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ON CENTRALIZERS IN SEPARABLE EXTENSIONS II

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0. The aim of this paper is to improve and generalize some results of the author's previous paper [8]. Therefore, all notations and terminologies are same as those in [7] and [8]. In [8] the author studied some commutator theory of H-separable extension $\Lambda|\Gamma$ in the case where $\Lambda \cong \Gamma \otimes_C \Delta$ with $\Delta (= V_\Lambda(\Gamma))$ central separable over C and $C =$ the center of $\Lambda =$ the center of Γ , and in the case where Λ is left or right Γ -f. g. (finitely generated) projective and $\Lambda|\Gamma$ satisfies the following condition (*)

- (*) 1) Λ is an H-separable extension of Γ such that ${}_1\Gamma_\Gamma \triangleleft \bigoplus_\Gamma \Lambda_\Gamma$.
- 2) $V_\Lambda(\Gamma) = C'$, where C' is the center of Γ .

(See Theorem 1.2, Corollary 1.4 and Theorem 1.3 [8]). In case $\Lambda|\Gamma$ satisfies the condition (*) 1), Λ is left Γ -f. g. projective if and only if Λ is right Γ -f. g. projective by Corollary 2 [9], hence we shall simply say that Λ is Γ -f. g. projective in this case. We note also that the condition (*) implies that $V_\Lambda(C') = \Gamma$ by Proposition 1.2 [7]. In this paper, we shall consider the case where Λ is left or right Γ -f. g. projective and Λ is an H-separable extension of Γ , and shall prove that there exists a one to one correspondence between the class of sub-rings B of Λ which is separable extensions of Γ and ${}_B B_B \triangleleft \bigoplus_B \Lambda_B$ and the class of separable C -subalgebras of Δ (Theorem 1). From this theorem, Corollary 1.4 and a more beautiful result than Theorem 1.3 [8] follows.

1. To obtain our main results we need the next lemma which appears in [6].

Lemma 1 (Corollary 1.2 [6]). *Let A be a ring, M a left A -module, $\Omega = \text{End}({}_A M)$ and $E = \text{End}(M_\Omega)$. Then if M is A -f. g. projective, $E \otimes_A M \cong M$ as E - Ω -module by the map: $e \otimes m \rightarrow em$ for $e \in E$ and $m \in M$.*

Proof. Since M is A -f. g. projective, we have natural isomorphisms

$$\begin{aligned} E \otimes_A M &= \text{Hom}(M_\Omega, M_\Omega) \otimes_A M \cong \text{Hom}(\text{Hom}({}_A M, {}_A M)_\Omega, M_\Omega) \\ &= \text{Hom}(\Omega_\Omega, M_\Omega) \cong M \end{aligned}$$

as E - Ω -module. The composition of the above isomorphisms is the required one.

For rings $\Gamma \subset B \subset \Lambda$, we shall say that $B \otimes_{\Gamma} \Lambda \rightarrow \Lambda$ splits if the map of $B \otimes_{\Gamma} \Lambda$ to Λ such that $b \otimes x \rightarrow bx$ for $b \in B$ and $x \in \Lambda$ splits as B - Λ -map. We also need Proposition 2.3 [8]. This proposition can be improved as follows

Proposition 1. *Let Λ be an H -separable extension of Γ . Then for any intermediate ring B between Γ and Λ such that ${}_B B_{\Gamma} \ll \bigoplus_B \Lambda_{\Gamma}$ and $B \otimes_{\Gamma} \Lambda \rightarrow \Lambda$ splits, ${}_D D \ll \bigoplus_D \Delta$ and $D \otimes_C \Delta \rightarrow \Delta$ splits, where $D = V_{\Lambda}(B)$. Conversely for any C -subalgebra D of Δ such that ${}_D D \ll \bigoplus_D \Delta$ and $D \otimes_C \Delta \rightarrow \Delta$ splits, ${}_B B_{\Gamma} \ll \bigoplus_B \Lambda_{\Gamma}$ and $B \otimes_{\Gamma} \Lambda \rightarrow \Lambda$ splits, where $B = V_{\Lambda}(D)$.*

Proof. The first part of this proposition have been proved in Proposition 2.3 [8]. Hence we need to prove only the second part without assuming that B is right Γ - $f.g.$ projective. Suppose that D is a C -subalgebra of Δ such that $D \otimes_C \Delta \rightarrow \Delta$ splits. Then $B = V_{\Lambda}(D) \cong \text{Hom}({}_D \Delta_{\Delta}, {}_D \Lambda_{\Delta}) \ll \bigoplus \text{Hom}({}_D D \otimes_C \Delta_{\Delta}, {}_D \Lambda_{\Delta}) \cong V_{\Lambda}(C)$ as B - $V_{\Lambda}(\Delta)$ -module. Hence ${}_B B_{\Gamma} \ll \bigoplus_B \Lambda_{\Gamma}$. Then, since Λ is H -separable over Γ and ${}_B B_{\Gamma} \ll \bigoplus_B \Lambda_{\Gamma}$, we have a B - Λ -isomorphism η of $B \otimes_{\Gamma} \Lambda$ to $\text{Hom}({}_D \Delta, {}_D \Lambda)$ such that $\eta(b \otimes x)(d) = bdx$ for $b \in B, d \in D$ and $x \in \Lambda$ by Proposition 1.3 [7]. Hence, we have a commutative diagram of B - Λ -maps

$$\begin{array}{ccc} B \otimes_{\Gamma} \Lambda & \xrightarrow{\quad \eta \quad} & \text{Hom}({}_D \Delta, {}_D \Lambda) \\ \downarrow & & \downarrow i_* \\ \Lambda & \xrightarrow{\quad j \quad} & \text{Hom}({}_D D, {}_D \Lambda) \end{array}$$

where j is the natural isomorphism and i_* is the one induced by the inclusion map $i: D \subset \Delta$. Then if ${}_D D \ll \bigoplus_D \Delta, i_* B$ - Λ -splits and $B \otimes_{\Gamma} \Lambda \rightarrow \Lambda$ splits.

Let Λ be a semisimple R -algebra in the sense of A. Hattori [2], that is, Λ is a weakly semisimple extension of $R \cdot 1$ in the sense of [3]. Then every finitely generated Λ -module which is R -projective is Λ -projective, and by Proposition 4.1 [1] if Σ is a finitely generated projective R -algebra which contains $\Lambda, {}_{\Lambda} \Lambda \ll \bigoplus_{\Lambda} \Sigma$ and $\Lambda_{\Lambda} \ll \bigoplus \Sigma_{\Lambda}$. It is also well known that a separable algebra is a semisimple algebra.

Proposition 2. *Let Λ be an H -separable extension of Γ . If (1) D is a separable C -subalgebra of Δ , or if (2) Δ is a separable C -algebra (e.g., if ${}_{\Gamma} \Gamma_{\Gamma} \ll \bigoplus_{\Gamma} \Lambda_{\Gamma}$) and D is a semisimple C -subalgebra of Δ , then $V_{\Lambda}(V_{\Lambda}(D)) = D, {}_B B_{\Gamma} \ll \bigoplus_B \Lambda_{\Gamma}$ and $B \otimes_{\Gamma} \Lambda \rightarrow \Lambda$ splits, where $B = V_{\Lambda}(D)$.*

Proof. Suppose (1). Then since $D \otimes_C D \rightarrow D$ splits as D - D -map, $D \otimes_C \Delta \rightarrow \Delta$ splits as D - Δ -map. Suppose (2). Then $D \otimes_C \Delta \rightarrow \Delta$ splits as D - C -map, since D is C -semisimple. Then $D \otimes_C \Delta \rightarrow \Delta$ splits as D - Δ -map, since Δ is C -separable. Thus in both cases, $D \otimes_C \Delta \rightarrow \Delta$ splits and ${}_D D \ll \bigoplus_D \Delta$. The latter

follows from Proposition 4.1 [1], since D is C -semisimple and Δ is C - $f.g.$ projective. Then $B \otimes_{\Gamma} \Lambda \rightarrow \Lambda$ splits and ${}_B B_{\Gamma} \prec \bigoplus_B \Lambda_{\Gamma}$ by Proposition 1. Hence Λ is H -separable over B by Proposition 2.2 [8]. Let $D' = V_{\Lambda}(B)$. Then there exists a ring isomorphism $\eta: D' \otimes_C \Lambda^0 \rightarrow \text{End}({}_B \Lambda)$ such that $\eta(d \otimes x^0)(y) = dyx$ for $x, y \in \Lambda, d \in D'$ (see Proposition 3.3 [5]). Then $D' \otimes_C \Lambda^0$ is the double centralizer of a left $D \otimes_C \Lambda^0$ -module Λ , since $B = \text{End}({}_D \Lambda_{\Lambda})$. While $D \otimes_C \Lambda \rightarrow \Lambda$ splits, since $D \otimes_C \Delta \rightarrow \Delta$ splits. This implies that Λ is left $D \otimes_C \Lambda^0$ - $f.g.$ projective. Then by lemma 1, $(D' \otimes_C \Lambda^0) \otimes_{D \otimes_C \Lambda^0} \Lambda \cong \Lambda$, hence $D' \otimes_D \Lambda^0 \cong \Lambda$. This isomorphism is given by $d \otimes x \rightarrow dx$ for $d \in D', x \in \Lambda$. Then for every $d \in D', d \otimes 1 = 1 \otimes d$ in $D' \otimes_D \Lambda$, since both are mapped to d by this isomorphism. On the other hand, since Λ is H -separable over B, D' is C - $f.g.$ projective, and D' is right D - $f.g.$ projective, since D is C -semisimple. Hence $D' \otimes_D D' \subset D' \otimes_D \Lambda$, and $d \otimes 1 = 1 \otimes d$ in $D' \otimes_D D'$ for every $d \in D'$. Since ${}_D D \prec \bigoplus_D D', D' = D \oplus A$ for some left D -submodule A of D' and $D' \otimes_D D' = D' \otimes_D D \oplus D' \otimes_D A$. Let x be an arbitrary element of D' and $x = d + a$ for $d \in D, a \in A$. Then $D' \otimes_D D' \ni x \otimes 1 = 1 \otimes x = 1 \otimes d + 1 \otimes a$, and $1 \otimes a = 0, x \otimes 1 = 1 \otimes d$. Thus $x = d \in D$. Thus $D' = D$. Thus $D = V_{\Lambda}(V_{\Lambda}(D))$.

The next proposition is a generalization of Proposition 1.5 [8].

Proposition 3. *Let Λ be an arbitrary R -algebra which is R - $f.g.$ projective. Then for any separable R -subalgebra Γ of Λ, Γ is a Γ - Γ -direct summand of Λ .*

Proof. Since Γ is R -separable, there exists $\sum r_i \otimes s_i \in (\Gamma \otimes_C \Gamma)^{\Gamma}$ such that $\sum r_i s_i = 1$. While, since Γ is R -semisimple, ${}_R \Gamma \prec \bigoplus_R \Gamma$. Let p be the left Γ -projection of Λ to Γ . Then the map p^* of Λ to Γ such that $p^*(x) = \sum p(xr_i) s_i$ for $x \in \Lambda$ is a Γ - Γ -map, and $p^*(r) = \sum p(rr_i) s_i = \sum r r_i s_i = r$ for every $r \in \Gamma$. Thus ${}_R \Gamma \prec \bigoplus_R \Lambda_{\Gamma}$.

Now we are ready to get our main theorem.

Theorem 1. *Let Λ be an H -separable extension of Γ . Then if Λ is left or right Γ - $f.g.$ projective, there exists a one to one correspondence $V: A \rightsquigarrow V_{\Lambda}(A)$ such that $V^2 = \text{identity}$ between the class of separable extensions B of Γ such that ${}_B B_B \prec \bigoplus_B \Lambda_B$ and the class of C -separable subalgebras of Δ .*

Proof. Let D be an arbitrary separable C -subalgebra of Δ and $B = V_{\Lambda}(D)$. Then ${}_B B_{\Gamma} \prec \bigoplus_B \Lambda_{\Gamma}$ and $V_{\Lambda}(B) = D$. This and Corollary 1.3 [8] imply that B is separable over Γ , since B is left or right Γ - $f.g.$ projective and ${}_D D_D \prec \bigoplus_D \Delta_D \cdot {}_B B_B \prec \bigoplus_B \Lambda_B$ follows from ${}_B B_{\Gamma} \prec \bigoplus_B \Lambda_{\Gamma}$ and the separability of B over Γ . On the other hand, if B is a separable extension of Γ such that ${}_B B_B \prec \bigoplus_B \Lambda_B$, then $D = V_{\Lambda}(B)$ is a separable C -algebra and $V_{\Lambda}(V_{\Lambda}(B)) = B$ by Proposition 1.4 [8].

Corollary 1. *Let Λ be an H -separable extension of Γ with the condition (*) of §0. Then if Λ is Γ - $f.g.$ projective, there exists a one to one correspondence*

$V: A \rightsquigarrow V_{\Lambda}(A)$ such that $V^2 = \text{identity}$ between the class of subrings of Λ which are H -separable extensions of Γ and the class of separable C -subalgebras of C' . In this case V corresponds each H -separable extension of Γ to its center.

Proof. Let B be any ring with $\Gamma \subset B \subset \Lambda$. Then $V_{\Lambda}(B) \subset V_{\Lambda}(\Gamma) = C' \subset B$, hence the center of $B = V_B(B) = B \cap V_{\Lambda}(B) = V_{\Lambda}(B)$. On the other hand, by Propositions 1.8 and 1.9 B is H -separable over Γ , if and only if ${}_B B_B \prec \bigoplus_B \Lambda_B$. Thus the assertion follows from Theorem 1.

REMARK. Ring extension $\Lambda | \Gamma$ which satisfy the condition $(*)$ and such that Λ is Γ - $f. g.$ projective really exists. Let Λ be a central separable C -algebra and Γ a C -separable subalgebra with its center $C' \neq C$. Then Λ is H -separable over Γ , ${}_r \Gamma_r \prec \bigoplus_r \Lambda_r$ and Λ is Γ - $f. g.$ projective. Let $\Lambda' = V_{\Lambda}(C')$. Then $\Lambda | \Lambda'$ satisfy the condition $(*)$ by Proposition 1.3 [8].

In [10] we considered ring extension $\Lambda | \Gamma$ which satisfy the following condition $(\#)$.

- ($\#$) (1) Λ is a separable extension of Γ such that $V_{\Lambda}(\Gamma) = C$.
- (2) Λ is Γ -centrally projective (i.e., ${}_r \Lambda_r \prec \bigoplus_r (\Gamma \oplus \dots \oplus \Gamma)_r$).

And we proved that if $\Lambda | \Gamma$ satisfy the condition $(\#)$, there exist one to one correspondences U and V between the class \mathfrak{A} of separable extensions B of Γ such that ${}_B B_B \prec \bigoplus_B \Lambda_B$ and the class \mathfrak{B} of separable C' -subalgebras of C , defined by $V: B \rightsquigarrow B \cap C$ and $U: R \rightsquigarrow R\Gamma$ for $B \in \mathfrak{A}$ and $R \in \mathfrak{B}$, with $UV = 1_{\mathfrak{A}}$ and $VU = 1_{\mathfrak{B}}$ (Theorem 8 [10]).

Let a ring extension $\Lambda | \Gamma$ satisfy the condition $(*)$ of §0. Then $\Omega = [\text{End}({}_r \Lambda)]^0 = C' \otimes_C \Lambda$ and C' is a commutative C -separable algebra and C - $f. g.$ projective. Then clearly, the center of $\Omega = C' = V_{\Omega}(\Lambda)$, and $\Omega | \Lambda$ satisfies the condition $(\#)$. Let \mathfrak{A} be the class of separable extensions Σ of Λ such that ${}_z \Sigma_z \prec \bigoplus_z \Omega_z$, \mathfrak{B} the class of separable C -subalgebras of C' , and let U and V be such that $U(R) = R\Lambda$ for $R \in \mathfrak{B}$ and $V(\Sigma) = \Sigma \cap C'$ for $\Sigma \in \mathfrak{A}$. Then by Theorem 8 [10], U and V provide one to one correspondences between \mathfrak{A} and \mathfrak{B} with $UV = 1_{\mathfrak{A}}$ and $VU = 1_{\mathfrak{B}}$. Furthermore, let \mathfrak{C} be the class of subrings of Λ which are H -separable extensions of Γ . Then by Corollary 1 we have.

Proposition 4. Let a ring extension $\Lambda | \Gamma$ satisfy the condition $(*)$ and Λ be Γ - $f. g.$ projective. Then if we define \mathfrak{A} , \mathfrak{B} and \mathfrak{C} as above, the correspondences $W: \mathfrak{A} \rightsquigarrow \mathfrak{C}$ such that $W(\Sigma) = V_{\Lambda}(\Sigma \cap (C' \otimes 1))$ for $\Sigma \in \mathfrak{A}$ and $T: \mathfrak{C} \rightsquigarrow \mathfrak{A}$ such that $T(B) = \text{End}({}_B \Lambda)$ for $B \in \mathfrak{C}$ are one to one with $WT = 1_{\mathfrak{C}}$ and $TW = 1_{\mathfrak{A}}$.

Proof. For $B \in \mathfrak{C}$, $V_{\Lambda}(B) \otimes_C \Lambda = \text{End}({}_B \Lambda)$. Then by Corollary 1 and Theorem 8[10], $TW = 1_{\mathfrak{A}}$ and $WT = 1_{\mathfrak{C}}$.

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