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ON THE NON-EXISTENCE OF L-SPACE SURGERY STRUCTURE

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Abstract

We illustrate homology 3-spheres which never yield any lens spaces by integral Dehn surgery by using Ozsváth and Szabó's contact invariant.

1. Introduction

Let Y be a closed oriented 3-manifold. In this paper we denote by $Y_r(K)$ the Dehn surgered manifold of a knot K in Y with slope r. Lens spaces can be obtained from the Dehn surgery of the unknot U with slope $-p/q$, i.e. $L(p, q) = S^3_{-p/q}(U)$.

In general it is difficult to determine when a lens space can be obtained by an integral surgery of a non-trivial knot K in S^3. There are some well-known non-trivial knots in S^3 yielding lens spaces by integral surgeries, for example torus knots, 2-cable knots of torus knots, and the $(-2, 3, 7)$-pretzel knot and so on.

If we generalize the ambient space of knots to homology 3-spheres, we can construct more lens spaces by integral Dehn surgery. For example in [1] R. Fintushel and R. Stern have asserted that a lens space $L(p, q)$ is obtained by an integral Dehn surgery on a homology 3-sphere Y if and only if there exists an integer x such that $q = \pm x^2 \bmod p$. Thus it is a quite natural problem to find constraints on homology 3-spheres and knots that realize lens space surgery for a given pair (p, x) satisfying the above condition by Fintushel and Stern.

The author in [11] has studied lens space surgery on L-space homology 3-spheres to find several families of knots in the Poincaré homology 3-sphere $\Sigma(2, 3, 5)$ yielding lens spaces by positive integral Dehn surgery. $\Sigma(2, 3, 5)$ and S^3 are L-space homology 3-spheres that Ozsváth–Szabó's correction term d have 2 and 0 respectively. The notions of L-space and d shall be defined in Section 2.

On the other hand in [11] we could not find L-space homology spheres with $d \neq 0, 2$ and with a certain definite range of p. This computation led us to the following question.

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QUESTION 1.1 (Conjecture 1.3 in [11]). Let Y be an L-space homology sphere with $d(Y) \neq 0, 2$. None of knots in Y constructs any lens space by positive integral Dehn surgery.

Restricting our attention to lens space surgery on $\overline{\Sigma(2, 3, 5)}$, whose correction term is -2, we consider the problem of the nonexistence of lens space surgery. We will prove the following.

Theorem 1.1. *$\overline{\Sigma(2, 3, 5)}$ does not yield any lens spaces by any positive integral Dehn surgeries.*

In [2] J.B. Etnyre and K. Honda have shown that there do not exist any positive tight contact structures over $\overline{\Sigma(2, 3, 5)}$. One of motivations of this paper is to relate lens space surgery and contact structure and to consider Question 1.1 from the contact topological view point. In fact these two non-existence properties are linked via Heegaard Floer homology, so that in Lemma 3.1 we can explain how these two phenomena are related.

On the other hand it is believed that all irreducible L-space homology 3-spheres are S^3, $\Sigma(2, 3, 5)$, or $\overline{\Sigma(2, 3, 5)}$.

We also consider lens space surgery on non-irreducible L-space homology sphere.

Theorem 1.2. *Let Y be any manifold in the set $\{\#^n \Sigma(2, 3, 5) \#^m \overline{\Sigma(2, 3, 5)}\}$. If Y yields lens space by positive integral Dehn surgery, then $m = 0$.*

We will require other techniques for proving non-existence of lens space surgery on $\#^n \Sigma(2, 3, 5)$ $(n \geq 2)$ furthermore.

2. Two preliminaries

In this section we define several notions of Dehn surgery and review some general theories of contact topology.

2.1. Lens and L-surgery structure. P. Ozsváth and Z. Szabó in [7, 8] defined the Heegaard Floer homologies $\widehat{HF}(Y, \mathfrak{s})$, $HF^\infty(Y, \mathfrak{s})$, $HF^+(Y, \mathfrak{s})$, $HF^-(Y, \mathfrak{s})$ for any closed oriented 3-manifold with a spin^c-structure $\mathfrak{s}$. The homologies are $\mathbb{Z}[U]$-modules, where U is the action that lowers the degree of the homologies by 2. We call a rational homology 3-sphere Y *L-space* if the Heegaard Floer homology for any spin^c-structure is isomorphic to that of S^3. It is well-known that the set of L-spaces contains all spherical manifolds and some hyperbolic manifolds.

We now assign the coefficients of any homology as $\mathbb{Z}_2$ hence $HF^+(Y, \mathfrak{s})$ is a $\mathbb{Z}_2[U]$-module. When Y is a rational homology 3-sphere, $HF^+(Y, \mathfrak{s})$ admits the absolute $\mathbb{Q}$-grading as in [6]. The *correction term* $d(Y, \mathfrak{s})$ is defined to be the minimal

grading of the non-torsion elements in the image by the natural map $\pi_*\colon HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s})$ defined in [7].

DEFINITION 2.1. Let Y be a closed oriented 3-manifold. We say that Y carries positive (negative) L-surgery structure, if there exist a positive (negative) integer p and a null-homologous knot $K \subset Y$ such that $Y_p(K)$ is an L-space. Moreover if the complement $Y - K$ is irreducible, we say that Y carries proper L-surgery structure.

In particular we say that Y carries positive (negative) lens surgery structure if $Y_p(K)$ is a lens space for a positive (negative) integer p.

If any connected-sum component of Y is not a lens space, the existence of lens space surgery structure on Y means the existence of proper L-surgery structure on Y.

For example S^3 carries both positive and negative lens surgery structure, and $\Sigma(2,3,5)$ positive lens surgery structure (see [11]). Theorem 1.1 means the non-existence of lens surgery structure on $\overline{\Sigma(2,3,5)}$. We will indeed prove the non-existence of positive proper L-surgery structure in Section 3.

Proposition 2.1. *$\overline{\Sigma(2,3,5)}$ does not carry positive proper L-surgery structure.*

We will prove Proposition 2.1 in Section 3. Here we prove Theorem 1.1.

Proof of Theorem 1.1. The assertion follows from Proposition 2.1 and the irreducibility of lens spaces. □

Note that $\overline{\Sigma(2,3,5)}$ carries non-proper positive L-surgery structure. For, the trivial 1-surgery on $\overline{\Sigma(2,3,5)}$ is $\overline{\Sigma(2,3,5)}$ itself obviously.

2.2. The contact invariant of Heegaard Floer homology. We will prepare fundamental tools of contact topology and review Ozsváth–Szabó's contact invariant, which is an invariant associated with a positive cooriented contact structure ξ over a closed oriented 3-manifold Y. This invariant is defined in [9].

Let Y be an oriented closed smooth 3-manifold and α a global 1-form on Y. If there exists a positive smooth function f such that $\alpha_p \wedge d\alpha_p = f(p)\,\mathrm{vol}_p$ holds then we call $(Y, \xi := \ker\alpha)$ *a positive cooriented contact structure* on Y. Here vol is the volume form on Y.

Let K be a fibered knot in Y and $\pi\colon Y - K \to S^1$ the fibration map. Then we call a triple (Y, K, π) *an open book decomposition* on Y.

Due to the results by W.P. Thurston and H.E. Winkelnkemper [13] and E. Giroux [3] there exists a one-to-one correspondence between contact structures up to isotopy

and open book decompositions up to positive stabilization. We denote the correspondence as follows:

$$\{\text{open book decompositions}\}/\text{positive stabilization} \leftrightarrow \{\text{contact structures}\}/\text{isotopy},$$
$$D = (Y, K, \pi) \mapsto \xi_D.$$

Contact 3-manifolds are classified into either of *overtwisted* and *tight*. We here omit the definitions of the notions of positive stabilization, overtwisted and tight. We refer the reader to [3, 5] for the details of these notions. Let (Y, ξ_D) be the contact structure associated with an open book decomposition $D = (Y, K, \pi)$ on Y. Over the fiber bundle $Y_0(K)$ there is the canonical contact structure ξ_0 satisfying $\langle c_1(\xi_0), [\hat{F}]\rangle = 2g(\hat{F})-2$, where $\hat{F} \subset Y_0(K)$ is the closed surface obtained by capping the fiber F of π.

The homomorphism $\hat{F}_W : \widehat{HF}(-Y_0(K), \mathfrak{t}(\bar{\xi}_0)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \widehat{HF}(-Y, \mathfrak{t}(\bar{\xi}))$ is the natural map by the 2-handle cobordism with the spin^c structure over the 4-manifold: $(-Y_0(K), \mathfrak{t}(\bar{\xi}_0)) \xrightarrow{W} (-Y, \mathfrak{t}(\bar{\xi}))$. Here the symbol $\mathfrak{t}(\,\cdot\,)$ is the spin^c structure associated with a contact structure. The notation of the overbar means the contact structure fitted to the reverse of the orientation over the underlying manifold. Let h be the generator in $\widehat{HF}(-Y_0(K), \mathfrak{t}(\bar{\xi}_0))$ whose image in $HF^+(-Y_0(K), \mathfrak{t}(\bar{\xi}_0)) \cong \mathbb{Z}_2$ by the natural map is the generator. We define the contact invariant $c(\xi)$ to be $\hat{F}_W(h)$.

Let $\iota(m)$: $\mathcal{F}(Y, K, S, m) \hookrightarrow \widehat{CF}(Y)$ be the knot filtration of the knot Floer homology associated with (Y, K), which is defined as the subcomplex in $\widehat{CF}(Y)$ with the filtration level $\leq m$. Here S is a Seifert surface of K. The *tau invariant* $\tau(K)$ by Ozsváth and Szabó is defined as the minimal integer among m's for which the induced map $\iota_*(m)$: $H_*(\mathcal{F}(Y, K, S, m)) \to \widehat{HF}(Y)$ is non-zero. Suppose that $(-Y, -K)$ is a fibered knot with a fibration π and with the fiber surface F. Then the contact invariant $c(\xi_D)$ for the open book decomposition $D = (-Y, -K, \pi)$, coincides with the image of the generator of $\mathbb{Z}_2 \cong H_*(\mathcal{F}(-Y, -K, F, -g))$ by the map $\iota_*(-g)$. Here g is the Seifert genus of K. The main property of $c(\xi)$ in this paper is the following:

Theorem 2.1 ([9]). *If a positive contact structure (Y, ξ) is overtwisted, then $c(\xi) = 0$.*

From this theorem $c(\xi) \neq 0$ implies tightness of ξ.

3. Proof of Proposition 2.1

Key lemma for the proof of Proposition 2.1 is the next one.

Lemma 3.1. *Let Y be an L-space homology 3-sphere. If Y carries positive proper L-surgery structure, then Y admits positive tight contact structure.*

Let Y be an L-space homology 3-sphere. If $Y_p(K)$ is an L-space for some knot K in Y and a positive integer p, then $\widehat{HFK}(Y, K, g) \cong \mathbb{Z}_2$ holds where g is the Seifert genus of K. This assertion is easily proved by replacing S^3 with an L-space homology sphere Y in the proof of Theorem 1.2 in [10]. Moreover from this fact and Y. Ni's result in [4], if $Y - K$ is irreducible then K is a fibered knot. As a result any knot K in an L-space homology 3-sphere carrying a proper L-surgery structure can make a contact structure on Y according to the method [13] of W.P. Thurston and H.E. Winkelnkemper. L-space surgery on any non-L-space homology 3-sphere is not always able to make a contact structure, since the knot K may be a non-fibered knot.

Assuming Lemma 3.1, we can prove Proposition 2.1.

Proof of Proposition 2.1. Suppose that $\overline{\Sigma(2, 3, 5)}$ carries a positive proper L-surgery structure. From Lemma 3.1 $\overline{\Sigma(2, 3, 5)}$ must admit a positive tight contact structure. However, by the result [2] $\overline{\Sigma(2, 3, 5)}$ does not admit any positive tight contact structure. $\overline{\Sigma(2, 3, 5)}$ does not, therefore, carry a positive proper L-surgery structure. □

We will prove Lemma 3.1.

Proof of Lemma 3.1. Let Y be an L-space homology 3-sphere and $Y_p(K)$ an L-space. From the fiberness of K we can make a contact structure over Y as above. Consider the following surgery exact triangle:

(1)
$$\begin{array}{ccc} HF^+(-Y) & \xleftarrow{\quad F_1 \quad} & HF^+(-Y_0(K),\ Q^{-1}[i]) \\ & {\scriptstyle F_3}\searrow \quad \nearrow{\scriptstyle F_2} & \\ & HF^+(-Y_p(K),\ [i]) & \end{array}$$

The map Q: $\mathrm{Spin}^c(-Y_0(K)) \cong \mathbb{Z} \to \mathrm{Spin}^c(-Y_p(K))$ between the sets of spin^c structures is defined in [6]. The notation $[i] \in \mathrm{Spin}^c(-Y_p(K))$ stands for the image $Q(i)$.

If $c_1(\mathfrak{t}(\bar{\xi}_0)) \neq 0$, then $HF^+(-Y_0(K), \mathfrak{t}(\bar{\xi}_0)) = HF^+(-Y_0(K), 1-g) \cong \mathbb{Z}_2$ is not included in the image of F_2 since F_2 is a U-equivariant map and $-Y_p(K)$ is L-space. Hence the restriction of F_1 to the $\mathfrak{t}(\bar{\xi}_0)$-component

$$HF^+(-Y_0(K), 1-g) \to HF^+(-Y)$$

is injective. The U-equivariant homomorphism F_1 maps the kernel of U in $HF^+(-Y_0(K), Q^{-1}[i])$ to the kernel of U in $HF^+(-Y)$. From the definition of h and

injectivity of F_1, $F_1(i_*(h))$ is non-zero and thereafter the commutative diagram

$$\begin{array}{ccc} h \in \widehat{HF}(-Y_0(K),\, 1-g) & \xrightarrow{\hat{F}_1} & \widehat{HF}(-Y) \\ \downarrow{\scriptstyle i_*} & & \downarrow \\ HF^+(-Y_0(K),\, 1-g) & \xrightarrow{F_1} & HF^+(-Y) \end{array}$$

means that $\hat{F}_1(h) = c(\xi_D)$ is also non-zero. From Theorem 2.1 ξ_D is, therefore, tight.

If $c_1(\mathfrak{t}(\bar{\xi}_0)) = 0$, then the genus of K is one. Then for non-zero i, $HF^+(-Y_0(K), i) \cong 0$ and $HF_{\mathrm{red}}(-Y_0(K),\, 0) \cong 0$. The knot Floer homology of K is

$$\widehat{HFK}(-Y,\, -K,\, i) \cong \begin{cases} \mathbb{Z}_2 & \text{for} \quad i = 0,\, \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can see that the tau invariant $\tau(-K)$ is -1 by the same method as [10]. Thus $\widehat{HFK}(-Y,\, -K,\, -1) \to \widehat{HF}(-Y)$ is injective. Hence the contact invariant $c(\xi_D)$ does not vanish. From Theorem 2.1 the contact structure ξ_D is tight. □

We prove the following corollary and Theorem 1.2.

Corollary 3.1. *The homology* 3*-sphere* $\Sigma(2,3,5)\#\overline{\Sigma(2,3,5)}$ *carries neither positive nor negative proper L-surgery structure.*

Proof. Since $\Sigma(2,3,5)\#\overline{\Sigma(2,3,5)}$ admits neither positive nor negative tight contact structure, Lemma 3.1 follows Corollary 3.1. □

Proof of Theorem 1.2. We consider the manifold $Y = \#^n \Sigma(2,3,5)\#^m \overline{\Sigma(2,3,5)}$. If $m > 0$, then Y does not admit positive tight contact structure. Therefore if Y carries a positive proper L-surgery structure, then m must be 0. □

We call a knot K in a homology 3-sphere Y *a lens space Berge knot* if an integral Dehn surgery of K is a lens space and the dual knot K' of K is the union of two arcs each of which is embedded in the meridian disk of the genus one Heegaard decomposition of the lens space (see Definition 1.7. in [10]).

The author has verified that many Brieskorn homology 3-spheres appear as the homology spheres Y yielding lens spaces. For example $\Sigma(2,3,6n \pm 1)$, $\Sigma(2,2q+1,2(2q+1)\pm 1)$ contain lens space Berge knots and yield infinite lens spaces for each of the homology spheres, see [12]. Ozsváth and Szabó have shown that any lens space Berge knot is fibered [10]. As a result many Brieskorn homology spheres carry proper L-surgery structure with contact structures associated with the lens space Berge knots. Here we raise a question which generalizes Proposition 2.1.

QUESTION 3.1. Any negatively oriented Brieskorn homology sphere $\overline{\Sigma(p, q, r)}$ does not carry positive proper L-surgery structure.

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