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ON EMBEDDABLE 1-CONVEX SPACES

VIOREL VÂJÂITU

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1. Introduction

Throughout this paper all complex spaces are assumed to be reduced and with countable topology.

Let X be a complex space. X is said to be *embeddable* if it can be realized as a complex analytic subset of $\mathbb{C}^m \times \mathbb{P}^n$ for some positive integers m and n. For instance, one checks that a complex curve of bounded Zariski dimension is embeddable.

We say that X is 1-*convex* if X is a modification at finitely many points of a Stein space Y, *i.e.*, there exist a compact analytic set $S \subset X$ without isolated points and a proper holomorphic map $\pi : X \longrightarrow Y$ such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ and π induces an isomorphism between $X \setminus S$ and $Y \setminus \pi(S)$. S is called the *exceptional set* of X and Y the *Remmert's reduction* of X. See [16] for further properties of 1-convex spaces.

A criterion of Schneider [18] says that a 1-*convex space X of bounded Zariski dimension is embeddable if, and only if, there is a holomorphic line bundle L over X such that $L|_S$ is ample*.

Using this, Bănică [3] proved that a 1-convex complex surface X of bounded Zariski dimension is *embeddable* provided that X does not admit compact two dimensional irreducible components. By extending this Colţoiu ([4], [5]) showed that every connected 1-convex manifold X with 1-dimensional exceptional set is embeddable if $\dim(X) > 3$. This is true also for threefolds X with some exceptions when the exceptional set contains a $\mathbb{P}^1$ ([5]).

In this short note we reconsider Colţoiu's example from another point of view. This is based on the following proposition which may be of independent interest.

Proposition 1. *Let $Y \subset \mathbb{P}^n$ be a hypersurface of degree d with isolated singularities, $\pi : M \longrightarrow Y$ a resolution of singularities, and $H \subset \mathbb{P}^n$ a hyperplane which avoids the singular locus of Y and such that $\Gamma := H \cap Y$ is smooth. Set $X := M \setminus \pi^{-1}(\Gamma)$. Then for $n \geq 4$ the following statements are equivalent*:
(a) *X is embeddable.*
(b) *X is Kähler.*
(c) *M is projective.*

By this and an example due to Moishezon [12] (see also [6]) we obtain:

Theorem 1. *There exists a* 1*-convex threefold* X *with exceptional set* $\mathbb{P}^1$ *such that* X *is not Kähler*; a fortiori X *is not embeddable.*

For the proof of Proposition 1 we use several short exact sequences, Bott's formula, Thom's isomorphism, and some facts on pluriharmonic functions.

Also employing recent results due to Fujiki [9] we prove (see the next section for definitions):

Theorem 2. *Let* $\pi : X \longrightarrow Y$ *be a finite holomorphic map of complex spaces with* X *of bounded Zariski dimension. If* X *is maximal and* Y *is Hodge, then it holds*:
(a) Y *compact implies* X *projective.*
(b) Y *is* 1*-convex implies* X *is* 1*-convex and embeddable.*

REMARK 1. Note that by [23], 1-convexity is invariant under finite holomorphic surjections. However, this does not hold for embeddability.

As a consequence of Theorem 2 we improve a well-known projectivity criterion due to Grauert [10] to:

Proposition 2. *Let* X *be a compact complex space. If* X *is Hodge and maximal, then* X *is projective.*

and the embeddability result due to Th. Peternell ([17], Theorem 2.6) to:

Proposition 3. *Let* X *be a* 1*-convex space of bounded Zariski dimension such that* X *is Hodge and maximal. Then* X *is embeddable.*

2. Continuous weakly pluriharmonic functions

Let X be a complex space. As usual, $\mathcal{P}_X$ denotes the sheaf of germs of pluriharmonic functions on X. Then the canonical map $\mathcal{O}_X \longrightarrow \mathcal{P}_X$ given by $f \mapsto \operatorname{Re} f$ induces a short exact sequence

$$(\star) \qquad 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{P}_X \longrightarrow 0.$$

Consider $\widehat{\mathcal{P}}_X$:= the sheaf of *continuous weakly pluriharmonic functions*, *i.e.*, for every open subset U of X, $\widehat{\mathcal{P}}_X(U)$ consists of those $h \in C^0(U, \mathbb{R})$ which are pluriharmonic on $\operatorname{Reg}(U)$.

Clearly $\mathcal{P}_X \subseteq \widehat{\mathcal{P}}_X$, and if $\widehat{\mathcal{O}}_X$ denotes the sheaf of continuous weakly holomorphic functions, we have a natural map $\widehat{\mathcal{O}}_X \longrightarrow \widehat{\mathcal{P}}_X$ given by $f \mapsto \operatorname{Re} f$.

Here we prove:

Proposition 4. *The canonical short sequence*

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{\mathcal{O}}_X \longrightarrow \widehat{\mathcal{P}}_X \longrightarrow 0,$$

is exact.

Proof. We check only the surjectivity of $\widehat{\mathcal{O}}_X \longrightarrow \widehat{\mathcal{P}}_X$. We do this in two steps.

STEP 1. Suppose X is normal. Let $\pi : M \longrightarrow X$ be a resolution of singularities. Then $\pi_\star \mathcal{P}_M = \mathcal{P}_X$ by Proposition 2.1 in [9]. Now, since on a complex manifold a continuous real-valued function φ is pluriharmonic if and only if φ and $-\varphi$ are plurisubharmonic we obtain that $\widehat{\mathcal{P}}_X = \mathcal{P}_X$, whence the desired surjectivity in view of $(\star)$.

STEP 2. The general case. Let $\nu : Y \longrightarrow X$ be the normalization of X. Let $x_o \in X$, U an open neighborhood of x_o, and $h \in \widehat{\mathcal{P}}_X(U)$. Then, by Step 1., $h \circ \nu \in \mathcal{P}_Y(\nu^{-1}(U))$. By Proposition 2.3 in [9] after shrinking $U \ni x_o$, there is $f \in \widehat{\mathcal{O}}_X(U)$ such that $\operatorname{Re} f = h$. Note that in *loc. cit.* this is done under the additional hypothesis $h \in C^\infty(U, \mathbb{R})$. But our case follows *mutatis mutandis*, whence the proposition. □

Recall ([7], pp. 122–126) that a complex space Z is said to be *maximal* if $\mathcal{O}_Z = \widehat{\mathcal{O}}_Z$ and that every complex space X admits a *maximalization* $\widehat{X}$, *i.e.*, $\widehat{X}$ is maximal and there is a holomorphic homeomorphism $\pi : \widehat{X} \longrightarrow X$ which induces a biholomorphic map between $\widehat{X} \setminus \pi^{-1}(M(X))$ and $X \setminus M(X)$, where $M(X)$ is the non-maximal locus of X, *i.e.*, $M(X) = \{x \in X\,;\, \mathcal{O}_{X,x} \neq \widehat{\mathcal{O}}_{X,x}\}$. Clearly every normal complex space is maximal. For this reason, maximal complex spaces are also called "weakly normal".

Corollary 1. *If X is maximal, then $\mathcal{P}_X = \widehat{\mathcal{P}}_X$.*

Corollary 2. *If X is normal, then every pluriharmonic function h on* $\operatorname{Reg}(X)$ *extends uniquely to a pluriharmonic function on X.*

Proof. Since h and $-h$ extend uniquely to plurisubharmonic functions φ and ψ on X, we get $\varphi = -\psi$. Hence φ is continuous, whence φ is pluriharmonic by Corollary 1. □

By a *d-closed, real* $(1,1)$-*form* (in the sense of Grauert [10]) on a complex space X we mean, a d-closed, real $(1,1)$-form ω on $\operatorname{Reg}(X)$ such that every point $x \in X$ admits an open neighborhood U on which there is $\varphi \in C^2(U, \mathbb{R})$ with $\omega = i\partial\overline{\partial}\varphi$ on $\operatorname{Reg}(U)$. This φ is called a *local potential function* for ω. We say that ω is *Kähler* if the local potentials may be chosen strongly plurisubharmonic.

Alternatively, by Moishezon [14] we define a d-closed, real $(1,1)$-form on X as a collection $\{(U_j, \varphi_j)\}_{j\in J}$ where $\{U_j\}_j$ is an open covering of X and $\varphi_j \in C^2(U_j, \mathbb{R})$ are such that $\varphi_j - \varphi_k$ is pluriharmonic. Two such collections $\{(U_j, \varphi_j)\}_{j\in J}$ and $\{(V_k, \psi_k)\}_{k\in K}$ define the same form if $\varphi_j - \psi_k$ is pluriharmonic on $U_j \cap V_k$ for all indices j and k.

Corollary 3. *For a maximal complex space X the above two notions of d-closed, real $(1,1)$-forms coincide in an obvious sense.*

Proof. This is immediate by Corollary 1. $\square$

To every d-closed, real $(1,1)$-form ω on X we associate canonically an element of $H^1(X, \widehat{\mathcal{P}}_X)$, which in turn goes into its *de Rham* class $[\omega] \in H^2(X, \mathbb{R})$ via the cohomology sequence from Proposition 4.

We say that ω is *integral* if its de Rham class belongs to $\operatorname{Im}(H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}))$.

One has the following (see [10], proof of Satz 3)

Lemma 1. *If ω is an integral form on a maximal space X, then there is a holomorphic line bundle $L \longrightarrow X$ together with a class C^2-hermitean metric on L whose Chern form is ω. In particular, if ω is Kähler, then L is positive.*

Let X be a complex space. X is said to be *Kähler* if X has a Kähler form (in the sense of Grauert). We say that X is *Hodge* if it admits a Kähler form which is integral.

Proposition 5. *Let $\pi : Y \longrightarrow X$ be a finite holomorphic map of complex spaces such that X is Hodge. Then Y is Hodge. In particular, the maximalization $\widehat{X}$ and the normalization $X^\star$ of X are Hodge, too.*

Proof. Let $\{(U_j, \psi_j)\}_j$, $U_j \Subset X$, defines a Kähler form ω on X. Let $V_j \Subset U_j$ such that $\{V_j\}_j$ is also a covering of X. Then by [22] for every $\delta \in C^0(X, \mathbb{R})$, $\delta > 0$, there exists $\psi \in C^\infty(Y, \mathbb{R})$, $0 < \psi < \delta$, such that $\sigma_j := \psi_j \circ \pi + \psi$ are strongly plurisubharmonic on $W_j := \pi^{-1}(V_j)$ for all j; hence $\{(W_j, \sigma_j)\}_j$ defines a Kähler form $\pi^\star\omega$ on Y. Of course $\pi^\star\omega$ depends on δ and ψ, but this is irrelevant for our discussion. Moreover, in view of a canonical commutative diagram and Proposition 4, if ω is integral, then $\pi^\star\omega$ is integral too. $\square$

Now Lemma 1 and the criteria of Grauert [10] and Schneider [18] give Theorem 2.

REMARK 2. There is a compact, normal, two dimensional complex space X with only one singularity such that $\operatorname{Reg}(X)$ is Kähler, and X is *not* Kähler. (This follows from [14] and [10].)

3. Proof of proposition 1

The only nontrivial implication is (b) $\Rightarrow$ (c) which we now consider. First we state:

CLAIM. The restriction map $H^1(M, \mathcal{P}_M) \longrightarrow H^1(X, \mathcal{P}_M)$ is surjective.

The proof of this will be done in several steps.

STEP 1. *For every abelian group G we have $H^1(\Gamma, G) = 0$.*

Indeed, by a theorem of Siu [19], as $Y \setminus \Gamma$ is a Stein subspace of $\mathbb{P}^n \setminus \Gamma$, it admits a Stein open neighborhood D; thus $\mathbb{P}^n \setminus \Gamma = D \cup (\mathbb{P}^n \setminus Y)$ is a union of two Stein open subsets. On the other hand, if an n-dimensional complex manifold Ω is a union of q Stein open subsets, then $H^i_c(\Omega, G) = 0$ for $i \leq n - q$. The assertion follows easily.

STEP 2. $H^2(Y, \mathcal{O}_Y) = 0$.

For this, we let $\mathcal{I}_Y$ be the coherent ideal sheaf of Y in $\mathbb{P}^n$. Then $\mathcal{I}_Y \simeq \mathcal{O}(-[Y])$, where $[Y]$ denotes the canonical line bundle associated to the divisor defined by Y.

Now Bott's formula gives the vanishing of $H^i(\mathbb{P}^n, \mathcal{O}(k))$ for integers i, k with $1 \leq i < n$, and by the long exact cohomology sequence associated to the short exact sequence $0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$, the assertion of Step 2 results immediately.

STEP 3. *The maps $H^1(M, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O})$ and $H^2(M, \mathcal{O}) \longrightarrow H^2(X, \mathcal{O})$ are surjective and injective respectively.*

Let V be an arbitrary open neighborhood of Γ in Y. Since $Y \setminus \Gamma$ is Stein, the Mayer-Vietoris sequence for $Y = (Y \setminus \Gamma) \cup V$ and Step 2 give that the maps $H^1(V, \mathcal{O}) \longrightarrow H^1(V \setminus \Gamma, \mathcal{O})$ and $H^2(V, \mathcal{O}) \longrightarrow H^2(V \setminus \Gamma, \mathcal{O})$ are surjective and injective respectively.

Assume now $V \subset \mathrm{Reg}(Y)$; hence $\pi^{-1}(V)$ is biholomorphic to V via π. This and the above discussion plus the Mayer-Vietoris sequence for $M = X \cup \pi^{-1}(V)$ completes the proof of Step 3.

STEP 4. *$H^2(M, G) \longrightarrow H^2(X, G)$ is surjective for every abelian group G.*

We view Γ as a smooth complex hypersurface in M. The inclusion $X \subset M$ gives rise to an exact cohomology sequence (coefficients in any abelian group G)

$$\cdots \longrightarrow H^i(M, X : G) \longrightarrow H^i(M; G) \longrightarrow H^i(X; G) \longrightarrow H^{i+1}(M, X; G) \longrightarrow \cdots$$

On the other hand since Γ is a non-singular complex hypersurface, a tubular neighborhood of Γ is diffeomorphic to a neighborhood of the 0-section of the normal bundle of Γ in M. This bundle being holomorphic is naturally oriented. We thus have, see [2], a Thom isomorphism:

$$H^i(M, X; G) \cong H^{i-2}(\Gamma; G),$$

whence the assertion of Step 4 using Step 1.

($\bullet$) The proof of the claim follows by diagram chasing using Steps 3 and 4 and the next commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^1(M,\mathcal{O}) & \longrightarrow & H^1(M,\mathcal{P}) & \longrightarrow & H^2(M,\mathbb{R}) & \longrightarrow & H^2(M,\mathcal{O}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(X,\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{P}) & \longrightarrow & H^2(X,\mathbb{R}) & \longrightarrow & H^2(X,\mathcal{O}), \end{array}$$

(•) For the proof of the proposition we let $\mathcal{K}^{1,1}_M$ be the sheaf of germs of real smooth $(1,1)$-forms on M which are d-closed. As usual, $\mathcal{E}_M$ represents the sheaf of germs of smooth real functions on M. The short exact sequence on M,

$$0 \longrightarrow \mathcal{P}_M \longrightarrow \mathcal{E}_M \longrightarrow \mathcal{K}^{1,1}_M \longrightarrow 0,$$

where the last non trivial map is given by $\varphi \mapsto \sqrt{-1}\partial\overline{\partial}\varphi$, induces a commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^o(M,\mathcal{E}_M) & \longrightarrow & H^o(M,\mathcal{K}^{1,1}_M) & \longrightarrow & H^1(M,\mathcal{P}_M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^o(X,\mathcal{E}_M) & \longrightarrow & H^o(X,\mathcal{K}^{1,1}_M) & \longrightarrow & H^1(X,\mathcal{P}_M) & \longrightarrow & 0. \end{array}$$

By diagram chasing and the above claim if ω is the Kähler form of X, then there are: a smooth, d-closed, real $(1,1)$-form α on M and a smooth real-valued function φ on X such that

$$(\P) \qquad \alpha|_X - \omega = \sqrt{-1}\partial\overline{\partial}\varphi.$$

Now, select $\chi \in C^\infty(X,\mathbb{R})$ which vanishes on a neighborhood Ω of $\pi^{-1}(\mathrm{Sing}(Y))$ and equals 1 outside a compact subset of X. By (¶), the smooth $(1,1)$-form $\omega + \sqrt{-1}\partial\overline{\partial}(\chi\varphi)$ on X extends trivially to a smooth, real, and d-closed $(1,1)$-form $\widehat{\omega}$ on M.

Let β be the canonical Kähler form on $\mathbb{P}^n$. For every $c > 0$ define a d-closed $(1,1)$-form $\widetilde{\omega}_c$ on M by setting:

$$\widetilde{\omega}_c := \widehat{\omega} + c\pi^*(\beta).$$

Clearly $\widetilde{\omega}_c$ restricted to Ω is positive definite for every $c > 0$. On the other hand, there is $c > 0$ sufficiently large such that $\widetilde{\omega}_c$ is positive definite near the compact set $M \setminus \Omega$. Thus M is Kähler. Since M is Moishezon, by [13] M is projective. □

REMARK 3. In [20] a similar version to our Proposition 1, without any smoothness assumption on $H \cap Y$ and with the additional assumption that $H^2(X,\mathcal{O}_X) = 0$, is stated.

Unfortunately, the "given proof" is wrong. See Colţoiu's pertinent comments [5] for this and many, many other fatal errors, which, to our unpleasant surprise, are used again in [21].

4. Proof of theorem 1

Let $Y \subset \mathbb{P}^4$ be a hypersurface of degree $d > 2$ having a nondegenerate quadratic point y_o as its only singularity [12]. Let $\sigma : V \longrightarrow \mathbb{P}^4$ be the quadratic transform with center y_o. Set $\Sigma := \sigma^{-1}(y_o)$, $W :=$ the proper transform of Y (W is a nonsingular hypersurface in V), and $T := \Sigma \cap W \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let S be one of the two factors and $\rho : T \longrightarrow S$ the corresponding projection.

If N denotes the normal bundle of T in W, the restriction of N to each of the fibres of ρ is the negative of the hyperplane bundle, so the criterion of Nakano and Fujiki applies ([8], [15]).

In other words W is obtained by blowing-up a non singular M along a rational non singular curve S. One obtains easily a holomorphic map $\pi : M \longrightarrow Y$ which resolve the singularity y_o of Y and $S = \pi^{-1}(y_o) \simeq \mathbb{P}^1$.

On the other hand, by [6], M is not Kähler if $d > 2$. Therefore, if we choose a linear hyperplane H in $\mathbf{P}^4$, $H \not\ni y_o$, such that $H \cap Y$ is smooth, then by Proposition 1, $X := M \setminus \pi^{-1}(Y \cap H)$ is the desired example. $\square$

REMARK 4. As a counterexample for embeddability this example is due to Colţoiu [5] where by a different method he obtained that $H^1(X, \mathcal{O}_X) = 0$ under the additional hypothesis that H intersects Y transversally.

Here we emphasize the non-Kähler property of the example.

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Institute of Mathematics of the Romanian Academy
P.O. Box 1–764, RO–70700 Bucharest, Romania
e-mail: vvajaitu@stoilow.imar.ro