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Osaka University
1. Introduction

Let $D = \{ z \in \mathbb{C}; |z| < 1 \}$ be the unit disc in $\mathbb{C}$ and $\partial D = \{ e^{it}; -\pi \leq t \leq \pi \}$ the boundary of $D$. For an integrable function $f$ (In this note a function will always mean a complex valued function) on $\partial D$ with respect to the normalized measure $\frac{1}{2\pi} dt$ on $\partial D$, we define the Poisson integral of $f$ by

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P(z, e^{it}) dt$$

for $z \in D$

where

$$P(re^{i\theta}, e^{it}) = \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2}$$

for $0 \leq r < 1$

and it is called the Poisson kernel of the unit disc $D$. $F$ is a $C^\infty$-function on $D$ and it is harmonic on $D$, that is $\Delta F = 0$ for the Laplace-Beltrami operator $\Delta$ on $C^\infty$-functions on $D$ with respect to the Poincaré metric on $D$.

Then the classical Fatou’s theorem asserts that for an integrable function $f$ on $\partial D$,

$$\lim_{r \to 1} F(re^{i\theta}) = f(e^{i\theta})$$

for almost every point $e^{i\theta}$ of $\partial D$ with respect to the measure $\frac{1}{2\pi} d\theta$.

Now let $G$ be any non-compact connected semi-simple Lie group with finite center, and let $K$ be a maximal compact subgroup of $G$. Then the homogeneous space $G/K$ is a symmetric space of non-compact type. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$ with respect to the Lie algebra $\mathfrak{k}$ of $K$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Fix an order on $\mathfrak{a}$ and let $\mathfrak{a}^+$ be the positive Weyl chamber of $\mathfrak{a}$ with respect to this order. Let $M$ be the centralizer of $\mathfrak{a}$ in $K$. Then the homogeneous space $K/M$ is the maximal boundary of $G/K$ in the sense of Furstenberg [2]. Let $\mu$ be the normalized
K-invariant measure on $K/M$ and $L^p(K/M)$ denote the $L^p$-space on $K/M$ with respect to the measure $\mu$. Let $P(gK, kM)$ be the Poisson kernel on $G/K \times K/M$ given by Korányi [11].

Knapp [7] has proved the following Fatou-type theorem which generalizes the classical Fatou's theorem: Suppose $G/K$ is a symmetric space of non-compact type of rank one. Then for $X \in \alpha^+$ and $f \in L'(K/M)$, it holds

$$\lim_{t \to \infty} \int_{K/M} f(kM)P(k, \exp tX \cdot K, kM)d\mu(kM) = f(k_0M)$$

for almost every point $k_0M$ of $K/M$ with respect to the measure $\mu$.

In the case of an arbitrary symmetric space $G/K$ of non-compact type, for $f \in L^\infty(K/M)$ and $X \in \alpha^+$, Helgason-Korányi [5] has proved a theorem of the same type as above on the boundary behavior of the Poisson integral of $f$.

In the classical Fatou’s theorem, the unit disc $\mathbb{D}$ is a symmetric bounded domain of tube type and the boundary $\mathcal{B}$ is the Bergman-Šilov boundary of $\mathbb{D}$. The purpose of the present paper is to prove for a symmetric bounded domain $\mathcal{D}$ of tube type and the Bergman-Šilov boundary $\mathcal{B}$ of $\mathcal{D}$, the Poisson integral of a function $f \in L'(\mathcal{B})$ converges to $f$ almost everywhere $\mathcal{B}$.

In general, Korányi [11] has defined the notion of the admissibly and unrestrictedly convergence. Knapp and Williamson [8] showed that the Poisson integral of a function $f$ in $L^\infty(K/M)$ converges to $f$ admissibly and unrestrictedly almost everywhere. Moreover, in the case of a Siegel domain in the sense of Pyatetskii-Šapiro [14] which is analytically isomorphic to a symmetric bounded domain $\mathcal{D}$, Stein and Weiss [16], [17], [19], have defined the notion of the restricted and admissible convergence. Let $B$ denote the Šilov boundary in the sense of Pyatetskii-Šapiro [14] of the Siegel domain. Then they showed that the Poisson integral of an integrable function $f$ on $B$ converges to $f$ admissibly and restrictedly almost everywhere on $B$. The generalized Cayley transform of Korányi-Wolf [12] carries the bounded symmetric domain $\mathcal{D}$ onto the Siegel domain and its inverse image of the Šilov boundary $B$ of the Siegel domain is open and dense in the Bergman-Šilov boundary $\mathcal{B}$ of the bounded domain. The inverse Cayley transform carries the $L^p$-space $L^p(B)$ of $B$ into the $L^p$-space $L^p(\mathcal{B})$ on $\mathcal{B}$, but not onto, unless $p = \infty$. Therefore Fatou’s theorem for symmetric bounded domains and that for Siegel domains are not equivalent.

In §2, for a symmetric bounded domain $\mathcal{D}$ we define the notion of the radial convergence of Poisson integrals of functions on the Bergman-Šilov boundary of $\mathcal{D}$ and formulate a Fatou-type theorem. In §3, we give an explicit formula and an estimate of the Poisson kernel of $\mathcal{D}$. In §4, for a symmetric bounded domain of tube type, we define a maximal function and establish an estimate of Poisson integrals by means of this maximal function. In §5, we prove a covering theorem of Vitali-type and a maximal theorem of Knapp-type and give the proof of Fatou’s
2. **Statement of Fatou's theorem**

Let $G$ be a connected semi-simple Lie group with finite center, $K$ a maximal compact subgroup of $G$. We assume that the quotient space $G/K$ is an irreducible hermitian symmetric space. Let $g$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$, respectively, and let $g = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $g$ with respect to $\mathfrak{k}$. Then $K$ has the same rank as $G$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Then $\mathfrak{t}$ is also a Cartan subalgebra of $g$. Let $g^c$, $\mathfrak{t}^c$, $\mathfrak{p}^c$ and $\mathfrak{t}^c$ be the complexifications of $g$, $\mathfrak{t}$, $\mathfrak{p}$ and $\mathfrak{t}$, respectively. Then the set $R$ of roots of $g^c$ with respect to $\mathfrak{t}^c$ can be decomposed into two disjoint sets $C = \{\alpha \in R; E^c_{\alpha} \in \mathfrak{t}^c\}$ and $P = \{\alpha \in R; E^c_\alpha \in \mathfrak{p}^c\}$, where $\{E_\alpha\}$ is a set of root vectors. A root of $C$ or $P$ is called compact or non-compact. Let $\mathfrak{p}^\pm$ be the subspace of $\mathfrak{p}^c$ corresponding to $(\pm \mathfrak{i})$-eigenspace of the complex structure tensor on the tangent space of $G/K$ at the origin $eK$.

We choose and fix an order $\preceq$ on roots in $R$ such that $\mathfrak{p}^+$, $\mathfrak{p}^-$ are spanned by the $E_\alpha$'s, $E_{-\alpha}$'s, respectively, where $\alpha$ runs through positive non-compact roots. Let $\Delta$ be the maximal set of strongly orthogonal non-compact positive roots of Harish-Chandra [4]. We choose root vectors $\{E_\alpha\}$ in such a way that $\tau E_\alpha = -E_{-\alpha}$ for the conjugation $\tau$ of $g^c$ with respect to the compact real from $g^c = \mathfrak{k} + \mathfrak{p}$ of $g^c$. For $\alpha \in R$, let $H_\alpha'$ be the unique element of $\mathfrak{t}$ satisfying $\alpha(H) = \langle H_\alpha', H \rangle$ for all $H \in \mathfrak{t}$, where $\langle , \rangle$ denotes the Killing form of $g^c$. For $\alpha \in \Delta$, we put $X_\alpha = E_\alpha + E_{-\alpha}$, $Y_\alpha = (-\mathfrak{i})(E_\alpha - E_{-\alpha})$ and $H_\alpha = \frac{2}{\langle H_\alpha', H_\alpha \rangle} H_\alpha'$. Let $\mathfrak{g}_\alpha$ denote the subalgebra of $\mathfrak{g}$ spanned by $\{iH_\alpha, X_\alpha^0, Y_\alpha^0\}$. Strong orthogonality of $\Delta$ implies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$ for $\alpha \neq \beta$. Let $\mathfrak{t}^+$ be the subalgebra of $\mathfrak{t}$ spanned by $\{iH_\alpha; \alpha \in \Delta\}$ and let $\mathfrak{t}^-$ be the orthogonal complement of $\mathfrak{t}^+$ in $\mathfrak{t}$ with respect to the Killing form $\langle , \rangle$. The vectors $X_\alpha^0, \alpha \in \Delta$, span a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and $\mathfrak{h} = \mathfrak{t}^+ + \mathfrak{a}$ is a Cartan subalgebra of $g$. Let $\mathfrak{h}^c$ be the complexification of $\mathfrak{h}$. $A$ and $H^c$ denote analytic subgroups of $G$ generated by $\mathfrak{a}$ and $\mathfrak{t}^+$, respectively.

Following Moore [13], we consider the Cayley transform $\tilde{c}$ of $g^c$ defined by $\tilde{c} = Ad \left( \exp \left( \frac{\pi}{4} \sum_{\alpha \in \Delta} (-\mathfrak{i}) Y_\alpha^0 \right) \right)$. Then $\tilde{c}$ transforms $X_\alpha^0 \mapsto -H_\alpha$, $H_\alpha \mapsto X_\alpha^0$ and $Y_\alpha^0 \mapsto Y_\alpha^0$ ($\alpha \in \Delta$) and $\tilde{c}$ leaves $\mathfrak{t}^+$ pointwise fixed. Hence $\tilde{c}$ maps $\mathfrak{t}^-$ onto $\mathfrak{a}$ and $\mathfrak{t}^c$ onto $\mathfrak{h}^c$, so that it maps $R$ onto the set $\Sigma$ of roots of $g^c$ with respect to $\mathfrak{h}^c$. Let $\sigma$ be the conjugation of $g^c$ with respect to $g$. $\sigma$ permutes roots of $\Sigma$ by $\sigma(\alpha)(H) = \overline{\alpha(\sigma(H))}$ for $\alpha \in \Sigma, H \in \mathfrak{h}^c$. 

Theorem for a symmetric domain of tube type. In §6, we prove inequalities of Hardy-Littlewood, making use of the maximal theorem.
We choose a following linear order $<$ on $\Sigma$ and fix it once and for all: (i) If $\alpha \in \Sigma$, $\alpha > 0$ and $\alpha$ does not vanish on $\alpha$, then $\sigma(\alpha) > 0$. (ii) If $\gamma \in \Delta$, then $\varepsilon(\gamma) > 0$. Then $\Sigma$ can be decomposed into three disjoint sets; $\Sigma^+ = \{ \alpha \in \Sigma; \alpha > 0, \sigma(\alpha) > 0 \}$, $\Sigma^- = -\Sigma^+$ and $\Sigma_0 = \{ \alpha \in \Sigma; \alpha = -\sigma(\alpha) \}$. $\sum_{\alpha \in \Sigma^+} CE_{\alpha}$ and $\sum_{\alpha \in \Sigma^-} CE_{\alpha}$ are both invariant under $\sigma$, where $\{ E_{\alpha} \}$ is a set of root vectors of $g^c$ with respect to $\mathfrak{h}$.$^c$. We put $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} CE_{\alpha} \cap g$ and $\mathfrak{h} = \sum_{\alpha \in \Sigma^-} CE_{\alpha} \cap g$, which are real forms of $\sum_{\alpha \in \Sigma^+} CE_{\alpha}$ and $\sum_{\alpha \in \Sigma^-} CE_{\alpha}$, respectively. Then $\mathfrak{n}$ and $\mathfrak{h}$ are nilpotent subalgebras of $g$. We obtain the Iwasawa decompositions $g = \mathfrak{f} + \mathfrak{a} + \mathfrak{n}$ and $G = KAN$, where $A$ and $N$ are analytic subgroups of $G$ generated by $\alpha$, $\mathfrak{n}$. So any $g \in G$ can uniquely decomposed as $g = k(g) \exp H(g) n(g)$, where $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$.

The restriction to $\alpha$ of a root of $\Sigma - \Sigma_0$ is called a restricted root and the order $>$ on $\Sigma$ induces a linear order $>$ on the set of restricted roots. Let $F$ be the fundamental system of restricted roots with respect to the order $>$. Let $X^0 = \sum_{\alpha \in E} X^0_\alpha$, and we put $E = \{ \alpha \in F; \alpha(X^0) = 0 \}$ and $\alpha(E) = \{ H \in \mathfrak{a}; \alpha(H) = 0 \}$ for all $\alpha \in E$. Then $\alpha(E)$ is spanned by $X^0$, and $g$ is the direct sum of eigen-spaces for $ad X^0$ on $g$. The sum of the positive (negative) eigen-spaces of $g$ is denoted by $\mathfrak{n}(E)(\bar{\mathfrak{n}}(E))$. Let $\mathfrak{b}(E)$ be the sum of non-negative eigen-spaces, $I$ the centralizer of $X^0$ in $\mathfrak{f}$, let $2\rho_E$ be the sum of restricted roots $\alpha$ with $\alpha(X^0) > 0$, with multiplicities counted.

The analytic subgroups of $G$ generated by $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$ will be denoted by $N(E)$, $\bar{N}(E)$. Let $L$ be the centralizer of $X^0$ in $K$ and $B(E)$ the normalizer of $\mathfrak{n}(E)$ in $G$. Then $I$, $\mathfrak{b}(E)$ are Lie algebras of $L$, $B(E)$ and we have the decompositions $B(E) = LAN$ and $\mathfrak{b}(E) = I + \mathfrak{a} + \mathfrak{n}$. From the Iwasawa decomposition $G = KAN$, $K/L$ is naturally identified with $G/B(E)$ as $K$-spaces. Let $\Phi$ be the holomorphic imbedding of Harish-Chandra [4] of $G/K$ into $\mathfrak{p}^-$ as a bounded domain in the complex vector space $\mathfrak{p}^-$ and let $\mathcal{D} = \Phi(G/K)$. Then the imbedding $\Phi$ is equivariant with respect to the natural action of $K$ on $G/K$ and the adjoint action of $K$ on $\mathfrak{p}^-$. Let $\mathcal{B}$ be the Bergman-Šilov boundary of the bounded domain $\mathcal{D}$ in $\mathfrak{p}^-$. Then it is known (Korényi-Wolf [12]) that $\sum_{\alpha \in \Sigma} E_{-\alpha} \in \mathcal{B}, K$ acts transitively on $\mathcal{B}$ by the adjoint action and $L$ becomes the isotropy subgroup of $K$ at $\sum_{\alpha \in \Sigma} E_{-\alpha}$. Thus the Bergman-Šilov boundary $\mathcal{B}$ is isomorphic to $K/L$.

Let $\mu_E$ be the normalized $K$-invariant measure on $K/L$ and $L^p(K/L)$ denote the $L^p$-space on $K/L$ with respect to the measure $\mu_E$. Then the Poisson kernel on $G/K \times K/L$ is defined by

$$P_E(gK, kL) = e^{-2p_H(H^{-1} k))} \quad \text{for } g \in G, k \in K$$
where $\exp H(g^{-1}k)$ is the $A$-component of $g^{-1}k$ in the Iwasawa decomposition. We define the Poisson integral of a function $f \in L^1(K/L)$ by

$$\int_{K/L} f(kL) P_E(gK, kL) d\mu_E(kL) \quad \text{for} \quad g \in G.$$  

The hermitian symmetric space $G/K$ of non-compact type is called of tube type if $(\mathfrak{t}, \mathfrak{t})$ is a symmetric pair, then $t^+$ is a Cartan subalgebra of $(\mathfrak{t}, \mathfrak{l})$ and eigenvalues of $ad\left(\frac{1}{2}X_\alpha\right)$ are $0, \pm 1$ (Korányi-Wolf [12]).

Now we can state our main theorem:

**Theorem 1.** Let $G/K$ be an irreducible hermitian symmetric space of tube type. Let $a_t = \exp tX_\alpha$ for a real number $t$. If $f \in L^1(K/L)$, then

$$\lim_{t \to \infty} \int_{K/L} f(kL) P_E(k_\alpha a_t K, kL) d\mu_E(kL) = f(k_\alpha L)$$

for almost every point $k_\alpha L$ of $K/L$ with respect to $\mu_E$.

We assumed the irreducibility of $G/K$ for the simplicity, but the generalization of Theorem 1 of general spaces of tube type is immediate.

### 3. Estimate of Poisson kernel

In this section we assume $G/K$ is an irreducible hermitian symmetric space, not necessarily of tube type.

**Proposition 1.** Let $a = \exp \sum \alpha \in A \ t_\alpha X_\alpha \in A$, $h = \exp \sum \theta_\alpha \frac{iH_\alpha}{2} \in H^-$. Then we have

$$P_E(aK, hL) = \prod_{\alpha \in A} P(\tanh t_\alpha, e^{i\theta_\alpha})^\mathfrak{g}(X_\alpha^0)$$

where $P(t, u)$ is a function on the product of the open interval $(-1, 1)$ and the circle $\mathbb{B} = \{u \in \mathbb{C}; |u| = 1\}$ defined by $P(r, u) = (1 - r^2)/|1 - ru|^2$. (We note that $P(r, u)$ coincides on $(-1, 1)$ with the Poisson kernel of the unit disc in $\mathbb{C}$.)

Proof. To calculate $e^{-2p}(H(a^{-1}h))$, we consider the Iwasawa decomposition of the element $a^{-1}h$ of $G$. We have $Y_\alpha + iH_\alpha \in \mathfrak{t}$ for $\alpha \in \Delta$ because we have $Y_\alpha + iH_\alpha = c(Y_\alpha^0 - iX_\alpha^0) = c((-i)(E_\alpha - E_{-\alpha}) - i(E_\alpha + E_{-\alpha})) = c(-2iE_\alpha) \in C\mathfrak{E}_\alpha$ and from the condition (ii) of the ordering $>_\Sigma$ on $\Sigma$, we obtain $Y_\alpha + iH_\alpha \in \mathfrak{g} \cap \sum_{\alpha \in \Sigma} C\mathfrak{E}_\alpha = \mathfrak{t}$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$ for $\alpha \neq \beta$, $\alpha, \beta \in \Delta$, it follows that

$$a^{-1}h = \prod_{\alpha \in \Delta} \exp(-tX_\alpha) \exp\left(\theta_\alpha \frac{iH_\alpha}{2}\right).$$

If we have the Iwasawa decomposition
$\exp (-t_a X^0_a) \exp \left( \theta_a \frac{iH^0_a}{2} \right) = \exp a_a \frac{iH^0_a}{2} \exp b_a X_a \exp (c_a (Y^0_a+iH_a))$

of each factor, we have

$$a^{-1}h = \exp \left( \sum_{\alpha \in \Delta} a_{\alpha} \frac{iH^0_\alpha}{2} \right) \exp \left( \sum_{\alpha \in \Delta} b_{\alpha} X_\alpha \right) \exp \left( \sum_{\alpha \in \Delta} c_{\alpha} (Y^0_\alpha+iH_\alpha) \right)$$

and thus $H(a^{-1}h) = \sum b_a X^0_a$. Now let

$$SU(1, 1) = \left\{ x \in M_2(\mathbb{C}); \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$X^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Then the Lie algebra $\mathfrak{su}(1, 1)$ of $SU(1, 1)$ is spanned by $X^0$, $iH$ and $Y^0+iH$ and the homomorphism $\phi_a$: $\mathfrak{su}(1, 1) \rightarrow \mathfrak{g}_a$ defined by

$$X^0 \mapsto X^0_a, \quad iH \mapsto iH^0_a, \quad Y^0+iH \mapsto Y^0_a+iH_a$$

can be extended to the homomorphism $\phi_a$: $SU(1, 1) \rightarrow G$. In $SU(1, 1)$ we have the decomposition

$$\exp (-tX^0) \exp \left( \theta \frac{iH^0}{2} \right) = \exp \left( a \frac{iH^0}{2} \right) \exp bX^0 \exp c(Y^0+iH)$$

with $b = \frac{1}{2} \log (\cosh^2 t - 2 \sinh t \cosh \theta + \sinh^2 t) = -\frac{1}{2} \log P(\tanh t, e^{i\theta})$. Applying the homomorphism $\phi_a$ on the both sides, we have

$$b_a = -\frac{1}{2} \log P(\tanh t_a, e^{i\theta_a}).$$

This implies the Proposition. Q.E.D.

Now we define for $0 < \rho < 1$,

$$\mathfrak{H}_\rho = \left\{ \exp \left( \sum_{\alpha \in \Delta} \theta_{\alpha} \frac{iH^0_\alpha}{2} \right) \in H^-; \ |\theta_{\alpha}| < \pi \rho, \text{ for any } \alpha \in \Delta \right\},$$

$$\mathfrak{B}_\rho = \{ lhL \in K/L; l \in L, h \in \mathfrak{H}_\rho \},$$

and for $\rho > 1$,

$$\mathfrak{B}_\rho = \{ lhL \in K/L; l \in L, h \in H^- \}.$$

In §4, we shall calculate the measure of $\mathfrak{B}_\rho$ with respect to $\mu_E$ for a space of tube type. We give an estimate of Poisson kernel on $\mathfrak{B}_\rho$ in the following.

**Proposition 2.** Let $a = \exp \sum_{\alpha \in \Delta} t_a X^0_a \in A$. Then we obtain an estimate of Poisson kernel as follows:
RADIAL CONVERGENCE OF POISSON INTEGRALS

(i) If $0 < \rho < 1$ and $\frac{1}{2} < \tanh t < 1$ for any $\alpha \in \Delta$, then

$$\sup_{h \in H^- \backslash G_0} P_E(aK, hL) \leq C_1 \prod_{\alpha \in \Delta} \left( \frac{1-\tanh t_\alpha}{\rho^2} \right)^{\rho \beta(\alpha, 0)}$$

(ii) $\sup_{h \in H^-} P_E(aK, hL) \leq C_1 \prod_{\alpha \in \Delta} \left( \frac{1}{1-\tanh t_\alpha} \right)^{\rho \beta(\alpha, 0)}$

where $C_1, C_2$ are constants independent on $a$ and $\rho$.

In particular, if $a_t = \exp tX^0$, then

(i) If $0 < \rho < 1$ and $\frac{1}{2} < \tanh t < 1$, then

$$\sup_{h \in H^- \backslash G_0} P_E(a_tK, hL) \leq C_1 \left( \frac{1-\tanh t}{\rho^2} \right)^{\rho \beta(\alpha, 0)}$$

(ii) $\sup_{h \in H^-} P_E(a_tK, hL) \leq C_2 \left( \frac{1}{1-\tanh t} \right)^{\rho \beta(\alpha, 0)}$

(We note that $\beta_1$ is equal to $\beta/\lambda$ if $G/K$ is of tube type).

Proof. We have (Korányi [10]) an estimate of the Poisson kernel for the unit disc in $C$ as follows:

(i) $\sup_{r \leq 1 \leq \rho \leq 1} |1-r \tau| |1-re^{-i\theta}|^{-2} \leq C_1 \frac{1-r}{\rho^2}$ if $\frac{1}{2} < r < 1$.

(ii) $\sup_{0 \leq 1 \leq \rho \leq 1} |1-r \tau| |1-re^{-i\theta}|^{-2} \leq C_2 \frac{1}{1-r}$ if $0 < r < 1$.

where $C_1', C_2'$ are constants. This together with Proposition 1 implies the first statement. If $a_t = \exp tX^0$, then we have $P_E(a_tK, hL) = P_E(a_tK, hL)$ for $h \in H^-$ and $l \in L$ since $L$ centralizes $X^0$ in $K$. This together with the first statement implies the second statement. Q.E.D.

4. Maximal function

Henceforth we shall assume that $G/K$ is an irreducible hermitian symmetric space of tube type. We consider the Poisson integral

$$\int_{K/L} f(kL)P_E(a_tK, kL)d\mu_E(kL)$$

for $a_t = \exp tX^0$ and an integrable function $f$ on $K/L$ with respect to $\mu_E$.

Since $K/L$ is a symmetric space, we may use the following integral formula for $K/L$ (Harish-Chandra [4]): For each continuous function $f$ on $K/L$, we have
\[ \int_{k/L} f(kL) d\mu(kL) = c \int_{H^-} \left( \int_{L/Z(t^-)} f(lhL) dl \right) |D(h)| dh \]

where \( c \) is a constant independent of \( f \), \( Z_L(t^-) \) is the centralizer of \( t^- \) in \( L \), \( dh \) is a Haar measure on \( H^- \) and \( dl \) is a quotient measure on \( L/Z_L(t^-) \) induced from the normalized Haar measure \( dl \) on \( L \). Moreover

\[ D(h) = \prod_{\beta \in P_+^*} \sin \beta(iH) \quad \text{for} \quad h = \exp H, H \in t^- \]

where \( P_+^* = \{ \alpha \in C; \text{positive and } \alpha \mid t^- \neq 0 \} \).

Making use of this integral formula, we have the measure \( ||33\) of \( 33 \) with respect to \( \mu_E \) as follows:

\[ ||33|| = \int_{k/L} \chi_{33}(kL) d\mu_E(kL) = c \int_{H^-} \left( \int_{L/Z_L(t^-)} \chi_{33}(lhL) dl \right) |D(h)| dh \]

where \( \chi_{33} \) is the characteristic function of \( 33 \). The density \( D(h) \) of the integral is given as follows: Let \( \Delta = \{ \gamma_1, \cdots, \gamma_m \}, \gamma_1 - \gamma_2, \cdots, \gamma_1 - \gamma_m \), where \( m = \text{rank of } G/K \). For \( \alpha \in R \), let \( \pi(\alpha) \) be the restriction of \( \alpha \) to the complexification \( (t^-)^c \) of \( t^- \), but \( \pi(\gamma_i) \) will be denoted by \( \gamma_i \) for the brevity, since any root \( \beta \neq \gamma_i \) does not coincide with \( \pi(\gamma_i) \) on \( (t^-)^c \). Since \( G/K \) is of tube type, we have (Harish-Chandra [4], Korányi-Wolf [12]) for a positive compact root \( \beta \),

\[ \pi(\beta) = \begin{cases} 0 & \text{or} \\ \frac{1}{2} (\gamma_j - \gamma_i) & (i < j) \end{cases} \]

and for a positive non-compact root \( \beta \),

\[ \pi(\beta) = \begin{cases} \gamma_i & \text{or} \\ \frac{1}{2} (\gamma_j + \gamma_i) & (i < j) \end{cases} \]

Moreover the number \( r_{ij} \) \((i < j)\) of elements of \( \{ \beta \in P_+^*; \pi(\beta) = \frac{1}{2} (\gamma_j - \gamma_i) \} \) is the same as the number of positive non-compact roots \( \beta \) such that \( \pi(\beta) = \frac{1}{2} (\gamma_j + \gamma_i) \). It follows that

\[ D\left( \exp \sum \theta_i \frac{iH_i}{2} \right) = \prod_{1 \leq i < j \leq m} \left( \sin \frac{1}{2} (\theta_i - \theta_j) \right)^{r_{ij}}. \]

Now we obtain the following

**Lemma 1.** For \( 0 < \rho < 1 \), we have an estimate of the measure of \( 33 \):
where $C$ is a constant independent on $\rho$. For $\rho \geq 1$, we have $\|\mathfrak{g}_\rho\| = 1$ (from the definition of $\mathfrak{g}_\rho$).

Proof. From the above argument,

$$\|\mathfrak{g}_\rho\| = \int_{T^p} |D(h)| \, dh = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{i<j} |\sin \frac{1}{2} (\theta_i - \theta_j)|^{r_{ij}} \, d\theta_1 \cdots d\theta_m$$

$$\leq C(\pi^p) \sum_{i\leq j} r_{ij} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \, d\theta_1 \cdots d\theta_m \leq C\rho^{m+\sum_{i\leq j} r_{ij}/2}$$

because $|\sin \frac{1}{2} (\theta_i - \theta_j)| \leq \frac{1}{2} |\theta_i - \theta_j| \leq \pi\rho$.

On the other hand, $X^0 = \sum_{i=1}^m X^0_{ih}$ and

$$\rho_X(X^0) = (\tilde{e}^{-1} \rho_X)(\tilde{e}^{-1} X^0) = \frac{1}{2} \left( m + \sum_{i<j} r_{ij} (\gamma_j + \gamma_i) \right) (H_{\gamma_k})$$

$$= 1 + \sum_{i<j} r_{ij}.$$  

Hence $\rho_X(X^0) = m + \sum_{i<j} r_{ij}$, then the result follows. Q.E.D.

Definition. For an integrable function $f$ on $K/L$, we define a maximal function $f^*$ on $K/L$ by

$$f^*(k_0 L) = \sup_{0<\rho<1} \frac{1}{\|\mathfrak{g}_\rho\|} \int_{\mathfrak{g}_\rho} |f(k_0 kL)| \, d\mu_E(kL) \quad \text{for} \quad k_0 L \in K/L.$$  

The function $f^*$ on $K/L$ is measurable because the supremum over rational $\rho (0 < \rho < 1)$ gives the same answer.

Proposition 3. For an integrable function $f$ on $K/L$, we have an estimate of Poisson integral by means of the above maximal function:

$$\sup_{1<\tanh t<1} \int_{K/L} |f(kL)| P_E(k_0 a_k K, kL) \, d\mu_E(kL) \leq C' f^*(k_0 L)$$

for all $k_0 \in K$, where $a_k = \exp tX^0$ and $C'$ is a constant not depending on $f$ and $k_0 L$.

Proof. We fix first an arbitrary constant $\alpha > 0$ put $\delta = (1 - \tanh t)\alpha$ for $\frac{1}{2} < \tanh t < 1$. We may suppose $k_0 = e$ in view of the $K$-invariance of the measure $\mu_E$, replacing $f$ by the function $f^k_0$ defined by $f^k_0(kL) = f(k_0 kL)$. Then for $\frac{1}{2} < \tanh t < 1$, we have
Here we note that the summation of the second term in (5) is in fact finite sum because \( B_{2/\delta} = K/L \) for \( 2/\delta \geq 1 \).

The right hand side of (5) can be estimated as follows:

the first term \( \leq C_2 \left\{ \frac{1}{1 - \tanh t} \right\}^{p_{H^\alpha} (\cdot, 0)} \int_{B_{\delta}} |f(kL)| \ d\mu_E(kL) \) (by (2))

\[ \leq C_2 \left\{ \frac{1}{1 - \tanh t} \right\}^{p_{H^\alpha} (\cdot, 0)} ||B_{\delta}|| f^*(eL) \) (by the definition of \( f^* \))

\[ \leq C_2 C \left\{ \frac{1}{1 - \tanh t} \right\}^{p_{H^\alpha} (\cdot, 0)} \left( 1 + \frac{1}{2} \right)^{p_{H^\alpha} (\cdot, 0)} f^*(eL) \) (by (4))

\[ = C_2 C \alpha^{p_{H^\alpha} (\cdot, 0)} f^*(eL). \] (6)

the second term \( \leq \sum_{j=0}^{\infty} C_1 \left\{ \frac{1}{2 \delta} \right\}^{p_{H^\alpha} (\cdot, 0)} \int_{B_{2^{-1} \delta} \setminus B_{2^{-1} \delta}} |f(kL)| \ d\mu_E(kL) \) (by (1))

\[ \leq C_1 \sum_{j=0}^{\infty} \left\{ \frac{1}{2 \delta} \right\}^{p_{H^\alpha} (\cdot, 0)} ||B_{2^{-1} \delta}|| f^*(eL) \) (by the definition of \( f^* \))

\[ \leq C_1 C \sum_{j=0}^{\infty} \left\{ \frac{1}{2 \delta} \right\}^{p_{H^\alpha} (\cdot, 0)} \left( 2^{j+1} \delta \right)^{p_{H^\alpha} (\cdot, 0)} f^*(eL) \) (by (4))

\[ = C_1 C \left( \frac{2}{\alpha} \right)^{p_{H^\alpha} (\cdot, 0)} \left( \sum_{j=0}^{\infty} \left\{ \frac{1}{2 \delta} \right\}^{p_{H^\alpha} (\cdot, 0)} \right)^j f^*(eL) \] (7)

where the sum \( \sum_{j=0}^{\infty} \left\{ \frac{1}{2 \delta} \right\}^{p_{H^\alpha} (\cdot, 0)} \) converges to \( \frac{1}{1 - (1/2)^{p_{H^\alpha} (\cdot, 0)}} \).

Hence putting together (6) and (7) into (5), we obtain the inequality:

\[ \sup_{1/\delta < \tanh t < 1} \int_{K/L} |f(kL)| P_E(a_t K, kL) d\mu_E(kL) \leq \left( C_2 C \alpha^{p_{H^\alpha} (\cdot, 0)} + C_1 C \left( \frac{2}{\alpha} \right)^{p_{H^\alpha} (\cdot, 0)} \frac{1}{1 - (1/2)^{p_{H^\alpha} (\cdot, 0)}} \right) f^*(eL) \] Q.E.D.

5. Covering theorem and proof of Fatou's theorem

In this section we shall prove a covering theorem of Vitali type with respect to the family of sets of the form \( kB_{\rho}, \ 0 < \rho < 1, \ k \in K \) and prove a maximal theorem related to the maximal function \( f^* \) on \( K/L \).

Let \( q \) be the orthogonal complement of \( l \) in \( l \) with respect to \( \langle \cdot, \cdot \rangle \). Then \( q = Ad(L)l^- \) since \( K/L \) is a symmetric space. We define a map \( \psi: q \to p \) by
ψ(X) = \frac{1}{2} [X^0, X] for X ∈ q and putting p* = ψ(q), define a map j: p* → \bar{\Omega}(E) by
j(X) = X - \frac{1}{2} [X^0, X] for X ∈ p*. Then both ψ and j are L-equivariant isomorphisms (Takeuchi [18]). We have ψ(iHα) = Y_α^0 and j(Y_α^0) = Y_α^0 - iH_α for any α ∈ Δ so that j(ψ') is the subspace of \bar{\Omega}(E) spanned by \{Y_α^0 - iH_α; α ∈ Δ\}. Thus we have the following

**Lemma 2.** \( Ad(L)\{Y_α^0 - iH_α; α ∈ Δ\}_R = \bar{\Omega}(E) \)
where \( \{Y_α^0 - iH_α; α ∈ Δ\}_R \) is the subspace of \( \bar{\Omega}(E) \) spanned by \( \{Y_α^0 - iH_α; α ∈ Δ\} \).

Now we define an L-invariant norm \( || \cdot || \) on \( \bar{\Omega}(E) \) as follows. We define a K-invariant inner product on \( g \) by

\[(X, Y) = -\langle X, τ Y \rangle \quad \text{for} \quad X, Y ∈ g.\]

For \( Z ∈ \bar{\Omega}(E) \), let \( |Z| \) denote the operator norm of \( ad(j^{-1}Z) \) with respect to \( (\cdot, \cdot) \) and let \( ||Z|| = \frac{1}{2} |Z| \). Then (Takeuchi [18]) \( || \cdot || \) is a L-invariant norm on \( \bar{\Omega}(E) \) satisfying

\[ ||Z|| = \max_{α ∈ Δ} |a_α| \quad \text{for} \quad Z = \sum_{α ∈ Δ} a_α(Y_α^0 - iH_α). \]

For each \( δ > 0 \), let

\[ B_δ = \{ Z ∈ \bar{\Omega}(E); ||Z|| < δ \}, \]
\[ \bar{B}_δ = \{ k(\bar{n})L ∈ K/L; \bar{n} = \exp Z, Z ∈ B_δ \}. \]

where \( k(\bar{n}) \) is the K-component of \( \bar{n} \) in the Iwasawa decomposition.

**Lemma 3.** For \( 0 < ρ < 1 \), we have

\[ \mathcal{B}_ρ = \left\{ k(\bar{n})L ∈ K/L; \bar{n} = \exp Ad(l)\left( \sum_{α ∈ Δ} a_α(Y_α^0 - iH_α) \right), l ∈ L, \max_{α ∈ Δ} |a_α| < \frac{1}{2} \tan((π/2)ρ) \right\} \]

and therefore

\[ \mathcal{B}_ρ = \bar{B}_{1/2 \tan((π/2)ρ)}. \]

**Proof.** Recall the definition of \( \mathcal{B}_ρ \) for \( 0 < ρ < 1 \):

\[ \mathcal{B}_ρ = \left\{ lhL ∈ K/L; l ∈ L, h = \exp \left( \sum_{α ∈ Δ} \frac{iH_α}{2} \right), |θ_α| < πρ \right\}. \]

As in the proof of Proposition 1, we have

\[ \exp \left( \sum_{α ∈ Δ} \frac{i}{2} H_α \right) = k\left( \exp \left( -\frac{1}{2} \sum_{α ∈ Δ} \tan \left( \frac{1}{2}\theta_α \right)(Y_α^0 - iH_α) \right) \right) \quad \text{for} \quad |θ_α| < π. \]
Since $l n l^{-1} B(E) = l k(\bar{n}) B(E)$ for $l \in L$, $\bar{n} \in \bar{N}(E)$ and $G|B(E) \equiv g B(E) \mapsto k(g)L \in K/L$ is a bijection, we have $k(l n l^{-1}) L = l k(\bar{n}) L$. Then the statement follows.

Q.E.D.

The purpose of this section is to prove the following covering theorem;

**Theorem 2.** There is some constant $C'' > 0$ with the following property. If $U$ is any Borel set in $K/L$, and if to each point $kL$ in $U$ there is associated a set $kB_p$ (with $0 < p < 1$ depending on $k \in K$), then there is a countable disjoint subfamily of $\{kB_p\}$, say $k_j B_j$, such that

$$ C'' \sum_{j=1}^\infty \mu_B(k_j B_j) \geq \mu_B(U). $$

In view of Lemma 3, we may prove the following theorem in place of Theorem 2.

**Theorem 2'.** There is some constant $C'' > 0$ with the following property. If $U$ is any Borel set in $K/L$, and if to each point $kL$ in $U$ there is associated a set $kB_3$ (with $\delta > 0$ depending on $k \in K$), then there is a countable disjoint subfamily of $\{kB_3\}$, say $k_j B_j$, such that

$$ C'' \sum_{j=1}^\infty \mu_B(k_j B_j) \geq \mu_B(U). $$

The proof will proceed in the same way as Knapp's proof [7] of the covering theorem on Furstenberg's boundary $K/M$ of a symmetric space of rank one.

Any $n \subseteq \bar{N}(E)$ can be written uniquely in the form $n = \exp Z$, $Z \in \bar{H}(E)$. We write as $Z = \log n$. Then we define

$$ |n| = \| \log n \|. $$

We have $|\tilde{n} \exp t X^0/2| = e^{-t - e} |\tilde{n}|$ for $\tilde{n} \exp t X^0/2 = \left( \exp \frac{t X^0}{2} \right) \tilde{n} \exp \left( -\frac{t X^0}{2} \right)$ since $\bar{n}(E)$ is $(-1)$-eigenspace of $ad \frac{1}{2} X^0$.

**Lemma 4.** There exists a constant $C_3$ such that

$$ |n n'| \leq C_3 (|n| + |n'|) $$

for all $n, n' \subseteq \bar{N}(E)$.

Proof. The proof is quite same as that of Lemma 2.3 in Korányi [11]. Let $V_r = \{ n \subseteq \bar{N}(E); |n| \leq e^r \}$ for $t \in R$. The sets $V_r$ are compact and converge to $\bar{N}(E)$ as $t \to \infty$. Then there exists $r > 0$ such that $V_o \cdot V_r \subseteq V_r$. We put $C_3 = e^r$. By the above remark $V_r = V_o \exp(-t X^0/2)$. For $\tilde{n}, \tilde{n}' \subseteq \bar{N}(E)$ we write $|\tilde{n}| = e^r$, $|\tilde{n}'| = e^{r'}$, and let $\tau = \max \{ t, t' \}$. Then $\tilde{n} \tilde{n}' \subseteq V_r$, $V_{r+t} \subseteq VV = (V_o \cdot V_o \exp(-t X^0/2) \subseteq V_{r+t}$ and so $|\tilde{n} \tilde{n}'| \leq e^{r+t} \leq e^r (|\tilde{n}| + |\tilde{n}'|)$. Q.E.D.
Lemma 5. By $\tilde{N}(E)$-hull of $\exp(B_δ)$, we mean the union of all $\tilde{N}(E)$-translates of $\exp(B_δ)$ which have non-empty intersection with $\exp(B_δ)$. Then there is a constant $C_4$ such that for each $δ > 0$,

$$\tilde{N}(E)$$-hull of $\exp(B_δ) \subset \exp(B_{C_4δ})$.

Proof. Let $\tilde{n} \exp(B_δ) \cap \exp(B_δ) \neq \phi$ for $\tilde{n} \in \tilde{N}(E)$ and $\tilde{n} \tilde{n}_1 = \tilde{n}_2$ for $\tilde{n}_1, \tilde{n}_2 \in \exp(B_δ)$. Then $|\tilde{n}| = |\tilde{n}_1\tilde{n}_1^{-1}| \leq C_4(|\tilde{n}_1| + |\tilde{n}_2|) \leq 2C_4δ$ by Lemma 4. Hence for each $\tilde{n} \in \exp(B_δ)$, we have

$$|\tilde{n} \tilde{n}_1| \leq C_4(|\tilde{n}| + |\tilde{n}_2|) \leq C_4 (2C_4δ + δ) = (2C_4^2 + C_4)δ.$$

Therefore $C_4 = 2C_4^2 + C_4$ is a desired constant. Q.E.D.

The mapping $γ$ of $G$ onto $K/L$ which sends $g$ into $\gamma(g)L$ is an injective real analytic mapping of $\tilde{N}(E)$ onto a dense open subset of $K/L$. By the continuity of the action of $K$ on $K/L$, there exist open subsets $U \subset K$, $V \subset K/L$ with $e \in U$, $eL \in V$ such that $U\tilde{V} \subset \gamma(\tilde{N}(E)) \subset K/L$. We put $V = \gamma^{-1}(V) \subset \tilde{N}(E)$. The function $γ^{-1}$ is defined at each point of $\tilde{V}$ since $\tilde{V} = e\tilde{V} \subset \gamma(\tilde{N}(E))$. For $g \in G$ and $\tilde{n} \in \tilde{N}(E)$, we put

$$g \cdot \tilde{n} = γ^{-1}(g \cdot γ(\tilde{n}))$$

if the right hand side is defined. If $k \in U$ and $\tilde{n} \in V$, then $k \cdot γ(\tilde{n}) \in U\tilde{V}$ and $k \cdot \tilde{n} = γ^{-1}(k \cdot γ(\tilde{n}))$ is defined. We put $\tilde{n}(k) = γ^{-1}(kL)$ for $k \in U$. We consider the mapping $U \times V \rightarrow \tilde{N}(E)$ defined by

$$(k, \tilde{n}) \mapsto \tilde{n}(k)^{-1}(k \cdot \tilde{n}) \quad \text{for} \quad k \in U, \tilde{n} \in V.$$ (11)

Then we obtain the following Lemma, which, together with Lemma 5, is essential for proof of the covering theorem.

Lemma 6. There exist a neighborhood $W_1$ of $e$ in $\tilde{N}(E)$, a neighborhood $W_2$ of $e$ in $K$ and a constant $C_5 > 0$ such that if $k \in W_2$ and $\exp(B_δ) \subset W_2$, then $\tilde{n}(k)^{-1}(k \cdot \exp(B_δ)) \subset B_{C_4δ}$.

Proof. Let $ν$ be the dimension of $K$ and $d$ the dimension of $\tilde{N}(E)$. We fix any basis $\{X_i\}$ of $\tilde{n}(E)$ and define coordinates of $\tilde{N}(E)$ by

$$\exp(\sum_{i=1}^{d} x_i X_i) \mapsto (x_1, \ldots, x_d).$$

Restrict the coordinates to the open set $V \subset \tilde{N}(E)$ and choose an open coordinate neighborhood $U \subset U$ of $e$ in $K$ with local coordinates $(k_1, \ldots, k_d)$, $(k_j(e), \ldots, k_j(e)) = (0, \ldots, 0)$. We will describe the mapping (11) by these coordinates $x_i, k_j$. We choose neighborhoods $W_1, W_2$ such that $W_1 \subset V \cap \exp(B_δ)$, $W_2 \subset U$, $W_2$ has compact closure and these power series of coordinates of
\[ \tilde{n}(k) = (k \cdot \tilde{n}) \] converge in an open neighborhood of the closure of \( W_1 \times W_2 \). We can rearrange the terms of these power series to write the \( l \)-th coordinate of \( \tilde{n}(k)^{-1}(k \cdot \tilde{n}) \) as

\[
a_l(k) + \sum_{i=1}^{d} a_{i1}(k)x_i + \sum_{i,j} a_{ij}(\tilde{n}, k)x_ix_j, \quad l = 1, \ldots, d
\]

where \( a_l(k), a_{i1}(k) \) and \( a_{ij}(\tilde{n}, k) \) are real analytic functions of \( \tilde{n} \in W_1 \) and \( k \in W_2 \).

The terms \( a_l(k) \) vanish on \( W_2 < K \) since \( \tilde{n}(k)^{-1}(e \cdot k) = e \). There exist \( C_6, C_7 > 0 \) such that for each \( l, i, j \), \( (a_{ij}(\tilde{n}, k)) \leq C_6 \) on the compact closure of \( W_1 \times W_2 \) and \( \max |x_i| \leq C_7 \|X\| \) for \( X = \sum_{i=1}^{d} x_i X_i \in \tilde{n}(E) \). Then \( \sum_{i,j} a_{ij}(\tilde{n}, k)x_ix_j \leq C_6 C_7^2 \| \log \tilde{n} \| \) on the closure of \( W_1 \times W_2 \). Hence we obtain

\[
\tilde{n}(k)^{-1}(k \cdot \tilde{n}) = \exp \left( \sum a_{i1}(k)x_i + Z \right)
\]

where \( \|Z\| \leq C_6 C_7^2 \| \log \tilde{n} \| \) for \( \tilde{n} \in W_1 \) and \( k \in W_2 \).

For fixed \( k \in W_2 \), the matrix \( (a_{i1}(k)) \) is the Jacobian matrix of the transformation

\[
\tilde{n} \mapsto \tilde{n}(k)^{-1}(k \cdot \tilde{n}) \quad \text{for} \quad \tilde{n} \in \tilde{N}(E).
\] (12)

Since \( k \in W_2 \subset U \subset U \) and \( (U)(eL) \subset \gamma(\tilde{N}(E)) \), we can write \( \tilde{n}(k) = \gamma^{-1}(k \cdot L) = k \) by uniquely determined \( b \in B(E) \) because the restriction of \( \gamma \) to \( \tilde{N}(E) \) is an injection.

Then the mapping (12) is the same as the mapping

\[
\tilde{n} \mapsto b^{-1} \cdot \tilde{n} \quad \text{for} \quad \tilde{n} \in \tilde{N}(E).
\] (13)

In fact, \( \gamma^{-1} \) is defined on \( k \tilde{n}B(E) \) for \( k \in W_2 \) and \( \tilde{n} \in W_1 \) and we have \( b^{-1} \cdot b \tilde{n}B(E) = b^{-1} k^{-1} \gamma^{-1}(k \tilde{n}B(E))B(E) = \gamma(\tilde{n}(k)^{-1} \gamma^{-1}(k \tilde{n}B(E))) = \gamma(\tilde{n}(k)^{-1} \gamma^{-1}(k \cdot \gamma(\tilde{n}))) = \gamma(\tilde{n}(k)^{-1}(k \cdot \tilde{n})) \).

The differential of the mapping (13) at \( e \in \tilde{N}(E) \) is given by

\[
X \mapsto P_{\tilde{n}(E)} Ad(b^{-1})X \quad \text{for} \quad x \in \tilde{n}(E)
\]

where \( P_{\tilde{n}(E)} \) is the projection of \( g \) onto \( \tilde{n}(E) \) along the decomposition \( g = \tilde{n}(E) + b(E) \), since the mapping (13) is the composite of the conjugation of \( b^{-1} \), the quotient map \( G \to G/B(E) \) and the map \( \gamma^{-1} \).

Now we consider the operator \( P_{\tilde{n}(E)} Ad(b^{-1}) \). The restriction of \( P_{\tilde{n}(E)} Ad(b^{-1}) \) to \( \tilde{n}(E) \) is a bounded operator on \( \tilde{n}(E) \) with respect to the norm \( \| \| \). Let \( \| P_{\tilde{n}(E)} Ad(b^{-1}) \|_{\tilde{n}(E)} \) be the operator norm of \( P_{\tilde{n}(E)} Ad(b^{-1}) \) on \( \tilde{n}(E) \). Then since the closure \( \tilde{W}_2 \) of \( W_2 \) is compact, \( C_8 = \sup_{k \in \tilde{W}_2} \| P_{\tilde{n}(E)} Ad(b^{-1}) \|_{\tilde{n}(E)} \) is finite and we have \( \| P_{\tilde{n}(E)} Ad(b^{-1})X \| \leq C_8 \|X\| \) for all \( X \in \tilde{n}(E) \) and \( k \in W_2 \).

Consequently we have for \( \tilde{n} \in W_1 \) and \( k \in W_2 \).
\[ ||\log (\bar{n}(k)^{-1}(k \cdot \bar{n}))|| = ||a_{4N}(k)x_i + Z|| \leq (C_5 C_7 + C_6)||\log \bar{n}||.\]

Therefore we conclude
\[
\bar{n}(k)^{-1}(k \cdot \exp(B_8)) \subseteq B_{C_5}, \quad C_8 = C_5 C_7 + C_6
\]
for \(\exp(B_4) \subseteq W_1\) and \(k \in W_2\).

By \(K\)-hull of \(\bar{B}_8\), we mean the union of all \(K\)-translates of \(\bar{B}_8\) which have non-empty intersection with \(\bar{B}_8\).

**Proposition 4.** \(\sup_{\delta < \delta_0} \frac{\mu_E(K\text{-hull of } \bar{B}_8)}{\mu_E(\bar{B}_8)} < \infty\)

Proof. Let \(W_1\) and \(W_2\) be neighborhoods as in Lemma 6. Let \(k \in W_2\), \(k \cdot \exp(B_8) \cap \exp(B_3) = \phi\) and \(\exp(B_3) \subseteq W_1\). Then \(\bar{n}(k) \exp(B_{C_5}) \cap \exp(B_{C_5}) \subseteq \bar{n}(k)[\bar{n}(k)^{-1}(k \cdot \exp(B_8))] \cap \exp(B_3) = k \cdot \exp(B_3) \cap \exp(B_3) = \phi\). Lemma 5 shows that \(k \cdot \exp(B_8) = \bar{n}(k)[\bar{n}(k)^{-1}(k \cdot \exp(B_8))] \subseteq \bar{n}(k) \exp(B_{C_5}) \cap \exp(B_{C_5})\). Hence we have \(k \bar{B}_8 \subseteq B_{C_5}\) with \(C_8 = C_5 C_7\).

There exists a number \(\delta_0 > 0\) such that \(\exp(\delta_0)\) is included in \(W_1\) for any \(\delta < \delta_0\). We may prove that
\[
\sup_{\delta < \delta_0} \frac{\mu_E(K\text{-hull of } \bar{B}_8)}{\mu_E(\bar{B}_8)} < \infty
\]
(14) since \(\mu_E(K/L) = 1\).

Now we assume that (14) is false. Then there exist a sequence \(0 < \delta_n < \delta_0\) and \(k_n \in K\) such that \(k_n \bar{B}_{\delta_n} \cap \bar{B}_{\delta_n} = \phi\) and \(k_n \bar{B}_{\delta_n} \subseteq \bar{B}_{C_5}\delta_n\). Since there exists a constant \(C_10\) such that \(\frac{\mu_E(\bar{B}_{\delta_n})}{\mu_E(\bar{B}_{\delta})} \leq C_10\) for each \(\delta < \delta_0\). Moreover we may assume \(\delta_n \rightarrow 0\) as \(n \rightarrow \infty\) since \(\mu_E(K/L) = 1\). Let \(\sigma\) be the quotient mapping of \(K\) onto \(K/L\). Since \(k \bar{B}_8 = k \bar{B}_8\) for \(l \in L\) and \(k \in K\), it follows from the first argument that if \(k \in \sigma^{-1}(\sigma(W_3))\), \(\delta < \delta_0\) and \(k \bar{B}_{\delta_n} \cap \bar{B}_{\delta_n} = \phi\), then \(k \bar{B}_8 \subseteq B_{C_5}\). Therefore \(\sigma(k_n)\) is not in the neighborhood \(\sigma(W_3)\) of \(eL\). We may suppose \(k_n\) converges to some point \(k_0 \in K\) with \(\sigma(k_0) = eL\) since \(K\) is compact. If \(p_n \in k_n \bar{B}_{\delta_n} \cap \bar{B}_{\delta_n}\), \(p_n\) converges to \(eL\) since \(\bar{B}_{\delta_n}\) shrinks to \(eL\) as \(n \rightarrow \infty\). But \(p_n = k_n q_n\) with \(q_n \in \bar{B}_{\delta_n}\), \(q_n \rightarrow eL\) as \(n \rightarrow \infty\). Therefore we obtain \(eL = k_0 eL\) or \(\sigma(k_0) = eL\), a contradiction.

Q.E.D.

**Proof of Theorem 2'.** We put
\[
C'' = \sup_{-\infty < \delta < \infty} \frac{\mu_E(K\text{-hull of } \bar{B}_{\delta})}{\mu_E(\bar{B}_{\delta-1})}.
\]

Then Proposition 4 implies that \(1 < C'' < \infty\). Let \(T_1 = \sup \{t_k; k \bar{B}_{\delta}\text{'s is associated to }kL \subseteq U\}\). If \(T_1 = + \infty\), then we can find a set \(k \bar{B}_{\delta}\text{'s with measure as close to }1\) as we like, and the conclusion of the theorem follows since \(1 < C'' < \infty\). We
assume from now on that $T_i < \infty$. We construct $R_n$, $T_n$ and $k_n\overline{B}_{\varepsilon^*}$ in the following process: Let $R_1$ be the family \{\(k\overline{B}_{\varepsilon^*}\)\} of all associated sets. Taking a set $k_1\overline{B}_{\varepsilon^*} \in R_1$ with $T_1-1 < t_1 < T_1$, we put $R_2 = \{k\overline{B}_{\varepsilon^*} \in R_1; k\overline{B}_{\varepsilon^*} \cap k_1\overline{B}_{\varepsilon^*} = \phi\}$. If $R_2 = \phi$, then our process is over. If $R_2 \neq \phi$, then we put $T_2 = \sup \{t_2; k\overline{B}_{\varepsilon^*} \in R_2; k\overline{B}_{\varepsilon^*} \cap k_2\overline{B}_{\varepsilon^*} = \phi\}$ and our process is continued inductively.

If $V_n$ is the union of the members of $R_n - R_{n+1}$ and $V_0$ is the union of the members of $R_1$, then $V_0 = \bigcup_{n=1}^{\infty} V_n$. Since $U \subset V_0$, we obtain $\mu_E(U) \leq \sum_{n=1}^{\infty} \mu_E(V_n)$.

The proof will be complete if we show that $\mu_E(V_n) \leq C'' \mu_E(k_n\overline{B}_{\varepsilon^*})$. Let $k\overline{B}_{\varepsilon^*} \in R_n - R_{n+1}$. Then $T_n > t_n$ and $k\overline{B}_{\varepsilon^*} \cap k_n\overline{B}_{\varepsilon^*} = \phi$. Thus $k\overline{B}_{\varepsilon^*} \cap k_n\overline{B}_{\varepsilon^*} = \phi$, $k_n^{-1}k\overline{B}_{\varepsilon^*} \cap k_n\overline{B}_{\varepsilon^*} = \phi$, $k_n^{-1}k\overline{B}_{\varepsilon^*} \subseteq$ (K-hull of $\overline{B}_{\varepsilon^*}$) and $k\overline{B}_{\varepsilon^*} \subset k_n$ (K-hull of $\overline{B}_{\varepsilon^*}$). Hence $V_n \subset k_n$ (K-hull of $B_{\varepsilon^*}$). From the definition of $C''$ and the inequality $T_n - 1 < t_n$, we obtain $\mu_E(V_n) \leq \mu_E(k_n$ (K-hull of $\overline{B}_{\varepsilon^*}$)) \leq C'' \mu_E(k_n\overline{B}_{\varepsilon^*})$. Q.E.D.

REMARK. From the definition of the maximal function $f^*$ for an integrable function $f$ on $K/L$ and Lemma 3, we have

$$f^*(k_0, L) = \sup_{0 < \varepsilon < \infty} \frac{1}{\mu_E(\overline{B}_{\varepsilon^*})} \int_{\overline{B}_{\varepsilon^*}} |f(k_0, kL)| \, d\mu_E(kL) \quad \text{for} \quad k_0 L \subseteq K/L.$$

**Theorem 3.** (Maximal theorem)

For an integrable function $f$ on $K/L$ and any real number $\xi > 0$, we obtain the following inequalities:

(i) $\mu_E \{kL \in K/L; f^*(kL) > \xi\} \leq \frac{C''}{\xi} \int_{K/L} |f(kL)| \, d\mu_E(kL)$ \hspace{1cm} (15)

(ii) $\mu_E \{kL \in K/L; f^*(kL) > \xi\} \leq \frac{2C''}{\xi} \int_{f(kL) > \xi} |f(kL)| \, d\mu_E(kL)$ \hspace{1cm} (16)

where $C''$ is the same constant as in Theorem 2'.

Proof. Let $U = \{kL \in K/L; f^*(kL) > \xi\}$. From the above Remark, for each $k_0 L \subseteq U$ there exists $\overline{B}_{\varepsilon_{k_0}}$ such that

$$\int_{k_0 \overline{B}_{\varepsilon_{k_0}}} |f(kL)| \, d\mu_E(kL) \geq \xi \mu_E(\overline{B}_{\varepsilon_{k_0}}) = \xi \mu_E(k_0 \overline{B}_{\varepsilon_{k_0}}).$$

Theorem 2' says that there exists a disjoint subfamily $\{k_j \overline{B}_{\delta_j}\}$ of $\{k_0 \overline{B}_{\varepsilon_{k_0}}; k_0 L \subseteq U\}$ such that $C'' \sum_{j=1}^{\infty} \mu_E(k_j \overline{B}_{\delta_j}) \geq \mu_E(U)$. Therefore

$$\int_{K/L} |f(kL)| \, d\mu_E(kL) \geq \sum_{j=1}^{\infty} \int_{k_j \overline{B}_{\delta_j}} |f(kL)| \, d\mu_E(kL) \geq \xi \sum_{j=1}^{\infty} \mu_E(k_j \overline{B}_{\delta_j}) \geq \frac{\xi}{C''} \mu_E(U).$$
and the inequality (i) follows. For the proof of (ii), we define an integrable function \( h \) on \( K/L \) by

\[
h(kL) = \begin{cases} 
    f(kL) & \text{if } |f(kL)| \geq \frac{1}{2} \xi \\
    0 & \text{otherwise.}
\end{cases}
\]

Then \( h^*(kL) + \frac{1}{2} \xi \geq f^*(kL) \). Hence by (i)

\[
\mu_E(U) \leq \mu_E\left\{ kL; h^*(kL) \geq \xi \right\} \leq \frac{2C''}{\xi} \int_{K/L} |h(kL)| \, d\mu_E(kL) 
\]

\[
= \frac{2C''}{\xi} \int_{\{f(kL)\geq \xi\}} |f(kL)| \, d\mu_E(kL) .
\]

Q.E.D.

Proof of Theorem 1. For any \( \varepsilon > 0 \), we can write as \( f = f_1 + f_2 \) where \( f_1 \) is continuous and \( f_2 \in L^1(K/L) \) with \( L^1 \) norm \( ||f_2|| < \varepsilon^2 \). Let \( h_1, h_2 \) and \( h \) be the Poisson integrals of \( f_1, f_2 \) and \( f \), respectively. Since \( f_1 \) is continuous, we can choose (Körányi-Helgason [5]) \( T > 0 \) large enough such that \( t > T \) implies

\[
|h_1(k\alpha, K) - f_1(kL)| < \varepsilon \quad \text{for all } k \in K
\]

where \( \alpha = \exp t X^0 \). If \( U_1 = \{ kL \in K/L; |f_1(kL) - f(kL)| \geq \varepsilon \} \subseteq K/L; |f_2(kL)| \geq \varepsilon \}, \) then \( \mu_E(U_1) \leq \varepsilon \) since \( E \mu_E(U_1) \leq ||f_3|| \leq \varepsilon^2 \). Therefore except in the set \( U_1 \) of measure \( \leq \varepsilon \),

\[
|h_1(k\alpha, K) - f_1(kL)| < 2\varepsilon \quad \text{for } t > T .
\]

Let \( U_2 = \{ kL \in K/L; f_2^*(kL) > \varepsilon \} \) where \( C' \) is a constant in Proposition 3. Then we have by Theorem 3 (i)

\[
\mu_E(U_2) \leq \frac{C'C''}{\varepsilon} \int_{K/L} |f_2(kL)| \, d\mu_E(kL) \leq \frac{C'C''}{\varepsilon} \cdot \varepsilon^2 = C'C''\varepsilon .
\]

Hence we have by Proposition 3

\[
|h_2(k\alpha, K)| \leq \varepsilon \quad \text{for all } t > \tanh^{-1}\left( \frac{1}{2} \right)
\]

except in the set \( U_2 \) of measure \( \leq C'C''\varepsilon \). Therefore, except in the set \( U_1 \cup U_2 \) of measure \( (C'C'' + 1)\varepsilon \),

\[
|h(k\alpha, K) - f(kL)| < 3\varepsilon \quad \text{for } t > \max\left( T, \tanh^{-1}\left( \frac{1}{2} \right) \right) .
\]

Replacing \( \varepsilon \) by \( 2^{-n}\varepsilon \) and taking \( U_1^{(n)} \) and \( U_2^{(n)} \) in place of \( U_1 \) and \( U_2 \), let \( U \) be the union of all \( U_1^{(n)} \cup U_2^{(n)}, \quad n = 1, 2, \ldots . \) Then we have
\[
\lim_{t \to \infty} h(ka_t, K) = f(kL)
\]

except in the set \( U \) of measure \( 
\leq 2(C'C''+1)\varepsilon \). Since \( \varepsilon \) is arbitrary,

\[
\lim_{t \to \infty} h(ka_t, K) = f(kL)
\]

almost everywhere on \( K/L \) with respect to \( \mu_E \). Q.E.D.

6. Inequalities of Hardy-Littlewood

In this section, we shall prove inequalities of Hardy-Littlewood in the same way as Rauch’s proof [15] of the inequalities for hermitian hyperbolic spaces. We assume again that \( G/K \) is an irreducible hermitian symmetric space of tube type. For a function \( f \) on \( K/L \), we define a real valued non-negative function \( \log^+ |f| \) on \( K/L \) by

\[
(\log^+ |f|)(kL) = \begin{cases}
\log |f(kL)| & \text{if } |f(kL)| \geq 1 \\
0 & \text{otherwise}.
\end{cases}
\]

For a measurable function \( \varphi \) on \( K/L \), we define a decreasing function \( \mu_\varphi \) on \( R^+=[0, \infty) \) by

\[
\mu_\varphi(\xi) = \mu_E(kL \in K/L; |\varphi(kL)| > \xi) \quad \text{for } \xi \in R^+.
\]

Then for any non-negative increasing function \( s \) on \( R^+ \) we obtain

\[
\int_{K/L} s(|\varphi(kL)|) d\mu_E(kL) = -\int_0^\infty s(\xi) d\mu_\varphi(\xi) \quad \text{(17)}
\]

where the right hand side means the Lebesgue-Stieltjes integral with respect to \( \mu_\varphi \).

Proposition 5. There exist positive constants \( C \), \( \alpha \) and \( \beta \) such that

(i) if \( p > 1 \), \( \int_{K/L} |f^*(kL)|^p d\mu_E(kL) \leq C \|f\|_p^p \) for all \( f \in L^p(K/L) \)

(ii) if \( p = 1 \), \( \int_{K/L} |f^*(kL)| d\mu_E(kL) \leq \alpha \int_{K/L} |f(kL)| \log^+ (|f(kL)|) d\mu_E + \beta \) for all functions \( f \) such that \( f \log^+ f \in L^1(K/L) \).

Proof. Since we have from Theorem 3 (ii) and (17)

\[
\mu_\varphi(\xi) \leq \frac{2C''}{\xi} \int_{|f(kL)| \geq \xi} |d\mu_E(kL)| = -\frac{2C''}{\xi} \int_0^\infty xd\mu_\varphi(x) \quad \text{for } \xi > 0,
\]

we have

\[
\int_0^\infty \mu_\varphi(\xi) \xi^{p-1} d\xi \leq -2C'' \int_0^\infty \xi^{p-2} d\xi \int_0^\infty xd\mu_\varphi(x) = -2C'' \int_0^\infty xd\mu_\varphi(x) \int_0^\infty \xi^{p-2} d\xi.
\]
Let $p > 1$. Then
\[
\int_0^\infty \mu_r(\xi)\xi^{p-1} d\xi = -\frac{2^{p-1}}{p-1} (2C') \int_0^\infty x^p d\mu (x)
= \frac{2^{p-1}}{p-1} (2C') \int_{K/L} |f(kL)|^p d\mu_E(kL) < \infty.
\]
Hence we obtain
\[
\lim_{\xi \to 0} \int_\xi^{2\xi} \mu_r(x)x^{p-1} dx = 0.
\]
Since $\mu_r$ is a decreasing function on $R^+$, we have
\[
\mu_r(2\xi)\xi^{p-1} = \mu_r(\xi)\xi^{p-1} \leq \int_\xi^{2\xi} \mu_r(x)x^{p-1} dx.
\]
Therefore, we obtain
\[
\mu_r(2\xi)\xi^{p-1} = \frac{2^{p-1}}{p-1} (2C') \int_{K/L} |f(kL)|^p d\mu_E(kL).
\]
If $p = 1$, then we have
\[
\int_1^\infty \mu_r(x) dx \leq 2C' \int_1^\infty y \mu_r(y) \frac{dy}{x} = -2C' \int_1^\infty \frac{dy}{x} \log (2y) \mu_r(y)
\leq 2C' \int_{K/L} |f(kL)| \log^+ (|f(kL)|) d\mu_E(kL)
+ 2C' \log 2 \int_{K/L} |f(kL)| d\mu_E(kL).
\]
Since $|f| \leq 1 + |f| \log^+ |f|$, we have
\[
\int_{K/L} |f(kL)| d\mu_E(kL) \leq \mu_E(K/L) + \int_{K/L} |f(kL)| \log^+ (|f(kL)|) d\mu_E(kL)
\]
and
\[
\int_1^\infty \mu_r(x) dx \leq \mu_E(K/L).
\]
Since
\[
\int_{K/L} |f^*(kL)| d\mu_E(kL) = -\int_0^\infty x d\mu_r(x) = \int_0^\infty \mu_r(x) dx
\]
\[
\int_1^\infty \mu_r(x) dx + \int_1^\infty \mu_r(x) dx,
\]
the second inequality follows. Q.E.D.

**DEFINITION.** For an integrable function \( f \) on \( K/L \), we define a function \( f_* \) on \( K/L \) by

\[
f_*(k_0L) = \sup_{t < \text{tanh}^{-1} t \leq 1} \int_{K/L} |f(kL)| P_E(k_0 a_t K, kL) d\mu_E(kL) \quad \text{for} \quad k_0L \in K/L
\]

where \( a_t = \exp t X^0 \). Since \( L \) centralizes \( X^0 \), \( f_* \) is a well defined function on \( K/L \). Since the supremum over rational \( t \) gives the same answer, \( f_* \) is a measurable function on \( K/L \).

**Theorem 4.** (Inequalities of Hardy-Littlewood) **There exist constants** \( C_{13}, \alpha' \) and \( \beta' \) **such that**

(i) if \( p > 1 \),

\[
\int_{K/L} |f_*(kL)|^p d\mu_E(kL) \leq C_{13} \| f \|^p_F \quad \text{for all} \quad f \in L^p(K/L)
\]

(ii) if \( p = 1 \),

\[
\int_{K/L} |f_*(kL)| d\mu_E(kL) \leq \alpha' \int_{K/L} |f(kL)| \log^+ (|f(kL)|) d\mu_E(kL) + \beta'
\]

for all \( f \) such that \( f \log^+ |f| \in L'(K/L) \).

Proof. These are immediate consequences of Propositions 3 and 5. Q.E.D.

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**References**


