Yoshizawa, M.
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ON SOME SHARPLY T-TRANSITIVE SETS

MITSUO YOSHIZAWA

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Let S_k be the symmetric group on a set $\Omega=\{1, 2, \cdots, k\}$ and t be an integer with $t\geqq 2$. A sharply t-transitive set G on Ω is a subset of S_k with the property that for every two ordered t-tuples $\alpha_1, \cdots, \alpha_t$ and $\beta_1, \cdots, \beta_t$ of elements in $\Omega(\alpha_i\neq\alpha_j, \beta_i\neq\beta_j$ for $i\neq j)$ there uniquely exists $g\in G$ which takes α_i into β_i: $(\alpha_i)g=\beta_i(i=1, \cdots, t)$. If $t=k-1$, G is S_k. So from now on we assume $t<k$. Although the sharply t-transitive groups were classified by Jordan and Zassenhaus (cf. [1]), it seems difficult to classify the sharply t-transitive sets. Now we define a distance d in S_k as follows: For two elements g_1 and g_2 in S_k,

$$d(g_1, g_2) = |\{\alpha\in\Omega\colon (\alpha)g_1\neq(\alpha)g_2\}| .$$

Then (S_k, d) is a metric space and we have the following two propositions.

Proposition 1. *Let g be an element in a sharply t-transitive set G on $\Omega(|\Omega|=k)$ and $x_i(0\leqq i\leqq k)$ denote the number of elements $g'\in G$ satisfying $d(g, g')=k-i$. Then the following equality holds for $i=0, 1, \cdots, t-1$:*

$$x_i = \sum_{j=i}^{t-1}\binom{j}{i}\binom{k}{j}\{(k-j)(k-j-1)\cdots(k-t+1)-1\}(-1)^{j+i} .$$

In particular x_i's are uniquely determined independent of the choice of an element g in G.

Proof. Counting in two ways the number of the set $\{(g', (\alpha_1, \cdots, \alpha_i\})\colon g'$ an element $\neq g$, $\{\alpha_1, \cdots, \alpha_i\}\subseteq\Omega$, $\alpha_u\neq\alpha_v$ for $u\neq v$, $(\alpha_j)g=(\alpha_j)g'$ for $j=1, \cdots, i.\}$ gives the following equality for $i=0, 1, \cdots, t-1$:

$$x_i+\binom{i+1}{i}x_{i+1}+\cdots+\binom{t-1}{i}x_{t-1} = \binom{k}{i}\{(k-i)(k-i-1)\cdots(k-t+1)-1\} .$$

Hence we have

$$M\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{t-1}\end{pmatrix} = \begin{pmatrix} \binom{k}{0}\{k(k-1)\cdots(k-t+1)-1\} \\ \binom{k}{1}\{(k-1)(k-2)\cdots(k-t+1)-1\} \\ \vdots \\ \binom{k}{t-1}\{(k-t+1)-1\}\end{pmatrix},$$

where $M=\left(a_{ij}\right)$ is the $t\times t$ matrix with $a_{ij}=\binom{j}{i}(0\leqq i, j\leqq t-1)$. Since the inverse matrix $M^{-1}=\left(b_{ij}\right)$ is expressed by $b_{ij}=\binom{j}{i}(-1)^{i+j}$, we get the result.

Proposition 2. *Let g_1 and g_2 be elements in S_k. Then $d(g_1, g_2)=d(gg_1, gg_2)$ $=d(g_1g, g_2g)$ holds for any element g in S_k.*

We call a sharply t-transitive set G schematic if it forms an association scheme [2] with the relations determined by the distance. That is, if $d(g_1, g_2)=k-h(g_1, g_2\in G)$, then the number $f(i, j, h)=|\{g\in G: d(g, g_1)=k-i, d(g, g_2)=k-j\}|$ does not depend on the choice of g_1 and g_2 (with $d(g_1, g_2)=k-h$), but it depends on $i, j, h(0\leqq i, j, h\leqq k)$. We define $f(i, j, h)=0$ when there exist no elements g_1 and g_2 with $d(g_1, g_2)=k-h$. We find that A_5 (the alternating group of degree five) and any sharply two-transitive set are schematic (cf. [3, Theorem 5.25]). Another example is given by

Proposition 3. *$PSL(2,8)$ is schematic sharply three-transitive set.*

Proof. PSL(2, 8) is a sharply three-transitive group on a set Ω of nine letters. We may assume PSL(2, 8)$=\langle$a, b, c: a$=$(1 2 3 4 5 6 7), b$=$(1 8) (2 4) (3 7) (5 6), c$=$(2 7) (3 6) (4 5) (8 9)$\rangle=G$ with $\Omega=\{1, 2, \cdots, 9\}$ (cf. [4]). Let g and g_1 be elements in G with $d(g, g_1)=9-h$. Let us set $f(i, j; g, g_1)=|\{g'\in G: d(g', g)=9-i, d(g', g_1)=9-j\}|$. We want to show that $f(i, j; g, g_1)$ depends on i, j, h, but it does not depend on the choice of g and g_1. By Proposition 2 we may assume $g_1=e$ (the identity). Since $h=1$ if and only if g is an involution and since all involutions are conjugate to one another, we may assume $h=0$ or 2. By the Sylow's theorem if $h=0$ or 2, then g is conjugate to $u^n(1\leqq n\leqq 8)$ or $a^n(1\leqq n\leqq 6)$ respectively, where $u=a^3bc=(1\ 7\ 2\ 3\ 4\ 6\ 5\ 9\ 8)$. Now $a, a^2, \cdots$, and a^6 are conjugate to one another and u, u^2, u^4, u^5, u^7 and u^8 are also conjugate to one another in the automorphism group PL(2, 8) of PSL(2, 8). Hence we may assume $h=0$, and it is sufficient to show that $f(i, j; u, e)=f(i, j; u^3, e)$ holds for each i and j $(0\leqq i, j\leqq 2)$. But it can easily be found by computer calculations. Really if we set $f(i, j; u^n, e)=u_{ij}$ $(n=1, 3)$, we can get

$$\left(u_{ij}\right)=\begin{pmatrix} 88 & 27 & 108 \\ 27 & 9 & 27 \\ 108 & 27 & 81 \end{pmatrix} \quad (0\leqq i, j\leqq 2)\,.$$

Our main result is

Theorem. *If a sharply t-transitive set G on $\Omega(|\Omega|=k>t\geqq 2)$ is schematic, then $2t-1\leqq k$.*

Proof. First we remark that S_4 (the symmetric group of degree four) is not schematic. Hence the theorem holds for $t=2, 3$. Let us suppose that there exists a schematic sharply t-transitive set G on $\Omega(|\Omega|=k>t)$ with $t\geqq 4$ and $k\leqq 2t-2$. Let us set $n=k-t$. Then by Proposition 1, there exist two elements g_1 and g_2 in G with $d(g_1, g_2)=n+1$. If we set $\Gamma=\{\alpha_1, \cdots, \alpha_{n+1}\}=\{\alpha\in\Omega: (\alpha)g_1 \neq (\alpha)g_2\}$, then we have

$$f(t-1, t-n-2, t-1) = |\{g\in G: d(g, g_1) = n+1, d(g, g_2) = 2n+2\}|$$
$$= |\{g\in G: (\alpha_i)g = (\alpha_i)g_1\ (i = 1, \cdots, n+1),\ |\{\beta\in\Omega-\Gamma: (\beta)g\neq(\beta)g_1\}| = n+1\}|$$
$$= \binom{t-1}{n+1} n.$$

Since $f(t-1, t-1, t-n-2)=f(t-1, t-n-2, t-1)x_{t-1}/x_{t-n-2}$ holds (cf. [2]), we have the following by Proposition 1:

$$f(t-1, t-1, t-n-2)$$
$$= \frac{\binom{t-1}{n+1} n \binom{t+n}{n+1} n}{\sum_{i=t-n-2}^{t-1} \binom{i}{t-n-2}\binom{t+n}{i}\{(t+n-i)(t+n-i-1)\cdots(n+1)-1\}(-1)^{i+t-n+2}}.$$

On the denominator of the above we have

$$\binom{i}{t-n-2}\binom{t+n}{i}\{(t+n-i)(t+n-i-1)\cdots(n+1)-1\}$$
$$>\binom{i+1}{t-n-2}\binom{t+n}{i+1}\{(t+n-i-1)(t+n-i-2)\cdots(n+1)-1\}$$

for $i=t-n-2, t-n-1, \cdots, t-2$, because we have

$$\frac{\binom{i}{t-n-2}\binom{t+n}{i}\{(t+n-i)(t+n-i-1)\cdots(n+1)-1\}}{\binom{i+1}{t-n-2}\binom{t+n}{i+1}\{(t+n-i-1)(t+n-i-2)\cdots(n+1)-1\}}$$
$$>\frac{\binom{i}{t-n-2}\binom{t+n}{i}(t+n-i)}{\binom{i+1}{t-n-2}\binom{t+n}{i+1}} = i+1-t+n+2\geqq 1.$$

If $n=1$, then we have

$$f(t-1, t-1, t-3)<\left\{\binom{t-1}{2}\binom{t+1}{2}\right\}\Big/\left\{\binom{t-1}{t-3}\binom{t+1}{t-1}\right\} = 1,$$

which contradicts that $f(t-1, t-1, t-3)$ is a positive integer. Thus we have $n\geqq 2$. Hence,

$$f(t-1, t-1, t-n-2)<$$

$$< \frac{\binom{t-1}{n+1}\binom{t+n}{n+1}n^2}{\binom{t-n}{t-n-2}\binom{t+n}{t-n}\{(2n)\cdots(n+1)-1\}-\binom{t-n+1}{t-n-2}\binom{t+n}{t-n+1}\{(2n-1)\cdots(n+1)-1\}}$$

$$< \frac{\binom{t-1}{n+1}\binom{t+n}{n+1}n^2}{\binom{t-n}{2}\binom{t+n}{2n}(2n)\cdots(n+1)-\binom{t-n+1}{3}\binom{t+n}{2n-1}(2n-1)\cdots(n+1)}$$

$$= \frac{3(2n)!\,n^2}{(2n)\cdots(n+1)\,(n+1)!\,(n+1)!} = \frac{3n^2}{(n+1)\,(n+1)!} < 1\,,$$

a contradiction. Thus we complete the proof.

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Department of Mathematics
Josai University
1–1 Keyakidai, Sakado
350–02, Japan