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ON SOME SHARPLY T-TRANSITIVE SETS

MITSUO YOSHIZAWA

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Let $S_k$ be the symmetric group on a set $\Omega = \{1, 2, \ldots, k\}$ and $t$ be an integer with $t \geq 2$. A sharply $t$-transitive set $G$ on $\Omega$ is a subset of $S_k$ with the property that for every two ordered $t$-tuples $\alpha_1, \ldots, \alpha_t$ and $\beta_1, \ldots, \beta_t$ of elements in $\Omega (\alpha_i \neq \alpha_j, \beta_i \neq \beta_j$ for $i \neq j$) there uniquely exists $g \in G$ which takes $\alpha_i$ into $\beta_i$: $(\alpha_i)g = \beta_i (i = 1, \ldots, t)$. If $t = k - 1$, $G$ is $S_k$. So from now on we assume $t < k$. Although the sharply $t$-transitive groups were classified by Jordan and Zassenhaus (cf. [1]), it seems difficult to classify the sharply $t$-transitive sets. Now we define a distance $d$ in $S_k$ as follows: For two elements $g_1$ and $g_2$ in $S_k$,

$$d(g_1, g_2) = \sum_{\alpha, \beta \in \Omega} d(\alpha, \beta),$$

where $d(\alpha, \beta) = 0$ if $\alpha = \beta$, and $d(\alpha, \beta) = 1$ otherwise. Then $(S_k, d)$ is a metric space and we have the following two propositions.

**Proposition 1.** Let $g$ be an element in a sharply $t$-transitive set $G$ on $\Omega (|\Omega| = k)$ and $x_i (0 \leq i \leq k)$ denote the number of elements $g' \in G$ satisfying $d(g, g') = k - i$. Then the following equality holds for $i = 0, 1, \ldots, t - 1$:

$$x_i = \sum_{j=0}^{i-1} \binom{j}{i} \binom{k}{j} (k-j)(k-j-1) \cdots (k-t+1) - 1 (-1)^{i+j}.$$

In particular $x_i$'s are uniquely determined independent of the choice of an element $g$ in $G$.

Proof. Counting in two ways the number of the set \{(g', (\alpha_1, \ldots, \alpha_t)) : g' an element \neq g, \{\alpha_1, \ldots, \alpha_t\} \subseteq \Omega, \alpha_u \neq \alpha_v \text{ for } u \neq v, (\alpha_j)g = (\alpha_j)g' \text{ for } j = 1, \ldots, t \}$ gives the following equality for $i = 0, 1, \ldots, t - 1$:

$$x_0 + \binom{t+1}{2} x_{t+1} + \cdots + \binom{t-1}{t} x_{t-1} = \binom{k}{i} \sum_{j=0}^{t-i} \binom{k-j}{j} (k-j-1) \cdots (k-t+1) - 1.$$
where $M = \begin{pmatrix} a_{ij} \end{pmatrix}$ is the $t \times t$ matrix with $a_{ij} = \binom{j}{i} (0 \leq i, j \leq t-1)$. Since the inverse matrix $M^{-1} = \begin{pmatrix} b_{ij} \end{pmatrix}$ is expressed by $b_{ij} = \binom{j}{i}(-1)^{i+j}$, we get the result.

**Proposition 2.** Let $g_1$ and $g_2$ be elements in $S_t$. Then $d(g_1, g_2) = d(gg_1, gg_2) = d(g, g_2)$ holds for any element $g$ in $S_t$.

We call a sharply $t$-transitive set $G$ schematic if it forms an association scheme [2] with the relations determined by the distance. That is, if $d(g_1, g_2) = k - h$, then the number $f(i, j, h) = | \{ g \in G : d(g, g_1) = k - i, d(g, g_2) = k - j \} |$ does not depend on the choice of $g_1$ and $g_2$ (with $d(g_1, g_2) = k - h$), but it depends on $i, j, h (0 \leq i, j, h \leq k)$. We define $f(i, j, h) = 0$ when there exist no elements $g_1$ and $g_2$ with $d(g_1, g_2) = k - h$. We find that $A_5$ (the alternating group of degree five) and any sharply two-transitive set are schematic (cf. [3, Theorem 5.25]). Another example is given by

**Proposition 3.** $PSL(2, 8)$ is schematic sharply three-transitive set.

Proof. $PSL(2, 8)$ is a sharply three-transitive group on a set $\Omega$ of nine letters. We may assume $PSL(2, 8) = \langle a, b, c : a = (1 2 3 4 5 6 7), b = (1 8)(2 4)(3 7)(5 6), c = (2 7)(3 6)(4 5)(8 9) \rangle = G$ with $\Omega = \{1, 2, \ldots, 9\}$ (cf. [4]). Let $g$ and $g_1$ be elements in $G$ with $d(g, g_1) = 9 - h$. Let us set $f(i, j; g, g_1) = | \{ g' \in G : d(g', g_1) = 9 - i, d(g', g_1) = 9 - j \} |$. We want to show that $f(i, j; g, g_1)$ depends on $i, j, h$, but it does not depend on the choice of $g$ and $g_1$. By Proposition 2 we may assume $g_1 = e$ (the identity). Since $h = 1$ if and only if $g$ is an involution and since all involutions are conjugate to one another, we may assume $h = 0$ or $2$. By the Sylow's theorem if $h = 0$ or $2$, then $g$ is conjugate to $u^n(1 \leq n \leq 8)$ or $a^n(1 \leq n \leq 6)$ respectively, where $u = a^3bc = (1 7 2 3 4 6 5 9 8)$. Now $a, a^2, \ldots, a^6$ are conjugate to one another and $u, u^2, u^4, u^5, u^7$ and $u^6$ are also conjugate to one another in the automorphism group $PL(2, 8)$ of $PSL(2, 8)$. Hence we may assume $h = 0$, and it is sufficient to show that $f(i, j; u, e) = f(i, j; u^n, e)$ holds for each $i$ and $j$ ($0 \leq i, j \leq 2$). But it can easily be found by computer calculations. Really if we set $f(i, j; u^n, e) = u_{ij}$ ($n = 1, 3$), we can get

$$
\begin{pmatrix}
88 & 27 & 108 \\
27 & 9 & 27 \\
108 & 27 & 81
\end{pmatrix} \begin{pmatrix}
i \\
j
\end{pmatrix} \begin{array}{c}
(0 \leq i, j \leq 2) \end{array}
$$

Our main result is

**Theorem.** If a sharply $t$-transitive set $G$ on $\Omega (|\Omega| = k > t \geq 2)$ is schematic, then $2t - 1 \leq k$.

Proof. First we remark that $S_4$ (the symmetric group of degree four) is not
schematic. Hence the theorem holds for \( t=2, 3 \). Let us suppose that there exists a schematic sharply \( t \)\-transitive set \( G \) on \( \Omega (|\Omega|=k>t) \) with \( t\geq 4 \) and \( k\leq 2t-2 \). Let us set \( n=k-t \). Then by Proposition 1, there exist two elements \( g_1 \) and \( g_2 \) in \( G \) with \( d(g_1, g_2)=n+1 \). If we set \( \Gamma=\{\alpha_1, \ldots, \alpha_{n+1}\}=\{\alpha\in\Omega: (\alpha)g_1 \neq (\alpha)g_2\} \), then we have

\[
 f(t-1, t-n-2, t-1) = \left| \{g\in G: d(g, g_1) = n+1, d(g, g_2) = 2n+2\} \right| \\
= \left| \{g\in G: (\alpha_i)g = (\alpha_i)g_1 (i=1, \ldots, n+1), \{\beta\in\Omega-\Gamma: (\beta)g \neq (\beta)g_1\} = n+1\} \right| \\
= \binom{t-1}{n+1} n.
\]

Since \( f(t-1, t-1, t-n-2)=f(t-1, t-n-2, t-1)x_{t-1}/x_{t-n-2} \) holds (cf. [2]), we have the following by Proposition 1:

\[
 f(t-1, t-1, t-n-2) = \frac{(t-1)n(t+n)n}{\sum_{i=t-n-2}^{t-1} \binom{i}{t-n-2} \binom{t+n}{i} \{t+n-i\} \{t+n-2\} \cdots (n+1) \cdots 1} (-1)^{i+t-n+2}.
\]

On the denominator of the above we have

\[
 \binom{i}{t-n-2} \binom{t+n}{i} \{t+n-i\} \{t+n-2\} \cdots (n+1) \cdots 1
\]

for \( i=t-n-2, t-n-1, \ldots, t-2 \), because we have

\[
 \binom{i}{t-n-2} \binom{t+n}{i} \{t+n-i\} \{t+n-2\} \cdots (n+1) \cdots 1
\]

\[
 = i+1-t+n+2 \geq 1.
\]

If \( n=1 \), then we have

\[
 f(t-1, t-1, t-3) < \left\{ \binom{t-1}{2} \right\} \left\{ \binom{t+1}{2} \right\} / \left\{ \binom{t-1}{3} \right\} \left\{ \binom{t+1}{3} \right\} = 1,
\]

which contradicts that \( f(t-1, t-1, t-3) \) is a positive integer. Thus we have \( n \geq 2 \). Hence,

\[
 f(t-1, t-1, t-n-2) <
\]
\[ \frac{(t-1)(t+n)}{(n+1)(n+1)} \frac{n^2}{n^2} < \frac{(t-n)(t+n)}{(t-n-2)(t-n)} \frac{n^2}{n^2} \]

\[ \frac{(2n)(n+1)}{(t-n-2)(t-n)} \frac{n^2}{n^2} = \frac{3(2n)! n^3}{(2n)! (n+1)! (n+1)!} = \frac{3n^3}{(n+1)(n+1)!} < 1, \]

a contradiction. Thus we complete the proof.

References


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