(Y)C^l_{jk}(\partial/\partial u^l)_g \\ &= \sum_{j\cdot k\cdot l} u^j(g\cdot X)u^k(Y)C^l_{jk}(\partial/\partial u^l)_g \\ &= \sum_l u^l((g\cdot X)\cdot Y)(\partial/\partial u^l)_g \\ &= (X\cdot Y)_g=(\nabla_X^* Y). \end{aligned}$$

Therefore the inclusion is an affine mapping. Q. E. D.

Let us consider the case where the underlying associative algebra may not contain the identity. Let $\mathfrak{g}$ be an $\mathcal{A}$-Lie algebra over $\boldsymbol{R}$. We denote by $\tilde{\mathfrak{g}}$ the trivial extension of $\mathfrak{g}$ by adding the identity e; $\mathfrak{g}=R\cdot e+\mathfrak{g}$(§3). Since $\tilde{\mathfrak{g}}$ contains the identity e, the group $\tilde{\mathfrak{g}}^*$ of all invertible elements in $\tilde{\mathfrak{g}}$ is an algebraic group. Let G^* be the set of all invertible elements of $\tilde{\mathfrak{g}}$ which are contained in the subset $e+\mathfrak{g}=\{(e, x)\in\tilde{\mathfrak{g}}\,|\,X\in\mathfrak{g}\}$; $G^*=\tilde{\mathfrak{g}}^*\cap(e+\mathfrak{g})$. Clearly G^* is a real algebraic subgroup of $\tilde{\mathfrak{g}}^*$. Thus G^* has only finitely many (topological) connected components. Obviously we have the identification between the Lie algebra $\mathfrak{g}$ and the Lie algebra of G^*, corresponding to that between $\tilde{\mathfrak{g}}$ and the Lie algebra of $\tilde{\mathfrak{g}}^*$. Hence G^* is an $\mathcal{A}$-Lie group with Lie algebra $\mathfrak{g}$.

Since $\tilde{\mathfrak{g}}^*$ acts on the affine space $\tilde{\mathfrak{g}}$ on the left side affinely and effectively, so does G^*. Moreover G^* leaves the affine subspace $e+\mathfrak{g}$ of $\tilde{\mathfrak{g}}$ invariant, and hence G^* acts affinely on the affine subspace $e+\mathfrak{g}$ on the left side affinely and effectively. On the other hand G^* is a Lie subgroup of $\tilde{\mathfrak{g}}^*$ and the algebra of G^* is an associative subalgebra (ideal) $\mathfrak{g}$ of $\tilde{\mathfrak{g}}$. It follows from Proposition 3.1 that the inclusion of the $\mathcal{A}$-Lie group G^*, with the canonical flat affine connection, into the affine subspace $e+\mathfrak{g}$ of $\tilde{\mathfrak{g}}$ is an affine mapping.

The topological identity component of the Lie group G^* is called the *canonical* $\mathcal{A}$-Lie group of the $\mathcal{A}$-Lie algebra $\mathfrak{g}$. One can show easily that the canonical $\mathcal{A}$-Lie group of $\mathbf{aut}(A^n)$ is nothing but the connected component of the group $\mathbf{Aut}(A^n)$ of all affine transformations on A^n.

Theorem 4.2. *Let G be the canonical $\mathcal{A}$-Lie group of an $\mathcal{A}$-Lie algebra $\mathfrak{g}$. Then we have an affine diffeomorphism ι of the $\mathcal{A}$-Lie group G with the canonical flat affine connection onto a domain of the affine subspace $e+\mathfrak{g}$ of $\tilde{\mathfrak{g}}$ and a faithful affine representation φ of G on the affine subspace $e+\mathfrak{g}$ of $\tilde{\mathfrak{g}}$ such that*

$$\iota(g_1 \cdot g_2) = \varphi(g_1) \cdot \iota(g_2)$$

for $g_i \in G$.

The affine imbedding ι of G into $e+\mathfrak{g}$ is called *the canonical affine imbedding* and the faithful affine representation φ of G on $e+\mathfrak{g}$ is called *the canonical affine representation.*

The following propositions give characterizations of canonical $\mathcal{A}$-Lie groups.

Proposition 4.3. *An $\mathcal{A}$-Lie group G with algebra $\mathfrak{g}$ is the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$ if and only if G is affinely diffeomorphic to a domain in an affine space.*

Proof. Suppose G is affinely diffeomorphic to a domain in an affine space. Let G_1 be the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$ and $\tilde{G}$ the simply connected $\mathcal{A}$-Lie group with algebra $\mathfrak{g}$. $\tilde{G}$ is a universal covering space of G and G_1 with projection p and p_1. Then p and p_1 are affine immersions of $\tilde{G}$ into an affine space. By Proposition 1.1, there exists an affine diffeomorphism φ of G onto G_1 such that $\varphi \circ p = p_1$. Since p and p_1 are $\mathcal{A}$-homomorphisms, φ is also an $\mathcal{A}$-homomorphism and hence φ is an $\mathcal{A}$-isomorphism of G onto G_1. Therefore G is the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$. The converse is Theorem 4.2. Q. E. D.

Proposition 4.4. *Let G be a canonical $\mathcal{A}$-Lie group with algebra $\mathfrak{g}$, and H a connected Lie subgroup of G with algebra $\mathfrak{h}$ such that $\mathfrak{h}$ is an associative subalgebra of $\mathfrak{g}$. Then H is the canonical $\mathcal{A}$-Lie group of $\mathfrak{h}$.*

Proof. By Proposition 2.8, H is a connected affine submanifold of G. G is affinely diffeomorphic to a domain in an n-dimensional affine space A^n. Thus, H is a connected affine submanifold of A^n and hence H is a domain in an affine subspace of A^n. By Proposition 4.3, H is the canonical $\mathcal{A}$-Lie group with $\mathfrak{h}$. Q. E. D.

Corollary. *Let H be a connected Lie group with Lie algebra $\mathfrak{h}$ acting on an affine space A^n affinely and effectively. If the Lie algebra $\mathfrak{h}'$ of vector fields on A^n induced by $\mathfrak{h}$ is an associative subalgebra of* $\mathbf{aut}(A^n)$, *then H is the canonical $\mathcal{A}$-Lie group with $\mathfrak{h}$.*

Proof. Let G be the connected component of $\mathbf{Aut}\,(A'')$. Then G is the canonical $\mathcal{A}$-Lie group with the algebra $\mathfrak{g}$ of $\mathbf{Aut}\,(A'')$. The action of H on A'' defines an embedding of H into G and furthermore the condition that $\mathfrak{h}'$ is an associative subalgebra of $\mathbf{aut}\,(A'')$ implies that $\mathfrak{h}$ is an associative subalgebra of $\mathfrak{g}$. It follows from Proposition 4.4 that H is the canonical $\mathcal{A}$-Lie group with $\mathfrak{h}$.

Q. E. D.

Let $(\mathfrak{g}, \mathfrak{a})$ be an $\mathcal{A}$-pair of algebras and G the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$; $G=(G^*)_0\subset G^*=(\tilde{\mathfrak{g}})^*\cap(e+\mathfrak{g})$. Let A be $G\cap(e+\mathfrak{a})$. A is a subgroup of G.

Clearly A is a closed subgroup of G whose Lie algebra is $\mathfrak{a}$ under the identification of $\mathfrak{g}$ and the Lie algebra of G. Therefore (G, A) is an $\mathcal{A}$-pair of groups with algebras $(\mathfrak{g}, \mathfrak{a})$, which is called *the canonical $\mathcal{A}$-pair of groups with* $(\mathfrak{g}, \mathfrak{a})$. Let ι be the canonical affine imbedding of G into the affine subspace $e+\mathfrak{g}$ of $\tilde{\mathfrak{g}}$, and φ the canonical affine representation of G on $e+\mathfrak{g}$ in Theorem 4.2 such that $\iota(g_1g_2)=\varphi(g_1)\cdot\iota(g_2)$ for $g_i\in G$. We show that the affine representation φ of G induces an affine representation of G on the affine space $[e]+\mathfrak{g}/\mathfrak{a}$, where $[e]+\mathfrak{g}/\mathfrak{a}$ denotes the affine subspace of $\tilde{\mathfrak{g}}/\mathfrak{a}$ through $[e]$ associated to the vector subspace $\mathfrak{g}/\mathfrak{a}$ of $\tilde{\mathfrak{g}}/\mathfrak{a}$. Let $g=e+Z\in G\subset e+\mathfrak{g}$ and $X\in\mathfrak{g}$. Then

$$\varphi(g)(e+X+\mathfrak{a}) = e+Z+X+Z\cdot X+\mathfrak{a} = (\varphi(g)(e+X))+\mathfrak{a}$$

since $\mathfrak{a}$ is a left ideal of $\mathfrak{g}$. Therefore we have a unique affine automorphism, denoted by the same letter $\varphi(g)$, of $[e]+\mathfrak{g}/\mathfrak{a}$ such that

$$\begin{array}{ccccc} e+\mathfrak{g} & \xrightarrow[\varphi(g)]{} & e+\mathfrak{g} & \subset & \tilde{\mathfrak{g}} \\ \downarrow & & \downarrow & & \downarrow \\ [e]+\mathfrak{g}/\mathfrak{a} & \xrightarrow{\varphi(g)} & [e]+\mathfrak{g}/\mathfrak{a} & \subset & \tilde{\mathfrak{g}}/\mathfrak{a} \end{array}$$

is commutative. Clearly φ is an affine representation of G on the affine subspace $[e]+\mathfrak{g}/\mathfrak{a}$ of $\tilde{\mathfrak{g}}/\mathfrak{a}$.

Through the above representation φ of G, G acts on $[e]+\mathfrak{g}/\mathfrak{a}$ affinely. Suppose $g=e+Z\in G\subset e+\mathfrak{g}$ and $\varphi(g)[e]=[e]$. Then $[e+Z]=[e]$ and hence $Z\in\mathfrak{a}$. Therefore the isotropy subgroup of G at $[e]$ is $G\cap(e+\mathfrak{a})$, which is, by definition, A. Thus, we have an injective mapping ι of G/A into $[e]+\mathfrak{g}/\mathfrak{a}$ such that

$$\iota(gA) = \varphi(g)[e] \qquad \text{for } g\in G\,.$$

Since $\varphi(g)[e]=[\varphi(g)e]=[\iota(g)]$ for $g\in G$, we have the following commutative diagram:

$$\begin{array}{ccccc} G & \xrightarrow{\iota} & e+\mathfrak{g} & \subset & \tilde{\mathfrak{g}} \\ \downarrow & & \downarrow & & \downarrow \\ G/A & \xrightarrow{\iota} & [e]+\mathfrak{g}/\mathfrak{a} & \subset & \tilde{\mathfrak{g}}/\mathfrak{a}\,. \end{array}$$

Therefore the inclusion of an $\mathcal{A}$-space G/A into $[e]+\mathfrak{g}/\mathfrak{a}$ is an affine mapping.

Theorem 4.5. *Let (G, A) be the canonical $\mathcal{A}$-pair of groups with algebra $(\mathfrak{g}, \mathfrak{a})$. Then we have an affine diffeomorphism ι of the $\mathcal{A}$-space G/A onto a domain in the affine space $[e]+\mathfrak{g}/\mathfrak{a}$ and an affine representation φ of G on $[e]+\mathfrak{g}/\mathfrak{a}$ such that for $a\in G$, $gA\in G/A$,*

$$(T_a(gA)) = \varphi(a)\cdot\iota(gA)\,.$$

Moreover $(\mathfrak{g}, \mathfrak{a})$ is effective if and only if G acts on G/A effectively.

Proof. To complete the theorem, we prove the last assertion. Suppose $(\mathfrak{g}, \mathfrak{a})$ is effective. Let $g=e+X\in G\subset e+\mathfrak{g}$ such that $gaA=aA$ for all $aA\in G/A$. Since $A=G\cap(e+\mathfrak{a})$ and $\mathfrak{a}$ is a left ideal of $\mathfrak{g}$, $aA=G\cap(a+\mathfrak{a})$ for $a\in G$. Thus, $G\cap(ga+\mathfrak{a})=G\cap(a+\mathfrak{a})$, and hence $ga+\mathfrak{a}=a+\mathfrak{g}$ since G is an open subset of $e+\mathfrak{g}$. Thus $Xa\in\mathfrak{a}$ for any $a\in G$. Hence $X\in\mathfrak{a}$ and $X\mathfrak{g}\subset\mathfrak{a}$. Since $\mathfrak{a}$ is a left ideal of $\mathfrak{g}$ and $X\in\mathfrak{a}$, $X\cdot\mathfrak{g}+\mathfrak{g}\cdot X\cdot\mathfrak{g}$ is a two sided ideal of $\mathfrak{g}$ contained in $\mathfrak{a}$. Since $(\mathfrak{g}, \mathfrak{a})$ is effective, $X\cdot\mathfrak{g}=(0)$ and hence $\boldsymbol{R}\cdot X+\mathfrak{g}\cdot X$ is a two sided ideal contained in $\mathfrak{g}$. Therefore by the effectiveness of $(\mathfrak{g}, \mathfrak{a})$, $X=0$ and $g=e$. This proves the effectiveness of the action of G on G/A. The converse is trivial.

Q. E. D.

If (G, A) is a canonical $\mathcal{A}$-pair of groups with $(\mathfrak{g}, \mathfrak{a})$, then by Theorem 3.5 the $\mathcal{A}$-space G/A is affinely diffeomorphic to a domain in an affine space. This $\mathcal{A}$-space G/A is called *the $\mathcal{A}$-domain of* $(\mathfrak{g}, \mathfrak{a})$ and is denoted by $\Omega_{(\mathfrak{g},\mathfrak{a})}$

The following proposition gives a sufficient condition for an $\mathcal{A}$-space to be an $\mathcal{A}$-domain.

Proposition 4.6. *Let (G, A) be an $\mathcal{A}$-pair of groups with algebras $(\mathfrak{g}, \mathfrak{a})$ such that G is connected. If the $\mathcal{A}$-space is affinely diffeomorphic to a domain in an affine space, then G/A is affinely diffeomorphic to the $\mathcal{A}$-domain of $(\mathfrak{g}, \mathfrak{a})$. Furthermore if G acts on G/A effectively, then (G, A) is the canonical $\mathcal{A}$-pair of $(\mathfrak{g}, \mathfrak{a})$.*

Proof. Let (G_1, A_1) be the canonical $\mathcal{A}$-pair of $(\mathfrak{g}, \mathfrak{a})$, and G the simply connected $\mathcal{A}$-Lie group with algebra $\mathfrak{g}$, and $\tilde{A}$ the connected Lie subgroup of G with algebra $\mathfrak{a}$. Then $(\tilde{G}, \tilde{A})$ is an $\mathcal{A}$-pair of groups with algebras $(\mathfrak{g}, \mathfrak{a})$ since $\tilde{A}$ is a closed subgroup of $\tilde{G}$ by the corollary to Proposition 2.9. Let p and p_1 be the covering projections of $\tilde{G}$ onto G and G_1 respectively. Then p and p_1 induce affine mappings of $\tilde{G}/\tilde{A}$ onto G/A and G_1/A_1, respectively, which are covering projections. By Proposition 1.1, we have an affine diffeomorphism φ of G/A onto G_1/A_1 since p and p_1 are affine immersions of a simply connected affine manifold $\tilde{G}/\tilde{A}$ into an affine space. Therefore G/A is affinely diffeomorphic to the $\mathcal{A}$-domain G_1/A_1 of $(\mathfrak{g}, \mathfrak{a})$. Assume G acts on G/A effectively and G/A is affinely diffeomorphic to a domain D in an affine space A^n. We identify G/A

with D. The action of G on G/A is uniquely extended to that of G on $\boldsymbol{A}^n$ since G/A is a domain in $\boldsymbol{A}^n$. Clearly the action of G on $\boldsymbol{A}^n$ is effective. Since G/A is an $\mathcal{A}$-space, the condition in the corollary to Proposition 4.4 is satisfied and hence G is the canonical $\mathcal{A}$-Lie group with $\mathfrak{g}$. By a method similar to that of the proof for the first assertion, we can show that (G, A) is the canonical $\mathcal{A}$-pair of groups with $(\mathfrak{g}, \mathfrak{a})$. Q. E. D.

The following proposition gives us a sufficient condition for an affinely homogeneous domain in $\boldsymbol{A}^n$ to be an $\mathcal{A}$-domain.

Proposition 4.7. *Suppose a connected Lie group G acts on a domain D in $\boldsymbol{A}^n$ affinely, transitively and effectively. Let A be an isotropy subgroup of G at a point in D. If the Lie algebra $\mathfrak{g}'$ of vector fields on D induced by $\mathfrak{g}$ is an associative subalgebra of* **aut** (D), *the* $(\mathfrak{g}, \mathfrak{a})$ *is an $\mathcal{A}$-pair of algebras and (G, A) is the canonical $\mathcal{A}$-pair of* $(\mathfrak{g}, \mathfrak{a})$ *and hence $D=G/A$ is an $\mathcal{A}$-domain of* $(\mathfrak{g}, \mathfrak{a})$.

Proof. By Proposition 2.3, $(\mathfrak{g}, \mathfrak{a})$ is an $\mathcal{A}$-pair of algebras and the last assertion follows from Proposition 4.6. Q. E. D.

For later use, we state the following proposition. The proof, which is not shown here, is straightforward.

Proposition 4.8. *Let (G, A) be the canonical $\mathcal{A}$-pair of groups of* $(\mathfrak{g}, \mathfrak{a})$, *and $\mathfrak{n}$ a two sided ideal of $\mathfrak{g}$ contained in $\mathfrak{a}$. If (G, N) denotes the canonical $\mathcal{A}$-pair of groups of* $(\mathfrak{g}, \mathfrak{n})$, *then N is a normal subgroup of G, and $(G/N, A/N)$ is the canonical $\mathcal{A}$-pair of groups of* $(\mathfrak{g}/\mathfrak{n}, \mathfrak{a}/\mathfrak{n})$, *and $\Omega_{(\mathfrak{g}, \mathfrak{a})}$ is affinely diffeomorphic to* $\Omega_{(\mathfrak{g}/\mathfrak{n}, \mathfrak{a}/\mathfrak{n})}$.

5. The determination of $\mathcal{A}$-domains

In this section $\boldsymbol{F}$ denotes a division algebra over $\boldsymbol{R}$. That is, $\boldsymbol{F}$ is $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$.

Let $\mathfrak{g}$ be an $\mathcal{A}$-Lie algebra over $\boldsymbol{R}$ whose underlying associative algebra is simple. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(n; \boldsymbol{F})$ for some n and the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$ is the topological component of $GL(n; \boldsymbol{F})$. $GL^+(n; \boldsymbol{R})$ denotes the topological component of $GL(n; \boldsymbol{R})$. Let $\mathfrak{a}$ be a left ideal. By Lemma 3.2, without loss of generality we may assume that $\mathfrak{g}=\mathfrak{gl}(n; \boldsymbol{F})$ and $\mathfrak{a}=\mathfrak{a}_r(n; \boldsymbol{F})$. Let $A_r(n; \boldsymbol{F})$ be $GL(n: \boldsymbol{F})\cap(1_n+\mathfrak{a}_r(n; \boldsymbol{F}))$ where 1_n denotes the identity matrix. Then the Lie algebra of $A_r(n; \boldsymbol{F})$ is $\mathfrak{a}_r(n; \boldsymbol{F})$. And moreover $(GL^+(n; \boldsymbol{R}), A_r^+(n; \boldsymbol{R}))$, $(GL(n; \boldsymbol{C}), A_r(n; \boldsymbol{C}))$ and $(GL(n; \boldsymbol{H}), A_r(n; \boldsymbol{H}))$ are the canonical $\mathcal{A}$-pairs of groups of $(\mathfrak{g}, \mathfrak{a})$ if $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{H}$ respectively, where $A_r^+(n; \boldsymbol{R})=A_r(n; \boldsymbol{R})\cap GL^+(n; \boldsymbol{R})$. Let $\boldsymbol{V}_r(\boldsymbol{F}^n)$ denote the homogeneous space $GL(n; \boldsymbol{F})/A_r(n; \boldsymbol{F})$, which is called the $\boldsymbol{F}$-Stiefel manifold. We note that $GL^+(n; \boldsymbol{R})/A_r^+(n; \boldsymbol{R})=\boldsymbol{V}_r(\boldsymbol{R}^n)$ $(r\neq n)$. Therefore the $\boldsymbol{F}$-Stiefel manifold $\boldsymbol{V}_r(\boldsymbol{F}^n)$ *is* the $\mathcal{A}$-domain of $(\mathfrak{gl}(n; \boldsymbol{F}), \mathfrak{a}_r(n; \boldsymbol{F}))$. It follows from Theorem 4.5 that the $\mathcal{A}$-

domain $V_r(F^n)$ is affinely imebdded in an affine space $F^{r\cdot n}$ as follows: for $a=(a^i_j)\in GL(n;\, F)$

$$\begin{array}{ccc} V_r(F^n) & \longrightarrow & F^{r\cdot r} \\ \Uparrow & & \Uparrow \\ a \bmod A_r(n;\, F) & \rightarrow & \left(a^i_j \,\middle|\, \begin{array}{l} 1\leqq i\leqq n \\ 1\leqq i\leqq r \end{array}\right). \end{array}$$

We have the following assertion.

Theorem 5.1. *If* $(\mathfrak{g}, \mathfrak{a})$ *is an* $\mathcal{A}$*-pair of algebras and if* $\mathfrak{g}$ *is a semi-simple associative algebra, then the* $\mathcal{A}$*-domain of* $(\mathfrak{g}, \mathfrak{a})$ *is a direct product of Stiefel manifolds.*

Let $\mathfrak{g}$ be a semi-simple associative algebra over $\boldsymbol{R}$. Then $\mathfrak{g}$ can be decomposed as follows:

$$\mathfrak{g} = \mathfrak{S}_1 \oplus \mathfrak{S}_2 \quad \text{(direct sum)}$$

where $\mathfrak{S}_1$ is a direct sum of $\mathfrak{gl}(1;\, \boldsymbol{R})$ and no simple factor of $\mathfrak{S}_2$ is $\mathfrak{gl}(1;\, \boldsymbol{R})$. Let $\mathfrak{a}$ be a left ideal and $\mathfrak{a}=\mathfrak{a}_1\oplus\mathfrak{a}_2$ the corresponding decomposition of $\mathfrak{a}$. Then *if* $\Omega_{(\mathfrak{g}, \mathfrak{a})}$ *is convex, then* $\mathfrak{S}_2=\mathfrak{a}_2$ *and* $\Omega_{(\mathfrak{g}, \mathfrak{a})}$ *is affinely diffeomorphic to a direct product of half lines. And* $\Omega_{(\mathfrak{g}, \mathfrak{a})}$ *is not complete if* $\mathfrak{g}\neq\mathfrak{a}$.

Let $\mathfrak{g}$ be an $\mathcal{A}$-Lie algebra over $\boldsymbol{R}$ whose underlying associative algebra is nilpotent. $\tilde{\mathfrak{g}}$ denotes the trivial extension of $\mathfrak{g}$ by adding e. Using the notation in §4, we have the following:

Lemam 5.2. $e+\mathfrak{g}=G^*$ *if the assosiative algebra* $\mathfrak{g}$ *is nilpotent.*

Proof. By the definition of G^*, $e+\mathfrak{g}\supset G^*$. Let $e+X\in e+\mathfrak{g}$. Since X is a nilpotent element, $\sum_{k\geqq 0}(-1)^k X^k$ is a finite sum, which is in $e+\mathfrak{g}$. $(e+X)(\sum_{k\geqq 0}(-1)^k X^k)=e$. Thus $e+X\in\tilde{\mathfrak{g}}^*$. Hence we have $e+\mathfrak{g}\subset\tilde{\mathfrak{g}}^*\cap(e+\mathfrak{g})=G^*$.

Q. E. D.

The following assertion follows from Lemma 5.2 and Theorem 4.2.

Theorem 5.3. *Let* $(\mathfrak{g}, \mathfrak{a})$ *be an* $\mathcal{A}$*-pair of algebras such that* $\mathfrak{g}$ *is a nilpotent associative algebra. Then the* $\mathcal{A}$*-domain of* $(\mathfrak{g}, \mathfrak{a})$ *is affinely diffeomorphic to an affine space.*

Let $(\mathfrak{g}, \mathfrak{a})$ be an $\mathcal{A}$-pair of algebras and $\mathfrak{r}$ the radical of the associative algebra $\mathfrak{g}$. Since $\mathfrak{r}$ is a two sided ideal of the associative algebra $\mathfrak{g}$, $(\mathfrak{g}, \mathfrak{r})$ and $(\mathfrak{g}, \mathfrak{a}+\mathfrak{r})$ are $\mathcal{A}$-pairs of algebras. Let G denote the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$; G is the identity component of $G^*=\tilde{\mathfrak{g}}^*\cap(e+\mathfrak{g})$ where $\tilde{\mathfrak{g}}$ is the trivial extension of $\mathfrak{g}$ by adding the identity e. We denote by A and R the Lie subgroups of G with algebras $\mathfrak{a}$ and $\mathfrak{r}$ respectievely such that (G, A) and (G, R) are the canonical $\mathcal{A}$-pairs of $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{r})$ respectively. That is, $A=G\cap(e+\mathfrak{a})$ and $R=G\cap(e+\mathfrak{r})$. By Lemma 5.2, $R=e+\mathfrak{r}$. Since R is a normal subgroup of G, $A\cdot R$ is a subgroup of G. By the definition of A and R, $A\cdot R=(G\cap(e+\mathfrak{a}))\cdot(e+\mathfrak{r})=G\cap(e+\mathfrak{a}+\mathfrak{r})$, and hence $(G, A\cdot R)$ is the canonical $\mathcal{A}$-pair of $(\mathfrak{g}, \mathfrak{a}+\mathfrak{r})$. We have the following commutative diagram:

$$\begin{tikzcd}
\Omega_{(\mathfrak{g},\mathfrak{a})} = G/A \arrow[r] \arrow[d, "A\cdot R/A"] & {[e]+\mathfrak{g}/\mathfrak{a}} \arrow[d] \arrow[r, phantom, "\subset"] & \tilde{\mathfrak{g}}/\mathfrak{a} \arrow[d] \\
\Omega_{(\mathfrak{g},\mathfrak{a}+\mathfrak{r})} = G/A\cdot R \arrow[r] & {[e]+\mathfrak{g}/\mathfrak{a}+\mathfrak{r}} \arrow[r, phantom, "\subset"] & \tilde{\mathfrak{g}}/\mathfrak{a}+\mathfrak{r}
\end{tikzcd}$$

where the vertical mappings are the canonical projections and the horizontal mappings are defined in §4. First we are concerned with the fibre of the fibre bundle. The fibre $A\cdot R/A$ is an $\mathcal{A}$-space, which is an affine submanifold of G/A. Since G/A is a domain in an affine space, so is $A\cdot R/A$. On the other hand we have the following commutative diagram:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A\cdot R \arrow[d] \\
R/R\cap A \arrow[r] & A\cdot R/A
\end{tikzcd}$$

where the vertical mappings are the canonical projection and the above horizontal is the inclusion. Since the inclusion of R into $A\cdot R$ is an affine mapping, the bijective mapping of $R/R\cap A$ onto $A\cdot R/A$ is an affine diffeomorphism. It follows from Theorem 5.3 that $A\cdot R/A$ is an affine space. Let us consider $\Omega_{(\mathfrak{g},\mathfrak{a})}$ and $\Omega_{(\mathfrak{g},\mathfrak{a}+\mathfrak{r})}$ as domains in affine spaces $[e]+\mathfrak{g}/\mathfrak{a}$ and $[e]+\mathfrak{g}/\mathfrak{a}+\mathfrak{r}$ respectively. Let p be the projection of $[e]+\mathfrak{g}/\mathfrak{a}$ onto $[e]+\mathfrak{g}/\mathfrak{a}+\mathfrak{r}$. Then $\Omega_{(\mathfrak{g},\mathfrak{a})}=p^{-1}p(\Omega_{(\mathfrak{g},\mathfrak{a})})$ since each fibre of p is affinely diffeomorphic to an affine space $A\cdot R/A$. It follows easily that $\Omega_{(\mathfrak{g},\mathfrak{a})}$ is affinely diffeomorphic to the product affine manifold $\Omega_{(\mathfrak{g},\mathfrak{a}+\mathfrak{r})}\times A\cdot R/A$. On the other hand $\mathfrak{r}$ is a two sided ideal of $\mathfrak{g}$ and $\mathfrak{r}\subset\mathfrak{a}+\mathfrak{r}$. By Proposition 4.8, $\Omega_{(\mathfrak{g},\mathfrak{a}+\mathfrak{r})}$ is affinely diffeomorphic to $\Omega_{(\mathfrak{g}/\mathfrak{r},\mathfrak{a}+\mathfrak{r}/\mathfrak{r})}$. Since $\mathfrak{g}/\mathfrak{r}$ is a semi-simple associative algebra, by Theorem 5.1, $\Omega_{(\mathfrak{g}/\mathfrak{r},\mathfrak{a}+\mathfrak{r}/\mathfrak{r})}$ is affinely diffeomorphic to a direct product of Stiefel manifolds. Therefore we have the following:

Theorem 5.5. *An $\mathcal{A}$-domain of an $\mathcal{A}$-pair of algebras is affinely diffeomorphic to a product of Stiefel manifolds and an affine space.*

REMARK. Theorem 4.5 determines the underlying affine manifold of an $\mathcal{A}$-domain completely. That is to say, an $\mathcal{A}$-domain splits to Stiefel manifolds and an affine space. However, in general the action of group does not split. Namely, the description in Theorem 4.5 is not equivariant.

6. Compact regular homogeneous affine manifolds

Theorem 6.1. *Let D be a domain in $\boldsymbol{A}^n$ and Γ a discrete group acting on D properly discontinuously and freely as an affine transformation group. Suppose $M=\Gamma\backslash D$ is a compact homogeneous affine manifold and let $\mathfrak{g}=\mathbf{aut}(M)$ and $\mathfrak{a}=\{X\in g \mid X_0=0\}$ (O is a fixed point in M). Then*

1) *D is the $\mathcal{A}$-domain $\Omega_{(\mathfrak{g},\mathfrak{a})}$.*
2) *M is the $\mathcal{A}$-space G/A_1, where (G, A) is the canonical $\mathcal{A}$-pair of $(\mathfrak{g}, \mathfrak{a})$ and A_1 is a closed subgroup of G with algebra $\mathfrak{a}$ and $A_1\supset A$.*

Proof. A linear mapping σ of $\mathfrak{g}$ into $\mathbf{aut}(D)$ is defined as follows: for $X\in\mathfrak{g}$, $\sigma(X)$ is a unique vector field on D whose projection image is X. Clearly σ is well defined and injective. Then the image $\sigma(\mathfrak{g})$ is an associative subalgebra of $\mathbf{aut}(D)$. Let G be the connected Lie group generated by $\{\operatorname{Exp} t\tilde{X} \mid \tilde{X}\in\sigma(\mathfrak{g})\}$. G acts on D affinely, effectively and transitively. Hence by Proposition 4.7, (G, A) is the canonical $\mathcal{A}$-pair of $(\mathfrak{g}, \mathfrak{a})$ and $D=\Omega_{(\mathfrak{g},\mathfrak{a})}$ where A is the isotropy subgroup of G at $\tilde{O}$. Since, for $X\in\mathfrak{g}$, $\sigma(X)$ is Γ-invariant, the action of G on D induces that of G on $M=\Gamma\backslash D$. Let A_1 be the isotropy subgroup of G at O. Then $A_1\supset A$ and $M=G/A_1$. Q. E. D.

Let $(\mathfrak{g}, \mathfrak{a})$ be an $\mathcal{A}$-pair of algebras. Then it is easy to show that the normalizer of $\mathfrak{a}$ in the associative algebra $\mathfrak{g}$ is equal to that of $\mathfrak{a}$ in the Lie algebra $\mathfrak{g}$, since $\mathfrak{a}$ is a left ideal of the associative algebra $\mathfrak{g}$. We denote it by $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{r})$. Let $\mathfrak{g}$ be the radical of the associative algebra $\mathfrak{g}$. Then $\mathfrak{g}/\mathfrak{r}$ is a semi-simple associative algebra over R, which is a direct sum of $\mathfrak{gl}(n; \boldsymbol{F})$ ($\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$). Then $\mathfrak{g}/\mathfrak{r}=\mathfrak{s}_1\oplus\mathfrak{s}_2$ where $\mathfrak{s}_1$ is a direct sum of $\mathfrak{gl}(1; \boldsymbol{R})$ and $\mathfrak{s}_2$ contains no $\mathfrak{gl}(1; \boldsymbol{R})$ as a simple factor.

Lemma 6.2. *Suppose (G, A) is the canonical $\mathcal{A}$-pair of $(\mathfrak{g}, \mathfrak{a})$ and A_1 is a closed subgroup of G with algebra a such that G/A_1 is compact. Then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})\supset\mathfrak{r}$ and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})/\mathfrak{r}\supset\mathfrak{s}_1$.*

Proof. By definition, G is a topological component of an algebraic group G^*. Letting $A^*=G^*\cap(e+\mathfrak{a})$, $N_{G^*}(A^*)\supset A_1$. Since A^* is an algebraic subgroup of G^*, so is $N_{G^*}(A^*)=N$. G^*/N is compact. Therefore N contains a maximal solvable irreducible real algebraic subgroup of G^* [1]. In particular, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})\supset\mathfrak{r}$. By a similar argument on G^*/R^* and N/R^*, we can get $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})/\mathfrak{r}\supset\mathfrak{s}_1$ where $R^*=G^*\cap(e+\mathfrak{r})$. Q. E. D.

We recall that a convex or complete affine manifold is always regular.

Theorem 6.3. *Let M be a compact convex homogeneous affine manifold. Then M is an $\mathcal{A}$-space G/Γ where G is the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$ and is* *the topological component of an irreducible real triangulable algebraic group and* Γ *is a discrete subgroup of* G.

Proof. Since M is convex, M is regular and hence by Theorem 6.1, $M=G/A_1$ where (G, A) is the canonical $\mathcal{A}$-pair of $(\mathfrak{g}, \mathfrak{a})$ and A_1 is a closed subgroup of G with algebra $\mathfrak{a}$ and $A_1 \supset A$. Moreover $(\mathfrak{g}, \mathfrak{a})$ is effective. Then M is a quotient space of $\Omega_{(\mathfrak{g}, \mathfrak{a})}$ by some discrete group. Then $\Omega_{(\mathfrak{g}, \mathfrak{a})}$ is convex and hence, so is $\Omega_{(\mathfrak{g}, \mathfrak{a}+\mathfrak{r})}=\Omega_{(\mathfrak{g}/\mathfrak{r}, \mathfrak{a}+\mathfrak{r}/\mathfrak{r})}$. Let $\mathfrak{g}/\mathfrak{r}=\mathfrak{S}_1 \oplus \mathfrak{S}_2$ as before. By Lemma 6.2, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}) \supset \mathfrak{r}$ and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})/\mathfrak{r} \supset \mathfrak{S}_1$. On the other hand, since $\Omega_{(\mathfrak{g}/\mathfrak{r}, \mathfrak{a}+\mathfrak{r}/\mathfrak{r})}$ is convex, $\mathfrak{a}+\mathfrak{r}/\mathfrak{r} \supset \mathfrak{S}_2$ by §5. Obviously $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})/\mathfrak{r} \supset \mathfrak{a}+\mathfrak{r}/\mathfrak{r}$ and hence $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})/\mathfrak{r} \supset \mathfrak{S}_1 \oplus \mathfrak{S}_2=\mathfrak{g}/\mathfrak{r}$ and $\mathfrak{g}=\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$. Since $(\mathfrak{g}, \mathfrak{a})$ is effective, $\mathfrak{a}$ must be trivial. Therefore A_1 is a discrete subgroup of G. Namely $\Omega_{(\mathfrak{g},\mathfrak{a})}=G$ is convex. Thus, G is the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$ and the topological component of an irreducible real triangulable algebraic group. Q. E. D.

As a corollary to Theorem 6.3, we can show the following theorem.

Theorem 6.4. *Let M be a compact complete homogeneous affine manifold. Then M is an $\mathcal{A}$-space G/Γ where G is the canonical $\mathcal{A}$-Lie group of $\mathfrak{g}$ and $\mathfrak{g}$ is a nilpotent associative algebra and Γ is a discrete subgroup of G.*

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