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AN APPLICATION ON NAGAO'S LEMMA

Dedicated to Professor Hiroshi Nagao on his sixtieth birthday

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With time, the importance of Nagao's lemma has grown in modular representation theory of finite groups. In this note, we add another application.

Let G be a finite group, and let F be a field of characteristic $p > 0$.

For a subgroup H of G and a (right) FG -module V , we denote V^H the fixed-point-set of H in V , so that V^H is an $FN_G(H)$ -module. The trace map $Tr_H^G: V^H \rightarrow V^G$ is defined by $Tr_H^G(v) = \sum_g vg$, where g runs over a complete set of representatives of $H \backslash G$.

Main Theorem. *Let V be an indecomposable FG -module in a block B , and let P be a p -subgroup of G . Then each composition factor of the $FN_G(P)$ -module*

$$V(P) := V^P / \sum_{A < P} Tr_A^P(V^A),$$

where A runs over proper subgroups of P , belongs to a block b such that $b^G = B$.

REMARK. If $V(P) \neq 0$, then P is contained in a defect group of B .

Proof. of the theorem. Set $N = N_G(P)$. Let e be the centrally primitive idempotent of FG corresponding to B . Let $s: Z(FG) \rightarrow Z(FN)$ be the Brauer homomorphism. Then Nagao's lemma ([2], Chapter III, Theorem 7.5) states that

$$V_N = V_N s(e) \oplus W_1 \oplus \cdots \oplus W_n$$

as FN -modules, where each W_i is Q_i -projective FN -module for some p -subgroup Q_i of N with $P \not\cong Q_i$. Thus in order to prove the theorem, it will suffice to show that

$$W_i^P \subseteq \sum_{A < P} Tr_A^P(V^A),$$

where A runs over proper subgroups of P . But this follows directly from the following lemma, and so the theorem is proved.

Lemma. *Let N be a finite group with a normal p -subgroup P . Let W be a Q -projective FN -module, where $Q \cong P$. Then*

$$W^P = \sum_{A < P} \text{Tr}_A^P(W^A),$$

where A runs over proper subgroups of P .

Proof. In order to prove this lemma, we may assume that for some FQ -module U ,

$$W = \text{Ind}_Q^N(U).$$

Then by Mackey decomposition, we have that

$$W_P = \bigoplus_n \text{Ind}_P^P \text{Ind}_{P \cap Q^n}^P(U_{P \cap Q^n}^n),$$

where n runs over a complete set of representatives of $Q \backslash N/P$ and $Q^n = n^{-1}Qn$. Let n be an element of N and set $R = P \cap Q^n$, $X = U_R^n$. Since P is normal in N and Q is not contained in P , we have that R is a proper subgroup of P . Thus, in order to prove the lemma, it will suffice to show that

$$(\text{Ind}_R^P(X))^P \subseteq \text{Tr}_R^P(\text{Ind}_R^P(X)^R).$$

But this follows directly from an easy calculation (eq. [2] Chapter II Lemma 3.4). The lemma is proved.

REMARK. The main theorem can be proved also by the Brauer homomorphism of modules, which is defined by Broué and Puig [1]. Let B be a block of G and e a corresponding central primitive idempotent of FG . We define the Brauer homomorphism Br_P^V with respect to P by the canonical homomorphism $V^P \rightarrow V(P)$. Now let $s_P: Z(FG) \rightarrow Z(FC_G(P))$ be the classical Brauer homomorphism with respect to P . Then we can prove that $Br_P^V(ve) = Br_P^V(v)s_P(e)$ for the element v of V^P . The main theorem is immediate from this fact.

References

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- [2] W. Feit: *The representation theory of finite groups*, North-Holland, 1982.

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