

Title	On some sharply t-transitive sets
Author(s)	Yoshizawa, Mitsuo
Citation	Osaka Journal of Mathematics. 1987, 24(3), p. 461-464
Version Type	VoR
URL	https://doi.org/10.18910/4259
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Osaka University

ON SOME SHARPLY T-TRANSITIVE SETS

MITSUO YOSHIKAWA

(Received February 5, 1986)

Let S_k be the symmetric group on a set $\Omega = \{1, 2, \dots, k\}$ and t be an integer with $t \geq 2$. A sharply t -transitive set G on Ω is a subset of S_k with the property that for every two ordered t -tuples $\alpha_1, \dots, \alpha_t$ and β_1, \dots, β_t of elements in Ω ($\alpha_i \neq \alpha_j, \beta_i \neq \beta_j$ for $i \neq j$) there uniquely exists $g \in G$ which takes α_i into $\beta_i: (\alpha_i)g = \beta_i (i=1, \dots, t)$. If $t=k-1, G$ is S_k . So from now on we assume $t < k$. Although the sharply t -transitive groups were classified by Jordan and Zassenhaus (cf. [1]), it seems difficult to classify the sharply t -transitive sets. Now we define a distance d in S_k as follows: For two elements g_1 and g_2 in S_k ,

$$d(g_1, g_2) = |\{\alpha \in \Omega: (\alpha)g_1 \neq (\alpha)g_2\}|.$$

Then (S_k, d) is a metric space and we have the following two propositions.

Proposition 1. *Let g be an element in a sharply t -transitive set G on Ω ($|\Omega| = k$) and $x_i (0 \leq i \leq k)$ denote the number of elements $g' \in G$ satisfying $d(g, g') = k - i$. Then the following equality holds for $i = 0, 1, \dots, t-1$:*

$$x_i = \sum_{j=i}^{t-1} \binom{j}{i} \binom{k}{j} \{(k-j)(k-j-1) \cdots (k-t+1) - 1\} (-1)^{j+i}.$$

In particular x_i 's are uniquely determined independent of the choice of an element g in G .

Proof. Counting in two ways the number of the set $\{(g', (\alpha_1, \dots, \alpha_i)): g' \text{ an element } \neq g, \{\alpha_1, \dots, \alpha_i\} \subseteq \Omega, \alpha_u \neq \alpha_v \text{ for } u \neq v, (\alpha_j)g = (\alpha_j)g' \text{ for } j=1, \dots, i\}$ gives the following equality for $i = 0, 1, \dots, t-1$:

$$x_i + \binom{i+1}{i} x_{i+1} + \cdots + \binom{t-1}{i} x_{t-1} = \binom{k}{i} \{(k-i)(k-i-1) \cdots (k-t+1) - 1\}.$$

Hence we have

$$M \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} \binom{k}{0} \{k(k-1) \cdots (k-t+1) - 1\} \\ \binom{k}{1} \{(k-1)(k-2) \cdots (k-t+1) - 1\} \\ \vdots \\ \binom{k}{t-1} \{(k-t+1) - 1\} \end{pmatrix},$$

where $M = (a_{ij})$ is the $t \times t$ matrix with $a_{ij} = \binom{j}{i} (0 \leq i, j \leq t-1)$. Since the inverse matrix $M^{-1} = (b_{ij})$ is expressed by $b_{ij} = \binom{j}{i} (-1)^{i+j}$, we get the result.

Proposition 2. *Let g_1 and g_2 be elements in S_k . Then $d(g_1, g_2) = d(gg_1, gg_2) = d(g_1g, g_2g)$ holds for any element g in S_k .*

We call a sharply t -transitive set G schematic if it forms an association scheme [2] with the relations determined by the distance. That is, if $d(g_1, g_2) = k - h (g_1, g_2 \in G)$, then the number $f(i, j, h) = |\{g \in G : d(g, g_1) = k - i, d(g, g_2) = k - j\}|$ does not depend on the choice of g_1 and g_2 (with $d(g_1, g_2) = k - h$), but it depends on $i, j, h (0 \leq i, j, h \leq k)$. We define $f(i, j, h) = 0$ when there exist no elements g_1 and g_2 with $d(g_1, g_2) = k - h$. We find that A_5 (the alternating group of degree five) and any sharply two-transitive set are schematic (cf. [3, Theorem 5.25]). Another example is given by

Proposition 3. *$PSL(2, 8)$ is schematic sharply three-transitive set.*

Proof. $PSL(2, 8)$ is a sharply three-transitive group on a set Ω of nine letters. We may assume $PSL(2, 8) = \langle a, b, c : a = (1\ 2\ 3\ 4\ 5\ 6\ 7), b = (1\ 8)(2\ 4)(3\ 7)(5\ 6), c = (2\ 7)(3\ 6)(4\ 5)(8\ 9) \rangle = G$ with $\Omega = \{1, 2, \dots, 9\}$ (cf. [4]). Let g and g_1 be elements in G with $d(g, g_1) = 9 - h$. Let us set $f(i, j; g, g_1) = |\{g' \in G : d(g', g) = 9 - i, d(g', g_1) = 9 - j\}|$. We want to show that $f(i, j; g, g_1)$ depends on i, j, h , but it does not depend on the choice of g and g_1 . By Proposition 2 we may assume $g_1 = e$ (the identity). Since $h = 1$ if and only if g is an involution and since all involutions are conjugate to one another, we may assume $h = 0$ or 2 . By the Sylow's theorem if $h = 0$ or 2 , then g is conjugate to $u^n (1 \leq n \leq 8)$ or $a^n (1 \leq n \leq 6)$ respectively, where $u = a^3bc = (1\ 7\ 2\ 3\ 4\ 6\ 5\ 9\ 8)$. Now $a, a^2, \dots,$ and a^6 are conjugate to one another and u, u^2, u^4, u^5, u^7 and u^8 are also conjugate to one another in the automorphism group $PL(2, 8)$ of $PSL(2, 8)$. Hence we may assume $h = 0$, and it is sufficient to show that $f(i, j; u, e) = f(i, j; u^3, e)$ holds for each i and $j (0 \leq i, j \leq 2)$. But it can easily be found by computer calculations. Really if we set $f(i, j; u^n, e) = u_{ij} (n = 1, 3)$, we can get

$$\begin{pmatrix} u_{ij} \end{pmatrix} = \begin{pmatrix} 88 & 27 & 108 \\ 27 & 9 & 27 \\ 108 & 27 & 81 \end{pmatrix} \quad (0 \leq i, j \leq 2).$$

Our main result is

Theorem. *If a sharply t -transitive set G on $\Omega (|\Omega| = k > t \geq 2)$ is schematic, then $2t - 1 \leq k$.*

Proof. First we remark that S_4 (the symmetric group of degree four) is not

schematic. Hence the theorem holds for $t=2, 3$. Let us suppose that there exists a schematic sharply t -transitive set G on Ω ($|\Omega|=k>t$) with $t\geq 4$ and $k\leq 2t-2$. Let us set $n=k-t$. Then by Proposition 1, there exist two elements g_1 and g_2 in G with $d(g_1, g_2)=n+1$. If we set $\Gamma=\{\alpha_1, \dots, \alpha_{n+1}\}=\{\alpha\in\Omega: (\alpha)g_1\neq(\alpha)g_2\}$, then we have

$$\begin{aligned} f(t-1, t-n-2, t-1) &= |\{g\in G: d(g, g_1) = n+1, d(g, g_2) = 2n+2\}| \\ &= |\{g\in G: (\alpha_i)g = (\alpha_i)g_1 (i = 1, \dots, n+1), |\{\beta\in\Omega-\Gamma: (\beta)g\neq(\beta)g_1\}| = n+1\}| \\ &= \binom{t-1}{n+1}n. \end{aligned}$$

Since $f(t-1, t-1, t-n-2)=f(t-1, t-n-2, t-1)x_{t-1}/x_{t-n-2}$ holds (cf. [2]), we have the following by Proposition 1:

$$\begin{aligned} &f(t-1, t-1, t-n-2) \\ &= \frac{\binom{t-1}{n+1}n\binom{t+n}{n+1}n}{\sum_{i=t-n-2}^{t-1} \binom{i}{t-n-2} \binom{t+n}{i} \{(t+n-i)(t+n-i-1)\dots(n+1)-1\} (-1)^{i+t-n+2}}. \end{aligned}$$

On the denominator of the above we have

$$\begin{aligned} &\binom{i}{t-n-2} \binom{t+n}{i} \{(t+n-i)(t+n-i-1)\dots(n+1)-1\} \\ &> \binom{i+1}{t-n-2} \binom{t+n}{i+1} \{(t+n-i-1)(t+n-i-2)\dots(n+1)-1\} \end{aligned}$$

for $i=t-n-2, t-n-1, \dots, t-2$, because we have

$$\begin{aligned} &\frac{\binom{i}{t-n-2} \binom{t+n}{i} \{(t+n-i)(t+n-i-1)\dots(n+1)-1\}}{\binom{i+1}{t-n-2} \binom{t+n}{i+1} \{(t+n-i-1)(t+n-i-2)\dots(n+1)-1\}} \\ &> \frac{\binom{i}{t-n-2} \binom{t+n}{i} (t+n-i)}{\binom{i+1}{t-n-2} \binom{t+n}{i+1}} = i+1-t+n+2\geq 1. \end{aligned}$$

If $n=1$, then we have

$$f(t-1, t-1, t-3) < \left\{ \binom{t-1}{2} \binom{t+1}{2} \right\} / \left\{ \binom{t-1}{t-3} \binom{t+1}{t-1} \right\} = 1,$$

which contradicts that $f(t-1, t-1, t-3)$ is a positive integer. Thus we have $n\geq 2$. Hence,

$$f(t-1, t-1, t-n-2) <$$

$$\begin{aligned}
&< \frac{\binom{t-1}{n+1} \binom{t+n}{n+1} n^2}{\binom{t-n}{t-n-2} \binom{t+n}{t-n} \{(2n) \cdots (n+1) - 1\} - \binom{t-n+1}{t-n-2} \binom{t+n}{t-n+1} \{(2n-1) \cdots (n+1) - 1\}} \\
&< \frac{\binom{t-1}{n+1} \binom{t+n}{n+1} n^2}{\binom{t-n}{2} \binom{t+n}{2n} (2n) \cdots (n+1) - \binom{t-n+1}{3} \binom{t+n}{2n-1} (2n-1) \cdots (n+1)} \\
&= \frac{3(2n)! n^2}{(2n) \cdots (n+1) (n+1)! (n+1)!} = \frac{3n^2}{(n+1) (n+1)!} < 1,
\end{aligned}$$

a contradiction. Thus we complete the proof.

References

- [1] N. Blackburn and B. Huppert: *Finite groups III*, Springer-Verlag, Berlin/Heidelberg/New-York, 1982.
- [2] R.C. Bose and D.M. Mesner: *On linear associative algebras corresponding to association schemes of partially balanced designs*, *Ann. Math. Statist.* **30** (1959), 21-38.
- [3] P. Delsarte: *An algebraic approach to the association schemes of coding theory*, *Philips Res. Rep. Suppl.* **10** (1973).
- [4] C.C. Sims: *Computational methods in permutation groups*, *Computational Problems in Abstract Algebra*, (Oxford 1967), (J. Leech, Ed.), 169-184, Pergamon Press, Oxford/London/Edinburgh, 1970.

Department of Mathematics
 Josai University
 1-1 Keyakidai, Sakado
 350-02, Japan