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# RADIAL CONVERGENCE OF POISSON INTEGRALS ON SYMMETRIC BOUNDED DOMAINS OF TUBE TYPE

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## 1. Introduction

Let $\mathscr{D}=\{z\in \boldsymbol{C};\ |z|<1\}$ be the unit disc in $\boldsymbol{C}$ and $\mathscr{B}=\{e^{it};\ -\pi\leq t\leq\pi\}$ the boundary of $\mathscr{D}$. For an integrable function $f$ (In this note a function will always mean a complex valued function) on $\mathscr{B}$ with respect to the normalized measure $\frac{1}{2\pi}dt$ on $\mathscr{B}$, we define the Poisson integral of $f$ by

$$F(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)P(z,e^{it})dt \qquad \text{for} \quad z\in\mathscr{D}$$

where

$$P(re^{i\theta},e^{it})=\frac{1-r^2}{1-2r\cos(\theta-t)+r^2} \qquad \text{for} \quad 0\leq r<1$$

and it is called the Poisson kernel of the unit disc $\mathscr{D}$. $F$ is a $C^\infty$-function on $\mathscr{D}$ and it is harmonic on $\mathscr{D}$, that is $\Delta F=0$ for the Laplace-Beltrami operator $\Delta$ on $C^\infty$-functions on $\mathscr{D}$ with respect to the Poincaré metric on $\mathscr{D}$.

Then the classical Fatou's theorem asserts that for an integrable function $f$ on $\mathscr{B}$,

$$\lim_{r\uparrow 1}F(re^{i\theta})=f(e^{i\theta})$$

for almost every point $e^{i\theta}$ of $\mathscr{B}$ with respect to the measure $\frac{1}{2\pi}d\theta$.

Now let $G$ be any non-compact connected semi-simple Lie group with finite center, and let $K$ be a maximal compact subgroup of $G$. Then the homogeneous space $G/K$ is a symmetric space of non-compact type. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$ with respect to the Lie algebra $\mathfrak{k}$ of $K$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Fix an order on $\mathfrak{a}$ and let $\mathfrak{a}^+$ be the positive Weyl chamber of $\mathfrak{a}$ with respect to this order. Let $M$ be the centralizer of $\mathfrak{a}$ in $K$. Then the homogeneous space $K/M$ is the maximal boundary of $G/K$ in the sense of Furstenberg [2]. Let $\mu$ be the normalized

$K$-invariant measure on $K/M$ and $L^p(K/M)$ denote the $L^p$-space on $K/M$ with respect to the measure $\mu$. Let $P(gK, kM)$ be the Poisson kernel on $G/K \times K/M$ geven by Korányi [11].

Knapp [7] has proved the following Fatou-type theorem which generalizes the classical Fatou's theorem: Suppose $G/K$ is a symmetric space of non-compact type of rank one. Then for $X \in \mathfrak{a}^+$ and $f \in L^1(K/M)$, it holds

$$\lim_{t\to\infty} \int_{K/M} f(kM)P(k_0 \exp tX\cdot K, kM)d\mu(kM) = f(k_0M)$$

for almost every point $k_0M$ of $K/M$ with respect to the measure $\mu$.

In the case of an arbitrary symmetric space $G/K$ of non-compact type, for $f \in L^\infty(K/M)$ and $X \in \mathfrak{a}^+$, Helgason-Korányi [5] has proved a theorem of the same type as above on the boundary behavior of the Poisson integral of $f$.

In the classical Fatou's theorem, the unit disc $\mathscr{D}$ is a symmetric bounded domain of tube type and the boundary $\mathscr{B}$ is the Bergman-Šilov boundary of $\mathscr{D}$. The purpose of the present paper is to prove for a symmetric bounded domain $\mathscr{D}$ of tube type and the Bergman-Šilov boundary $\mathscr{B}$ of $\mathscr{D}$, the Poisson integral of a function $f \in L^1(\mathscr{B})$ converges to $f$ almost everywhere $\mathscr{B}$.

In general, Korányi [11] has defined the notion of the admissibly and unrestrictedly convergence. Knapp and Williamson [8] showed that the Poisson integral of a function $f$ in $L^\infty(K/M)$ converges to $f$ admissibly and unrestrictedly almost everywhere. Moreover, in the case of a Siegel domain in the sense of Pyatetskii-Šapiro [14] which is analytically isomorphic to a symmetric bounded domain $\mathscr{D}$, Stein and Weiss [16], [17], [19], have defined the notion of the restricted and admissible convergence. Let $B$ denote the Šilov boundary in the sense of Pyatetskii-Šapiro [14] of the Siegel domain. Then they showed that the Poisson integral of an integrable function $f$ on $B$ converges to $f$ admissibly and restrictedly almost everywhere on $B$. The generalized Cayley transform of Korányi-Wolf [12] carries the bounded symmetric domain $\mathscr{D}$ onto the Siegel domain and its inverse image of the Šilov boundary $B$ of the Siegel domain is open and dense in the Bergman-Šilov boundary $\mathscr{B}$ of the bounded domain. The inverse Cayley transform carries the $L^p$-space $L^p(B)$ of $B$ into the $L^p$-space $L^p(\mathscr{B})$ on $\mathscr{B}$, but not onto, unless $p=\infty$. Therefore Fatou's theorem for symmetric bounded domains and that for Siegel domains are not equivalent.

In §2, for a symmetic bounded domain $\mathscr{D}$ we define the notion of the radial convergence of Poisson integrals of functions on the Bergman-Šilov boundary of $\mathscr{D}$ and formulate a Fatou-type theorem. In §3, we give an explicit formula and an estimate of the Poisson kernel of $\mathscr{D}$. In §4, for a symmetric bounded domain of tube type, we define a maximal function and establish an estimate of Poisson integrals by means of this maximal function. In §5, we prove a covering theorem of Vitali-type and a maximal theorem of Knapp-type and give the proof of Fatou's theorem for a symmetric domain of tube type. In §6, we prove inequalities of Hardy-Littlewood, making use of the maximal theorem.

## 2. Statement of Fatou's theorem

Let $G$ be a connected semi-simple Lie group with finite center, $K$ a maximal compact subgroup of $G$. We assume that the quotient space $G/K$ is an irreducible hermitian symmetric space. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$, respectively, and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Then $K$ has the same rank as $G$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Then $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^C$, $\mathfrak{k}^C$, $\mathfrak{p}^C$ and $\mathfrak{t}^C$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{t}$, respectively. Then the set $R$ of roots of $\mathfrak{g}^C$ with respect to $\mathfrak{t}^C$ can be decomposed into two disjoint sets $C=\{\alpha\in R; E_\alpha\in\mathfrak{k}^C\}$ and $P=\{\alpha\in R; E_\alpha\in\mathfrak{p}^C\}$, where $\{E_\alpha\}$ is a set of root vectors. A root of $C$ or $P$ is called compact or non-compact. Let $\mathfrak{p}^{\pm}$ be the subspace of $\mathfrak{p}^C$ corresponding to $(\pm i)$-eigenspace of the complex structure tensor on the tangent space of $G/K$ at the origin $eK$. We choose and fix an order $\succ$ on roots in $R$ such that $\mathfrak{p}^+$, $\mathfrak{p}^-$ are spanned by the $E_\alpha$'s, $E_{-\alpha}$'s, respectively, where $\alpha$ runs through positive non-compact roots. Let $\Delta$ be the maximal set of strongly orthogonal non-compact positive roots of Harish-Chandra [4]. We choose root vectors $\{E_\alpha\}$ in such a way that $\tau E_\alpha=-E_{-\alpha}$ for the conjugation $\tau$ of $\mathfrak{g}^C$ with respect to the compact real from $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$ of $\mathfrak{g}^C$. For $\alpha\in R$, let $H_\alpha'$ be the unique element of $i\mathfrak{t}$ satisfying $\alpha(H)=\langle H_\alpha', H\rangle$ for all $H\in\mathfrak{t}$, where $\langle\,,\rangle$ denotes the Killing form of $\mathfrak{g}^C$. For $\alpha\in\Delta$, we put $X_\alpha^0=E_\alpha+E_{-\alpha}$, $Y_\alpha^0=(-i)(E_\alpha-E_{-\alpha})$ and $H_\alpha=\dfrac{2}{\langle H_\alpha', H_\alpha'\rangle}H_\alpha'$. Let $\mathfrak{g}_\alpha$ denote the subalgebra of $\mathfrak{g}$ spanned by $\{iH_\alpha, X_\alpha^0, Y_\alpha^0\}$. Strong orthogonality of $\Delta$ implies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]=\{0\}$ for $\alpha\neq\beta$. Let $\mathfrak{t}^-$ be the subalgebra of $\mathfrak{t}_\alpha$ spanned by $\{iH_\alpha;\ \alpha\in\Delta\}$ and let $\mathfrak{t}^+$ be the orthogonal complement of $\mathfrak{t}^-$ in $\mathfrak{t}$ with respect to the Killing form $\langle\,,\rangle$. The vectors $X_\alpha^0$, $\alpha\in\Delta$, span a maximal abelian subalgbra $\mathfrak{a}$ of $\mathfrak{p}$ and $\mathfrak{h}=\mathfrak{t}^++\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}^C$ be the complexification of $\mathfrak{h}$. $A$ and $H^-$ denote analytic subgroups of $G$ generated by $\mathfrak{a}$ and $\mathfrak{t}^-$, respectively.

Following Moore [13], we consider the Cayley transform $\tilde{c}$ of $\mathfrak{g}^C$ defined by $\tilde{c}=Ad\left(\exp\left(\frac{\pi}{4}\sum_{\alpha\in\Delta}(-i)Y_\alpha^0\right)\right)$. Then $\tilde{c}$ transforms

$$X_\alpha^0\mapsto -H_\alpha,\ H_\alpha\mapsto X_\alpha^0 \text{ and } Y_\alpha^0\mapsto Y_\alpha^0 \qquad (\alpha\in\Delta)$$

and $\tilde{c}$ leaves $\mathfrak{t}^+$ pointwise fixed. Hence $\tilde{c}$ maps $i\mathfrak{t}^-$ onto $\mathfrak{a}$ and $\mathfrak{t}^C$ onto $\mathfrak{h}^C$, so that it maps $R$ onto the set $\Sigma$ of roots of $\mathfrak{g}^C$ with repsect to $\mathfrak{h}^C$. Let $\sigma$ be the conjugation of $\mathfrak{g}^C$ with repsect to $\mathfrak{g}$. $\sigma$ permutes roots of $\Sigma$ by

$$\sigma(\alpha)(H)=\overline{\alpha(\sigma(H))} \qquad \text{for} \quad \alpha\in\Sigma,\ H\in\mathfrak{h}^C.$$

We choose a following linear order $<$ on $\Sigma$ and fix it once and for all: (i) If $\alpha\in\Sigma$, $\alpha>0$ and $\alpha$ does not vanish on $\mathfrak{a}$, then $\sigma(\alpha)>0$. (ii) If $\gamma\in\Delta$, then $\tilde{c}(\gamma)>0$. Then $\Sigma$ can be decomposed into three disjoint sets; $\Sigma^+=\{\alpha\in\Sigma;\ \alpha>0,\ \sigma(\alpha)>0\}$, $\Sigma^-=-\Sigma^+$ and $\Sigma_0=\{\alpha\in\Sigma;\ \alpha=-\sigma(\alpha)\}$, $\sum_{\alpha\in\Sigma^+}\boldsymbol{C}\tilde{E}_\alpha$ and $\sum_{\alpha\in\Sigma^-}\boldsymbol{C}\tilde{E}_\alpha$ are both invariant under $\sigma$, where $\{\tilde{E}_\alpha\}$ is a set of root vectors of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$. We put $\mathfrak{n}=\sum_{\alpha\in\Sigma^+}\boldsymbol{C}\tilde{E}_\alpha\cap\mathfrak{g}$ and $\bar{\mathfrak{n}}=\sum_{\alpha\in\Sigma^-}\boldsymbol{C}\tilde{E}_\alpha\cap\mathfrak{g}$, which are real forms of $\sum_{\alpha\in\Sigma^+}\boldsymbol{C}\tilde{E}_\alpha$ and $\sum_{\alpha\in\Sigma^-}\boldsymbol{C}\tilde{E}_\alpha$, respectively. Then $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are nilpotent subalgebras of $\mathfrak{g}$. We obtain the Iwasawa decompositions $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ and $G=KAN$, where $A$ and $N$ are analytic subgroups of $G$ generated by $\mathfrak{a}$, $\mathfrak{n}$. So any $g\in G$ can uniquely decomposed as $g=k(g)\exp H(g)n(g)$, where $k(g)\in K$, $H(g)\in\mathfrak{a}$ and $n(g)\in N$.

The restriction to $\mathfrak{a}$ of a root of $\Sigma-\Sigma_0$ is called a restricted root and the order $>$ on $\Sigma$ induces a linear order $>$ on the set of restricted roots. Let $F$ be the fundamental system of restricted roots with respect to the order $>$. Let $X^0=\sum_{\alpha\in\Delta}X^0_\alpha$, and we put $E=\{\alpha\in F;\ \alpha(X^0)=0\}$ and $\mathfrak{a}(E)=\{H\in\mathfrak{a};\ \alpha(H)=0$ for all $\alpha\in E\}$. Then $\mathfrak{a}(E)$ is spanned by $X^0$, and $\mathfrak{g}$ is the direct sum of eigen-spaces for ad $X^0$ on $\mathfrak{g}$. The sum of the positive (negative) eigen-spaces of $\mathfrak{g}$ is denoted by $\mathfrak{n}(E)(\bar{\mathfrak{n}}(E))$. Let $\mathfrak{b}(E)$ be the sum of non-negative eigen-spaces, $\mathfrak{l}$ the centralizer of $X^0$ in $\mathfrak{k}$, let $2\rho_E$ be the sum of restriected roots $\alpha$ with $\alpha(X^0)>0$, with multiplicties counted.

The analytic subgroups of $G$ generated by $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$ will be denoted by $N(E)$, $\bar{N}(E)$. Let $L$ be the centralizer of $X^0$ in $K$ and $B(E)$ the normalizer of $\mathfrak{n}(E)$ in $G$. Then $\mathfrak{l}$, $\mathfrak{b}(E)$ are Lie algebras of $L$, $B(E)$ and we have the decompositions $B(E)=LAN$ and $\mathfrak{b}(E)=\mathfrak{l}+\mathfrak{a}+\mathfrak{n}$. From the Iwasawa decomposition $G=KAN$, $K/L$ is naturally identified with $G/B(E)$ as $K$-spaces. Let $\Phi$ be the holomorphic imbedding of Harish-Chandra [4] of $G/K$ into $\mathfrak{p}^-$ as a bounded domain in the complex vector space $\mathfrak{p}^-$ and let $\mathcal{D}=\Phi(G/K)$. Then the imbedding $\Phi$ is equivariant with respect to the natural action of $K$ on $G/K$ and the adjoint action of $K$ on $\mathfrak{p}^-$. Let $\mathcal{B}$ be the Bergman-Šilov boundary of the bounded domain $\mathcal{D}$ in $\mathfrak{p}^-$. Then it is known (Korányi-Wolf [12]) that $\sum_{\alpha\in\Delta}E_{-\alpha}\in\mathcal{B}$, $K$ acts transitively on $\mathcal{B}$ by the adjoint action and $L$ becomes the isotropy subgroup of $K$ at $\sum_{\alpha\in\Delta}E_{-\alpha}$. Thus the Bergman-Šilov boundary $\mathcal{B}$ is isomorphic to $K/L$.

Let $\mu_E$ be the normalized $K$-invariant measure on $K/L$ and $L^p(K/L)$ denote the $L^p$-space on $K/L$ with respect to the measure $\mu_E$. Then the *Poisson kernel* on $G/K\times K/L$ is defined by

$$P_E(gK,\ kL)=e^{-2\rho_E(H(g^{-1}k))}\qquad\text{for}\quad g\in G,\ k\in K$$

where $\exp H(g^{-1}k)$ is the $A$-component of $g^{-1}k$ in the Iwasawa decomposition. We define the *Poisson integral* of a function $f\in L^1(K/L)$ by

$$\int_{K/L} f(kL)P_E(gK,\ kL)d\mu_E(kL) \qquad \text{for}\quad g\in G.$$

The hermitian symmetric space $G/K$ of non-compact type is called *of tube type* if $(\mathfrak{k}, \mathfrak{l})$ is a symmetric pair, then $\mathfrak{t}^-$ is a Cartan subalgebra of $(\mathfrak{k}, \mathfrak{l})$ and eigenvalues of $ad\left(\frac{1}{2}X_0\right)$ are 0, $\pm 1$ (Korányi-Wolf [12]).

Now we can state our main theorem:

**Theorem 1.** *Let $G/K$ be an irreducible hermitian symmetric space of tube type. Let $a_t = \exp tX^0$ for a real number $t$. If $f\in L^1(K/L)$, then*

$$\lim_{t\to\infty}\int_{K/L} f(kL)P_E(k_0 a_t K,\ kL)d\mu_E(kL) = f(k_0L)$$

*for almost every point $k_0L$ of $K/L$ with respect to $\mu_E$.*

We assumed the irreducibility of $G/K$ for the simplicity, but the generalization of Theorem 1 of general spaces of tube type is immediate.

## 3. Estimate of Poisson kernel

In this section we assume $G/K$ is an irreducible hermitian symmetric space, not necessarily of tube type.

**Proposition 1.** *Let $a=\exp\sum_{\alpha\in\Delta} t_\alpha X_\alpha^0 \in A$, $h=\exp\sum_{\alpha\in\Delta}\theta_\alpha \frac{iH_\alpha}{2}\in H^-$. Then we have*

$$P_E(aK,\ hL) = \prod_{\alpha\in\Delta} P(\tanh t_\alpha,\ e^{i\theta_\alpha})^{\rho_E(X_\alpha^0)}$$

*where $P(t, u)$ is a function on the product of the open interval $(-1, 1)$ and the circle $\mathcal{B}=\{u\in \boldsymbol{C};\ |u|=1\}$ defined by $P(r, u)=(1-r^2)|1-r\bar{u}|^{-2}$.* (We note that $P(r, u)$ coincides on $(-1, 1)$ with the Poisson kernel of the unit disc in $\boldsymbol{C}$.)

Proof. To calculate $e^{-2\rho_E(H(a^{-1}h))}$, we consider the Iwasawa decomposition of the element $a^{-1}h$ of $G$. We have $Y_\alpha^0+iH_\alpha\in\mathfrak{n}$ for $\alpha\in\Delta$ because we have $Y_\alpha^0+iH_\alpha=\tilde{c}(Y_\alpha^0-iX_\alpha^0)=\tilde{c}\{(-i)(E_\alpha-E_{-\alpha})-i(E_\alpha+E_{-\alpha})\}=\tilde{c}(-2iE_\alpha)\in \boldsymbol{C}\widetilde{E}_{\tilde{c}\alpha}$ and from the condition (ii) of the ordering $>$ on $\Sigma$, we obtain $Y_\alpha^0+iH_\alpha\in\mathfrak{g}\cap\sum_{\alpha\in\Sigma^+}\boldsymbol{C}\widetilde{E}_\alpha$ $=\mathfrak{n}$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]=\{0\}$ for $\alpha\neq\beta$, $\alpha$, $\beta\in\Delta$, it follows that

$$a^{-1}h = \prod_{\alpha\in\Delta}\exp(-t\ X_\alpha)\exp\left(\theta_\alpha\frac{iH_\alpha}{2}\right).$$

If we have the Iwasawa decompostion

$$\exp(-t_\alpha X_\alpha^0)\exp\left(\theta_\alpha \frac{iH_\alpha}{2}\right) = \exp a_\alpha \frac{iH_\alpha}{2} \exp b_\alpha X_\alpha^0 \exp(c_\alpha(Y_\alpha^0 + iH_\alpha))$$

of each factor, we have

$$a^{-1}h = \exp\left(\sum_{\alpha\in\Delta} a_\alpha \frac{iH_\alpha}{2}\right) \exp\left(\sum_{\alpha\in\Delta} b_\alpha X_\alpha^0\right) \exp\left(\sum_{\alpha\in\Delta} c_\alpha(Y_\alpha^0 + iH_\alpha)\right)$$

and thus $H(a^{-1}h) = \sum_{\alpha\in\Delta} b_\alpha X_\alpha^0$. Now let

$$SU(1, 1) = \left\{x \in M_2(\boldsymbol{C});\ {}^t\bar{x}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}x = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right\}$$

$$X^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad Y^0 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

Then the Lie algebra $\mathfrak{su}(1, 1)$ of $SU(1, 1)$ is spanned by $X^0$, $iH$ and $Y^0+iH$ and the homomorphism $\phi_\alpha$: $\mathfrak{su}(1, 1) \to \mathfrak{g}_\alpha$ defined by

$$X^0 \mapsto X_\alpha^0,\quad iH \mapsto iH_\alpha^0,\quad Y^0+iH \mapsto Y_\alpha^0 + iH_\alpha$$

can be extended to the homomorphism $\phi_\alpha$: $SU(1, 1) \to G$. In $SU(1, 1)$ we have the decomposition

$$\exp(-tX^0)\exp\left(\theta\frac{iH}{2}\right) = \exp\left(a\frac{iH}{2}\right)\exp bX^0 \exp c(Y^0+iH)$$

with $b = \frac{1}{2}\log(\mathrm{ch}^2 t - 2\mathrm{ch}t\,\mathrm{sh}t\cos\theta + \mathrm{sh}^2 t) = -\frac{1}{2}\log P(\tanh t, e^{i\theta})$. Applying the homorphism $\phi_\alpha$ on the both sides, we have

$$b_\alpha = -\frac{1}{2}\log P(\tanh t_\alpha, e^{i\theta_\alpha}).$$

This implies the Proposition. Q.E.D.

Now we define for $0 < \rho \leqslant 1$,

$$\mathfrak{H}_\rho = \left\{\exp\left(\sum_{\alpha\in\Delta}\theta_\alpha \frac{iH_\alpha}{2}\right) \in H^-;\ |\theta_\alpha| < \pi\rho,\ \text{for any } \alpha\in\Delta\right\},$$

$$\mathfrak{B}_\rho = \{lhL \in K/L;\ l\in L,\ h\in\mathfrak{H}_\rho\},$$

and for $\rho > 1$,

$$\mathfrak{B}_\rho = \{lhL \in K/L;\ l\in L,\ h\in H^-\}.$$

In §4, we shall calculate the measure of $\mathfrak{B}_\rho$ with respect to $\mu_E$ for a space of tube type. We give an estimate of Poisson kernel on $\mathfrak{B}_\rho$ in the following.

**Proposition 2.** *Let* $a = \exp\sum_{\alpha\in\Delta} t_\alpha X_\alpha^0 \in A$. *Then we obtain an estimate of Poisson kernel as follows:*

(i) *If* $0<\rho<1$ *and* $\frac{1}{2}<\tanh t_\alpha<1$ *for any* $\alpha\in\Delta$, *then*

$$\sup_{h\in H^- -\mathfrak{H}_\rho} P_E(aK, hL)\leqslant C_1\prod_{\alpha\in\Delta}\left(\frac{1-\tanh t_\alpha}{\rho^2}\right)^{\rho_E(X_\alpha^0)}$$

(ii) $$\sup_{h\in H^-} P_E(aK, hL)\leqslant C_3\prod_{\alpha\in\Delta}\left(\frac{1}{1-\tanh t_\alpha}\right)^{\rho_E(X_\alpha^0)}$$

*where $C_1$, $C_2$ are constants independent on $a$ and $\rho$.*

*In particular, if $a_t=\exp tX^0$, then*

(i) *If* $0<\rho<1$ *and* $\frac{1}{2}<\tanh t<1$, *then*

$$\sup_{kL\in\mathfrak{B}_1-\mathfrak{B}_\rho} P_E(a_tK, kL)\leqslant C_1\left(\frac{1-\tanh t}{\rho^2}\right)^{\rho_E(X^0)} \tag{1}$$

(ii) $$\sup_{kL\in\mathfrak{B}_1} P_E(a_tK, kL)\leqslant C_3\left(\frac{1}{1-\tanh t}\right)^{\rho_E(X^0)} \tag{2}$$

(We note that $\mathfrak{B}_1$ is equal to $K/L$ if $G/K$ is of tube type).

Proof. We have (Korânyi [10]) an estimate of the Poisson kernel for the unit disc in $\boldsymbol{C}$ as follows:

(i) $$\sup_{\pi\rho\leqslant|\theta|\leqslant\pi}(1-r^2)|1-re^{-i\theta}|^{-2}\leqslant C_1'\frac{1-r}{\rho^2} \qquad \text{if } \frac{1}{2}<r<1\,.$$

(ii) $$\sup_{0\leqslant|\theta|\leqslant\pi}(1-r^2)|1-re^{-i\theta}|^{-2}\leqslant C_2'\frac{1}{1-r} \qquad \text{if } 0<r<1\,.$$

where $C_1'$, $C_2'$, are constants. This together with Proposition 1 implies the first statement. If $a_t=\exp tX^0$, then we have $P_E(a_tK, lhL)=P_E(a_tK, hL)$ for $h\in H^-$ and $l\in L$ since $L$ centralizes $X^0$ in $K$. This together with the first statement implies the second statement. Q.E.D.

## 4. Maximal function

Henceforth we shall assume that $G/K$ is an irreducible hermitian symmetric space of tube type. We consider the Poisson integral

$$\int_{K/L} f(kL)P_E(a_tK, kL)d\mu_E(kL) \tag{3}$$

for $a_t=\exp tX^0$ and an integrable function $f$ on $K/L$ with respect to $\mu_E$.

Since $K/L$ is a symmetric space, we may use the following integral formula for $K/L$ (Harish-Chandra [4]): For each continuous function $f$ on $K/L$, we have

$$\int_{K/L} f(kL)d\mu_E(kL) = c\int_{H^-}\left(\int_{L/Z_L(\mathfrak{t}^-)} f(lhL)d\bar{l}\right)|D(h)|dh$$

where $c$ is a constant independent on $f$, $Z_L(\mathfrak{t}^-)$ is the centralizer of $\mathfrak{t}^-$ in $L$, $dh$ is a Haar measure on $H^-$ and $d\bar{l}$ is a quotient measure on $L/Z_L(\mathfrak{t}^-)$ induced from the normalized Haar measure $dl$ on $L$. Moreover

$$D(h) = \prod_{\beta\in P_+^k} \sin\beta(iH) \quad \text{for} \quad h = \exp H, H\in\mathfrak{t}^-$$

where $P_+^k = \{\alpha\in C;\text{ positive and } \alpha|\mathfrak{t}^- \neq 0\}$.

Making use of this integral formula, we have the measure $||\mathfrak{B}_\rho||$ of $\mathfrak{B}_\rho$ with respect to $\mu_E$ as follows:

$$\begin{aligned}||\mathfrak{B}_\rho|| &= \int_{K/L} \chi_{\mathfrak{B}_\rho}(kL)d\mu_E(kL) = c\int_{H^-}\left(\int_{L/Z_L(\mathfrak{t}^-)} \chi_{\mathfrak{B}_\rho}(lhL)d\bar{l}\right)|D(h)|dh \\ &= c\int_{\mathfrak{B}_\rho} |D(h)|dh\end{aligned}$$

where $\chi_{\mathfrak{B}_\rho}$ is the characteristic function of $\mathfrak{B}_\rho$. The density $D(h)$ of the integral is given as follows: Let $\Delta=\{\gamma_1,\cdots,\gamma_m\}$, $\gamma_1 \succ \gamma_2 \succ \cdots \succ \gamma_m$, where $m$=rank of $G/K$. For $\alpha\in R$, let $\pi(\alpha)$ be the restriction of $\alpha$ to the complexification $(\mathfrak{t}^-)^C$ of $\mathfrak{t}^-$, but $\pi(\gamma_i)$ will be denoted by $\gamma_i$ for the brevity, since any root $\beta\neq\gamma_i$ does not coincide with $\pi(\gamma_i)$ on $(\mathfrak{t}^-)^C$. Since $G/K$ is of tube type, we have (Harish-Chandra [4], Korányi-Wolf [12]) for a positive compact root $\beta$,

$$\pi(\beta) = \begin{cases} 0 & \text{or} \\ \frac{1}{2}(\gamma_j-\gamma_i) & (i<j) \end{cases}$$

and for a positive non-compact root $\beta$,

$$\pi(\beta) = \begin{cases} \gamma_i & \text{or} \\ \frac{1}{2}(\gamma_j+\gamma_i) & (i<j). \end{cases}$$

Moreover the number $r_{ij}$ $(i<j)$ of elements of $\left\{\beta\in P_+^k;\ \pi(\beta)=\frac{1}{2}(\gamma_j-\gamma_i)\right\}$ is the same as the number of positive non-compact roots $\beta$ such that $\pi(\beta)=\frac{1}{2}(\gamma_j+\gamma_i)$. It follows that

$$D\left(\exp\sum\theta_\alpha\frac{iH_\alpha}{2}\right) = \prod_{1\leq i<j\leq m}\left\{\sin\frac{1}{2}(\theta_i-\theta_j)\right\}^{r_{ij}}.$$

Now we obtain the following

**Lemma 1.** *For $0<\rho<1$, we have an estimate of the measure of $\mathfrak{B}_\rho$:*

$$||\mathfrak{B}_\rho|| \leqslant C\rho^{\rho_E(X^0)} \tag{4}$$

*where $C$ is a constant independent on $\rho$. For $\rho \geqslant 1$, we have* $||\mathfrak{B}_\rho||=1$ (from the definition of $\mathfrak{B}_\rho$).

Proof. From the above argument,

$$||\mathfrak{B}_\rho|| = c\int_{\mathfrak{B}_\rho} |D(h)|\,dh = c\int_{-\pi\rho}^{\pi\rho}\cdots\int_{-\pi\rho}^{\pi\rho}\prod_{i<j}|\sin\frac{1}{2}(\theta_i-\theta_j)|^{r_{ij}}d\theta_1\cdots d\theta_m$$
$$\leqslant c(\pi\rho)^{\sum_{i<j}r_{ij}}\int_{-\pi\rho}^{\pi\rho}\cdots\int_{-\pi\rho}^{\pi\rho}d\theta_1\cdots d\theta_m \leqslant C\rho^{m+\sum_{i<j}c_{ij}}(C=c\pi^{m+\sum_{i<j}r_{ij}}2^m)$$

because $|\sin\frac{1}{2}(\theta_i-\theta_j)| \leqslant \frac{1}{2}|\theta_i-\theta_j| \leqslant \pi\rho$.

On the other hand, $X^0=\sum_{k=1}^{m}X^0_{\gamma_k}$ and

$$\rho_E(X^0_{\gamma_k}) = (\tilde{c}^{-1}\rho_E)(\tilde{c}^{-1}X^0_{\gamma_k}) = \frac{1}{2}\Big(\sum_{i=1}^{m}\gamma_i+\sum_{i<j}\frac{r_{ij}}{2}(\gamma_j+\gamma_i)\Big)(H_{\gamma_k})$$
$$= 1+\sum_{i<k}r_{ik}\,.$$

Hence $\rho_E(X^0)=m+\sum_{1\leqslant i<j\leqslant m}r_{ij}$, then the result follows. Q.E.D.

DEFINITION. For an integrable function $f$ on $K/L$, we define a *maximal function* $f^*$ on $K/L$ by

$$f^*(k_0L) = \sup_{0<\rho<1}\frac{1}{||\mathfrak{B}_\rho||}\int_{\mathfrak{B}_\rho}|f(k_0kL)|\,d\mu_E(kL) \qquad \text{for} \quad k_0L\in K/L\,.$$

The function $f^*$ on $K/L$ is measurable because the supremum over rational $\rho$ $(0<\rho<1)$ gives the same answer.

**Proposition 3.** *For an integrable function $f$ on $K/L$, we have an estimate of Poisson integral by means of the above maximal function:*

$$\sup_{\frac{1}{2}<\tanh t<1}\int_{K/L}|f(kL)|P_E(k_0a_tK,\,kL)d\mu_E(kL) \leqslant C'f^*(k_0L)$$

*for all $k_0\in K$, where $a_t=\exp tX^0$ and $C'$ is a constant not depending on $f$ and $k_0L$.*

Proof. We fix first an arbitrary constant $\alpha>0$ put $\delta=(1-\tanh t)\alpha$ for $\frac{1}{2}<\tanh t<1$. We may suppose $k_0=e$ in view of the $K$-invariance of the measure $\mu_E$, replacing $f$ by the function $f^{k_0}$ defined by $f^{k_0}(kL)=f(k_0kL)$. Then for $\frac{1}{2}<\tanh t<1$, we have

$$\int_{K/L} |f(kL)| P_E(a_tK, kL) d\mu_E(kL) = \int_{\mathfrak{B}_1} |f(kL)| P_E(a_tK, kL) d\mu_E(kL)$$

$$\leqslant \int_{\mathfrak{B}_\delta} |f(kL)| P_E(a_tK, kL) d\mu_E(kL) + \sum_{j=0}^{\infty} \int_{\mathfrak{B}_{2^{j+1}\delta} - \mathfrak{B}_{2^j\delta}} |f(kL)| P_E(a_tK, kL) d\mu_E(kL). \tag{5}$$

Here we note that the summation of the second term in (5) is in fact finite sum because $\mathfrak{B}_{2^j\delta} = K/L$ for $2^j\delta \geqq 1$.

The right hand side of (5) can be estimated as follows:

$$\text{the first term} \leqslant C_2 \left\{\frac{1}{1-\tanh t}\right\}^{\rho_E(X^0)} \int_{\mathfrak{B}_\delta} |f(kL)| d\mu_E(kL) \quad \text{(by (2))}$$

$$\leqslant C_2 \left\{\frac{1}{1-\tanh t}\right\}^{\rho_E(X^0)} \|\mathfrak{B}_\delta\| f^*(eL) \quad \text{(by the definition of } f^*)$$

$$\leqslant C_2 C \left\{\frac{1}{1-\tanh t}\right\}^{\rho_E(X^0)} \delta^{\rho_E(X^0)} f^*(eL) \quad \text{(by (4))}$$

$$= C_2 C \alpha^{\rho_E(X^0)} f^*(eL). \tag{6}$$

$$\text{the second term} \leqslant \sum_{j=0}^{\infty} C_1 \left\{\frac{1-\tanh t}{(2^j\delta)^2}\right\}^{\rho_E(X^0)} \int_{\mathfrak{B}_{2^{j+1}\delta} - \mathfrak{B}_{2^j\delta}} |f(kL)| d\mu_E(kL) \quad \text{(by (1))}$$

$$\leqslant C_1 \sum_{j=0}^{\infty} \left\{\frac{1-\tanh t}{(2^j\delta)^2}\right\}^{\rho_E(X^0)} \|\mathfrak{B}_{2^{j+1}\delta}\| f^*(eL) \quad \text{(by the definition of } f^*)$$

$$\leqslant C_1 C \sum_{j=0}^{\infty} \left\{\frac{1-\tanh t}{(2^j\delta)^2}\right\}^{\rho_E(X^0)} (2^{j+1}\delta)^{\rho_E(X^0)} f^*(eL) \quad \text{(by (4))}$$

$$= C_1 C \left(\frac{2}{\alpha}\right)^{\rho_E(X^0)} \left(\sum_{j=0}^{\infty} \left\{\frac{1}{2^{\rho_E(X^0)}}\right\}^j\right) f^*(eL) \tag{7}$$

where the sum $\sum_{j=0}^{\infty} \left\{\frac{1}{2^{\rho_E(X^0)}}\right\}^j$ converges to $\frac{1}{1-(1/2)^{\rho_E(X^0)}}$.

Hence putting together (6) and (7) into (5), we obtain the inequality:

$$\sup_{1/2 < \tanh t < 1} \int_{K/L} |f(kL)| P_E(a_tK, kL) d\mu_E(kL)$$

$$\leqslant \left\{C_2 C \alpha^{\rho_E(X^0)} + C_1 C \left(\frac{2}{\alpha}\right)^{\rho_E(X^0)} \frac{1}{1-(1/2)^{\rho_E(X^0)}}\right\} f^*(eL) \qquad \text{Q.E.D.}$$

## 5. Covering theorem and proof of Fatou's theorem

In this section we shall prove a covering theorem of Vitali type with respect to the family of sets of the form $k\mathfrak{B}_\rho$, $0<\rho<1$, $k \in K$ and prove a maximal theorem related to the maximal function $f^*$ on $K/L$.

Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{k}$ with respect to $\langle\ ,\ \rangle$. Then $\mathfrak{q} = Ad(L)\mathfrak{t}^-$ since $K/L$ is a symmetric space. We define a map $\psi: \mathfrak{q} \to \mathfrak{p}$ by

$\psi(X)=\frac{1}{2}[X^0, X]$ for $X\in\mathfrak{q}$ and putting $\mathfrak{p}^*=\psi(\mathfrak{q})$, define a map $j\colon \mathfrak{p}^*\to\bar{\mathfrak{n}}(E)$ by $j(X)=X-\frac{1}{2}[X^0, X]$ for $X\in\mathfrak{p}^*$. Then both $\psi$ and $j$ are $L$-equivariant isomorphisms (Takeuchi [18]). We have $\psi(iH_\alpha)=Y_\alpha^0$ and $j(Y_\alpha^0)=Y_\alpha^0-iH_\alpha$ for any $\alpha\in\Delta$ so that $j\psi(\mathfrak{t}^-)$ is the subspace of $\bar{\mathfrak{n}}(E)$ spanned by $\{Y_\alpha^0-iH_\alpha;\ \alpha\in\Delta\}$. Thus we have the following

**Lemma 2.** $\quad Ad(L)\{Y_\alpha^0-iH_\alpha\colon \alpha\in\Delta\}_R=\bar{\mathfrak{n}}(E)$

*where* $\{Y_\alpha^0-iH_\alpha\colon \alpha\in\Delta\}_R$ *is the subspace of* $\bar{\mathfrak{n}}(E)$ *spanned by* $\{Y_\alpha^0-iH_\alpha\colon \alpha\in\Delta\}$.

Now we define an $L$-invariant norm $\|\ \|$ on $\bar{\mathfrak{n}}(E)$ as follows. We define a $K$-invariant inner product on $\mathfrak{g}$ by

$$(X, Y)=-\langle X, \tau Y\rangle \qquad \text{for}\quad X, Y\in\mathfrak{g}.$$

For $Z\in\bar{\mathfrak{n}}(E)$, let $|Z|$ denote the operator norm of $ad(j^{-1}Z)$ with respect of $(\ ,\ )$ and let $\|Z\|=\frac{1}{2}|Z|$. Then (Takeuchi [18]) $\|\ \|$ is a $L$-invariant norm on $\bar{\mathfrak{n}}(E)$ satisfying

$$\|Z\|=\max_{\alpha\in\Delta}|a_\alpha| \qquad \text{for}\quad Z=\sum_{\alpha\in\Delta}a_\alpha(Y_\alpha^0-iH_\alpha).$$

For each $\delta>0$, let

$$B_\delta=\{Z\in\bar{\mathfrak{n}}(E);\ \|Z\|<\delta\}$$
$$\bar{B}_\delta=\{k(\bar{n})L\in K/L;\ \bar{n}=\exp Z,\ Z\in B_\delta\}$$

where $k(\bar{n})$ is the $K$-component of $\bar{n}$ in the Iwasawa decomposition.

**Lemma 3.** *For* $0<\rho<1$, *we have*

$$\mathfrak{B}_\rho=\Big\{k(\bar{n})L\in K/L;\ \bar{n}=\exp Ad(l)\big(\sum_{\alpha\in\Delta}a_\alpha(Y_\alpha^0-iH_\alpha)\big),\ l\in L,\ \max_{\alpha\in\Delta}|a_\alpha|<\frac{1}{2}\tan((\pi/2)\rho)\Big\}$$

*and therefore*

$$\mathfrak{B}_\rho=\bar{B}_{1/2\tan((\pi/2)\rho)}.$$

Proof. Recall the definition of $\mathfrak{B}_\rho$ for $0<\rho<1$:

$$\mathfrak{B}_\rho=\Big\{lhL\in K/L;\ l\in L,\ h=\exp\Big(\sum_{\alpha\in\Delta}\theta_\alpha\frac{iH_\alpha}{2}\Big),\ |\theta_\alpha|<\pi\rho\Big\}.$$

As in the proof of Proposition 1, we have

$$\exp\Big(\sum_{\alpha\in\Delta}\theta_\alpha\frac{i}{2}H_\alpha\Big)=k\Big(\exp\Big(-\frac{1}{2}\sum_{\alpha\in\Delta}\tan\Big(\frac{1}{2}\theta_\alpha\Big)(Y_\alpha^0-iH_\alpha)\Big)\Big) \qquad \text{for}\quad |\theta_\alpha|<\pi.$$

Since $l\bar{n}l^{-1}B(E)=lk(\bar{n})B(E)$ for $l\in L$, $\bar{n}\in\bar{N}(E)$ and $G/B(E)\ni gB(E)\mapsto k(g)L \in K/L$ is a bijection, we have $k(l\bar{n}l^{-1})L=lk(\bar{n})L$. Then the statement follows. Q.E.D.

The purpose of this section is to prove the following covering theorem;

**Theorem 2.** *There is some constant $C''>0$ with the following property. If $U$ is any Borel set in $K/L$, and if to each point $kL$ in $U$ there is associated a set $k\mathfrak{B}_\rho$ (with $0<\rho<1$ depending on $k\in K$), then there is a countable disjoint subfamily of $\{k\mathfrak{B}_\rho\}$, say $k_j\mathfrak{B}_j$, such that*

$$C''\sum_{j=1}^{\infty}\mu_E(k_j\mathfrak{B}_j)\geqslant\mu_E(U)\,.$$

In view of Lemma 3, we may prove the following theorem in place of Theorem 2.

**Theorem 2′.** *There is some constant $C''>0$ with the following property. If $U$ is any Borel set in $K/L$, and if to each point $kL$ in $U$ there is associated a set $k\bar{B}_\delta$ (with $\delta>0$ depending on $k\in K$), then there is a countable disjoint subfamily of $\{k\bar{B}_\delta\}$, say $k_j\bar{B}_j$, such that*

$$C''\sum_{j=1}^{\infty}\mu_E(k_j\bar{B}_j)\geqslant\mu_E(U)\,.$$

The proof will proceed in the same way as Knapp's proof [7] of the covering theorem on Furstenberg's boundary $K/M$ of a symmetric space of rank one.

Any $\bar{n}\in\bar{N}(E)$ can be written uniquely in the form $\bar{n}=\exp Z$, $Z\in\bar{\mathfrak{n}}(E)$. We write as $Z=\log\bar{n}$. Then we define

$$|\bar{n}|=||\log\bar{n}||\,.$$

We have $|\bar{n}^{\exp t(X^0/2)}|=e^{-tX^0}|\bar{n}|$ for $\bar{n}^{\exp t(X^0/2)}=\left(\exp t\frac{X^0}{2}\right)\bar{n}\exp\left(-t\frac{X^0}{2}\right)$ since $\bar{\mathfrak{n}}(E)$ is $(-1)$-eigenspace of $ad\frac{1}{2}X^0$.

**Lemma 4.** *There exists a constant $C_3$ such that*

$$|\bar{n}\,\bar{n}'|\leqslant C_3(|\bar{n}|+|\bar{n}'|)$$

*for all $\bar{n}$, $\bar{n}'\in\bar{N}(E)$.*

Proof. The proof is quite same as that of Lemma 2.3 in Korányi [11]. Let $V_t=\{\bar{n}\in\bar{N}(E);\ |\bar{n}|\leqslant e^t\}$ for $t\in\boldsymbol{R}$. The sets $V_t$ are compact and converge to $\bar{N}(E)$ as $t\to\infty$. Then there exists $r>0$ such that $V_0\cdot V_0\subset V_r$. We put $C_3=e^r$. By the above remark $V_t=V_0^{\exp\{-t(X^0/2)\}}$. For $\bar{n}$, $\bar{n}'\in\bar{N}(E)$ we write $|\bar{n}|=e^t$, $|\bar{n}'|=e^{t'}$, and let $\tau=\operatorname{Max}\{t,t'\}$. Then $\bar{n}\,\bar{n}'\in V_t$, $V_{t'}\subset VV=(V_0\cdot V_0)^{\exp(-X^0/2)}\subset V_{\tau+r}$ and so $|\bar{n}\,\bar{n}'|\leqslant e^{\tau+r}\leqslant e^r(|\bar{n}|+|\bar{n}'|)$. Q.E.D.

**Lemma 5.** *By $\bar{N}(E)$-hull of* $\exp(B_\delta)$, *we mean the union of all $\bar{N}(E)$-translates of* $\exp(B_\delta)$ *which have non-empty intersection with* $\exp(B_\delta)$. *Then there is a constant $C_4$ such that for each $\delta>0$,*

$$\bar{N}(E)\text{-hull of } \exp(B_\delta)\subset\exp(B_{C_4\delta}).$$

Proof. Let $\bar{n}\exp(B_\delta)\cap\exp(B_\delta)\neq\phi$ for $\bar{n}\in\bar{N}(E)$ and $\bar{n}\,\bar{n}_1=\bar{n}_2$ for $\bar{n}_1, \bar{n}_2\in\exp(B_\delta)$. Then $|\bar{n}|=|\bar{n}_2\bar{n}_1^{-1}|\leqslant C_3(|\bar{n}_1|+|\bar{n}_2|)\leqslant 2C_3\delta$ by Lemma 4. Hence for each $\bar{n}_3\in\exp(B_\delta)$, we have

$$|\bar{n}\,\bar{n}_3|\leqslant C_3(|\bar{n}|+|\bar{n}_3|)\leqslant C_3(2C_3\delta+\delta)=(2C_3^2+C_3)\delta.$$

Therefore $C_4=2C_3^2+C_3$ is a desired constant. Q.E.D.

The mapping $\gamma$ of $G$ onto $K/L$ which sends $g$ into $k(g)L$ is an injective real analytic mapping of $\bar{N}(E)$ onto a dense open subset of $K/L$. By the continuity of the action of $K$ on $K/L$, there exist open subsets $U\subset K$, $\tilde{V}\subset K/L$ with $e\in U$, $eL\in\tilde{V}$ such that $U\tilde{V}\subset\gamma(\bar{N}(E))\subset K/L$. We put $V=\gamma^{-1}(\tilde{V})\subset\bar{N}(E)$. The function $\gamma^{-1}$ is defined at each point of $\tilde{V}$ since $\tilde{V}=e\tilde{V}\subset\gamma(\bar{N}(E))$. For $g\in G$ and $\bar{n}\in\bar{N}(E)$, we put

$$g\cdot\bar{n}=\gamma^{-1}(g\cdot\gamma(\bar{n}))$$

if the right hand side is defined. If $k\in U$ and $\bar{n}\in V$, then $k\cdot\gamma(\bar{n})\in U\tilde{V}$ and $k\cdot\bar{n}=\gamma^{-1}(k\cdot\gamma(\bar{n}))$ is defined. We put $\bar{n}(k)=\gamma^{-1}(kL)$ for $k\in U$. We consider the mapping $U\times V\to\bar{N}(E)$ defined by

$$(k, \bar{n})\mapsto\bar{n}(k)^{-1}(k\cdot\bar{n})\qquad\text{for}\quad k\in U,\ \bar{n}\in V. \tag{11}$$

Then we obtain the following Lemma, which, together with Lemma 5, is essential for proof of the covering theorem.

**Lemma 6.** *There exist a neighborhood $W_1$ of $e$ in $\bar{N}(E)$, a neighborhood $W_2$ of $e$ in $K$ and a constant $C_5>0$ such that if $k\in W_2$ and* $\exp(B_\delta)\subset W_1$, *then* $\bar{n}(k)^{-1}(k\cdot\exp(B_\delta))\subset B_{C_5\delta}$.

Proof. Let $\nu$ be the dimension of $K$ and $d$ the dimension of $\bar{N}(E)$. We fix any basis $\{X_i\}$ of $\bar{\mathfrak{n}}(E)$ and define coordinates of $\bar{N}(E)$ by

$$\exp\left(\sum_{j=1}^{d}x_iX_i\right)\mapsto(x_1, \cdots, x_d).$$

Restrict the coordinates to the open set $V\subset\bar{N}(E)$ and choose an open coordinate neighborhood $U_1\subset U$ of $e$ in $K$ with local coordinates $(k_1, \cdots, k_\nu)$, $(k_1(e), \cdots, k_\nu(e))=(0, \cdots, 0)$. We will describe the mapping (11) by these coordinates $x_i$, $k_j$. We choose neighborhoods $W_1$, $W_2$ such that $W_1\subset V\cap\exp(B_1)$, $W_2\subset U_1$, $W_2$ has compact closure and these power series of coordinates of

$\bar{n}(k)^{-1}(k\cdot\bar{n})$ converge in an open neighborhood of the closure of $W_1\times W_2$. We can rearrange the terms of these power series to write the $l$-th coordinate of $\bar{n}(k)^{-1}(k\cdot\bar{n})$ as

$$a_l(k)+\sum_{i=1}^{d} a_{li}(k)x_i+\sum_{i,j}^{d} a_{lij}(\bar{n},k)x_ix_j\,,\qquad l=1,\cdots,d$$

where $a_l(k)$, $a_{li}(k)$ and $a_{lij}(\bar{n},k)$ are real analytic functions of $\bar{n}\in W_1$ and $k\in W_2$.

The terms $a_l(k)$ vanish on $W_2\subset K$ since $\bar{n}(k)^{-1}(e\cdot k)=e$. There exist $C_6, C_7>0$ such that for each $l, i, j$, $|(a_{lij}(\bar{n},k)|\leqslant C_6$ on the compact closure of $W_1\times W_2$ and $\max_{1\leqslant i\leqslant d}|x_i|\leqslant C_7||X||$ for $X=\sum_{i=1}^{d}x_iX_i\in\bar{\mathfrak{n}}(E)$. Then $|\sum_{i,j}a_{lij}(\bar{n},k)x_ix_j|\leqslant C_6C_7^2||\log\bar{n}||$ on the closure of $W_1\times W_2$. Hence we obtain

$$\bar{n}(k)^{-1}(k\cdot\bar{n})=\exp\left(\sum a_{li}(k)x_i+Z\right)$$

where $||Z||\leqslant C_6C_7^2||\log\bar{n}||^2$ for $\bar{n}\in W_1$ and $k\in W_2$.

For fixed $k\in W_2$, the matrix $(a_{li}(k))$ is the Jacobian matrix of the transformation

$$\bar{n}\mapsto\bar{n}(k)^{-1}(k\cdot\bar{n})\qquad\text{for}\quad\bar{n}\in\bar{N}(E)\,. \tag{12}$$

Since $k\in W_2\subset U_1\subset U$ and $(U)(eL)\subset\gamma(\bar{N}(E))$, we can write $\bar{n}(k)=\gamma^{-1}(kL)=kb$ by uniquely determined $b\in\bar{B}(E)$ because the restriction of $\gamma$ to $\bar{N}(E)$ is an injection.

Then the mapping (12) is the same as the mapping

$$\bar{n}\mapsto b^{-1}\cdot\bar{n}\qquad\text{for}\quad\bar{n}\in\bar{N}(E) \tag{13}$$

In fact, $\gamma^{-1}$ is defined on $k\bar{n}B(E)$ for $k\in W_2$ and $\bar{n}\in W_1$ and we have $b^{-1}\bar{n}B(E)=b^{-1}k^{-1}\gamma^{-1}(k\bar{n}B(E))B(E)=\bar{n}(k)^{-1}\gamma^{-1}(k\bar{n}B(E))B(E)=\gamma(\bar{n}(k)^{-1}\gamma^{-1}(k\bar{n}B(E)))=\gamma(\bar{n}(k)^{-1}\gamma^{-1}(k\cdot\gamma(\bar{n})))=\gamma(\bar{n}(k))^{-1}(k\cdot\bar{n}))$.

The differential of the mapping (13) at $e\in\bar{N}(E)$ is given by

$$X\mapsto P_{\bar{\mathfrak{n}}(E)}Ad(b^{-1})X\qquad\text{for}\quad x\in\bar{\mathfrak{n}}(E)$$

where $P_{\bar{\mathfrak{n}}(E)}$ is the projection of $\mathfrak{g}$ onto $\bar{\mathfrak{n}}(E)$ along the decomposition $\mathfrak{g}=\bar{\mathfrak{n}}(E)+\mathfrak{b}(E)$, since the mapping (13) is the composite of the conjugation of $b^{-1}$, the quotient map $G\to G/B(E)$ and the map $\gamma^{-1}$.

Now we consider the operator $P_{\bar{\mathfrak{n}}(E)}Ad(b^{-1})$. The restriction of $P_{\bar{\mathfrak{n}}(E)}Ad(b^{-1})$ to $\bar{\mathfrak{n}}(E)$ is a bounded operator on $\bar{\mathfrak{n}}(E)$ with respect to the norm $||\ ||$. Let $||P_{\bar{\mathfrak{n}}(E)}Ad(b^{-1})|_{\bar{\mathfrak{n}}(E)}||$ be the operator norm of $P_{\bar{\mathfrak{n}}(E)}Ad(b^{-1})$ on $\bar{\mathfrak{n}}(E)$. Then since the closure $\bar{W}_2$ of $W_2$ is compact, $C_8=\sup_{\substack{k\in\bar{W}_2\\ \bar{n}(k)=kb}}||P_{\bar{\mathfrak{n}}(E)}Ad(b^{-1})|_{\bar{\mathfrak{n}}(E)}||$ is finite and we have $||P_{\bar{\mathfrak{n}}(E)}Ad(b^{-1})X||\leqslant C_8||X||$ for all $X\in\bar{\mathfrak{n}}(E)$ and $k\in W_2$.

Consequently we have for $\bar{n}\in W_1$ and $k\in W_2$,

$$||\log(\bar{n}(k)^{-1}(k\cdot\bar{n}))|| = ||a_{li}(k)x_i+Z|| \leqslant (C_6C_7^2+C_8)||\log \bar{n}||.$$

Therefore we conclude

$$\bar{n}(k)^{-1}(k\cdot\exp(B_\delta)) \subset B_{C_5\delta}, \quad C_5 = C_6C_7^2+C_8$$

for $\exp(B_\delta)\subset W_1$ and $k\in W_2$. Q.E.D.

By *K-hull* of $\bar{B}_\delta$, we mean the union of all $K$-translates of $\bar{B}_\delta$ which have non-empty intersection with $\bar{B}_\delta$.

**Proposition 4.** $\sup_{0<\delta<\infty} \frac{\mu_E(K\text{-hull of } \bar{B}_\delta)}{\mu_E(\bar{B}_\delta)} < \infty$

Proof. Let $W_1$ and $W_2$ be neighborhoods as in Lemma 6. Let $k\in W_2$, $k\cdot\exp(B_\delta)\cap\exp(B_\delta)\neq\phi$ and $\exp(B_\delta)\subset W_1$. Then $\bar{n}(k)\exp(B_{C_5\delta})\cap\exp(B_{C_5\delta})\subset \bar{n}(k)[\bar{n}(k)^{-1}(k\cdot\exp(B_\delta))]\cap\exp(B_\delta)=k\cdot\exp(B_\delta)\cap\exp(B_\delta)\neq\phi$. Lemma 5 shows that $k\cdot\exp(B_\delta)=\bar{n}(k)[\bar{n}(k)^{-1}(k\cdot\exp(B_\delta))]\subset\bar{n}(k)\exp(B_{C_5\delta})\subset\exp(B_{C_4C_5\delta})$. Hence we have $k\bar{B}_\delta\subset\bar{B}_{C_9\delta}$ with $C_9=C_4C_5$.

There exists a number $\delta_0>0$ such that $\exp(B_\delta)$ is included in $W_1$ for any $\delta<\delta_0$. We may prove that

$$\sup_{\delta\leqslant\delta_0} \frac{\mu_E(K\text{-hull of } \bar{B}_\delta)}{\mu_E(\bar{B}_\delta)} < \infty \tag{14}$$

since $\mu_E(K/L)=1$.

Now we assume that (14) is false. Then there exist a sequence $0<\delta_n\leqslant\delta_0$ and $k_n\in K$ such that $k_n\bar{B}_{\delta_n}\cap\bar{B}_{\delta_n}\neq\phi$ and $k_n\bar{B}_{\delta_n}\not\subset\bar{B}_{C_9\delta_n}$ since there exists a constant $C_{10}$ such that $\frac{\mu_E(\bar{B}_{C_9})}{\mu_E(\bar{B}_\delta)}\leqslant C_{10}$ for each $\delta\leqslant\delta_0$. Moreover we may assume $\delta_n\to 0$ as $n\to\infty$ since $\mu_E(K/L)=1$. Let $\sigma$ be the quotient mapping of $K$ onto $K/L$. Since $k\bar{B}_\delta=kl\bar{B}_\delta$ for $l\in L$ and $k\in K$, it follows from the first argument that if $k\in\sigma^{-1}(\sigma(W_2))$, $\delta\leqslant\delta_0$ and $k\bar{B}_\delta\cap\bar{B}_\delta\neq\phi$, then $k\bar{B}_\delta\subset\bar{B}_{C_9\delta}$. Therefore $\sigma(k_n)$ is not in the neighborhood $\sigma(W_2)$ of $eL$. We may suppose $k_n$ converges to some point $k_0\in K$ with $\sigma(k_0)\neq eL$ since $K$ is compact. If $p_n\in k_n\bar{B}_{\delta_n}\cap\bar{B}_{\delta_n}$, $p_n$ converges to $eL$ since $\bar{B}_{\delta_n}$ shrinks to $eL$ as $n\to\infty$. But $p_n=k_nq_n$ with $q_n\in\bar{B}_{\delta_n}$, $q_n\to eL$ as $n\to\infty$. Therefore we obtain $eL=k_0eL$ or $\sigma(k_0)=eL$, a contradiction. Q.E.D.

Proof of Theorem 2′. We put

$$C'' = \sup_{-\infty<t<\infty} \frac{\mu_E(K\text{-hull of } \bar{B}_{e^t})}{\mu_E(\bar{B}_{e^{t-1}})}.$$

Then Proposition 4 implies that $1<C''<\infty$. Let $T_1=\sup\{t_k;\, k\bar{B}_{e^{t_k}}$ is associated to $kL\in U\}$. If $T_1=+\infty$, then we can find a set $k\cdot\bar{B}_{e^{t_k}}$ with measure as close to 1 as we like, and the conclusion of the theorem follows since $1<C''<\infty$. We assume from now on that $T_1<\infty$. We construct $R_n$, $T_n$ and $k_n\bar{B}_{e^{t_n}}$ in the following process: Let $R_1$ be the family $\{k\bar{B}_{e^{t_k}}\}$ of all associated sets. Taking a set $k_1\bar{B}_{e^{t_1}}\in R_1$ with $T_1-1\leqslant t_1\leqslant T_1$, we put $R_2=\{k\bar{B}_{e^{t_k}}\in R_1;\ k\bar{B}_{e^{t_k}}\cap k_1\bar{B}_{e^{t_1}}=\phi\}$. If $R_2=\phi$, then our process is over. If $R_2\neq\phi$, then we put $T_2=\sup\{t_k;\ k\bar{B}_{e^{t_k}}\in R_2\}$. Taking a set $k_2\bar{B}_{e^{t_2}}\in R_2$ with $T_2-1\leqslant t_2\leqslant T_2$, we put $R_3=\{k\bar{B}_{e^{t_k}}\in R_2;\ k\bar{B}_{e^{t_k}}\cap k_2\bar{B}_{e^{t_2}}=\phi\}$ and our process is continued inductively.

If $V_n$ is the union of the members of $R_n-R_{n+1}$ and $V_0$ is the union of the members of $R_1$, then $V_0=\bigcup_{n=1}^{\infty}V_n$. Since $U\subset V_0$, we obtain $\mu_E(U)\leqslant\sum_{n=1}^{\infty}\mu_E(V_n)$. The proof will be complete if we show that $\mu_E(V_n)\leqslant C''\mu_E(k_n\bar{B}_{e^{t_n}})$. Let $k\bar{B}_{e^{t_k}}\in R_n-R_{n+1}$. Then $T_n\geqslant t_k$ and $k\bar{B}_{e^{t_k}}\cap k_n\bar{B}_{e^{t_n}}\neq\phi$. Thus $k\bar{B}_{e^{T_n}}\cap k_n\bar{B}_{e^{T_n}}\neq\phi$, $k_n^{-1}k\bar{B}_{e^{T_n}}\cap\bar{B}_{e^{T_n}}\neq\phi$, $k_n^{-1}k\bar{B}_{e^{T_n}}\subset K$-hull of $\bar{B}_{e^{T_n}}$ and $k\bar{B}_{e^{t_k}}\subset k_n$ (K-hull of $\bar{B}_{e^{T_n}}$). Hence $V_n\subset k_n$ ($K$-hull of $B_{e^{T_k}}$). From the definition of $C''$ and the inequality $T_n-1\leqslant t_n$, we obtain $\mu_E(V_n)\leqslant\mu_E(k_n(K\text{-hull of }\bar{B}_{e^{T_n}}))\leqslant C''\mu_E(\bar{B}_{e^{T_n-1}})\leqslant C''\mu_E(B_{e^{t_n}})=C''\mu_E(k_n\bar{B}_{e^{t_n}})$. Q.E.D.

REMARK. From the definition of the maximal function $f^*$ for an integrable function $f$ on $K/L$ and Lemma 3, we have

$$f^*(k_0L)=\sup_{0<\delta<\infty}\frac{1}{\mu_E(\bar{B}_\delta)}\int_{\bar{B}_\delta}|f(k_0kL)|\,d\mu_E(kL)\qquad\text{for}\quad k_0L\in K/L\,.$$

**Theorem 3.** (Maximal theorem)

*For an integrable function $f$ on $K/L$ and any real number $\xi>0$, we obtain the following inequalities:*

$$\text{(i)}\quad \mu_E\{kL\in K/L;\,f^*(kL)>\xi\}\leqslant\frac{C''}{\xi}\int_{K/L}|f(kL)|\,d\mu_E(kL)\tag{15}$$

$$\text{(ii)}\quad \mu_E\{kL\in K/L;\,f^*(kL)>\xi\}\leqslant\frac{2C''}{\xi}\int_{|f(kL)|\geqslant\frac{1}{2}\xi}|f(kL)|\,d\mu_E(kL)\tag{16}$$

*where $C''$ is the same constant as in Theorem 2′.*

Proof. Let $U=\{kL\in K/L;\,f^*(kL)>\xi\}$. From the above Remark, for each $k_0L\in U$ there exists $\bar{B}_{\delta_0}$ such that

$$\int_{k_0\bar{B}_{\delta_0}}|f(kL)|\,d\mu_E(kL)\geqslant\xi\mu_E(\bar{B}_{\delta_0})=\xi\mu_E(k_0\bar{B}_{\delta_0})\,.$$

Theorem 2′ says that there exists a disjoint subfamily $\{k_j\bar{B}_{\delta_j}\}$ of $\{k_0\bar{B}_{\delta_0};\ k_0L\in U\}$ such that $C''\sum_{j=1}\mu_E(k_j\bar{B}_{\delta_j})\geqslant\mu_E(U)$. Therefore

$$\int_{K/L}|f(kL)|\,d\mu_E(kL)\geqslant\sum_{j=1}^{\infty}\int_{k_j\bar{B}_{\delta_j}}|f(kL)|\,d\mu_E(kL)\geqslant\xi\sum_{j=1}^{\infty}\mu_E(k_j\bar{B}_{\delta_j})\geqslant\frac{\xi}{C''}\mu_E(U)$$

and the inequality (i) follows. For the proof of (ii), we define an integrable function $h$ on $K/L$ by

$$h(kL) = \begin{cases} f(kL) & \text{if } |f(kL)| \geqslant \frac{1}{2}\xi \\ 0 & \text{otherwise.} \end{cases}$$

Then $h^*(kL)+\frac{1}{2}\xi \geqslant f^*(kL)$. Hence by (i)

$$\mu_E(U) \leqslant \mu_E\left\{kL;\, h^*(kL) > \frac{1}{2}\xi\right\} \leqslant \frac{2C''}{\xi}\int_{K/L} |h(kL)|\, d\mu_E(kL)$$
$$= \frac{2C''}{\xi}\int_{|f(KL)|\geqslant\frac{1}{2}\xi}\!\!\int |f(kL)|\, d\mu_E(kL)\,. \qquad \text{Q.E.D.}$$

Proof of Theorem 1. For any $\varepsilon_1>0$, we can write as $f=f_1+f_2$ where $f_1$ is continuous and $f_2 \in L^1(K/L)$ with $L^1$-norm $\|f_2\|_1<\varepsilon^2$. Let $h_1$, $h_2$ and $h$ be the Poisson integrals of $f_1$, $f_2$ and $f$, respectively. Since $f_1$ is continuous, we can choose (Korányi-Helgason [5]) $T>0$ large enough such that $t>T$ implies

$$|h_1(ka_tK)-f_1(kL)|<\varepsilon \qquad \text{for all} \quad k\in K$$

where $a_t=\exp tX^0$. If $U_1=\{kL\in K/L;\ |f_1(kL)-f(kL)|\geqslant\varepsilon\}=\{kL\in K/L;\ |f_2(kL)|\geqslant\varepsilon\}$, then $\mu_E(U_1)<\varepsilon$ since $\varepsilon\mu_E(U_1)\leqslant\|f_2\|_1<\varepsilon^2$. Therefore except in the set $U_1$ of measure $<\varepsilon$,

$$|h_1(ka_tK)-f(kL)|<2\varepsilon \qquad \text{for} \quad t>T\,.$$

Let $U_2=\left\{kL\in K/L;\, f_2^*(kL)>\frac{\varepsilon}{C'}\right\}$ where $C'$ is a constant in Proposition 3. Then we have by Theorem 3 (i)

$$\mu_E(U_2)\leqslant\frac{C'C''}{\varepsilon}\int_{K/L}|f_2(kL)|\,d\mu_E(kL)\leqslant\frac{C'C''}{\varepsilon}\cdot\varepsilon^2=C'C''\varepsilon\,.$$

Hence we have by Proposition 3

$$|h_2(ka_tK)|\leqslant\varepsilon \qquad \text{for all} \quad t>\tanh^{-1}\left(\frac{1}{2}\right)$$

except in the set $U_2$ of measure $\leqslant C'C''\varepsilon$. Therefore, except in the set $U_1\cup U_2$ of measure $(C'C''+1)\varepsilon$,

$$|h(ka_tK)-f(kL)|<3\varepsilon \qquad \text{for} \quad t>\max\left(T,\tanh^{-1}\left(\frac{1}{2}\right)\right).$$

Replacing $\varepsilon$ by $2^{-n}\varepsilon$ and taking $U_1^{(n)}$ and $U_2^{(n)}$ in place of $U_1$ and $U_2$, let $U$ be the union of all $U_1^{(n)}\cup U_2^{(n)}$, $n=1, 2, \cdots$. Then we have

$$\lim_{t\to\infty} h(ka_tK) = f(kL)$$

except in the set $U$ of measure $\leqslant 2(C'C''+1)\varepsilon$. Since $\varepsilon$ is arbitrary,

$$\lim_{t\to\infty} h(ka_tK) = f(kL)$$

almost everywhere on $K/L$ with respect to $\mu_E$. Q.E.D.

## 6. Inequalities of Hardy-Littlewood

In this section, we shall prove inequlities of Hardy-Littlewood in the same way as Rauch's proof [15] of the inequalities for hermitian hyperbolic spaces. We assume again that $G/K$ is an irreducible hermitian symmetric space of tube type. For a function $f$ on $K/L$, we define a real valued non-negative function $\log^+|f|$ on $K/L$ by

$$(\log^+|f|)(kL) = \begin{cases} \log|f(kL)| & \text{if } |f(kL)|\geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

For a measurable function $\varphi$ on $K/L$, we define a decreasing function $\mu_\varphi$ on $\boldsymbol{R}^+=[0, \infty)$ by

$$\mu_\varphi(\xi) = \mu_E\{kL\in K/L;\ |\varphi(kL)|>\xi\} \qquad \text{for} \quad \xi\in\boldsymbol{R}^+.$$

Then for any non-negative increasing function $s$ on $\boldsymbol{R}^+$ we obtain

$$\int_{K/L} s(|\varphi(kL)|)d\mu_E(kL) = -\int_0^\infty s(\xi)d\mu_\varphi(\xi) \tag{17}$$

where the right hand side means the Lebesgue-Stieltjes integral with respect to $\mu_\varphi$.

**Proposition 5.** *There exist positive constants $C_{11}$, $\alpha$ and $\beta$ such that (i) if $p>1$, $\int_{K/L}|f^*(kL)|^p d\mu_E(kL)\leqslant C_{11}\|f\|_p^p$ for all $f\in L^p(K/L)$ (ii) if $p=1$, $\int_{K/L}|f^*(kL)|d\mu_E(kL)\leqslant \alpha\int_{K/L}|f(kL)|\log^+(|f(kL)|)d\mu_E+\beta$ for all functions $f$ such that $f\log^+ f\in L^1(K/L)$.*

Proof. Since we have from Theorem 3 (ii) and (17)

$$\mu_{f^*}(\xi)\leqslant\frac{2C''}{\xi}\int_{|f(KL)|\geqslant\frac{1}{2}\xi}|d\mu_E|(kL) = -\frac{2C''}{\xi}\int_{\frac{1}{2}\xi}^\infty x d\mu_f(x) \qquad \text{for} \quad \xi>0,$$

we have

$$\int_0^\infty \mu_{f^*}(\xi)\xi^{p-1}d\xi\leqslant -2C''\int_0^\infty \xi^{p-2}d\xi\int_{\frac{1}{2}\xi}^\infty x d\mu_f(x) = -2C''\int_0^\infty x d\mu_f(x)\int_0^{2x}\xi^{p-2}d\xi.$$

Let $p>1$. Then

$$\int_0^\infty \mu_{f^*}(\xi)\xi^{p-1}d\xi = -\frac{2^{p-1}}{p-1}(2C'')\int_0^\infty x^p d\mu_f(x)$$
$$= \frac{2^{p-1}}{p-1}(2C'')\int_{K/L} |f(kL)|^p d\mu_E(kL) < \infty .$$

Hence we obtain

$$\lim_{\xi\to 0}\int_\xi^{2\xi} \mu_{f^*}(x)x^{p-1}dx = 0 .$$

Since $\mu_{f^*}$ is a decreasing function on $\boldsymbol{R}^+$, we have

$$\mu_{f^*}(2\xi)\xi^p\frac{2^p-1}{p} = \mu_{f^*}(2\xi)\int_\xi^{2\xi} x^{p-1}dx \leqslant \int_\xi^{2\xi} \mu_{f^*}(x)x^{p-1}dx .$$

Therefore $\lim_{\xi\to\infty}\mu_{f^*}(2\xi)\xi^p=0$, and making use of integration by parts of Lebesgue-Stieltjes integral, we obtain

$$\int_{K/L} f^*(kL)^p d\mu_E(kL) = -\int_0^\infty x^p d\mu_{f^*}(x) = \int_0^\infty \mu_{f^*}(x)px^{p-1}dx$$
$$\leqslant \frac{p}{p-1}2^{p-1}(2C'')\int_{K/L} |f(kL)|^p d\mu_E(kL) .$$

If $p=1$, then we have

$$\int_1^\infty \mu_{f^*}(x)dx \leqslant -2C''\int_1^\infty y d\mu_f(y)\int_1^{2y}\frac{dx}{x} = -2C''\int_1^\infty y\log(2y)d\mu_f(y)$$
$$\leqslant 2C''\int_{K/L} |f(kL)|\log^+(|f(kL)|)d\mu_E(kL)$$
$$+2C''\log 2\int_{K/L} |f(kL)|d\mu_E(kL) .$$

Since $|f|\leqslant 1+|f|\log^+|f|$, we have

$$\int_{K/L} |f(kL)|d\mu_E(kL) \leqslant \mu_E(K/L)+\int_{K/L} |f(kL)|\log^+(|f(kL)|)d\mu_E(kL)$$

and

$$\int_0^1 \mu_{f^*}(x)dx \leqslant \mu_E(K/L) .$$

Since

$$\int_{K/L} |f^*(kL)|d\mu_E(kL) = -\int_0^\infty x d\mu_{f^*}(x) = \int_0^\infty \mu_{f^*}(x)dx$$
$$= \int_0^1 \mu_{f^*}(x)dx + \int_1^\infty \mu_{f^*}(x)dx ,$$

the second inequality follows. Q.E.D.

DEFINITION. For an integrable function $f$ on $K/L$, we define a function $f_*$ on $K/L$ by

$$f_*(k_0L) \doteq \sup_{\frac{1}{2}<\tanh t<1} \int_{K/L} |f(kL)| P_E(k_0 a_t K, kL) d\mu_E(kL) \qquad \text{for} \quad k_0L \in K/L$$

where $a_t = \exp tX^0$. Since $L$ centralizes $X^0$, $f_*$ is a well defined function on $K/L$. Since the supremum over rational $t$ gives the same answer, $f_*$ is a measurable function on $K/L$.

**Theorem 4.** (Inequalities of Hardy-Littlewood) *There exist constants $C_{13}$, $\alpha'$ and $\beta'$ such that*

(i) if $p>1$, $\int_{K/L} |f_*(kL)|^p d\mu_E(kL) \leqslant C_{13} \|f\|_p^p$ *for all* $f \in L^p(K/L)$

(ii) if $p=1$, $\int_{K/L} |f_*(kL)| d\mu_E(kL) \leqslant \alpha' \int_{K/L} |f(kL)| \log^+(|f(kL)|) d\mu_E(kL) + \beta'$

*for all $f$ such that $f \log^+ |f| \in L^1(K/L)$.*

Proof. These are immediate consequences of Propositions 3 and 5. Q.E.D.

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## References


[1] S. Bochner: *Classes of holomorphic functions of several variables in circular domains,* Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 721–723.

[2] H. Furstenberg: *A poisson formula for semi-simple Lie groups,* Ann. of Math. (2) **77** (1963), 335–386.

[3] G.H. Hardy and J.E. Littlewood: *A maxiaml theorem with function-theoretic applications,* Acta Math. **54** (1930), 81–116.

[4] Harish-Chandra: *Representations of semi-simple Lie groups*: VI, Amer. J. Math. **78** (1956), 564–628.

[5] S. Helgason and A. Korányi: *A Fatou-type theorem for harmonic functions for symmetric spaces,* Bull, Amer. Math. Soc. **74** (1968), 258–263

[6] S. Helgason: *A duality for symmetric space with applications to group representations,* Advance in Math. **5** (1970), 1–154.

[7] A.W. Knapp: *Fatou's theorem for symmetric spaces*: I, Ann. of Math. (2) **88** (1968), 106–127.

[8] A.W Knapp and R.E. Williamson: *Poisson integrals and semisiple Lie groups,* J. Analyse Math. **24** (1971) 53–76.

[9] A. Korányi: *The Poisson integral for generalized halfplanes and bounded symmetric domains,* Ann. of Math. (2) **82** (1965), 332–250.

[10] ———: *Harmonic functions on hermitian hyperbolic space,* Trans. Amer. Math. Soc. **135** (1969), 50–7516.

[11] A. Korányi: *Boundary behavior of Poisson integrals on symmetric spaces,* Trans. Amer. Math. Soc. **140** (1969), 393–409.

[12] ———and J.A. Wolf: *Realization of hermitian symmetric spaces as generalized half-planes,* Ann. of Math. (2) **81** (1965), 265–288.

[13] C.C. Moore: *Compactifications of symmetric spaces,* Amer. J. Math. **86** (1964), 201–218, 358–378.

[14] I.I. Pjateckii-Šapiro: Geometry of Classical Domains and Theory of Automorphic Functions, Moscow, 1961.

[15] H.E. Rauch: *Harmonic and ayalytic functions of several variables and the maximal theorem of Hardy and Littlewood,* Canad. J. Math. **8** (1956), 171–183.

[16] E.M. Stein and N.J. Weiss: *On the convergence of Poisson integrals,* Trans. Amer Math. Soc. **140** (1969), 35–54.

[17] E.M. Stein, G. Weiss and M. Weiss: *$H^p$ class of holomorphic functions in tube domains,* Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1035–1039.

[18] M. Takeuchi: *Cell decompositions and Morse equalities on certain symmetric spaces,* J. Fac. Sci. Univ. Tokyo Sect. I. **XII** (1965), 81–192.

[19] N.J. Weiss: *Almost everywhere convergence of Poisson integral on generalized half-planes,* Bull. Amer. Math. Soc. **74** (1968), 533–537.

[20] N.J. Weiss: *Almost everywhere convergence of Poisson integrals on tube domain, over cones,* Trans. Amer. Math. Soc. **129** (1967), 283–307.

[21] A. Zygmund: Trigonometric Series, 2nd ed. Cambrige Univ. Press, New York, 1959.