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<th><strong>Title</strong></th>
<th>An application on Nagao's lemma</th>
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With time, the importance of Nagao’s lemma has grown in modular representation theory of finite groups. In this note, we add another application.

Let $G$ be a finite group, and let $F$ be a field of characteristic $p > 0$. For a subgroup $H$ of $G$ and a (right) $F^G$-module $V$, we denote $V^H$ the fixed-point-set of $H$ in $V$, so that $V^H$ is an $FN_G(H)$-module. The trace map $\text{Tr}_H^G: V^H \to V^G$ is defined by $\text{Tr}_H^G(v) = \sum g v g$, where $g$ runs over a complete set of representatives of $H \setminus G$.

**Main Theorem.** Let $V$ be an indecomposable $F^G$-module in a block $B$, and let $P$ be a $p$-subgroup of $G$. Then each composition factor of the $FN_G(P)$-module $AJP$, where $A$ runs over proper subgroups of $P$, belongs to a block $b$ such that $b^G = B$.

**Remark.** If $V(P) \neq 0$, then $P$ is contained in a defect group of $B$.

**Proof.** of the theorem. Set $N = N_G(P)$. Let $e$ be the centrally primitive idempotent of $F^G$ corresponding to $B$. Let $s: Z(FG) \to Z(FN)$ be the Brauer homomorphism. Then Nagao’s lemma ([2], Chapter III, Theorem 7.5) states that

$$V_N = V_N s(e) \oplus W_1 \oplus \cdots \oplus W_n$$

as $FN$-modules, where each $W_i$ is $Q_i$-projective $FN$-module for some $p$-subgroup $Q_i$ of $N$ with $P \nmid Q_i$. Thus in order to prove the theorem, it will suffice to show that

$$W_i^P \subseteq \sum_{A \nmid P} \text{Tr}^P_A(V^A),$$

where $A$ runs over proper subgroups of $P$. But this follows directly from the following lemma, and so the theorem is proved.

**Lemma.** Let $N$ be a finite group with a normal $p$-subgroup $P$. Let $W$ be a $Q$-projective $FN$-module, where $Q \nmid P$. Then
where $A$ runs over proper subgroups of $P$.

Proof. In order to prove this lemma, we may assume that for some $FQ$-module $U$, 

\[ W = \text{Ind}_Q^P (U). \]

Then by Mackey decomposition, we have that

\[ W = \bigoplus_n \text{Ind}_{P^Q_n}^P (U_{P^Q_n}). \]

where $n$ runs over a complete set of representatives of $Q\setminus N \cap P$ and $Q^s = n^{-1}Qn$. Let $n$ be an element of $N$ and set $R = P \cap Q^s$, $X = U_{R}$. Since $P$ is normal in $N$ and $Q$ is not contained in $P$, we have that $R$ is a proper subgroup of $P$. Thus, in order to prove the lemma, it will suffice to show that

\[ (\text{Ind}_R^P (X))^p \subseteq \text{Tr}_R^P (\text{Ind}_R^P (X)^p). \]

But this follows directly from an easy calculation (eq. [2] Chapter II Lemma 3.4). The lemma is proved.

Remark. The main theorem can be proved also by the Brauer homomorphism of modules, which is defined by Broué and Puig [1]. Let $B$ be a block of $G$ and $e$ a corresponding central primitive idempotent of $FG$. We define the Brauer homomorphism $Br_P^G$ with respect to $P$ by the canonical homomorphism $V^p \rightarrow V(P)$. Now let $s_P : Z(FG) \rightarrow Z(FC_G(P))$ be the classical Brauer homomorphism with respect to $P$. Then we can prove that $Br_P^G(ve) = Br_P^G(v)s_P(e)$ for the element $v$ of $V^p$. The main theorem is immediate from this fact.

References


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