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ON MODULES WITH LIFTING PROPERTIES

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We have studied the lifting property on a direct sum of completely indecomposable and cyclic hollow modules over a ring R in [8].

In this note, we shall define the lifting property of decompositions with finite direct summands in §1 and give characterizations of this in terms of endomorphism rings of R-modules in §2. We shall study, in §3, R-modules with lifting properties and show that they are very closed to R-modules with direct decomposition of completely indecomposable and cyclic hollow modules when R is right noetherian.

We shall give the dual results for the extending property and its applications in the forthcoming papers.

1. Definitions

Throughout this paper we assume that a ring R contains an identity and every R-module M is a unitary right R-module. We recall here definitions in [8].

If $\operatorname{End}_{\mathbb{R}}(M)$ is a local ring, we call M a completely indecomposable module. We denote the Jacobson radical and an injective envelope of M by J(M) and E(M), respectively. By \overline{M} we denote M/J(M). If N is a submodule of M and N/J(N) is canonically monomorphic into M/J(M), then we mean \overline{N} both N/J(N) and the image of N/J(N) into M/J(M).

If J(M) is a unique maximal and small submodule in M, we call M a cyclic hollow module (actually M is cyclic). If, for each simple submodule A of \overline{M} , there exists a completely indecomposable and cyclic hollow direct summand M_1 of M such that $\overline{M}_1=A$, then we say M has the lifting property of simple modules (modulo radical). More generally, if for any direct summand B of \overline{M} , there exists a direct summand M' of M such that $\overline{M}'=B$, we say M has the lifting property of direct summands (modulo radical). Finally if, for any finite decomposition of \overline{M} ; $\overline{M}=C_1\oplus C_2\oplus \cdots \oplus C_n$, there exists a decomposition of M; $M=M_1\oplus M_2\oplus \cdots$ $\oplus M_n$ such that $\overline{M}_i=C_i$, we say M has the lifting property of decompositions with finite direct summands (modulo radical). If the above property is satisfied for any direct decompositions, we say M has the lifting property of decompositions (modulo radical).

We recall the definition in [8].

(E-I) Every epimorphism of M onto itself is an isomorphism.

Let $\{M_{\alpha}\}_{I}$ be a set of completely indecomposable modules. We assume each M_{α} satisfies (E-I). We shall define a relation \leq in $\{M_{\alpha}\}_{I}$. If $M_{\alpha} \approx M_{\beta}$, we put $M_{\alpha} \equiv M_{\beta}$. If there exists an epimorphism of M_{α} to M_{β} , we put $M_{\alpha} \geq M_{\beta}$. Then \leq defines a partial order in $\{M_{\alpha}\}_{I}$. We refer the reader for other definitions to [8]. We call briefly lifting property modulo racical lifting property.

2. Lifting property on direct sums

Let $\{M_{\alpha}\}_{I}$ be a set of completely indecomposable modules and $M = \sum_{I} \bigoplus M_{\alpha}$. In this section we shall study the lifting property of M when each M_{α} is a cyclic hollow module.

First we shall quote a well known property on semi-perfect module [11].

Proposition 1. Let P be a projective module. We assume that P|J(P) is semisimple and J(P) is small in P. Then the following conditions are equivalent:

- 1) P is semi-perfect.
- 2) P has the lifting property of simple modules.
- 3) P has the lifting property of decompositions.

Proof. It is clear from [11], Theorem 4.3 or [5], Lemma 6 and [6], Proposition 1.

We shall give an example which shows that the assumption on J(P) is necessary in the above proposition.

Let R be the ring of infinite lower-triagular and column finite matrices over a field K and let $\{e_{ii}\}$ be the set of matrix units. Then $P = \sum_{I}^{\infty} \bigoplus e_{ii}R$ is a projective module with P/J(P) semisimple. Since $\overline{e_{ii}R} \not\approx \overline{e_{jj}R}$ for $i \neq j$, P has trivially the lifting property of decompositions. However, J(P) is not small in P by [7] and so P is not semi-perfect.

In the above proposition, the lifting property of simple modules implies that P has a decomposition $P = \sum_{I} \bigoplus P_{\sigma}$ and the P_{σ} are cyclic hollow and completely indecomposable modules. However this fact is not true in general for any module N with N/J(N) semi-simple. We shall study this problem in the next section.

Thus, we assume from now on that $M = \sum_{I} \bigoplus M_{\alpha}$ and the M_{α} are cyclic hollow and completely indecomposables. We give one remark on any *R*-module M'.

Proposition 2. Let M' be an R-module. Let N_1 and N_2 be completely indecomposable direct summands of M'. If $\overline{N}_1 = \overline{N}_2$ in \overline{M}' , N_1 is isomorphic to N_2 .

Proof. Let $M' = N_1 \oplus N'_1 = N_2 \oplus N'_2$. Since N_1 has the exchange property

by [12], Proposition 1, $M' = N_1 \oplus N_2 \oplus N_2''$ $(N_2' \subseteq N_2')$ or $M' = N_1 \oplus N_2'$. Since $\overline{N}_1 = \overline{N}_2$, we have the latter decomposition and so $N_1 \approx N_2$.

First we consider a special case. Let $\{M_i\}_{\kappa}$ be the subset of $\{M_{\sigma}\}_i$ such that $\overline{M}_i \rightleftharpoons \overline{M}_{\sigma}$ for any $\alpha \pm i$. Then $\overline{M} = \sum_{\kappa} \oplus \overline{M}_i \oplus \sum_{I-\kappa} \oplus \overline{M}_{\sigma}$ is a (special) decomposition of \overline{M} . Since $\overline{M}_i \rightleftharpoons \overline{M}_j$ for $i \pm j$ in K, $\sum_{\kappa} \oplus M_i$ has the lifting property and so M has the lifting property if so does $\sum_{I-\kappa} \oplus M_{\beta}$. Hence, when we study the lifting property, we may assume, without loss of generality, the following.

(#) For each α in I, there exists $\beta(\alpha) \neq \alpha$ in I such that $\overline{M}_{\beta(\alpha)} \approx \overline{M}_{\alpha}$.

Let M be as above. Put $S_R = \operatorname{End}_R(M)$, $\overline{S} = \operatorname{End}_R(\overline{M})$ and $J_0(S) = \operatorname{Hom}_R(M, J(M))$. Then every element in S (resp. \overline{S}) is expressed as an infinite column finite matrix with entries $f_{\alpha\beta} \in \operatorname{Hom}_R(M_\beta, M_\alpha)$ (resp. $\operatorname{Hom}_R(\overline{M}_\beta, \overline{M}_\alpha)$). It is clear $\overline{S} \supseteq S/J_0(S)$. We shall take a partition of I; $I = \bigcup I_i$ such that for any α, β in $I_i \ \overline{M}_\alpha \approx \overline{M}_\beta$ and for any $\alpha' \in I_i$ and $\beta' \in I_j(i \neq j)$, $\overline{M}_{\alpha'} \approx \overline{M}_{\beta'}$. We assume here that each M_α satisfies (E-I) and M has the lifting property of simple modules. Then the subset $\{M_Y\}_{I_i}$ is a linearly ordered set with respect to the relation \leqslant by [8], Theorem 2 for each i. Further, we obtain a partition of $I_i = \bigcup_{j \in K_i} I_{ij}$ such that for α, β in $I_{ij} \ M_\alpha \approx M_\alpha$ and for $\gamma \in I_{ij}, \ \delta \in I_{ik} (j < k) \ M_\gamma > M_\delta$. For the sake of simplicity we assume $K_i = (\dots, \xi, \dots, \eta, \dots)$ and $M_\delta > M_\epsilon$ for $\delta \in I_{i\xi}$ and $\varepsilon \in I_{i\eta}$.

Now in general $\bar{S} \approx \prod_i \operatorname{End}_R(\sum_{I_i} \oplus \bar{M}_{\gamma})$ and $\operatorname{End}_R(\sum_{I_i} \oplus \bar{M}_{\gamma})$ is the ring of column finite matrices over the division ring $\Delta_i = \operatorname{End}_R(\bar{M}_{\gamma})$. For the partition $I_i = \bigcup I_{ij}$, by $T(\dots, I_{i\xi}, \dots, I_{i\eta}, \dots)$ we denote a subring of $\operatorname{End}_R(\sum_{I_i} \oplus \bar{M}_{\gamma})$ which consists of all lower tri-angular matrices



where $\Delta'_i(\xi) = \Delta_i$ if $|I_{i\xi}| =$ the cardinal of $I_{i\xi} \ge 2$ and $\Delta'_i(\xi) \approx \operatorname{End}_R(M_{\gamma})/\operatorname{Hom}_R(M_{\gamma}, J(M_{\gamma})) \subseteq \Delta_i$ if $|I_{i\xi}| = 1$.

We can restate [8], Theorem 2 as follows.

Proposition 3. Let $\{M_{\alpha}\}_{I}$ and M be as above. We assume that each M_{α}

satisfies (E-I). Then M has the lifting property of simple modules if and only if $S/J_0(S)$ is a subring $\prod_i T(\dots, I_{i\xi}, \dots, I_{i\epsilon}, \dots)$ in \overline{S} via the natural monomorphism $S/J_0(S) \rightarrow \overline{S}$.

REMARK. Since there exists a cyclic hollow module Q such that $\operatorname{End}_{R}(Q) \rightarrow$ $\operatorname{End}(\overline{Q}_{R}, \overline{Q}) \rightarrow 0$ is not exact, we need the restriction $\Delta'_{\ell}(\xi)$.

We shall consider the case $\bar{S}=S/J_0(S)$ after the following theorem.

Theorem 1. Let $\{M_{\alpha}\}_{I}$ be a set of completely indecomposable and cyclic hollow modules. We put $M = \sum_{I} \bigoplus M_{\alpha}$ and assume $\{M_{\alpha}\}_{I}$ is a semi-T-nilpotent set

[3]. Then the following conditions are equivalent:

- 1) M has the lifting property of simple modules.
- 2) M has the lifting property of direct summands.
- 3) Every direct summand of M has the above property.

Proof. 2) \rightarrow 3). (see [8], the proof of Proposition 2). Let $M=T_1\oplus T_2$ and $\overline{T}_1=A\oplus B$. We put $C=A\oplus \overline{T}_2$. Then from 2) we obtain direct summands K_i such that $M=K_1\oplus K_2$ and $\overline{K}_1=C$. Since $\{M_{\alpha}\}_I$ is semi-T-nilpotent, K_1 has the exchange property in M by [10], Theorem. Hence, $M=K_1\oplus T'_1\oplus T'_2$; $T'_i\subseteq T_i$. Since $\overline{T}'_2\subseteq \overline{T}_2\subseteq C=\overline{K}_1$, $\overline{T}'_2=0$ and so $T'_2=0$ by [10] and the assumption. Accordingly, $M=K_1\oplus T'_1$ and $T_1=T'_1\oplus (T_1\cap K_1)$. On the other hand, since $\overline{M}=\overline{K}_1\oplus\overline{T}'_1=C\oplus\overline{T}'_1$, $A\cap \overline{T}'_1=0$ and $\overline{T}_1\cup \overline{K}_1\subseteq\overline{T}_1\cap \overline{K}_1=\overline{T}_1\cap (A\oplus\overline{T}_2)=A$. Hence, $A=\overline{T_1\cap K_1}\oplus (A\cap\overline{T}'_1)=\overline{T_1\cap K_1}$ and $T_1\cap K_1$ is a direct summand of T_1 . 3) \rightarrow 1). It is clear.

1) \rightarrow 2). Since \overline{M} is semi-simple, we may assume $\overline{M} = A \oplus B$ and $A = \sum_{\mathfrak{x}} \oplus A_{\mathfrak{a}}$ with $A_{\mathfrak{a}}$ simple. We assume X is a well ordered set. We show by induction that for each $\alpha > \beta \ge \kappa$ there exists a summand N_{κ} of M such that $\sum_{\kappa < \beta} \oplus N_{\kappa}$ is a direct summand of M, $\{N_{\kappa}\}_{\kappa < \beta} \subseteq \{N_{\kappa'}\}_{\kappa' < \beta'}$ if $\beta \le \beta'$ and $\overline{\sum_{\kappa < \beta} \oplus N_{\kappa}} = \sum_{\kappa < \beta} \oplus A_{\kappa}$. If $\alpha = 1$, we have $N_1 = M_1$ by 1). Since $\sum_{\beta < \varpi} \oplus N_{\beta}$ is locally direct summand of M, by [9], Lemma 3 and [10], Theorem, it is a direct summand of M; $M = \sum_{\beta < \varpi} \oplus N_{\beta} \oplus$ N and $\overline{\sum_{\beta < \varpi} \oplus N_{\beta}} = \sum_{\beta < \varpi} \oplus A_{\beta}$. Put $A' = \sum_{\beta < \varpi} \oplus A_{\beta} \oplus A_{\omega}$. Then $A' = \sum_{\beta < \varpi} \oplus A_{\beta} \oplus (A' \cap \overline{N})$ and $A' \cap \overline{N} \approx A_{\varpi}$ is simple. Hence, there exists a direct summand N_{ϖ} of Nsuch that $\overline{N}_{\varpi} = A' \cap \overline{N}$ by [8], Proposition 2. Therefore, $\sum_{\beta < \varpi} \oplus N_{\beta} \oplus N_{\varpi}$ is a direct summand of M and $\overline{\sum_{\beta < \varpi} \oplus N_{\beta}} \oplus \overline{N}_{\varpi} = \sum_{\beta < \varpi} \oplus A_{\beta} \oplus (A' \cap \overline{N}) = A'$.

In the following theorem we consider an R-module M which has less assumptions.

Theorem 2. Let $\{M_{\alpha}\}_{I}$ be a set of *R*-modules with $M_{\alpha}/J(M_{\alpha})$ simple (not

necessarily either hollow or completely indecomposable). We put $M = \sum_{T} \bigoplus M_{a}$. Then the following conditions are equivalent:

- 1) M has the lifting property of decompositions with finite direct summands.
- 2) M has the lifting property of decompositions with two direct summands.
- 3) Hom_R $(\overline{M}_{\alpha}, \overline{M}_{\beta})$ is lifted to Hom_R (M_{α}, M_{β}) for $\alpha \neq \beta$ in I.

4) $S=S/J_0(S)$ (under the assumption (#)), where $S=\operatorname{End}_R(M)$, $\overline{S}=\operatorname{End}_R(\overline{M})$ and $J_0(S)=\operatorname{Hom}_R(M, J(M))$.

Proof. 1) \rightarrow 2). It is clear.

2) \rightarrow 3). Let f be in Hom_R ($\overline{M}_1, \overline{M}_2$) and $\overline{M}_1(f) = \{\overline{x} + f(\overline{x}) | x \in M_1\}$. Then $\overline{M} = \overline{M}_1(f) \oplus (\overline{M}_2 \oplus \overline{M}_3 \oplus \cdots)$. By 2) we obtain a decomposition of M; $M = N_1$ $\oplus N_2$ such that $\overline{N}_1 = \overline{M}_1(f)$ and $\overline{N}_2 = (\overline{M}_2 \oplus \overline{M}_3 \oplus \cdots)$. Let $\pi_i: M \to N_i$ and $\rho_i: M \to M_i$ projections with respect to the decompositions $M = \sum_{i=1}^{2} \bigoplus N_i$ and $M = \sum_{r} \bigoplus M_{a}$, respectively. Put $g = \rho_2 \pi_1 | M_1 \in \operatorname{Hom}_R(M_1, M_2)$. Let *m* be in M_1 . Then $m = \sum_{i=1}^{2} \pi_i(m)$ and $\pi_1(m) = \sum_{i} \rho_{ai}(\pi_1(m))$. Hence, since $\overline{m} = \sum_{i=1}^{n} \overline{\pi_i(m)}$ and $\overline{\pi_1(m)} = \overline{x} + f(\overline{x})$ for some $x \in M_1$, $\overline{m} = \overline{x}$. Accordingly, $\overline{g}(\overline{m}) = \overline{g(m)} = \overline{g(m)}$ $\rho_2 \pi_1(m) = \rho_2(\bar{x} + f(\bar{x})) = f(\bar{x}) = f(\bar{m})$. Hence, f is lifted to g. 3) \rightarrow 4). We assume (#). Let $\varphi: \overline{M}_{\alpha} \approx \overline{M}_{\rho(\alpha)}$ and $\alpha \neq \rho(\alpha)$. For any $g \in \operatorname{End}_{R}(\overline{M}_{\alpha})$, $g = \varphi^{-1}(\varphi g)$. Then φ^{-1} , φg are lifted to $H \in \operatorname{Hom}_{R}(M_{\rho(a)}, M_{a})$ and $H \in$ $\operatorname{Hom}_{R}(M_{\alpha}, M_{\rho(\alpha)})$ by 3), respectively. Hence, g is lifted to FH. Therefore, $S = S/J_0(S)$ by 3). 4) \rightarrow 1). First we shall show 4) \rightarrow 2). Let $\overline{M} = A \oplus B$ and $\overline{M} = A \oplus \sum_{n} \oplus \overline{M}_{\delta}$. Now, $M = \sum_{I=P} \bigoplus M_{\gamma} \bigoplus \sum_{P} \bigoplus M_{\delta}$. Let π be the projection of \overline{M} onto $\sum_{P} \bigoplus \overline{M}_{\delta}$ with respect to the decomposition $\bar{M}=A\oplus\sum_{s}\oplus\bar{M}_{s}$. Since $\bar{S}=S/J_{0}(S)$, we can choose, from the representation of column finite matrices, a homomorphism $f: M \to \sum_{p} \bigoplus M_{\delta}$ such that $\bar{f} = -\pi$. We put $M_{1}(f) = \{x + f(x) \mid x \in \sum_{I=P} \bigoplus M\} \subseteq M$. Then $M = M_1(f) \oplus \sum_{n \in \mathbb{N}} \oplus M_{\delta}$ and $\overline{M_1(f)} = A$. Next we take the projection π' of \overline{M} onto A with respect to the decomposition $\overline{M} = A \oplus B$. Since $A = \overline{M_1(f)}$, there exists $g: M \to M_1(f)$ such that $g = -\pi'$ (note $M_1(f) \approx \sum_{I=P} \bigoplus M_{\gamma}$ and $M = M_1(f) \oplus$ $\sum_{p} \oplus M_{\delta}$). Put $M_{2}(g) = \{y + g(y) | y \in \sum_{p} \oplus M_{\delta}\}$. Then $M = M_{1}(f) \oplus M_{2}(g)$ and $\overline{M_2(g)} = B$. Next we shall show 4) \rightarrow 1). Since $M_1(f) \approx \sum_{r-p} \bigoplus M_r$ and $\overline{M_2(g)} \approx$ $\sum_{R} \oplus M_{\delta}$, we can prove it by induction on number of summand (note that 4) is satisfied for the direct summand $M_2(g)$).

REMARK. We know, from the proof and the remark before (#), that $1)\sim 3$ are equivalent without (#).

Corollary 1. Let $\{M_{\alpha}\}_{I}$ and M be as above. We assume further each M_{α} is completely indecomposable and $\{M_{\alpha}\}_{I}$ is semi-T-nilpotent. Then the above conditions 1)~4) are equivalent to

5) M has the lifting property of direct decopositions.

Proof. 5) \rightarrow 1). It is clear.

1) \rightarrow 5). We shall use the same method in the proof 1) \rightarrow 2) of Theorem 1. Let $\overline{M} = \sum_{L} \oplus B_{\gamma}$. We shall prove it by transfinite induction on L. We assume L is a well ordered set. We assume that for each $\rho \leq \beta < \alpha$ there exist direct summands N_{ρ} such that $M = \sum_{\rho < \beta} \oplus N_{\rho} \oplus M'$ and $\overline{N}_{\rho} = B_{\rho}$. We note that every direct summand of M is a direct sum of completely indecomposable modules by [10], Theorem. Since $T = \sum_{\rho < \alpha} \oplus N_{\rho}$ is a locally direct summand, T is a direct summand of M by [9], say $M = T \oplus M'$. Now $\overline{M} = \sum_{\rho < \alpha} \oplus B_{\rho} \oplus \sum_{\alpha < \gamma} \oplus B_{\gamma}, M = T \oplus M'$ and $\overline{T} = \sum_{\rho < \alpha} \oplus \overline{N}_{\gamma} = \sum_{\rho < \alpha} \oplus B_{\rho}$. Then we know from the proof 4) \rightarrow 1) of the theorem that there exists a decomposition $M = T \oplus M'(g)$ and $\overline{M'(g)} = \sum_{\alpha < \gamma} \oplus B_{\gamma}$. M'(g) has also the lifting property of decompositions with two direct summands by the theorem. Hence, we obtain a direct summand N_{α} of M such that $\overline{N}_{\alpha} = B_{\alpha}$ and $M = \sum_{\alpha < \alpha} \oplus N_{\beta} \oplus M''$.

If M_{α} is a cyclic hollow module with (E-I), then $\operatorname{Hom}_{\mathbb{R}}(M_{\alpha}, J(M_{\alpha})) = J(\operatorname{End}_{\mathbb{R}}(M_{\alpha}))$. Hence, under the condition in Theorem 2, $J_0(S) \subseteq J(S)$ if and only if $\{M_{\alpha}\}_I$ is a semi-*T*-inlpotent set by [5], Proposition 2. If J(M) is small in M, $\{M_{\alpha}\}_I$ is semi-*T*-nilpotent by [7], Corollary 1 to Proposition 1.

Corollary 2. Let $\{M_{\alpha}\}_{I}$ and M be as above. We assume each M_{α} is a cyclic hollow module with (E-I). We assume one of the following.

i) $J_0(S) \subseteq J(S)$.

ii) $\{M_{\alpha}\}_{I}$ is a semi-T-nilpotent set.

iii) J(M) is small in M (e.g. R is right perfect).

Then the conditions in Theorem 2 are equivalent to 5).

Proof. Let $\overline{M} = \sum_{I} \bigoplus A_{\alpha}$ with A_{α} simple and let N_{α} be a direct summand of M such that $\overline{N}_{\alpha} = A_{\alpha}$. Then the inclusion map $i: \sum_{I} N_{\alpha} \to M$ modulo J(M)gives an isomorphism to \overline{M} . Hence, i is an isomorphism if $J_{0}(S) \subseteq J(S)$ (cf. [3]).

Corollary 3. We further assume in Theorem 2 that $M_{\alpha} \approx M_1$ for all α in I. Then the conditions in Theorem 2 are equivalent to

6) M has the lifting property of simple modules.

3. Modules with lifting property

In this section, we shall study *R*-modules *M* with lifting property and M/J(M) semi-simple.

Theorem 3. Let M be an R-module with M|J(M) semi-simple. We assume that M has the lifting property of simple module and satisfies one of the following conditions.

1) M has the lifting property of decomposition with two direct summands.

2) $\overline{S} = S/J_0(S)$, where $S = \operatorname{End}_R(M)$, $\overline{S} = \operatorname{End}_R(\overline{M})$ and $J_0(S) = \operatorname{Hom}_R(M, J(M))$.

3) For any cyclic hollow and completely indecomposable direct summand N of M and an element f in $\operatorname{Hom}_{\mathbb{R}}(N, M)$, if $\overline{f}: \overline{N} \to \overline{M}$ is a monomorphism, so is f.

4) For any two direct summands M_1 , M_2 as in 3) and $f \in \text{Hom}_R(M_1, M_2)$, if \bar{f} is an epimorphism, then f is an isomorphism.

Then M contains a submodule M' satisfying the following.

a) $M' = \sum \bigoplus M_{\alpha}$; the M_{α} are cyclic hollow and completely indecomposables.

b) M' has the lifting property of simple modules and decompositions with finite direct summands.

c) $\sum_{I} \oplus M_{\sigma}$ is a locally direct summand of M.

d) M = M' + J(M).

Conversely, if M contains a submodule M' above, M has the lifting property of simple modules. Furthermore, if each M_{α} satisfies (E-I), then M satisfies 3) and 4).

Proof. We assume M has the lifting property of simple modules. Let N be the set of submodules N of M such that $N = \sum_{T} \bigoplus N_{\gamma}$; the N_{γ} are cyclic hollow and completely indecomposable and N is a locally direct summand of M (with respect to $\sum_{T} \bigoplus N_{\gamma}$). Since M has the lifting property of simple modules, N is not empty. We can define a partial order in N by the members of direct summands of N. Then we can find a maximal element, say $M' = \sum_{T} \bigoplus N_{\alpha}$, in N by Zorn's lemma. Since N is a locally direct summand, $\overline{M}' \subseteq \overline{M}$.

Case 1). We put $\overline{M} = \overline{M}' \oplus K$ and show K=0. If $K \neq 0$, we have a simple submodule A such that $\overline{M} = \overline{M}' \oplus K' \oplus A$. Since M has the lifting property of decompositions with two direct summands, $M = L \oplus N$ and $\overline{L} = \overline{M}' \oplus K'$, $\overline{N} = A$. Let J be any finite subset of I. Then $M = \sum_{T} \oplus N_{Y} \oplus P$. Since $\sum_{T} \oplus N_{Y}$ has the exchange property by [1], Lemma 3.10 and [12], Proposition 1, $M = \sum_{T} \oplus N_{Y} \oplus L' \oplus N'$, where $L' = L' \oplus L''$ and $N = N' \oplus N''$. Since $\overline{N} = \overline{N}' \oplus \overline{N}'' = A$, N' = 0 or $\overline{N}'' = 0$. If $\overline{N}' = 0$, $\overline{M} = \sum_{T} \oplus \overline{N}_{Y} \oplus \overline{L}' \subseteq \overline{L}$. Hence $\overline{N}' \neq 0$, and so $\overline{N}'' = 0$. On the other hand, N'' is isomorphic to a direct unmand of $\sum_{T} \oplus N_{Y}$.

Hence N''=0 and so N=N'. Therefore, $\sum_{J} \oplus N_{\gamma} \oplus N$ is a direct summand of M, which contradicts the maximality of M' in N. Therefore, M=M'+J(M). M' has the lifting property of simple modules by [8], Theorem 2 and Proposition 2. Next we shall show that M' has the lifting property of decompositions with finite direct summands. Let N_1 , N_2 be in $\{N_{\alpha}\}_I$, then $N_1 \oplus N_2$ is a direct summand of M by c), say $M=N_1 \oplus N_2 \oplus M^*$. We can apply the argument in the proof of Theorem 2 to the decomposition $M=N_1 \oplus N_2 \oplus M^*$, instead of $M=\sum_I \oplus M_{\alpha}$. Hence, $\operatorname{Hom}_R(\bar{N}_1, \bar{N}_2)$ is lifted to $\operatorname{Hom}_R(N_1, N_2)$. Therefore, M' has the lifting property of decompositions with finite direct summands by Theorem 2 and has the lifting property of simple modules by [8], Theorem 2 and Proposition 2.

Case 2). We also show K=0. Put $N_0 = \sum_{J} \bigoplus N_i$ for a finite subset J of I. Then $M=N_0 \oplus P$ and $\overline{M}=\overline{N}_0 \oplus \overline{P}=\overline{N}_0 \oplus \sum_{I=J} \oplus \overline{N}_{\gamma} \oplus K$. Let π' be the projection of \overline{M} onto \overline{N}_0 with respect to the latter decomposition and $\pi=\pi' | \overline{P}$. Since $\overline{S}=S/J_0(S)$ and

$$S = \begin{pmatrix} \operatorname{Hom}_{R}(N_{0}, N_{0}) & \operatorname{Hom}_{R}(P, N_{0}) \\ \operatorname{Hom}_{R}(N_{0}, P) & \operatorname{Hom}_{R}(P, P) \end{pmatrix},$$

there exists $f \in \operatorname{Hom}_{\mathbb{R}}(P, N_0)$ with $\overline{f} = -\pi$. Then $M = P(f) \oplus N_0$ and $\overline{P(f)} = \sum \oplus \overline{N}_{\delta} \oplus K$ (cf. the proof of Theorem 2). We assume $K \neq 0$ and K_{α} is a simple component of K. Since M has the lifting property of simple modules, there exists a direct summand M_{α} of M such that $\overline{M}_{\alpha} = K_{\alpha}$. M_{α} has the exchange property and so $M = M_{\alpha} \oplus P(f)' \oplus N_0$ and $P(f)' \subseteq P(f)$, since $\overline{M}_{\alpha} \subseteq \overline{P(f)}$. Hence, $M_{\alpha} \oplus N_0$ is a direct summand of M for any finite subset J of I. Hence, $M_{\alpha} \oplus N_0$ is a locally direct summand of M, which is a contradiction. Therefore, K=0 and $\overline{M} = \overline{M}'$. It is clear that every direct summand of M satisfies the condition 2). Hence, M' has the lifting property of simple modules and decompositions with finite direct summands by Theorem 2 (note the remark before (\sharp)).

Cases 3) or 4). Let M_{α} be a cyclic hollow and completely indecomposable direct summand of M with $\overline{M}_{\alpha} = A_{\alpha}$. Put $M' = \sum_{I} M_{\alpha}$. Then $\overline{M}' = \overline{M}$. We shall show $\sum_{I} M_{\alpha}$ is a direct sum (cf. [8], Theorem 1). Let $\sum_{i=1}^{n} M_{i} = M(n)$ be any finite sum in $\sum_{I} M_{\alpha}$. We show $M(n) = \sum_{i=1}^{n} \bigoplus M_{i}$ and M(n) is a direct summand of M by induction on n. If n=1, it is clear. We assume $M=M(n-1)\oplus T$. Let π be the projection of M onto T and $f=\pi | M_{n}$. Since $f(M_{n}) \oplus J(T), \overline{f(M_{n})}$ is a simple component of \overline{T} and T has the lifting property by [8], Proposition 2 and so there exists a cyclic hollow and completely indecomposable direct summand T_{1} of T such that $\overline{T}_{1}=\overline{f(M_{n})}$. Let $T=T_{1}\oplus T_{2}$ and π_{1} the projection of T onto T_{1} . Put $g=\pi_{1}f$. Then $\overline{g(M_{n})}=\overline{T}_{1}$. Hence, \overline{g} is a monomorphism and so g is a mono-

morphism by 3) or 4). Accordingly, $M = M_n \oplus \ker \pi_1 \pi = M_n \oplus T_2 \oplus M(n-1) =$ $M(n) \oplus T_2$ and $M(n) = \sum_{i=1}^{n} \oplus M_i$. Cases 3) and 4) imply that $\{M_{\alpha}\}_I$ satisfies (E-I). Hence, M' has the lifting property of decompositions with finite direct summands by Theorems 1 and 2. Conversely, we assume M has the submodule M'. Since $M' = \sum_{T} \bigoplus M_{\sigma}$ is a locally direct summand of M, $\overline{M} = \sum_{T} \bigoplus \overline{M}_{\sigma} = \overline{M}'$. Let A be a simple submodule of \overline{M} . Then there exists a finite subset J of I such that $A \subseteq \sum_{I} \bigoplus \overline{M}_{\gamma}$. Since M' has the lifting property of simple module, $\sum_{I} \bigoplus M_{\gamma}$ contains a direct summand N with $\overline{N} = A$ by [8], Proposition 2 and so N is also a direct summand of M. Finally we assume that each M_{α} satisfies (E-I). Let N_{α} be any direct summand of M as in 3). Since $\overline{M} = \overline{M}'$, there exists a direct summand M_{α} of M' such that $\overline{M}_{\alpha} = \overline{N}_{\alpha}$. Hence, $M_{\alpha} \approx N_{\alpha}$ by Proposition 2. Accordingly, 4) is satisfied by Theorem 2. Let f be as in 3). Since $\overline{M} = \overline{M}'$ and M' has the lifting property of simple modules, there exists a direct summand T_{α} with $T_{\alpha} = f(N)$. T_{α} is cyclic and so T_{α} is contained in a direct summand $\sum \bigoplus M_{\sigma}$ of M by c). Hence, T_{σ} is a direct summand of M. Let π be the projection of M onto T_{α} . Then $\pi f: N_{\alpha} \to T_{\alpha}$ is an isomorphism by the above and Theorem 2. Hence, f is a monomorphism.

Corollary. Let R be a right artinian ring and M an R-module. Then M has the lifting property of decompositions if and only if M has the lifting property of decompositions with two direct summands.

Proof. If R is right artinian, every R-module N with N/J(N) simple is a cyclic hollow and completely indecomposable module. Hence, if M has the lifting property of decompositions with two direct summands, then M has the lifting property of simple modules and so $M = \sum_{T} \bigoplus M_{\alpha}$ by the theorem, where the M_{α} are cyclic hollow modules. Hence, M has the lifting property of decompositions by Corollary 2 to Theorem 2.

REMARKS 1. We note that if J(M) is small in M, M=M' in Theorem 3. If further each M_{α} satisfies (E-I), then all conditions 1)~4) in Theorem 3 are equivalent when M has the lifting property of simple modules.

2. We assume $M=M'\oplus K$ with K=J(K). Then M satisfies 1) and 2) if M' satisfies a) and b). We do not know whether this fact is true or not without assumptions.

3. Let Z be the ring of integers and p a prime. Put $M = \sum_i \bigoplus Z/p^i \bigoplus E(Z/p)$ and $N = (Z/p^i)^{(I)} \bigoplus E(Z/p)$, where $(Z/p^i)^{(I)}$ is the direct sum of |I|-copies of Z/p^i . Then M has the lifting property of simple modules but not of decompositions and N has the lifting property of decompositions.

Next we shall study R-modules satisfying the lifting property of simple

modules.

Proposition 4. Let M be an R-module. We assume that M is countably generated, M/J(M) is semi-simple and J(M) is small in M. Then if M has the lifting property of simple modules, $M = \sum_{i=1}^{\infty} \bigoplus M_i$ with M_i indecomposable.

Proof. Let $\{m_1, m_2, \dots, m_n, \dots\}$ be a set of generators. Since J(M) is small, we may assume $m_i \notin J(M)$ for all *i*. Further we may assume $\overline{m_iR}$ is simple. We assume there exists a set of indecomposable direct summands M_i such that $M(n) = \sum_{i=1}^{t_n} \oplus M_i$ is a direct summand of M and $\sum_{i=1}^{n} \overline{m_iR} \subseteq \overline{M(n)}$. Let $M = M(n) \oplus T$ and $m_{n+1} = x+t$; $x \in M(n)$, $t \in T$. If $t \notin J(T)$, there exists an indecomposable direct summand $M_{t_{n+1}}$ of T such that $\overline{M}_{t_{n+1}} = \overline{tR}$ by [8], Proposition 2. Hence, there exists a direct summand M(n+1) of M such that $\overline{M(n+1)} \supseteq \sum_{i=1}^{n+1} \overline{m_iR}$. Accordingly, $\bigcup_n M(n) = \sum_{i=1}^{\infty} \oplus M_{t_i} = M$, since J(M) is small in M.

Lemma 1. Let M be an R-module and let $\sum_{I_1} \oplus N_{\alpha_1}, \sum_{I_2} \oplus N_{\alpha_2}, \cdots$ be submodules in M. We assume that the N_{α_i} are completely indecomposable and $\sum_{I_i} \oplus N_{\alpha_i}$ is a locally direct summand of M for all i. If $N_{\alpha_i} \approx N_{\alpha_j}$ for any $\alpha_i \in I_i$ and any $\alpha_j \in I_j, \sum_{I_i} \sum_{I_i} \oplus N_{\alpha_i}$ is a locally direct summand of M.

Proof. Let J_i be any finite subset of I_i . Then $M = \sum_{J_i} \bigoplus N_{\alpha'} \bigoplus M(i)$. Since $\sum_{J_1} \bigoplus N_{\alpha'_1}$ has the exchange property, $M = \sum_{J_1} \bigoplus N_{\alpha'_1} \bigoplus \sum_{J_2} \bigoplus N_{\alpha'_2} \bigoplus M^*(2)$ where $M^*(2) \subseteq M(2)$, since $N_{\alpha'_1} \approx N_{\alpha'_2}$. Repeating this argument, we know $\sum_i \sum_{I_i} \bigoplus N_{\alpha_i}$ is a locally direct summand of M.

Corollary. Let M be an R-module such that $M/J(M) = \sum_{I} \bigoplus A_{\alpha}$ with A_{α} simple and $A_{\alpha} \approx A_{\beta}$ for $\alpha \pm \beta \in I$. Then M has the lifting property of simple modules if and only if M contains a submodule M' which has the lifting property of simples modules and satisfies a), c) and d).

Proof. If M has the lifting property of simple modules, we obtain a cyclic hollow and completely indecomposable module M_{σ} with $\overline{M}_{\sigma} = A_{\sigma}$. Then $M_{\sigma} \approx M_{\beta}$ if $\alpha \pm \beta$. Hence, $\sum_{I} M_{\sigma} = M'$ satisfies a), c) and d) by Lemma 1. The remaining part is clear.

Theorem 4. Let M be an R-module with M/J(M) semi-simple. We assume every cyclic direct summand of M is noetherian (e.g. R is right noetherian). Then M has the lifting property of simple modules if and only if M contains a submodule M' such that a) $M' = \sum \bigoplus M_{\alpha}$; the M_{α} are cyclic hollow and completely indecomposa-

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ble, b) M' has the lifting property of simple modules, c) $\sum_{I} \oplus M_{\alpha}$ is a locally direct summand of M and d) M = M' + J(M).

Proof. "If" part is clear from Theorem 3. We assume M has the lifting property of simple modules. Let M_1 , M_2 be completely indecomposable and cyclic hollow direct summands of M, say $M = M_1 \oplus N_1 = M_2 \oplus N_2$. Then $M_1 \approx M_2$ or $M = M_1 \oplus M_2 \oplus N_2'$ by Lemma 1. In the latter case, since $M_1 \oplus M_2$ has the lifting property of simple modules by [8], Proposition 2, $M_1 \leq M_2$ or $M_1 \ge M_2$ if $\overline{M}_1 \approx \overline{M}_2$ by [8], Theorem 2. Therefore, we may assume in any cases $M_1 \ge M_2$ or $M_1 > M_2$ whenever $\overline{M}_1 \approx \overline{M}_2$. We note that every cyclic direct summand satisfies (E-I) by the assumption. Now let $\{M_{\gamma}\}_{L}$ be a representative set of cyclic hollow and completely indecomposable direct summands of M with $\overline{M}_{\gamma} \approx \overline{M}_{\gamma_0}$ for all $\gamma \in L(\gamma_0 \text{ is fixed})$. Then $\{M_{\gamma}\}_L$ is a linearly ordered set with respect to \geq . Since M_{γ} is noetherian, we have the minimal member M_1 and a finite chain $M_{\gamma} > M_{\gamma-1} > \cdots > M_1$. Hence, L is well ordered and so we may assume $\{M_{\gamma}\}_{L} = \{M_{n}\}$ with $M_{n} > M_{n-1}$. We define the set N_{1} of submodules of M: $N_1 = \{T = \sum_{\alpha \in T'} \bigoplus M_{1\alpha} | T \text{ is a locally direct summano of } M \text{ and } M_{1\alpha} \approx M_1$ for all $\alpha \in I_1'$. Let $T_1 = \sum_{I_1} \bigoplus M_{1\omega}$ be a maximal member in N_1 . Then $\overline{M} = \overline{T} \oplus K_1$ and $\overline{T} = \sum_{I_1} \bigoplus \overline{M}_{1\omega}$. Let A_1 be a simple submodule of \overline{M} not contained in $ar{T}_1$. Since M has the lifting property of simple modules, there exists a completely indecomposable and cyclic hollow direct summand N_1 of M such that $ar{N}_1=A_1$. We shall show $N_1 pprox M_1$. Let J_1 be any finite subset of I_1 . Then $M = \sum_{\substack{J_1 \\ J_1}} \bigoplus M_{1\alpha} \bigoplus M'. \text{ Let } \pi_{M'}: M \to M' \text{ be the projection. Since } \overline{N}_1 \subset \overline{\sum_{J_1}} \bigoplus \overline{M}_1,$ $\overline{N} \approx \overline{\pi_{M'}(N_1)} \text{ and so } M' \text{ contains a cyclic hollow direct summand } N'_1 \text{ with }$ $\bar{N}'_1 = \overline{\pi_{M'}(N_1)}$ say $M' = N'_1 \oplus M''$ by [8], Proposition 2. Then $M = \sum \oplus M_{1a} \oplus$ $M'' \oplus N'_1$. Let $\pi: M \to N'_1$ and $\pi': M' \to N'_1$ be the projections, respectively. Then $\overline{\pi(N_1)} = \overline{\pi'(\pi_{M'}(N_1))} = \pi'(\overline{N}_1) = \overline{N}_1'$. Hence, $\pi \mid N_1 \to N_1'$ is an epimorphism. If $N_1 \approx M_1$, $N_1' \approx M_1$ since M_1 is minimal and so $\pi | N_1|$ is an isomorphism by (E-I). Therefore, $M = N_1 \oplus \ker \pi = N_1 \oplus \sum_{I_1} \oplus M_{1a} \oplus M''$. Since J_1 is any finite subset of $I_1, N_1 \oplus T_1$ is a locally direct summand of M and $N_1 \approx M_1$, which contradicts the maximality of T_1 in N_1 . Hence, $N_1 \not\approx M_1$. Now $\overline{M} = \overline{T}_1 \oplus K_1$ and $M_2 \oplus T_1$ is a locally direct summand of M by Lemma 1. Let $N_2 = \{T =$ $\sum_{I_1} \bigoplus M_{1a} \oplus \sum_{I'_2} \bigoplus M_{2\beta} \text{ is a locally direct summand of } M \text{ and } M_{2\beta} \approx M_2 \}. \text{ Let } T_2$ be a maximal member in N_2 . Then $\overline{M} = \overline{T} \oplus \sum_{I_2} \oplus \overline{M}_{2\beta} \oplus \overline{K}_2$. Let A_2 be a simple submodule not contained in \overline{T}_2 and N_2 a cyclic hollow direct summand of M with $\overline{N}_2 = A_2$. Since $A_2 \not\equiv \overline{T}_1$, $N_2 \not\approx M_1$ by the above. Let J_1 and J_2 be

any finite subsets of I_1 and I_2 , respectively, and $M = \sum_{I_1} \oplus M_{1a} \oplus \sum_{I_2} \oplus M_{2\beta} \oplus M^*$. Let $\pi_{M^*}: M \to M^*$ be the projection. Since $\overline{N}_2 = A_2 \oplus \overline{T}_2$ and $\overline{\sum_{I_1} \oplus M_{1a}} \oplus \overline{\sum_{I_2} \oplus M_{2\beta}} \subseteq \overline{T}_2, \overline{\pi_{M^*}(N_2)} \oplus \overline{T}_2$. Let N'_2 be a cyclic hollow direct summand of M^* with $\overline{N'_2} = \pi_{M^*}(N_2)$. Since $\overline{\pi_{M^*}(N_2)} \oplus \overline{T}_1$, $N'_2 \approx M_1$ from the above. We obtain the epimorphism $\pi \mid N_2 \to N'_2$ similarly to the above. If, since $N'_2 \approx M_1$, $N_2 \approx M_2$, we have a contradiction. Hence, $N_2 \approx M_i$ (i=1,2). Since L is well ordered, using inductively the above, we obtain a locally direct summand $T(\gamma_0) = \sum_{I_1} \oplus M_{1a_1} \oplus \sum_{I_2} \oplus M_{2a_2} \oplus \cdots \sum_{I_n} \oplus M_{na_n} \oplus \cdots$, which has the lifting property of simple modules and each M_{ia_i} is a cyclic hollow and completely indecomposable direct summand of M. Let $\{S_{\gamma}\}_P$ be the representative set of simple modules in \overline{M} . For each S_{γ} we can obtain $T(\gamma)$ as above. Then $\sum_P T(\gamma)$ is a locally direct summand of M by Lemma 1 and $\overline{M} = \sum_P \oplus \overline{T(\gamma)}$ from the above argument. Thus $M' = \sum \oplus T(\gamma)$ is the desired submodule of M by [8], Theorem 2.

Corollary. Let R be a right artinian ring and M an R-module. Then M has the lifting property of simple modules if and only if $M = \sum_{T} \bigoplus M_{\alpha}$ with M_{α} indecomposable hollow and $\{M_{\alpha}\}$ satisfies the conditions in [8], Theorem 2.

Proof. If R is right artinian, J(M) is small in M and M/J(M) is semi-simple.

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