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Osaka University
AN APPLICATION ON NAGAO'S LEMMA

Dedicated to Professor Hirosi Nagao on his sixtieth birthday

TADASI IKEDA, HIROKI SASAKI AND TOMOYUKI YOSHIDA

(Received July 5, 1984)

With time, the importance of Nagao's lemma has grown in modular representation theory of finite groups. In this note, we add another application.

Let $G$ be a finite group, and let $F$ be a field of characteristic $p > 0$. For a subgroup $H$ of $G$ and a (right) $i^G$-module $V$, we denote $V^H$ the fixed-point-set of $H$ in $V$, so that $V^H$ is an $FN_G(H)$-module. The trace map $Tr^G_H: V^H \rightarrow V^G$ is defined by $Tr^G_H(v) = \sum g v_g$, where $g$ runs over a complete set of representatives of $H \backslash G$.

Main Theorem. Let $V$ be an indecomposable $FG$-module in a block $B$, and let $P$ be a $p$-subgroup of $G$. Then each composition factor of the $FN_G(P)$-module $A^P$ where $A$ runs over proper subgroups of $P$, belongs to a block $b$ such that $b^G = B$.

Remark. If $V(P) \neq 0$, then $P$ is contained in a defect group of $B$.

Proof of the theorem. Set $N = N_G(P)$. Let $e$ be the centrally primitive idempotent of $FG$ corresponding to $B$. Let $s: Z(FG) \rightarrow Z(FN)$ be the Brauer homomorphism. Then Nagao's lemma ([2], Chapter III, Theorem 7.5) states that

$$V_N = V_N s(e) \oplus W_1 \oplus \cdots \oplus W_n$$

as $FN$-modules, where each $W_i$ is $Q_i$-projective $FN$-module for some $p$-subgroup $Q_i$ of $N$ with $P \not\subseteq Q_i$. Thus in order to prove the theorem, it will suffice to show that

$$W_i^P \subseteq \sum_{A \subset P} Tr^P_A(V^A),$$

where $A$ runs over proper subgroups of $P$. But this follows directly from the following lemma, and so the theorem is proved.

Lemma. Let $N$ be a finite group with a normal $p$-subgroup $P$. Let $W$ be a $Q$-projective $FN$-module, where $Q \not\supseteq P$. Then
where $A$ runs over proper subgroups of $P$.

Proof. In order to prove this lemma, we may assume that for some $FQ$-module $U$,

$$W = \text{Ind}^Q_U(U).$$

Then by Mackey decomposition, we have that

$$W_P = \bigoplus_n \text{Ind}_P^{Q_n}(U_P^{Q_n}),$$

where $n$ runs over a complete set of representatives of $Q \backslash N / P$ and $Q^n = n^{-1}Qn$. Let $n$ be an element of $N$ and set $R = P \cap Q^n$, $X = U_R^n$. Since $P$ is normal in $N$ and $Q$ is not contained in $P$, we have that $R$ is a proper subgroup of $P$. Thus, in order to prove the lemma, it will suffice to show that

$$(\text{Ind}_R^P(X))^P \subseteq \text{Tr}_R^P(\text{Ind}_R^P(X)^P).$$

But this follows directly from an easy calculation (eq. [2] Chapter II Lemma 3.4). The lemma is proved.

Remark. The main theorem can be proved also by the Brauer homomorphism of modules, which is defined by Broué and Puig [1]. Let $B$ be a block of $G$ and $e$ a corresponding central primitive idempotent of $FG$. We define the Brauer homomorphism $Br_P^V$ with respect to $P$ by the canonical homomorphism $V^p \rightarrow V(P)$. Now let $s_P : Z(FG) \rightarrow Z(FC_G(P))$ be the classical Brauer homomorphism with respect to $P$. Then we can prove that $Br_P^V(ve) = Br_P^V(v)s_P(e)$ for the element $v$ of $V^p$. The main theorem is immediate from this fact.

References


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