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ON CENTRALIZERS IN SEPARABLE EXTENSIONS II

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0. The aim of this paper is to improve and generalize some results of the author's previous paper [8]. Therefore, all notations and terminologies are same as those in [7] and [8]. In [8] the author studied some commutor theory of H-separable extension $\Lambda|\Gamma$ in the case where $\Lambda\simeq\Gamma\otimes_C\Delta$ with $\Delta(=V_\Lambda(\Gamma))$ central separable over C and C=the center of Λ=the center of Γ,and in the case where Λ is left or right Γ-$f.\ g.$ (finitely generated) projective and $\Lambda|\Gamma$ satisfies the following condition (*)

(*) 1) Λ is an H-separable extension of Γ such that ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$.
2) $V_\Lambda(\Gamma)=C'$, where C' is the center of Γ.

(See Theorem 1.2, Corollary 1.4 and Theorem 1.3 [8]). In case $\Lambda|\Gamma$ satisfies the condition (*) 1), Λ is left Γ-$f.\ g.$ projective if and only if Λ is right Γ-$f.\ g.$ projective by Corollary 2 [9], hence we shall simply say that Λ is Γ-$f.\ g.$ projective in this case. We note also that the condition (*) implies that $V_\Lambda(C')=\Gamma$ by Proposition 1.2 [7]. In this paper, we shall consider the case where Λ is left or right Γ-$f.\ g.$ projective and Λ is an H-separable extension of Γ, and shall prove that there exists a one to one correspondence between the class of subrings B of Λ which is separable extensions of Γ and ${}_BB_B<\oplus{}_B\Lambda_B$ and the class of separable C-subalgebras of Δ (Theorem 1). From this theorem, Corollary 1.4 and a more beautiful result than Thoerem 1.3 [8] follows.

1. To obtain our main results we need the next lemma which appears in [6].

Lemma 1 (Corollary 1.2 [6]). *Let A be a ring, M a left A-module, $\Omega=End({}_AM)$ and $E=End(M_\Omega)$. Then if M is A-$f.\ g.$ projective, $E\otimes_AM\simeq M$ as E-Ω-module by the map: $e\otimes m\to em$ for $e\in E$ and $m\in M$.*

Proof. Since M is A-$f.\ g.$ projective, we have natural isomorphisms

$$E\otimes_AM=\mathrm{Hom}(M_\Omega,M_\Omega)\otimes_AM\simeq\mathrm{Hom}(\mathrm{Hom}({}_AM,{}_AM)_\Omega,M_\Omega)$$
$$=\mathrm{Hom}(\Omega_\Omega,M_\Omega)\simeq M$$

as E–Ω–module. The composition of the above isomorphisms is the required one.

For rings $\Gamma\subset B\subset\Lambda$, we shall say that $B\otimes_\Gamma\Lambda\to\Lambda$ splits if the map of $B\otimes_\Gamma\Lambda$ to Λ such that $b\otimes x\to bx$ for $b\in B$ and $x\in\Lambda$ splits as B–Λ–map. We also need Proposition 2.3 [8]. This proposition can be improved as follows

Proposition 1. *Let Λ be an H–separable extension of Γ. Then for any intermediate ring B between Γ and Λ such that ${}_BB_\Gamma\langle\oplus{}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits, ${}_DD\langle\oplus{}_D\Delta$ and $D\otimes_C\Delta\to\Delta$ splits, where $D=V_\Lambda(B)$. Conversely for any C-subalgebra D of Δ such that ${}_DD\langle\oplus{}_D\Delta$ and $D\otimes_C\Delta\to\Delta$ splits, ${}_BB_\Gamma\langle\oplus{}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits, where $B=V_\Lambda(D)$.*

Proof. The first part of this proposition have been proved in Proposition 2.3 [8]. Hence we need to prove only the second part without assuming that B is right Γ–*f. g.* projective. Suppose that D is a C–subalgebra of Δ such that $D\otimes_C\Delta\to\Delta$ splits. Then $B=V_\Lambda(D)\cong\mathrm{Hom}({}_D\Delta_\Delta,\ {}_D\Lambda_\Delta)\langle\oplus\mathrm{Hom}({}_DD\otimes_C\Delta_\Delta,\ {}_D\Lambda_\Delta)$ $\cong V_\Lambda(C)$ as B–$V_\Lambda(\Delta)$–module. Hence ${}_BB_\Gamma\langle\oplus{}_B\Lambda_\Gamma$. Then, since Λ is H–separable over Γ and ${}_BB_\Gamma\langle\oplus{}_B\Lambda_\Gamma$, we have a B–Λ–isomorphism η of $B\otimes_\Gamma\Lambda$ to $\mathrm{Hom}({}_D\Delta,\ {}_D\Lambda)$ such that $\eta(b\otimes x)(d)=bdx$ for $b\in B$, $d\in D$ and $x\in\Lambda$ by Proposition 1.3 [7]. Hence, we have a commutative diagram of B–Λ–maps

$$\begin{array}{ccc} B\otimes_\Gamma\Lambda & \xrightarrow{\ \eta\ } & \mathrm{Hom}({}_D\Delta,\ {}_D\Lambda) \\ \downarrow & & \downarrow i_* \\ \Lambda & \xrightarrow[\ j\ ]{} & \mathrm{Hom}({}_DD,\ {}_D\Lambda) \end{array}$$

where j is the natural isomorphism and i_* is the one induced by the inclusion map i: $D\subset\Delta$. Then if ${}_DD\langle\oplus{}_D\Delta$, i_* B–Λ–splits and $B\otimes_\Gamma\Lambda\to\Lambda$ splits.

Let Λ be a semisimple R–algebra in the sense of A. Hattori [2], that is, Λ is a weakly semisimple extension of $R\cdot 1$ in the sense of [3]. Then every finitely generated Λ–module which is R–projective is Λ–projective, and by Proposition 4.1 [1] if Σ is a finitely generated projective R–algebra which contains Λ, ${}_\Lambda\Lambda\langle\oplus{}_\Lambda\Sigma$ and $\Lambda_\Lambda\langle\oplus\Sigma_\Lambda$. It is also well known that a separable algebra is a semisimple algebra.

Proposition 2. *Let Λ be an H–separable extension of Γ. If* (1) *D is a separable C–subalgebra of Δ, or if* (2) *Δ is a separable C–algebra (e. g., if ${}_\Gamma\Gamma_\Gamma\langle\oplus{}_\Gamma\Lambda_\Gamma$) and D is a semisimple C–subalgebra of Δ, then $V_\Lambda(V_\Lambda(D))=D$, ${}_BB_\Gamma\langle\oplus{}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits, where $B=V_\Lambda(D)$.*

Proof. Suppose (1). Then since $D\otimes_CD\to D$ splits as D–D–map, $D\otimes_C\Delta$ $\to\Delta$ splits as D–Δ–map. Suppose (2). Then $D\otimes_C\Delta\to\Delta$ splits as D–C–map, since D is C–semisimple. Then $D\otimes_C\Delta\to\Delta$ splits as D–Δ–map, since Δ is C–separable. Thus in both cases, $D\otimes_C\Delta\to\Delta$ splits and ${}_DD\langle\oplus{}_D\Delta$. The latter follows from Proposition 4.1 [1], since D is C-semisimple and Δ is C-$f.g.$ projective. Then $B\otimes_{\Gamma}\Lambda\to\Lambda$ splits and ${}_BB_{\Gamma}\langle\oplus_B\Lambda_{\Gamma}$ by Proposition 1. Hence Λ is H-separable over B by Proposition 2.2 [8]. Let $D'=V_{\Lambda}(B)$. Then there exists a ring isomorphism η: $D'\otimes_C\Lambda^0\to\mathrm{End}({}_B\Lambda)$ such that $\eta(d\otimes x^0)(y)=dyx$ for $x, y\in\Lambda$, $d\in D'$ (see Proposition 3.3 [5]). Then $D'\otimes_C\Lambda^0$ is the double centralizer of a left $D\otimes_C\Lambda^0$-module Λ, since $B=\mathrm{End}({}_D\Lambda_{\Lambda})$. While $D\otimes_C\Lambda\to\Lambda$ splits, since $D\otimes_C\Delta\to\Delta$ splits. This implies that Λ is left $D\otimes_C\Lambda^0$-$f.g.$ projective. Then by lemma 1, $(D'\otimes_C\Lambda^0)\otimes_{D\otimes_C\Lambda^0}\Lambda\cong\Lambda$, hence $D'\otimes_D\Lambda^0\cong\Lambda$. This isomorphism is given by $d\otimes x\to dx$ for $d\in D'$, $x\in\Lambda$. Then for every $d\in D'$, $d\otimes 1=1\otimes d$ in $D'\otimes_D\Lambda$, since both are mapped to d by this isomorphism. On the other hand, since Λ is H-separable over B, D' is C-$f.g.$ projective, and D' is right D-$f.g.$ projective, since D is C-semisimple. Hence $D'\otimes_D D'\subset D'\otimes_D\Lambda$, and $d\otimes 1=1\otimes d$ in $D'\otimes_D D'$ for every $d\in D'$. Since ${}_DD\langle\oplus_D D'$, $D'=D\oplus A$ for some left D-submodule A of D' and $D'\otimes_D D'=D'\otimes_D D\oplus D'\otimes_D A$. Let x be an arbitrary element of D' and $x=d+a$ for $d\in D$, and $a\in A$. Then $D'\otimes_D D'\ni x\otimes 1=1\otimes x=1\otimes d+1\otimes a$, and $1\otimes a=0$, $x\otimes 1=1\otimes d$. Thus $x=d\in D$. Thus $D'=D$. Thus $D=V_{\Lambda}(V_{\Lambda}(D))$.

The next proposition is a generalization of Proposition 1.5 [8].

Proposition 3. *Let Λ be an arbitrary R-algebra which is R-$f.g.$ projective. Then for any separable R-subalgebra Γ of Λ, Γ is a Γ-Γ-direct summand of Λ.*

Proof. Since Γ is R-separable, there exists $\Sigma r_i\otimes s_i\in(\Gamma\otimes_C\Gamma)^{\Gamma}$ such that $\Sigma r_is_i=1$. While, since Γ is R-semisimple, ${}_{\Gamma}\Gamma\langle\oplus_{\Gamma}\Lambda$. Let p be the left Γ-projedtion of Λ to Γ. Then the map p^* of Λ to Γ such that $p^*(x)=\Sigma p(xr_i)s_i$ for $x\in\Lambda$ is a Γ-Γ-map, and $p^*(r)=\Sigma p(rr_i)s_i=\Sigma rr_is_i=r$ for every $r\in\Gamma$. Thus ${}_{\Gamma}\Gamma_{\Gamma}\langle\oplus_{\Gamma}\Lambda_{\Gamma}$.

Now we are ready to get our main theorem.

Theorem 1. *Let Λ be an H-separable extension of Γ. Then if Λ is left or right Γ-$f.g.$ projective, there exists a one to one correspondence $V: A \rightsquigarrow V_{\Lambda}(A)$ such that V^2=identity between the class of separable extensions B of Γ such that ${}_BB_B\langle\oplus_B\Lambda_B$ and the class of C-separable subalgebras of Δ.*

Proof. Let D be an arbitrary separable C-subalgebra of Δ and $B=V_{\Lambda}(D)$. Then ${}_BB_{\Gamma}\langle\oplus_B\Lambda_{\Gamma}$ and $V_{\Lambda}(B)=D$. This and Corollary 1.3 [8] imply that B is separable over Γ, since B is left or right Γ-$f.g.$ projective and ${}_DD_D\langle\oplus_D\Delta_D$. ${}_BB_B\langle\oplus_B\Lambda_B$ follows from ${}_BB_{\Gamma}\langle\oplus_B\Lambda_{\Gamma}$ and the separability of B over Γ. On the other hand, if B is a separable extension of Γ such that ${}_BB_B\langle\oplus_B\Lambda_B$, then $D=V_{\Lambda}(B)$ is a separable C-algebra and $V_{\Lambda}(V_{\Lambda}(B))=B$ by Proposition 1.4 [8].

Corollary 1. *Let Λ be an H-separable extension of Γ with the condition (*) of §0. Then if Λ is Γ-$f.g.$ projective, there exists a one to one correspondence*

$V: A \leadsto V_\Lambda(A)$ such that V^2=identity between the class of subrings of Λ which are H-separable extensions of Γ and the class of separable C-subalgebras of C'. In this case V corresponds each H-separable extension of Γ to its center.

Proof. Let B be any ring with $\Gamma \subset B \subset \Lambda$. Then $V_\Lambda(B) \subset V_\Lambda(\Gamma) = C' \subset B$, hence the center of $B = V_B(B) = B \cap V_\Lambda(B) = V_\Lambda(B)$. On the other hand, by Propositions 1.8 and 1.9 B is H-separable over Γ, if and only if ${}_B B_B \langle \oplus {}_B \Lambda_B$. Thus the assertion follows from Theorem 1.

REMARK. Ring extension $\Lambda | \Gamma$ which satisfy the condition (*) and such that Λ is Γ-*f. g.* projective really exists. Let Λ be a central separable C-algebra and Γ a C-separable subalgebra with its center $C' \neq C$. Then Λ is H-separable over Γ, ${}_\Gamma \Gamma_\Gamma \langle \oplus {}_\Gamma \Lambda_\Gamma$ and Λ is Γ-*f. g.* projective. Let $\Lambda' = V_\Lambda(C')$. Then $\Lambda | \Lambda'$ satisfy the condition (*) by Proposition 1.3 [8].

In [10] we considered ring extension $\Lambda | \Gamma$ which satisfy the following condition (#).

(#) (1) Λ is a separable extension of Γ such that $V_\Lambda(\Gamma) = C$.
(2) Λ is Γ-centrally projective (i.e., ${}_\Gamma \Lambda_\Gamma \langle \oplus {}_\Gamma(\Gamma \oplus \cdots \oplus \Gamma)_\Gamma$).

And we proved that if $\Lambda | \Gamma$ satisfy the condition (#), there exist one to one correspondences U and V between the class $\mathfrak{A}$ of separable extensions B of Γ such that ${}_B B_B \langle \oplus {}_B \Lambda_B$ and the class $\mathfrak{B}$ of separable C'-subalgebras of C, defined by $V: B \leadsto B \cap C$ and $U: R \leadsto R\Gamma$ for $B \in \mathfrak{A}$ and $R \in \mathfrak{B}$, with $UV = 1_\mathfrak{A}$ and $VU = 1_\mathfrak{B}$ (Theorem 8 [10]).

Let a ring extension $\Lambda | \Gamma$ satisfy the condition (*) of §0. Then $\Omega = [\mathrm{End}({}_\Gamma \Lambda)]^0 = C' \otimes_C \Lambda$ and C' is a commutative C-separable algebra and C-*f. g.* projective. Then clearly, the center of $\Omega = C' = V_\Omega(\Lambda)$, and $\Omega | \Lambda$ satisfies the condition (#). Let $\mathfrak{A}$ be the class of separable extensions Σ of Λ such that ${}_\Sigma \Sigma_\Sigma \langle \oplus {}_\Sigma \Omega_\Sigma$, $\mathfrak{B}$ the class of separable C-subalgebras of C', and let U and V be such that $U(R) = R\Lambda$ for $R \in \mathfrak{B}$ and $V(\Sigma) = \Sigma \cap C'$ for $\Sigma \in \mathfrak{A}$. Then by Theorem 8 [10], U and V provide one to one correspondences between $\mathfrak{A}$ and $\mathfrak{B}$ with $UV = 1_\mathfrak{A}$ and $VU = 1_\mathfrak{B}$. Furthermore, let $\mathfrak{C}$ be the class of subrings of Λ which are H-separable extensions of Γ. Then by Corollary 1 we have.

Proposition 4. *Let a ring extension $\Lambda | \Gamma$ satisfy the condition (*) and Λ be Γ-f. g. projective. Then if we define $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ as above, the correspondences $W: \mathfrak{A} \leadsto \mathfrak{C}$ such that $W(\Sigma) = V_\Lambda(\Sigma \cap (C' \otimes 1))$ for $\Sigma \in \mathfrak{A}$ and $T: \mathfrak{C} \leadsto \mathfrak{A}$ such that $T(B) = End({}_B \Lambda)$ for $B \in \mathfrak{C}$ are one to one with $WT = 1_\mathfrak{C}$ and $TW = 1_\mathfrak{A}$.*

Proof. For $B \in \mathfrak{C}$, $V_\Lambda(B) \otimes_C \Lambda = \mathrm{End}({}_B \Lambda)$. Then by Corollary 1 and Theorem 8[10], $TW = 1_\mathfrak{A}$ and $WT = 1_\mathfrak{C}$.

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