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A CHARACTERIZATION OF THE ALTERNATING GROUPS OF DEGREES SIX AND SEVEN¹⁾

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1. Introduction

The purpose of this paper is to prove the following theorem.

Theorem. *Let \mathfrak{G} be a doubly transitive group on the set $\Omega = \{1, 2, \dots, n\}$. If the stabilizer \mathfrak{R} of the set of points 1 and 2 is isomorphic to the alternating group of degree four, then one of the followings holds :*

- (1) $n=6$ and \mathfrak{G} is \mathfrak{A}_6 ,
- (2) $n=15$ and \mathfrak{G} is \mathfrak{A}_7 ,
- (3) $n=16$ and \mathfrak{G} is $A(2, 4)$,
- (4) $\mathfrak{G} = C_{\mathfrak{G}}(J)O(\mathfrak{G})$ for some involution J .

Here \mathfrak{A}_m denotes the alternating group of degree m and $A(t, q)$ denotes the group of all affine transformations of the t -dimensional affine geometry $AG(t, q)$ over the field of q -elements.

Notation. Let \mathfrak{X} and \mathfrak{Y} be the subset of \mathfrak{G} . $\mathfrak{F}(\mathfrak{X})$ will denote the set of all the fixed points of \mathfrak{X} and $\alpha(\mathfrak{X})$ is the number of points in $\mathfrak{F}(\mathfrak{X})$. $\mathfrak{X} \sim \mathfrak{Y}$ means that \mathfrak{X} is conjugate to \mathfrak{Y} in \mathfrak{G} . All other notation is standard.

2. Preliminaries

Since \mathfrak{R} is \mathfrak{A}_4 , \mathfrak{R} is generated by the elements K and τ subject to the following relations:

$$K^3 = \tau^2 = (K\tau)^3 = 1 \quad (2.1)$$

Put $\tau_1 = K^{-1}\tau K$ and $\mathfrak{B} = \langle \tau, \tau_1 \rangle$. Then \mathfrak{B} is a four group and a Sylow 2-subgroup of \mathfrak{R} . Let \mathfrak{H} be the stabilizer of the point 1. Since \mathfrak{G} is doubly transitive on Ω , it contains an involution I with the cycle structure $(1, 2)\dots$ which normalizes \mathfrak{R} and we may assume that $[I, \tau] = 1$. Thus we have the following decomposition of \mathfrak{G} :

$$\mathfrak{G} = \mathfrak{H} \cup \mathfrak{H}I\mathfrak{H} \quad (2.2)$$

1) We thank Professor H. Nagao for pointing out a gap of our original proof.

Let $g(2)$, $h(2)$ and d denote the number of involutions in \mathfrak{G} , \mathfrak{H} , and the coset $\mathfrak{H}IH$ for $H \in \mathfrak{H}$, respectively. Then d is the number of elements in \mathfrak{R} inverted by I , that is, the number of involutions in \mathfrak{G} with the cycle structure $(1, 2)\cdots$ and the following equality is obtained from (2.2):

$$g(2) = h(2) + d(n-1) \tag{2.3}$$

Lemma 1. *One of the followings holds :*

- (1) $I\tau_1 I = \tau\tau_1, IKI = K^{-1}, d=6,$
 $I \sim IK \sim IK^2 \sim I\tau K\tau \sim I\tau K^2\tau \sim I\tau.$
- (2) $[I, \mathfrak{B}] = 1, IKI = \tau K\tau, d=4,$
 $I\tau \sim I\tau_1 \sim I\tau\tau_1.$
- (3) $[I, \mathfrak{R}] = 1, d=4,$
 $I\tau \sim I\tau_1 \sim I\tau\tau_1.$

Proof. Since the automorphism group of \mathfrak{R} is the symmetric group of degree four we may assume that the action of I on \mathfrak{R} is (1), (2), or (3) by (2.1). Assume that the case (1) holds. Now $\langle I, K \rangle$ and $\langle I, \tau K\tau \rangle$ are dihedral groups of order 6. Therefore $I \sim IK \sim IK^2$ in $\langle I, K \rangle$ and $I \sim I\tau K\tau \sim I\tau K^2\tau$ in $\langle I, \tau K\tau \rangle$. Thus the result follows in this case. Note that in this case every involution is conjugate to τ in \mathfrak{G} . The cases (2) and (3) are trivial. This proves our lemma.

Let τ keep i ($i \geq 2$) points of Ω unchanged. So we may put $\mathfrak{S}(\tau) = \{1, 2, \dots, i\}$. The group $C_{\mathfrak{G}}(\tau)$ acts on $\mathfrak{S}(\tau)$ and the kernel of this permutation representation is \mathfrak{B} or $\langle \tau \rangle$ because $C_{\mathfrak{G}}(\tau) \cap \mathfrak{R} = \mathfrak{B}$. By a theorem of Witt [6; p. 105], $|C_{\mathfrak{G}}(\tau)| = 4i(i-1)$. Hence there exist $(\mathfrak{G} : C_{\mathfrak{G}}(\tau)) = 3(n-1)n/(i-1)i$ involutions in \mathfrak{G} each of which is conjugate to τ .

At first, let us assume that n is odd. Let $h^*(2)$ be the number of involutions in \mathfrak{H} leaving only the point 1 fixed. Then from (2.3) the following equality is obtained:

$$h^*(2)n + 3(n-1)n/(i-1)i = 3(n-1)/(i-1) + h^*(2) + d(n-1) \tag{2.4}$$

Put $\beta = d - h^*(2)$. It follows from (2.4) that $n = i(\beta i - \beta + 3)/3$. This implies that i is odd.

Next let us assume that n is even. Let $g^*(2)$ be the number of involutions in \mathfrak{G} which are semi-regular on Ω . Then corresponding to (2.4) the following equality is obtained from (2.3):

$$g^*(2) + 3(n-1)n/(i-1) = 3(n-1)/(i-1) + d(n-1) \tag{2.5}$$

Put $\beta = d - g^*(2)/(n-1)$. Then by (2.5) we have $n = i(\beta i - \beta + 3)/3$. This implies that i is even.

In both cases, by the definition of β , β is the number of involutions with

the cycle structure $(1, 2)\cdots$ each of which is conjugate to τ . Since \mathfrak{G} is doubly transitive on Ω , we have $\beta > 0$.

Lemma 2. $\beta = 1, 3, 4,$ or 6 .

Proof. The result follows immediately from Lemma 1.

Lemma 3. *If $\alpha(\tau) = \alpha(\mathfrak{B})$, then $\mathfrak{B} \cap G^{-1}\mathfrak{B}G = 1$ or \mathfrak{B} for every element G in \mathfrak{G} .*

Proof. If $\mathfrak{B} \cap G^{-1}\mathfrak{B}G$ contains τ , then $\mathfrak{F}(\tau)$ contains $\mathfrak{F}(\mathfrak{B})$ and $\mathfrak{F}(G^{-1}\mathfrak{B}G)$, and thus $\mathfrak{F}(\tau) = \mathfrak{F}(\mathfrak{B}) = \mathfrak{F}(G^{-1}\mathfrak{B}G)$. This implies that \mathfrak{B} and $G^{-1}\mathfrak{B}G$ are contained in \mathfrak{K} and so $\mathfrak{B} = G^{-1}\mathfrak{B}G$. This proves our lemma.

Lemma 4. *If $\alpha(\tau) > \alpha(\mathfrak{B})$, then one of the followings holds :*

- (1) $i = 6$ and $C_{\mathfrak{G}}(\tau)/\langle \tau \rangle$ is \mathfrak{A}_5 ,
- (2) $i = 28$, $\alpha(\mathfrak{B}) = 4$ and $C_{\mathfrak{G}}(\tau)/\langle \tau \rangle$ is $PGL(2, 8)$,
- (3) $i = p^{2m}$ for some prime p , $\alpha(\mathfrak{B}) = p^m$ and $C_{\mathfrak{G}}(\tau)/\langle \tau \rangle$ contains a regular normal subgroup. Moreover if p is odd, then there exists a unique involution in $C_{\mathfrak{G}}(\tau)/\langle \tau \rangle$ which fixes only one point on $\mathfrak{F}(\tau)$.

Proof. Since $C_{\mathfrak{G}}(\tau)/\langle \tau \rangle$ is doubly transitive on $\mathfrak{F}(\tau)$ of degree i and order $2(i-1)i$, the results follow from Ito's theorem [7] and its proof.

Lemma 5. *If $\alpha(\tau) < \alpha(\mathfrak{B})$, then $\beta = 3, 4,$ or 6 .*

Proof. There exist two points j and k in $\mathfrak{F}(\tau) - \mathfrak{F}(\mathfrak{B})$ such that $\tau_1 = (j, k)\cdots$. Hence $\tau\tau_1 = (j, k)\cdots$ and Lemma 2 yields $\beta = 3, 4,$ or 6 since \mathfrak{G} is doubly transitive on Ω .

3. The case n is odd

In the following if $h^*(2) > 0$, then without loss of generality we may assume that $\alpha(I) = 1$.

Lemma 6. *If $h^*(2) = 1$, then $\mathfrak{G} = C_{\mathfrak{G}}(I)O(\mathfrak{G})$.*

Proof. Let \mathfrak{S} be a Sylow 2-subgroup of \mathfrak{G} containing I . By our assumption $\{G^{-1}IG; G \in \mathfrak{G}\} \cap \mathfrak{S} = \{I\}$. It follows from the Z^* -theorem of Glauberman [6; p. 628] that $\langle I \rangle O(\mathfrak{G})$ is a normal subgroup of \mathfrak{G} and then Frattini argument implies that $\mathfrak{G} = C_{\mathfrak{G}}(I)O(\mathfrak{G})$. This proves our lemma.

Lemma 7. *If $\alpha(\tau) = \alpha(\mathfrak{B})$ and $h^*(2) = 3$, then there exists no group satisfying the condition of our theorem.*

Proof. Put $\mathfrak{F}(I) = \{k\}$ and let \mathfrak{G}_k be the stabilizer of a point k in \mathfrak{G} . Then \mathfrak{G}_k contains $C_{\mathfrak{G}}(I)$ and $\mathfrak{F}(I\tau_1) = \mathfrak{F}(I\tau\tau_1) = \{k\}$. It follows from $h^*(2) = 3$

that $\langle I\tau_1, I\tau\tau_1, I \rangle = \langle I, \mathfrak{B} \rangle$ is normal in \mathfrak{G}_k . Now $\langle I, \mathfrak{B} \rangle$ is half transitive on $\Omega - \{k\}$. On the other hand I acts on $\mathfrak{F}(\mathfrak{B})$, and I -orbits on $\mathfrak{F}(\mathfrak{B})$ are of length 2 and \mathfrak{B} -orbits on $\Omega - \mathfrak{F}(\mathfrak{B})$ are of length 4 by our assumption. Thus we get a contradiction. The proof is complete.

Now $h^*(2)=0$ and then τ is a central involution in some Sylow 2-subgroup of \mathfrak{G} . Let \mathfrak{S} be a Sylow 2-subgroup of \mathfrak{G} containing $\langle I, \mathfrak{B} \rangle$ and contained in $C_{\mathfrak{G}}(\tau)$.

Lemma 8. *If $h^*(2)=0$, then $\alpha(\tau)=\alpha(\mathfrak{B})$.*

Proof. Assume by way of contradiction that $\alpha(\tau) < \alpha(\mathfrak{B})$. Then $|C_{\mathfrak{G}}(\tau)| = 4i(i-1)$ and $|N_{\mathfrak{G}}(\mathfrak{B})| = 4\sqrt{i}(\sqrt{i}-1)$ by Lemma 4 since $C_{\mathfrak{G}}(\tau)$ and $N_{\mathfrak{G}}(\mathfrak{B})$ is doubly transitive on $\mathfrak{F}(\tau)$ and $\mathfrak{F}(\mathfrak{B})$, respectively. Hence $N_{\mathfrak{G}}(\mathfrak{B})$ does not contain a Sylow 2-subgroup of \mathfrak{G} . We first show that (1) in Lemma 1 holds in this case. Suppose that (2) or (3) in Lemma 1 holds. Then $\mathfrak{G}_{\{1,2\}} = \langle I \rangle \mathfrak{R}$ has no element of order four and hence any 2-element with a 2-cycle must be an involution since \mathfrak{G} is doubly transitive. Then since $h^*(2)=0$ \mathfrak{G} has no element of order four and so \mathfrak{S} is elementary abelian. Then $N_{\mathfrak{G}}(\mathfrak{B})$ contains \mathfrak{S} , a contradiction. Thus (1) in Lemma 1 must hold. Then $\mathfrak{G}_{\{1,2\}}$ is the symmetric group of degree four, $\beta=d=6$ and $n=i(2i-1)$. In particular, $O(\mathfrak{G})=1$. Four groups in $\mathfrak{G}_{\{1,2\}}$ form two conjugate classes and their representatives are \mathfrak{B} and $\langle I, \tau \rangle$. Now we regard \mathfrak{G} as a transitive permutation group on the set of the unordered pairs of the points of Ω . Then $\mathfrak{G}_{\{1,2\}}$ is the stabilizer of the pair $\{1, 2\}$. If \mathfrak{B} is not conjugate to $\langle I, \tau \rangle$ in \mathfrak{G} , \mathfrak{B} satisfies the assumption of a theorem of Witt [6; p. 150], and hence $N_{\mathfrak{G}}(\mathfrak{B})$ is transitive on the pairs which \mathfrak{B} fixes. This forces \mathfrak{B} to have no orbit of length 2 on Ω since $N_{\mathfrak{G}}(\mathfrak{B})$ fixes $\mathfrak{F}(\mathfrak{B})$ as a whole. This implies that $\alpha(\tau)=\alpha(\mathfrak{B})$, contrary to the assumption. Thus $\mathfrak{B} \sim \langle I, \tau \rangle$ in \mathfrak{G} . On the other hand since $h^*(2)=0$, any four group has an orbit of length 2 and hence is conjugate to \mathfrak{B} . Then if \mathfrak{S} is not of a maximal class, $N_{\mathfrak{G}}(\mathfrak{B})$ contains a Sylow 2-subgroup of \mathfrak{G} (See [2; p. 215]) which is a contradiction. Thus \mathfrak{S} must be of a maximal class and hence dihedral or semi-dihedral. Since $C_{\mathfrak{G}}(\tau)/\langle \tau \rangle$ has a dihedral Sylow 2-subgroup $\mathfrak{S}/\langle \tau \rangle$, a result of Gorenstein-Walter [4] and Lemma 4 imply that $C_{\mathfrak{G}}(\tau)/\langle \tau \rangle$ is 2'-closed and so is $C_{\mathfrak{G}}(\tau)$. By theorems of Gorenstein [3] and Lüneburg [9] we get a contradiction. The proof is complete.

Lemma 9. *If $h^*(2)=0$, then $n=15$ and \mathfrak{G} is \mathfrak{A}_7 .*

Proof. Since $\alpha(\tau)=\alpha(\mathfrak{B})$ by Lemma 8, $C_{\mathfrak{G}}(\tau)/\mathfrak{B}$ is a complete Frobenius group of odd degree i and then $\mathfrak{S}/\mathfrak{B}$ is cyclic or generalized quaternion. Assume that $[I, \mathfrak{B}] \neq 1$. It follows that $(\mathfrak{S} : C_{\mathfrak{S}}(\mathfrak{B}))=2$ and $I\mathfrak{B}$ is a unique involution in $\mathfrak{S}/\mathfrak{B}$. Now $C_{\mathfrak{S}}(\mathfrak{B})=\mathfrak{B}$ and $\mathfrak{S}=\langle I, \mathfrak{B} \rangle$ is a dihedral group of order 8. Since $\beta=d=6$ by Lemma 1 and $n=i(2i-1)$, \mathfrak{G} contains no regular normal subgroup.

Applying theorems of Gorenstein-Walter [4] and Lüneburg [9], \mathfrak{G} is \mathfrak{A}_7 . On the other hand it is well known that \mathfrak{A}_7 has a doubly transitive permutation representation of degree 15 in which the stabilizer of two points is \mathfrak{A}_4 (See [6; p. 157]). Thus $n=15$ and \mathfrak{G} is \mathfrak{A}_7 . Next assume that $[I, \mathfrak{B}]=1$. Then by the same way as in the proof of Lemma 8, \mathfrak{C} is elementary abelian and hence $\mathfrak{C}=\langle I, \mathfrak{B} \rangle$. Now $C_{\mathfrak{G}}(\tau)$ is solvable and so theorems of Gorenstein [1] and Lüneburg [9] yield a contradiction. This proves our Lemma.

4. The case n is even

Lemma 10. *If $\alpha(\tau)=\alpha(\mathfrak{B})$, then $n=6$ and \mathfrak{G} is \mathfrak{A}_6 , or $n=16$ and \mathfrak{G} is $A(2, 4)$.*

Proof. Assume that \mathfrak{H} is 2-closed. Then \mathfrak{H} acts on $\mathfrak{F}(\mathfrak{B})$ and since \mathfrak{H} is transitive on $\Omega - \{1\}$, we have $\mathfrak{F}(\mathfrak{B})=\Omega$ which is impossible. By Lemma 3, \mathfrak{H} is a (TI) -group in the sense of Suzuki [11]. Since $\mathfrak{H}/O(\mathfrak{H})$ is also a (TI) -group, Suzuki's result [11; p. 69] implies that $\mathfrak{H}/O(\mathfrak{H})$ is $PSL(2, 4)$ and $O(\mathfrak{H})$ is contained in the center of \mathfrak{H} . On the other hand we have $|N_{\mathfrak{G}}(\mathfrak{B})|=12i(i-1)$ and $|C_{\mathfrak{H}}(\mathfrak{B})|=4(i-1)$. Now $|O(\mathfrak{H})|=i-1$ and $|\mathfrak{H}/O(\mathfrak{H})|=4(\beta i+3)=|PSL(2, 4)|=60$. Therefore $\beta i=12$. In our case since $C_{\mathfrak{G}}(\tau)\mathfrak{B}$ is a complete Frobenius group of even degree i , i is a power of 2 and then $i=2$ or 4. If $i=2$, then $n=6$ and \mathfrak{G} is \mathfrak{A}_6 . If $i=4$, then $n=16$. Since $\mathfrak{R}=\langle K, \mathfrak{B} \rangle$ and K is of order 3, $\alpha(\mathfrak{B})-\alpha(\mathfrak{R})$ is divisible by 3 and $\mathfrak{F}(\mathfrak{B})=\mathfrak{F}(\mathfrak{R})$. Applying a result of Witt [6; p. 150] to $N_{\mathfrak{G}}(\mathfrak{R})$ and $N_{\mathfrak{G}}(\langle K \rangle)$ we can get easily $\alpha(\mathfrak{R})=4$. Therefore \mathfrak{R} is semi-regular on $\Omega - \mathfrak{F}(\mathfrak{R})$ and thus \mathfrak{R} is transitive on $\Omega - \mathfrak{F}(\mathfrak{R})$. Since n is even it follows from Kantor's theorem [8] that \mathfrak{G} is isomorphic to a subgroup of $A(2, 4)$ or $A(4, 2)$. Assume that \mathfrak{G} is a subgroup of $A(4, 2)$. Let \mathfrak{N} be a regular normal subgroup of $A(4, 2)$. If $\mathfrak{G} \cap \mathfrak{N}=1$, then \mathfrak{G} is isomorphic to a subgroup of $GL(4, 2)$ which is impossible because $GL(4, 2)$ contains no subgroup of index 7. Hence $\mathfrak{G} \cap \mathfrak{N} \neq 1$ and then \mathfrak{G} contains \mathfrak{N} and \mathfrak{H} is isomorphic to $\mathfrak{G}/\mathfrak{N}$. Since $|O(\mathfrak{H})|=3$ and $O(\mathfrak{H})$ is contained in the center of \mathfrak{H} , \mathfrak{H} is $GL(2, 4)$. Thus $n=16$ and \mathfrak{G} is $A(2, 4)$. This proves our lemma.

Lemma 11. *If $\alpha(\tau) > \alpha(\mathfrak{B})$ and $i=6$ or 28, then there exists no group satisfying the condition of our theorem.*

Proof. Assume that $i=6$. Since $C_{\mathfrak{G}}(\tau)$ contains \mathfrak{B} , Schur's theorem [10] implies that $C_{\mathfrak{G}}(\tau)=\langle \tau \rangle \times \mathfrak{F}$ where \mathfrak{F} is \mathfrak{A}_5 by Lemma 4. It follows that $[I, \mathfrak{B}]=1$ and $d=4$. Now $|C_{\mathfrak{H}}(\tau)|=2^2 \cdot 5$ and $|\mathfrak{H}|=2^2 \cdot 3^3 \cdot 5$ or $2^2 \cdot 3 \cdot 5 \cdot 7$. Assume that $i=28$. Then Lemma 4 yields $\alpha(\mathfrak{B})=\alpha(\mathfrak{R})=4$ and using a result of Witt [6; p. 150], $|N_{\mathfrak{G}}(\mathfrak{R})|=|N_{\mathfrak{G}}(\mathfrak{B})|=144$, $N_{\mathfrak{G}}(\mathfrak{R})=\mathfrak{R} \times C_{\mathfrak{G}}(\mathfrak{R})$. It follows that $[I, \mathfrak{B}]=1$ and so $d=4$. Now $|C_{\mathfrak{H}}(\tau)|=2^2 \cdot 3^3$ and $|\mathfrak{H}|=2^2 \cdot 3^2 \cdot 5 \cdot 23$ or $2^2 \cdot 3^4 \cdot 29$. In both cases applying a theorem of Gorenstein-Watler [4], $\mathfrak{H}/O(\mathfrak{H})$

is isomorphic to a subgroup of $P\Gamma L(2, r)$ containing $PSL(2, r)$. Clearly this is impossible. This proves our lemma.

In the following we assume that the case (3) of Lemma 4 holds.

Lemma 12. *The group $\mathfrak{H}/O(\mathfrak{H})$ is $PSL(2, q)$ for some q .*

Proof. Since $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is a solvable doubly transitive group on $\mathfrak{F}(\tau)$ of degree 2^{2m} , it follows from a theorem of Huppert [5] that $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is contained in 1-dimensional semi-linear affine transformation group over the field of 2^{2m} -elements and then $C_{\mathfrak{H}}(\tau)$ has a cyclic normal 2-complement. Thus by a result of Gorenstein-Walter [4], $\mathfrak{H}/O(\mathfrak{H})$ is $PSL(2, q)$. This proves our lemma.

Lemma 13. *There exists no group with $\alpha(\tau)=2^{2m}$ and $\alpha(\mathfrak{B})=2^m$.*

Proof. Put $\bar{\mathfrak{H}}=\mathfrak{H}/O(\mathfrak{H})$. Then $\bar{\mathfrak{H}}$ is $PSL(2, q)$ with $q\equiv 3$ or $5 \pmod{8}$ by Lemma 12. Hence $C_{\bar{\mathfrak{H}}}(\bar{\mathfrak{B}})=\bar{\mathfrak{B}}$ and $C_{\bar{\mathfrak{H}}}(\bar{\tau})$ has a normal 2-complement of order $q+1/4$ or $q-1/4$ according as $q\equiv 3$ or $5 \pmod{8}$. On the other hand $|C_{\mathfrak{H}}(\tau)|=4(i-1)$ and $|C_{\mathfrak{H}}(\mathfrak{B})|=4(\sqrt{i}-1)$ since $|N_{\mathfrak{H}}(\mathfrak{B})|=12(\sqrt{i}-1)$. Then $|O(\mathfrak{H})\cap C_{\mathfrak{H}}(\mathfrak{B})|=(\sqrt{i}-1)$ and hence $|O(\mathfrak{H})\cap C_{\mathfrak{H}}(\tau)|=x(\sqrt{i}-1)$ with some odd integer x dividing $\sqrt{i}+1$. Then the formula of Brauer-Wielandt [12] yields $|O(\mathfrak{H})|=x^3(\sqrt{i}-1)$. Since $|\mathfrak{H}|=12(n-1)=4(\beta-1)(\beta i+3)$, we have

$$|\bar{\mathfrak{H}}|=4(\sqrt{i}+1)(\beta i+3)/x^3=(q+1)q(q-1)/2 \tag{4.1}$$

Since $C_{\mathfrak{H}}(\tau)$ has a cyclic normal 2-complement $\langle w \rangle$ of order $i-1$ and $C_{\bar{\mathfrak{H}}}(\bar{\tau})=\overline{C_{\mathfrak{H}}(\tau)}$, we have

$$o(\bar{w})=i-1/x(\sqrt{i}-1)=q\pm 1/4 \tag{4.2}$$

Then (4.1) yields

$$2(\beta i+3)/x^2=q(q\mp 1) \tag{4.3}$$

In particular, x is a common divisor of $\sqrt{i}+1$ and $\beta i+3$ where $\sqrt{i}=2^m$ and $\beta=3, 4$, or 6 by Lemma 5. Assume that $\beta=3$. Then $x=1$ or 3 since $\beta i+3=3\cdot 2^{2m}+3\equiv 6 \pmod{2^m+1}$. Now it is easy to see that (4.2) and (4.3) are incompatible. In the case where $\beta=4$ or 6 , the proofs are similar.

The proof of our theorem is complete.

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