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## ON PERIODIC $\beta$ -EXPANSIONS OF PISOT NUMBERS AND RAUZY FRACTALS

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### 0. Introduction

Let  $\lambda$  be the real maximum solution of the polynomial  $p(x) : k_1, k_2 \in \mathbf{N}$  and  $k_1 \geq k_2$  ( $k_1 \neq 0$ )

$$p(x) = x^3 - k_1x^2 - k_2x - 1.$$

The polynomial  $p(x)$  is given as the characteristic polynomial of the matrix  $M$ :

$$M = \begin{bmatrix} k_1 & k_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

And for each  $k_1, k_2$  the real cubic number  $\lambda$  is a Pisot number. A Pisot number is an algebraic integer whose conjugates other than itself have modulus less than one. Hence,

$$|\lambda'|, |\lambda''| < 1,$$

where  $\lambda', \lambda''$  are algebraic conjugates of  $\lambda$ . We denote the column and row eigenvectors of  $\lambda$  by

$$M \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad {}^t M \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix},$$

where  $t$  indicates the transpose.

Let  $T_\lambda : [0, 1) \rightarrow [0, 1)$  be the transformation given by

$$T_\lambda x = \lambda x - [\lambda x],$$

where  $[r]$  denotes the integer part of a real number  $r$ . Then each  $x \in [0, 1)$  is represented by

$$(*) \quad x = \sum_{k=1}^{\infty} \frac{b_k}{\lambda^k},$$

where  $b_k = [\lambda T_{\lambda}^{k-1} x]$ ,  $k = 1, 2, \dots$ , the expansion  $(*)$  of  $x$  is usually called  $\beta$ -expansion. In this paper, we call it  $\lambda$ -expansion.

Let  $\mathbf{Q}(\lambda)$  be the smallest extension field of rational numbers  $\mathbf{Q}$  containing  $\lambda$ . K. Schmidt showed the following result in [8].

**Theorem (Schmidt).** *A real number  $x$  is in  $\mathbf{Q}(\lambda) \cap [0, 1)$  if and only if  $\lambda$ -expansion of  $x$  is eventually periodic.*

In [1], Akiyama gives a sufficient condition of purely periodicity.

In this paper, we discuss when  $\lambda$ -expansion of  $x$  is purely periodic. For this purpose, we introduce the three dimensional domain  $\widehat{Y}$  with fractal boundary (see Fig. 1 and the definition in Section 2) and we say a real number  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$  is reduced if  $\rho(x) \in \widehat{Y}$  where  $\omega = 1/(1 + \alpha\gamma + \beta\delta)$ ,  $\rho(x)$  is given by

$$\rho(x) = \begin{cases} {}^t(x, x' + x'', (x' - x'')i) & \text{if } \mathbf{Q}(\lambda) \text{ is not a totally real cubic field,} \\ {}^t(x, x', x'') & \text{if } \mathbf{Q}(\lambda) \text{ is a totally real cubic field,} \end{cases}$$

and  $x'$  and  $x''$  denote algebraic conjugates of  $x$ . The main result of this paper is the following:

**Main Theorem.** *Let  $x$  be a real number in  $\mathbf{Q}(\lambda) \cap [0, 1)$ . Then  $\lambda$ -expansion of  $x$  is purely periodic if and only if  $\omega x$  is reduced.*

The main tool of the proof is a natural extension on the domain  $\widehat{Y}$  of the dynamical system  $([0, 1), T_{\lambda})$ , which is discussed in [7] and [9] originally. And the basic idea of the proof can be found in [4] and [5].

### 1. Dual transformation of $T_{\lambda}$

From the property of the eigenvector  ${}^t(1, \gamma, \delta)$  such that

$$k_1 + \gamma = \lambda \cdot 1 \quad k_2 + \delta = \lambda \cdot \gamma \quad 1 = \lambda \cdot \delta,$$

we see the transformation  $T_{\lambda}$  :

$$T_{\lambda}x = \lambda x \pmod{1}$$

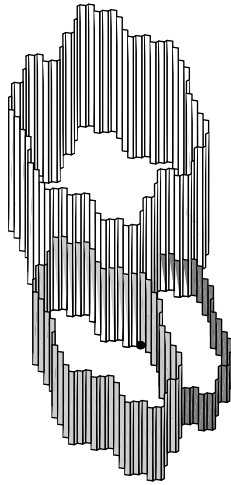


Fig. 1. Figure of  $\widehat{Y}$ .

is the  $\lambda$ -transformation with the shift of finite type. The sequences  $\{\{b_k\}_{k=1}^\infty\}$  of  $\lambda$ -expansion satisfies the following admissible condition:

- (1)  $0 \leq b_i \leq k_1$ ,
- (2) if  $b_i = k_1$  then  $b_{i+1} \leq k_2$ ,
- (3) if  $(b_i, b_{i+1}) = (k_1, k_2)$  then  $b_{i+2} = 0$ .

In other words, the admissible sequence  $(b_1, b_2, \dots, b_k, \dots)$  is given by the labeled graph  $G$  in Fig. 2.

Let  $W^* = \bigcup_{n=0}^\infty \{1, 2, 3\}^n$  be the free monoid of  $\{1, 2, 3\}$  and let us define the substitution  $\sigma_{k_1, k_2} : W^* \rightarrow W^*$  by

$$\sigma_{k_1, k_2} : \begin{array}{l} 1 \longrightarrow \overbrace{11 \dots 1}^{k_1} 2 \\ 2 \longrightarrow \overbrace{11 \dots 1}^{k_2} 3 \\ 3 \longrightarrow 1 \end{array} .$$

Then the matrix of the substitution  $\sigma_{k_1, k_2}$  is given by  $M$  and so it is called Pisot substitution. Moreover, the substitution  $\sigma_{k_1, k_2}$  satisfies the coincidence condition in [2]. Therefore, we have the following theorem.

**Theorem (Arnoux-Ito).** *Let  $\mathbf{P}$  be the contractive invariant plane with respect to the linear transformation  $M$ , which is given by*

$$\mathbf{P} = \{\mathbf{x} \in \mathbf{R}^3 \mid \langle \mathbf{x}, {}^t(1, \gamma, \delta) \rangle = 0\} .$$

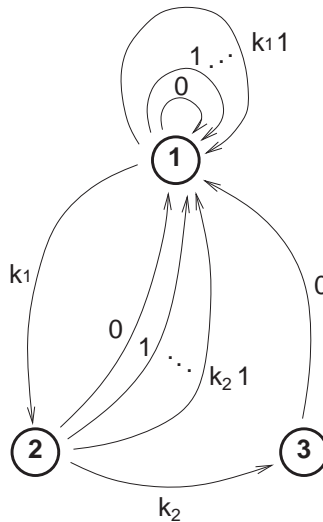


Fig. 2. Labeled graph  $G$ .

Then, there exist the closed domains  $X$  and  $X_i, i = 1, 2, 3$  on the plane  $\mathbf{P}$  satisfying the following properties:

The boundaries of  $X$  and  $X_i$  are fractal Jordan curves and

$$X = \bigcup_{i=1,2,3} X_i \quad (\text{disjoint}),$$

$$\bigcup_{\mathbf{z} \in \{\pi(m(\mathbf{e}_2 - \mathbf{e}_1) + n(\mathbf{e}_3 - \mathbf{e}_1)) \mid m, n \in \mathbf{Z}\}} (X + \pi \mathbf{z}) = \mathbf{P} \quad (\text{disjoint}),$$

and moreover

$$M^{-1}X_1 = \bigcup_{j=0}^{k_1-1} (X_1 - j\pi\mathbf{e}_3) \cup \bigcup_{j=0}^{k_2-1} (X_2 - j\pi\mathbf{e}_3) \cup X_3 \quad (\text{disjoint}),$$

$$M^{-1}X_2 = X_1 - k_1\pi\mathbf{e}_3, \quad M^{-1}X_3 = X_2 - k_2\pi\mathbf{e}_3,$$

where  $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$  is the canonical basis of  $\mathbf{R}^3$ ,  $\pi : \mathbf{R}^3 \rightarrow \mathbf{P}$  is the projection along  ${}^t(1, \alpha, \beta)$ , and  $A \cup B$  (disjoint) means that the interior of  $A$  and the interior of  $B$  are disjoint sets.

REMARK. By using the notation of [6], we will give a survey how the domains  $X, X_i, i = 1, 2, 3$  are obtained. From the substitution  $\sigma_{k_1, k_2}$ , let us give the map  $\Sigma_{k_1, k_2}$

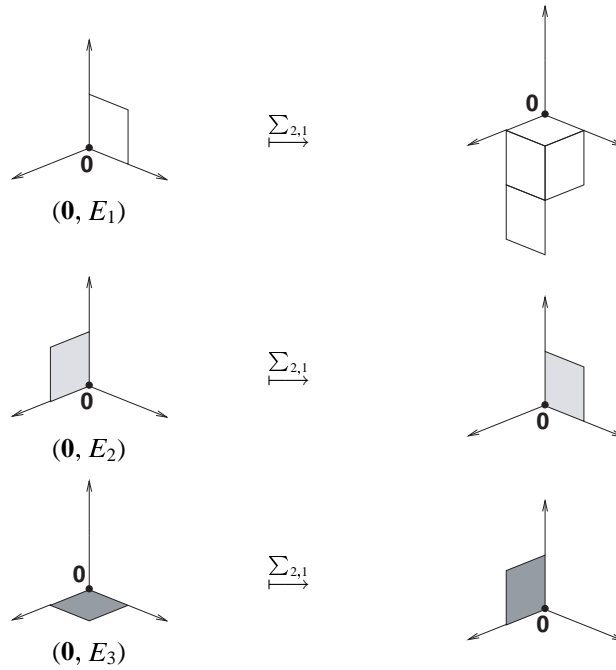


Fig. 3. Figure of  $\Sigma_{2,1}(0, E_i)$ ,  $i = 1, 2, 3$ .

on the family of patches of the stepped surface of  $\mathbf{P}$  by

$$\begin{aligned} \Sigma_{k_1, k_2} : \quad & \begin{aligned} (\mathbf{0}, E_1) &\longrightarrow (\mathbf{0}, E_3) + \sum_{k=1}^{k_1} (\mathbf{e}_1 - k\mathbf{e}_3, E_1) + \sum_{k=1}^{k_2} (\mathbf{e}_2 - k\mathbf{e}_3, E_2), \\ (\mathbf{0}, E_2) &\longrightarrow (\mathbf{0}, E_1) \\ (\mathbf{0}, E_3) &\longrightarrow (\mathbf{0}, E_2) \end{aligned} \\ \Sigma_{k_1, k_2}(\mathbf{x}, E_i) &:= M^{-1}\mathbf{x} + \Sigma_{k_1, k_2}(\mathbf{0}, E_i) \end{aligned}$$

(see Fig. 3). Then, the domains  $X$  and  $X_i, i = 1, 2, 3$  are given by

$$X = \lim_{n \rightarrow \infty} M^n \pi \left( \Sigma_{k_1, k_2} \right)^n \left( \bigcup_{i=1,2,3} (\mathbf{e}_i, E_i) \right),$$

and

$$X_i = \lim_{n \rightarrow \infty} M^n \pi \left( \Sigma_{k_1, k_2} \right)^n (\mathbf{e}_i, E_i)$$

(see Fig. 4).

The boundaries of the domains  $X, X_i, i = 1, 2, 3$  are given by the following manner: Let  $\theta_{k_1, k_2} : G \langle 1, 2, 3 \rangle \rightarrow G \langle 1, 2, 3 \rangle$  be the endomorphism of the free group of rank 3

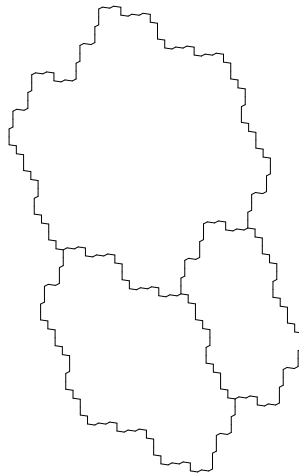


Fig. 4. Figure of  $X = \bigcup_{i=1,2,3} X_i$  on  $(k_1, k_2 = (1, 1))$ .

given by

$$\begin{aligned}
 & 1 \longrightarrow 3 \\
 \theta_{k_1, k_2} : 2 & \longrightarrow \overbrace{13^{-1}3^{-1} \dots 3^{-1}}^{k_1}, \\
 & 3 \longrightarrow \overbrace{23^{-1}3^{-1} \dots 3^{-1}}^{k_2}
 \end{aligned}$$

then the boundaries are given by

$$\begin{aligned}
 \partial X &= \lim_{n \rightarrow \infty} M^n \pi \left( \mathbf{f}_1^{(n)} + \mathcal{K} \left( \theta^n (21^{-1}32^{-1}13^{-1}) \right) \right), \\
 \partial X_i &= \lim_{n \rightarrow \infty} M^n \pi \left( \mathbf{f}_i^{(n)} + \mathcal{K} \left( \theta^n (j k j^{-1} k^{-1}) \right) \right),
 \end{aligned}$$

where  $(\mathbf{f}_1^{(n)}, \mathbf{f}_2^{(n)}, \mathbf{f}_3^{(n)}) = M^{-n}$ ,  $\mathcal{K}$  is the polygonal realization map on  $G \langle 1, 2, 3 \rangle$ , and  $\{i, j, k\} = \{1, 2, 3\}$  (see [6], [3] in detail).

From the fact in Theorem (Arnoux-Ito) and the property  $M\pi\mathbf{e}_3 = \pi M\mathbf{e}_3 = \pi\mathbf{e}_1$ , we know

$$X = \bigcup_{j=0}^{k_1} (MX_1 - j\pi\mathbf{e}_1) \cup \bigcup_{j=0}^{k_2} (MX_2 - j\pi\mathbf{e}_1) \cup MX_3 \quad (\text{disjoint}).$$

On the notation

$$\begin{aligned} P_j^{(1)} &= MX_1 - j\pi\mathbf{e}_1, & j = 0, 1, \dots, k_1, \\ P_j^{(2)} &= MX_2 - j\pi\mathbf{e}_1, & j = 0, 1, \dots, k_2, \\ P_0^{(3)} &= MX_3, \end{aligned}$$

the set  $\xi = \{P_0^{(1)}, \dots, P_{k_1}^{(1)}, P_0^{(2)}, \dots, P_{k_2}^{(2)}, P_0^{(3)}\}$  is a partition of  $X$ . Using the partition  $\xi$ , let us define the transformation  $T_\lambda^*$  on  $X$  by

$$T_\lambda^* \mathbf{x} = M^{-1} \mathbf{x} + b^* \pi \mathbf{e}_3 \quad \text{if } \mathbf{x} \in P_{b^*}^{(i)} \text{ for some } i \text{ and } b^*.$$

Then, for each  $\mathbf{x} \in X$  we have the following sequence  $(b_1^*, b_2^*, \dots)$  by

$$T_\lambda^{*k-1} \mathbf{x} \in P_{b_k^*}^{(i)} \quad \text{for some } i,$$

and we have the expansion: for each  $\mathbf{x} \in X$ ,

$$\mathbf{x} = - \sum_{k=1}^{\infty} b_k^* M^{k-1} \pi \mathbf{e}_1.$$

We see that the sequence  $(b_1^*, b_2^*, \dots)$  is obtained from the labeled graph  $G^*$ , which is dual of the graph  $G$  (see Fig. 5). Therefore, we say the transformation  $T_\lambda^*$  is a *dual* transformaiton of  $T_\lambda$ .

Let us define the three dimensional domains  $\widehat{X} = \bigcup_{i=1,2,3} \widehat{X}_i$  as follows: for  $i = 1, 2, 3$

$$\widehat{X}_i := \left\{ t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + \mathbf{x} \mid 0 \leq t < t_i^0, \mathbf{x} \in X_i \right\},$$

where  $(t_1^0, t_2^0, t_3^0) = (1, \gamma, \delta)$  and  $\omega = 1/(1 + \alpha\gamma + \beta\delta)$  (see Fig. 6). Let us define the transformation  $\widehat{T}_\lambda : \widehat{X} \rightarrow \mathbf{R}^3$  by

$$\widehat{T}_\lambda \left( t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + \mathbf{x} \right) := \left( \lambda t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} - [\lambda t]\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + M\mathbf{x} - [\lambda t]\pi\mathbf{e}_1 \right).$$

Then we have the proposition.

**Proposition 1.1.** *The transformation  $\widehat{T}_\lambda$  is surjective and a.e. injective transformation on  $\widehat{X}$ .*



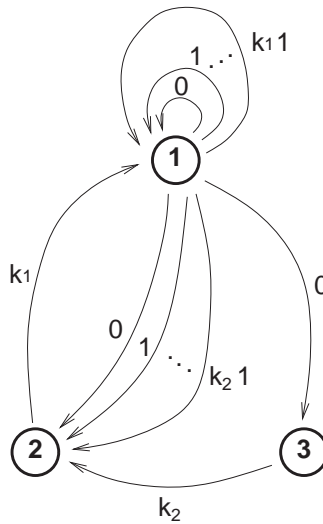


Fig. 5. Dual graph  $G^*$ .

Proof. By the Theorem (Arnoux-Ito), the domains  $X_i, i = 1, 2, 3$  are decomposed in the following way:

$$\begin{aligned}
 X_1 &= \bigcup_{j=0}^{k_1-1} (MX_1 - j\pi\mathbf{e}_1) \cup \bigcup_{j=0}^{k_2-1} (MX_2 - j\pi\mathbf{e}_1) \cup MX_3, \\
 X_2 &= MX_1 - k_1\pi\mathbf{e}_1, \\
 X_3 &= MX_2 - k_2\pi\mathbf{e}_1.
 \end{aligned}$$

On the other hand, the sets  $\widehat{X}_i, i = 1, 2, 3$  are transformed by  $M$

$$M\widehat{X}_i = \left\{ \lambda t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + M\mathbf{x} \mid 0 \leq t < t_i^0, \mathbf{x} \in X_i \right\}.$$

By using the fact that  $\lambda \cdot 1 = k_1 + \gamma$ ,  $\lambda \cdot \gamma = k_2 + \delta$ , and  $\lambda \cdot \delta = 1$ , we cut the cylinder  $M\widehat{X}_1$  to  $k_1$  pieces of the length  $\omega$  and one piece of length  $\gamma\omega$ . Analogously, we cut the cylinder  $M\widehat{X}_2$  to  $k_2$  pieces of the length  $\omega$  and one piece of length  $\delta\omega$ . Then, applying  $\widehat{T}_\lambda$  shows that  $\widehat{T}_\lambda$  is surjective and injective except the boundary on  $\widehat{X}$  (see Fig. 6). □

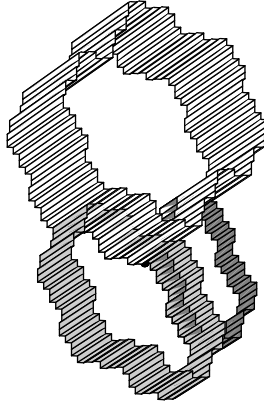


Fig. 6. Figure of  $\widehat{X}$ .

**2. Preliminaries from algebra**

We know the vector  $\langle 1, \alpha, \beta \rangle$  is the basis of  $\mathbf{Q}(\lambda)$ , that is, for any  $x \in \mathbf{Q}(\lambda)$  there exist rational numbers  $c_0, c_1$  and  $c_2$  such that

$$x = c_0 + c_1\alpha + c_2\beta,$$

and we denote  $x'$  and  $x''$  which are algebraic conjugates of  $x$ , that is,

$$\begin{aligned} x' &= c_0 + c_1\alpha' + c_2\beta' \in \mathbf{Q}(\lambda'), \\ x'' &= c_0 + c_1\alpha'' + c_2\beta'' \in \mathbf{Q}(\lambda''). \end{aligned}$$

First, let us assume that the cubic field is not totally real. We will begin with introducing two maps  $\eta : \mathbf{Q}(\lambda) \rightarrow \mathbf{R} \times \mathbf{C}^2$  and  $\rho : \mathbf{Q}(\lambda) \rightarrow \mathbf{R}^3$  by

$$\eta(x) := \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix} \quad \text{and} \quad \rho(x) := \begin{pmatrix} x \\ x' + x'' \\ (x' - x'')i \end{pmatrix}.$$

We get a few primitive lemmas and corollaries.

**Lemma 2.1.** *Let*

$$P := \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{bmatrix} \quad \text{and} \quad Q := [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2],$$

where

$$\mathbf{u}_0 := \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \mathbf{u}_1 := \frac{1}{2} \left( \begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix} + \begin{pmatrix} 1 \\ \alpha'' \\ \beta'' \end{pmatrix} \right), \mathbf{u}_2 := \frac{1}{2i} \left( \begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix} - \begin{pmatrix} 1 \\ \alpha'' \\ \beta'' \end{pmatrix} \right).$$

Then, we have

(1) for any  $x \in \mathbf{Q}(\lambda)$

$$P(\eta(x)) = Q(\rho(x)),$$

(2) the inverse matrix of  $P$  is given by

$$P^{-1} = \begin{bmatrix} \omega & \mu & \nu \\ \omega' & \mu' & \nu' \\ \omega'' & \mu'' & \nu'' \end{bmatrix},$$

where

$$\begin{aligned} \omega &= \frac{1}{D} \begin{vmatrix} \alpha' & \alpha'' \\ \beta' & \beta'' \end{vmatrix}, & \mu &= \frac{1}{D} (\beta' - \beta''), \\ \nu &= \frac{1}{D} (\alpha'' - \alpha'), & D &= \det P \end{aligned}$$

(In Corollary 2.3, we see  $\omega = 1/(1 + \alpha\gamma + \beta\delta)$ ).

Proof. (1) is easily obtained.

(2) By Cramer's rule, we have

$$(P^{-1})_{11} = \omega, \quad (P^{-1})_{12} = \mu, \quad \text{and} \quad (P^{-1})_{13} = \nu.$$

In Corollary 2.3, we can see that  $\omega, \mu$ , and  $\nu$  are elements of  $\mathbf{Q}(\lambda)$ . Consider the matrix  $P \cdot {}^t P$ :

$$P \cdot {}^t P = \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{bmatrix} \begin{bmatrix} 1 & \alpha & \beta \\ 1 & \alpha' & \beta' \\ 1 & \alpha'' & \beta'' \end{bmatrix} = \begin{bmatrix} 3 & Tr(\alpha) & Tr(\beta) \\ Tr(\alpha) & Tr(\alpha^2) & Tr(\alpha\beta) \\ Tr(\beta) & Tr(\alpha\beta) & Tr(\beta^2) \end{bmatrix},$$

where

$$Tr(\theta) = \theta + \theta' + \theta'',$$

for any algebraic number  $\theta$ . Since  $Tr(\theta)$  is rational, each element of  $P \cdot {}^t P$  is a rational number. Then there exists a right elementary transformation  $U$  whose elements are rational numbers such that

$$(P \cdot {}^t P) \cdot U = I,$$

where  $I$  indicates the identity matrix. So that,

$$P \cdot ({}^t P \cdot U) = I.$$

Therefore we know

$$P^{-1} = {}^t P \cdot U = \begin{bmatrix} \omega & \mu & \nu \\ \omega' & \mu' & \nu' \\ \omega'' & \mu'' & \nu'' \end{bmatrix}.$$

□

**Lemma 2.2.** *The inverse matrix of  $Q$  is given by*

$$Q^{-1} = \begin{bmatrix} \omega & \mu & \nu \\ \omega' + \omega'' & \mu' + \mu'' & \nu' + \nu'' \\ i(\omega' - \omega'') & i(\mu' - \mu'') & i(\nu' - \nu'') \end{bmatrix}.$$

Therefore, the canonical basis  $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$  of  $\mathbf{R}^3$  is given by

$$\begin{aligned} \mathbf{e}_1 &= \omega \mathbf{u}_0 + (\omega' + \omega'') \mathbf{u}_1 + i(\omega' - \omega'') \mathbf{u}_2, \\ \mathbf{e}_2 &= \mu \mathbf{u}_0 + (\mu' + \mu'') \mathbf{u}_1 + i(\mu' - \mu'') \mathbf{u}_2, \\ \mathbf{e}_3 &= \nu \mathbf{u}_0 + (\nu' + \nu'') \mathbf{u}_1 + i(\nu' - \nu'') \mathbf{u}_2. \end{aligned}$$

**Corollary 2.3.** *The projections of  $\mathbf{e}_i, i = 1, 2, 3$  by  $\pi$  are given by*

$$\begin{aligned} \pi \mathbf{e}_1 &= (\omega' + \omega'') \mathbf{u}_1 + i(\omega' - \omega'') \mathbf{u}_2, \\ \pi \mathbf{e}_2 &= (\mu' + \mu'') \mathbf{u}_1 + i(\mu' - \mu'') \mathbf{u}_2, \\ \pi \mathbf{e}_3 &= (\nu' + \nu'') \mathbf{u}_1 + i(\nu' - \nu'') \mathbf{u}_2. \end{aligned}$$

Moreover, we have

$$\begin{aligned} Q^{-1} \pi \mathbf{e}_1 &= \begin{pmatrix} 0 \\ \omega' + \omega'' \\ i(\omega' - \omega'') \end{pmatrix}, \quad Q^{-1} \pi \mathbf{e}_2 = \begin{pmatrix} 0 \\ \mu' + \mu'' \\ i(\mu' - \mu'') \end{pmatrix}, \\ Q^{-1} \pi \mathbf{e}_3 &= \begin{pmatrix} 0 \\ \nu' + \nu'' \\ i(\nu' - \nu'') \end{pmatrix}, \quad \text{and } (\omega, \mu, \nu) = \frac{1}{1 + \alpha\gamma + \beta\delta} (1, \gamma, \delta). \end{aligned}$$

**Proof.** Recall that  $\pi$  is the projection on the plane  $\mathbf{P}$  along  $\mathbf{u}_0$ . The plane  $\mathbf{P}$  which is orthogonal  ${}^t(1, \gamma, \delta)$  is spanned by  $\mathbf{u}_1$  and  $\mathbf{u}_2$ . Hence, we can get the above

from Lemma 2.2. The second assertion is obtained by  $Q^{-1}\mathbf{u}_1 = \mathbf{e}_2$ ,  $Q^{-1}\mathbf{u}_2 = \mathbf{e}_3$ . Put

$$\mathbf{e}_i = \pi\mathbf{e}_i + c_i\mathbf{u}_0, \quad i = 1, 2, 3.$$

Then from the relation

$$\left\langle \mathbf{e}_i, \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} \right\rangle = \left\langle \pi\mathbf{e}_i, \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} \right\rangle + c_i \left\langle \mathbf{u}_0, \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} \right\rangle,$$

we have

$$(c_1, c_2, c_3) = \frac{1}{1 + \alpha\gamma + \beta\delta} (1, \gamma, \delta).$$

On the other hand, we know from Lemma 2.2 that

$$(c_1, c_2, c_3) = (\omega, \mu, \nu).$$

Therefore, we arrive at the conclusion. □

**Lemma 2.4.** *The following relation holds:*

$$MQ = Q \begin{bmatrix} \lambda & 0 & 0 \\ 0 & (\lambda' + \lambda'')/2 & -i(\lambda' - \lambda'')/2 \\ 0 & i(\lambda' - \lambda'')/2 & (\lambda' + \lambda'')/2 \end{bmatrix}.$$

*Proof.* The proof is easily obtained from the equations

$$\begin{aligned} M\mathbf{u}_0 &= \lambda\mathbf{u}_0, & M\mathbf{u}_1 &= \frac{\lambda'}{2}(\mathbf{u}_1 + i\mathbf{u}_2) + \frac{\lambda''}{2}(\mathbf{u}_1 - i\mathbf{u}_2), \\ M\mathbf{u}_2 &= \frac{\lambda'}{2i}(\mathbf{u}_1 + i\mathbf{u}_2) - \frac{\lambda''}{2i}(\mathbf{u}_1 - i\mathbf{u}_2), \end{aligned}$$

and the definitions of  $\mathbf{u}_1$  and  $\mathbf{u}_2$ . □

Secondly, let us assume that the cubic field is totally real. We use the same notation  $\rho$  as the map from  $\mathbf{Q}(\lambda)$  to  $\mathbf{R}^3$  by

$$\rho(x) := \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}.$$

**Lemma 2.5.** *Let*

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{bmatrix} = [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2].$$

*Then the inverse matrix of  $Q$  is given by*

$$Q^{-1} = \begin{bmatrix} \omega & \mu & \nu \\ \omega' & \mu' & \nu' \\ \omega'' & \mu'' & \nu'' \end{bmatrix},$$

*where  $\omega, \mu, \nu$  is given as (2) in Lemma 2.1.*

In stead of Lemma 2.2, Corollary 2.3, and Lemma 2.4, we have

**Lemma 2.6.** *The following formula hold:*

$$\pi \mathbf{e}_1 = \omega' \mathbf{u}_1 + \omega'' \mathbf{u}_2,$$

$$\pi \mathbf{e}_2 = \mu' \mathbf{u}_1 + \mu'' \mathbf{u}_2,$$

$$\pi \mathbf{e}_3 = \nu' \mathbf{u}_1 + \nu'' \mathbf{u}_2.$$

*Therefore, we have*

$$Q^{-1} \pi \mathbf{e}_1 = \begin{pmatrix} 0 \\ \omega' \\ \omega'' \end{pmatrix}, Q^{-1} \pi \mathbf{e}_2 = \begin{pmatrix} 0 \\ \mu' \\ \mu'' \end{pmatrix}, Q^{-1} \pi \mathbf{e}_3 = \begin{pmatrix} 0 \\ \nu' \\ \nu'' \end{pmatrix},$$

*and*

$$(\omega, \mu, \nu) = \frac{1}{1 + \alpha\gamma + \beta\delta} (1, \gamma, \delta).$$

*Moreover, we know trivially*

$$MQ = Q \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda'' \end{bmatrix}.$$

We have the following corollary in the both cases:

**Corollary 2.7.** *Let us define  $R_\lambda$  by*

$$R_\lambda := \begin{cases} \begin{bmatrix} (\lambda' + \lambda'')/2 & -i(\lambda' - \lambda'')/2 \\ i(\lambda' - \lambda'')/2 & (\lambda' + \lambda'')/2 \end{bmatrix} & \text{if } \mathbf{Q}(\lambda) \text{ is not totally real,} \\ \begin{bmatrix} \lambda' & 0 \\ 0 & \lambda'' \end{bmatrix} & \text{if } \mathbf{Q}(\lambda) \text{ is totally real,} \end{cases}$$

then we have

$$MQ = Q \left[ \begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & & \\ 0 & R_\lambda & \end{array} \right].$$

Let us define the domains  $\widehat{Y}$  and  $\widehat{Y}_i, i = 1, 2, 3$  as follows:

$$\widehat{Y} := Q^{-1}(\widehat{X}) \quad \text{and} \quad \widehat{Y}_i := Q^{-1}(\widehat{X}_i).$$

Then the domains  $\widehat{Y}$  and  $\widehat{Y}_i$  have explicit forms (see Fig. 1). From now on, remark that  $\widehat{Y}$  and  $\widehat{Y}_i$  are written as domains in  $\mathbf{R} \times \mathbf{R}^2$ .

**Lemma 2.8.** *The domains  $\widehat{Y}$  and  $\widehat{Y}_i, i = 1, 2, 3$  are given by*

$$\widehat{Y}_i = \left\{ \left( t\omega, -\sum_{k=1}^{\infty} b_k^* R_\lambda^{k-1} \mathbf{v} \right) \mid 0 \leq t < t_i^0, \right. \\ \left. (b_1^*, b_2^*, \dots) \text{ is an admissible sequence starting at } i \text{ in } G^* \right\},$$

where

$$\mathbf{v} = \begin{cases} \begin{pmatrix} \omega' + \omega'' \\ i(\omega' - \omega'') \end{pmatrix} & \text{if } \mathbf{Q}(\lambda) \text{ is not totally real,} \\ \begin{pmatrix} \omega' \\ \omega'' \end{pmatrix} & \text{if } \mathbf{Q}(\lambda) \text{ is totally real.} \end{cases}$$

*Proof.* From the definitions of  $\widehat{X}_i$  and  $\widehat{Y}_i, \widehat{Y}_i$  is given by

$$\widehat{Y}_i = \{ t\omega Q^{-1} \mathbf{u}_0 + Q^{-1} \mathbf{x} \mid 0 \leq t < t_i^0, \mathbf{x} \in X_i \}.$$

Using the formula of  $X_i$  and  $Q^{-1} \mathbf{u}_0 = \mathbf{e}_1$ , we have

$$\widehat{Y}_i = \left\{ t\omega \mathbf{e}_1 - \sum_{k=1}^{\infty} b_k^* Q^{-1} M^{k-1} \pi \mathbf{e}_1 \mid 0 \leq t < t_i^0, \right. \\ \left. (b_1^*, b_2^*, \dots) \text{ is an admissible sequence starting at } i \text{ in } G^* \right\}.$$

From the fact that

$$Q^{-1} M^k = \left[ \begin{array}{c|cc} \lambda^k & 0 & 0 \\ \hline 0 & R_\lambda^k & \\ 0 & & \end{array} \right] Q^{-1} \quad \text{and} \quad Q^{-1} \pi \mathbf{e}_1 = \mathbf{v},$$

we have the conclusion. □

Now, let us define the transformation  $\widehat{S}_\lambda$  on  $\widehat{Y}$  by

$$\widehat{S}_\lambda := Q^{-1} \circ \widehat{T}_\lambda \circ Q.$$

**Proposition 2.9.** *The transformation  $\widehat{S}_\lambda$  on  $\widehat{Y}$  is given explicitly by*

$$\widehat{S}_\lambda \left( t\omega, - \sum_{k=1}^{\infty} b_k^* R_\lambda^{k-1} \mathbf{v} \right) = \left( (\lambda t \omega - [\lambda t] \omega), - [\lambda t] \mathbf{v} - \sum_{k=1}^{\infty} b_k^* R_\lambda^k \mathbf{v} \right)$$

and  $\widehat{S}_\lambda$  on  $\widehat{Y}$  is surjective.

*Proof.* The proof is obtained from Proposition 1.1. □

### 3. Reduction theorem

Let  $\widehat{\mathbf{Z}} := [0, \omega) \times \mathbf{R}^2$  and let us define the transformation  $\widetilde{S}_\lambda$  on  $\widehat{\mathbf{Z}}$  by

$$\widetilde{S}_\lambda \left( x, \begin{pmatrix} y \\ z \end{pmatrix} \right) = \left( \left( \lambda x - \omega \left[ \frac{\lambda x}{\omega} \right] \right), - \left[ \frac{\lambda x}{\omega} \right] \mathbf{v} + R_\lambda \begin{pmatrix} y \\ z \end{pmatrix} \right).$$

Then, the restriction of the map  $\widetilde{S}_\lambda$  on the set  $\widehat{Y}$  coincides with  $\widehat{S}_\lambda$ .  
In stead of the transformation  $T_\lambda : [0, 1) \rightarrow [0, 1)$

$$T_\lambda x = \lambda x - [\lambda x],$$

let us introduce the following transformation  $T'_\lambda : [0, \omega) \rightarrow [0, \omega)$  by

$$T'_\lambda x = \lambda x - \omega \left[ \frac{\lambda x}{\omega} \right].$$



Then dynamical systems  $([0, 1), T_\lambda)$  and  $([0, \omega), T'_\lambda)$  are isomorphic by the map :  $x \rightarrow \omega x$  and for any  $x \in [0, \omega)$  can be expressed by

$$x = \omega \sum_{k=1}^{\infty} \frac{b_k}{\lambda^k},$$

where  $(b_1, b_2, \dots)$  is the admissible sequence of  $x/\omega \in [0, 1)$  by the transformation  $T_\lambda$ . And we can say that the transformation  $\widehat{S}_\lambda$  is the natural extension of  $T'_\lambda$  and  $T_\lambda$ .

Hereafter, we denote  $\rho(x)$  by the map from  $\mathbf{Q}(\lambda)$  to  $\mathbf{R} \times \mathbf{R}^2$ , that is,

$$\rho(x) = \begin{cases} \left( x, \begin{pmatrix} x' + x'' \\ (x' - x'') i \end{pmatrix} \right) & \text{if } \mathbf{Q}(\lambda) \text{ is not a totally real cubic field,} \\ \left( x, \begin{pmatrix} x' \\ x'' \end{pmatrix} \right) & \text{if } \mathbf{Q}(\lambda) \text{ is a totally real cubic field.} \end{cases}$$

**Lemma 3.1.** For any real number  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$ , we have

$$\widetilde{S}_\lambda \rho(x) = \rho(y),$$

where  $y = T'_\lambda x$ .

Proof. From the definitions of  $T'_\lambda x$ ,  $\widetilde{S}_\lambda$  and  $\rho$ , we know

$$\widetilde{S}_\lambda \rho(x) = \begin{cases} \left( \lambda x - \omega \left[ \frac{\lambda x}{\omega} \right], - \left[ \frac{\lambda x}{\omega} \right] \begin{pmatrix} \omega' + \omega'' \\ (\omega' - \omega'') i \end{pmatrix} + R_\lambda \begin{pmatrix} x' + x'' \\ (x' - x'') i \end{pmatrix} \right) & \text{if } \mathbf{Q}(\lambda) \text{ is not totally real,} \\ \left( \lambda x - \omega \left[ \frac{\lambda x}{\omega} \right], - \left[ \frac{\lambda x}{\omega} \right] \begin{pmatrix} \omega' \\ \omega'' \end{pmatrix} + R_\lambda \begin{pmatrix} x' \\ x'' \end{pmatrix} \right) & \text{if } \mathbf{Q}(\lambda) \text{ is totally real.} \end{cases}$$

On the other hand, by  $y = \lambda x - \omega [\lambda x/\omega]$ , we see  $\rho(y) = \widetilde{S}_\lambda \rho(x)$ . □

Let us introduce the concept of *reduced*.

**DEFINITION 3.2.** A real number  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$  is said to be reduced if  $\rho(x) \in \widehat{Y}$ .

**Lemma 3.3.** Let  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$  be reduced. Then

- (1)  $T'_\lambda x$  is reduced,
- (2) there exists  $x^*$  such that  $x^*$  is reduced and  $T'_\lambda x^* = x$ .

Proof. (1) is easily obtained from Lemma 3.1.  
 (2) From Proposition 2.9, the transformation  $\tilde{S}_\lambda|_{\tilde{Y}} = \widehat{S}_\lambda$  is surjective. Hence, there exists  $\mathbf{x}^* \in \widehat{Y}$  such that

$$\widehat{S}_\lambda(\mathbf{x}^*) = \rho(x).$$

We put

$$\mathbf{x}^* = \left( x^*, \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \right).$$

Then

$$T'_\lambda x^* = x.$$

Thus it suffice to show that

$$\mathbf{x}^* = \rho(x^*).$$

Here we only show this in the case  $\mathbf{Q}(\lambda)$  is not totally real field. In the case of totally real, it is easy to show this relation. From  $\widehat{S}_\lambda(\mathbf{x}^*) = \rho(x)$ , we have

$$\lambda x^* - \omega \left[ \frac{\lambda x^*}{\omega} \right] = x,$$

and

$$- \left[ \frac{\lambda x^*}{\omega} \right] \mathbf{v} + R_\lambda \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x' + x'' \\ (x' - x'')i \end{pmatrix}.$$

In the two equations above, we take algebraic conjugates of the former one and substitute it to  $-[(\lambda x^*/\omega)(\mathbf{v})]$  of the latter one. From the fact that  $\lambda' \neq \lambda''$ , we have

$$x_2 = x^{*'} + x^{*''} \quad \text{and} \quad x_3 = (x^{*'} - x^{*''})i.$$

We can get the result. □

**Lemma 3.4.** For  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$  we put

$$x = \frac{1}{q} \left( u + v \frac{1}{\lambda} + w \frac{1}{\lambda^2} \right), \quad q, u, v, w \in \mathbf{Z},$$

and

$$\omega = \frac{1}{q_0} \left( u_0 + v_0 \frac{1}{\lambda} + w_0 \frac{1}{\lambda^2} \right), \quad q_0, u_0, v_0, w_0 \in \mathbf{Z}.$$

Let  $T'_\lambda{}^k y = x$ , then there exist integers  $u_k, v_k$  and  $w_k$  such that

$$y = \frac{1}{qq_0} \left( u_k + v_k \frac{1}{\lambda} + w_k \frac{1}{\lambda^2} \right).$$

Proof. From  $T'_\lambda{}^k y = x$ ,  $y$  is represented by

$$y = \omega \left( \sum_{i=1}^k b_i \lambda^{-i} \right) + x \lambda^{-k}.$$

Therefore, using the equation  $1/\lambda^3 = 1 - k_1/\lambda - k_2/\lambda^2$ , we can get the above. □

We call  $qq_0$  the *quotient* of  $T'_\lambda{}^k(x)$ . We claim that the quotient is independent of  $k$ .

**Proposition 3.5.** *Let  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$  be reduced. Then  $\lambda$ -expansion of  $x/\omega$  is purely periodic, that is, there exists an integer  $k$  such that  $T'_\lambda{}^k x = x$ .*

Proof. We put

$$x = \frac{1}{q} \left( u + v \frac{1}{\lambda} + w \frac{1}{\lambda^2} \right), \quad q, u, v, w \in \mathbf{Z}.$$

Lemma 3.3 shows that there exists a sequence  $(x_0^*, x_1^*, \dots)$  such that  $x_i^*$  is reduced and  $T'_\lambda x_i^* = x_{i-1}^*$  for  $i \in \mathbf{N}$  where  $x_0^* := x$ . We know the finiteness of the cardinality of the set  $\{x_i^* \mid x_i^* \text{ is reduced and } T'_\lambda x_i^* = x_{i-1}^* \text{ for } i \in \mathbf{N}\}$  since  $\widehat{Y}$  is a bounded set and the quotient of  $T'_\lambda x$  is invariant. Hence, there exist integers  $j$  and  $k$  ( $j - k > 0$ ) such that

$$x_j^* = x_{j-k}^*.$$

Then we have

$$x_k^* = x_0^*.$$

Consequently, we get

$$T'_\lambda{}^k x = x. \quad \square$$

**Proposition 3.6.** *Let  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$ . Then there exists  $N_1 > 0$  such that  $T'_\lambda{}^N x$  is reduced for any  $N > N_1$ .*

Proof. For any  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$ , the point  $(x, {}^t(0, 0))$  is in  $\widehat{Y}$ . We consider the Euclidean distance  $d$  between  $\widetilde{S}_\lambda^k \rho(x)$  and  $\widetilde{S}_\lambda^k(x, {}^t(0, 0))$  for all  $k \in \mathbf{N}$ . The first

coordinates are equal to each other for all  $k \in \mathbf{N}$ . Hence, we have

$$d\left(\tilde{S}_\lambda^k(\rho(x)), \tilde{S}_\lambda^k(x, {}^t(0, 0))\right) \leq u^k d(\rho(x), (x, {}^t(0, 0)))$$

where

$$u = \max(|\lambda'|, |\lambda''|).$$

On the other hand, from the fact  $(x, {}^t(0, 0)) \in \hat{Y}$  and  $\tilde{S}_\lambda|_{\hat{Y}} = \hat{S}_\lambda$  we know

$$\tilde{S}_\lambda^k(x, {}^t(0, 0)) \in \hat{Y}$$

for all  $k$ . Therefore  $\hat{S}_\lambda^k \rho(x)$  must exponentially comes near the set  $\hat{Y}$ . Since the quotient of  $T_\lambda'^k x$  is also invariant, using Lemma 3.1, we have

$$\tilde{S}_\lambda^N \rho(x) = \rho\left(T_\lambda'^N x\right) \in \hat{Y}$$

for sufficiently large  $N$ . Then  $T_\lambda'^N x$  is reduced. And, from Lemma 3.3 (1) we can get the above. □

We can get the following result:

**Theorem 3.7.** *Let  $x \in [0, \omega)$ , then*

- (1)  $x \in \mathbf{Q}(\lambda)$  if and only if  $\lambda$ -expansion of  $x/\omega$  is eventually periodic,
- (2)  $x \in \mathbf{Q}(\lambda)$  is reduced if and only if  $\lambda$ -expansion of  $x/\omega$  is purely periodic.

*Proof.* (1) Assume that  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$ . By Proposition 3.6, there exists  $N > 0$  such that  $T_\lambda'^N x$  is reduced. Proposition 3.5 says that  $T_\lambda'^N x/\omega = T_\lambda^N(x/\omega)$  has a purely periodic  $\lambda$ -expansion. Hence,  $\lambda$ -expansion of  $x/\omega$  is eventually periodic. The opposite direction is trivial.

(2) Necessity is obtained by Proposition 3.5. Conversely, assume that  $\lambda$ -expansion of  $x/\omega$  is purely periodic. From (1), we see  $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$ . According to Proposition 3.6, there exists  $N > 0$  such that  $T_\lambda'^N x$  is reduced. Therefore, we know that  $x$  is reduced by Lemma 3.3 (1) because of purely periodicity. □

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