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ON PERIODIC β-EXPANSIONS OF PISOT NUMBERS AND RAUZY FRACTALS

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0. Introduction

Let λ be the real maximum solution of the polynomial $p(x)$: $k_1, k_2 \in \mathbf{N}$ and $k_1 \geq k_2$ $(k_1 \neq 0)$

$$p(x) = x^3 - k_1 x^2 - k_2 x - 1.$$

The polynomial $p(x)$ is given as the characteristic polynomial of the matrix M:

$$M = \begin{bmatrix} k_1 & k_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

And for each k_1, k_2 the real cubic number λ is a Pisot number. A Pisot number is an algebraic integer whose conjugates other than itself have modulus less than one. Hence,

$$|\lambda'|, |\lambda''| < 1,$$

where λ', λ'' are algebraic conjugates of λ. We denote the column and row eigenvectors of λ by

$$M \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad {}^tM \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix},$$

where t indicates the transpose.
Let $T_\lambda : [0, 1) \to [0, 1)$ be the transformation given by

$$T_\lambda x = \lambda x - [\lambda x],$$

where $[r]$ denotes the integer part of a real number r.
Then each $x \in [0,1)$ is represented by

$$(*) \qquad x = \sum_{k=1}^{\infty} \frac{b_k}{\lambda^k},$$

where $b_k = \left[\lambda T_\lambda^{k-1} x\right]$, $k = 1, 2, \ldots$, the expansion $(*)$ of x is usually called β-expansion. In this paper, we call it λ-expansion.

Let $\mathbf{Q}(\lambda)$ be the smallest extension field of rational numbers $\mathbf{Q}$ containing λ. K. Shmidt showed the following result in [8].

Theorem (**Schmidt**). *A real number x is in $\mathbf{Q}(\lambda) \cap [0,1)$ if and only if λ-expansion of x is eventually periodic.*

In [1], Akiyama gives a sufficient condition of purely periodicity.

In this paper, we discuss when λ-expansion of x is purely periodic. For this purpose, we introduce the three dimensional domain $\widehat{Y}$ with fractal boundary (see Fig. 1 and the definition in Section 2) and we say a real number $x \in \mathbf{Q}(\lambda) \cap [0,\omega)$ is reduced if $\rho(x) \in \widehat{Y}$ where $\omega = 1/(1 + \alpha\gamma + \beta\delta)$, $\rho(x)$ is given by

$$\rho(x) = \begin{cases} {}^t\left(x, x' + x'', \left(x' - x''\right) i\right) & \text{if } \mathbf{Q}(\lambda) \text{ is not a totally real cubic field,} \\ {}^t\left(x, x', x''\right) & \text{if } \mathbf{Q}(\lambda) \text{ is a totally real cubic field,} \end{cases}$$

and x' and x'' denote algebraic conjugates of x.
The main result of this paper is the following:

Main Theorem. *Let x be a real number in $\mathbf{Q}(\lambda) \cap [0,1)$. Then λ-expansion of x is purely periodic if and only if ωx is reduced.*

The main tool of the proof is a natural extension on the domain $\widehat{Y}$ of the dynamical system $([0,1), T_\lambda)$, which is discussed in [7] and [9] originally. And the basic idea of the proof can be found in [4] and [5].

1. Dual transformation of T_λ

From the property of the eigenvector ${}^t(1, \gamma, \delta)$ such that

$$k_1 + \gamma = \lambda \cdot 1 \qquad k_2 + \delta = \lambda \cdot \gamma \qquad 1 = \lambda \cdot \delta,$$

we see the transformation T_λ :

$$T_\lambda x = \lambda x \pmod 1$$

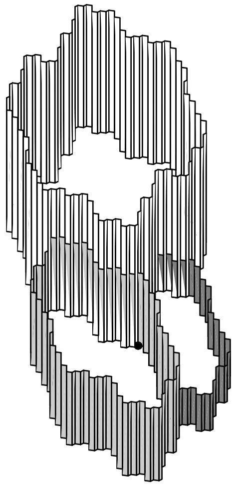

Fig. 1. Figure of $\widehat{Y}$.

is the λ-transformation with the shift of finite type. The sequences $\left\{\{b_k\}_{k=1}^{\infty}\right\}$ of λ-expansion satisfies the following admissible condition:

(1) $0 \leq b_i \leq k_1$,

(2) if $b_i = k_1$ then $b_{i+1} \leq k_2$,

(3) if $(b_i, b_{i+1}) = (k_1, k_2)$ then $b_{i+2} = 0$.

In other words, the admissible sequence $(b_1, b_2, \ldots, b_k, \ldots)$ is given by the labeled graph G in Fig. 2.

Let $W^* = \bigcup_{n=0}^{\infty} \{1, 2, 3\}$ be the free monoid of $\{1, 2, 3\}$ and let us define the substitution $\sigma_{k_1,k_2} : W^* \to W^*$ by

$$\sigma_{k_1,k_2} : \begin{array}{l} 1 \longrightarrow \overbrace{11\ldots1}^{k_1}2 \\ 2 \longrightarrow \overbrace{11\ldots1}^{k_2}3 \\ 3 \longrightarrow 1 \end{array}.$$

Then the matrix of the substitution σ_{k_1,k_2} is given by M and so it is called Pisot substitution. Moreover, the substitution σ_{k_1,k_2} satisfies the coincidence condition in [2]. Therefore, we have the following theorem.

Theorem (**Arnoux-Ito**). *Let* $\mathbf{P}$ *be the contractive invariant plane with respect to the linear transformation* M*, which is given by*

$$\mathbf{P} = \left\{\mathbf{x} \in \mathbf{R}^3 \ \mid \ \langle \mathbf{x}, \ {}^t(1, \gamma, \delta)\rangle = 0\right\}.$$

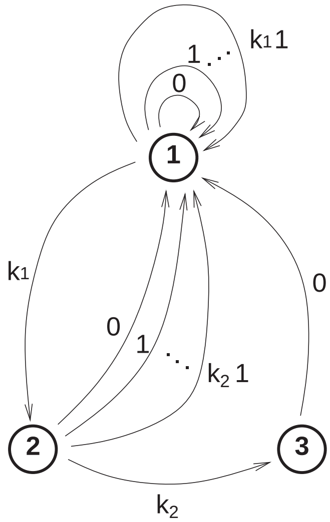

Fig. 2. Labeled graph G.

Then, there exist the closed domains X *and* $X_i, i = 1, 2, 3$ *on the plane* $\mathbf{P}$ *satisfying the following properties*:
The boundaries of X *and* X_i *are fractal Jordan curves and*

$$X = \bigcup_{i=1,2,3} X_i \quad \textit{(disjoint)},$$

$$\bigcup_{\mathbf{z}\in\{\pi(m(\mathbf{e}_2-\mathbf{e}_1)+n(\mathbf{e}_3-\mathbf{e}_1))\ |\ m,n\in\mathbf{Z}\}} (X + \pi\mathbf{z}) = \mathbf{P} \quad \textit{(disjoint)},$$

and moreover

$$M^{-1}X_1 = \bigcup_{j=0}^{k_1-1}(X_1 - j\pi\mathbf{e}_3) \cup \bigcup_{j=0}^{k_2-1}(X_2 - j\pi\mathbf{e}_3) \cup X_3 \quad \textit{(disjoint)},$$

$$M^{-1}X_2 = X_1 - k_1\pi\mathbf{e}_3, \qquad M^{-1}X_3 = X_2 - k_2\pi\mathbf{e}_3,$$

where $\langle\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle$ *is the canonical basis of* $\mathbf{R}^3$, $\pi : \mathbf{R}^3 \to \mathbf{P}$ *is the projection along* ${}^t(1, \alpha, \beta)$, *and* $A \cup B$ (disjoint) *means that the interior of* A *and the interior of* B *are disjoint sets.*

REMARK. By using the notation of [6], we will give a survey how the domains X, $X_i, i = 1, 2, 3$ are obtained. From the substitution σ_{k_1,k_2}, let us give the map Σ_{k_1,k_2}

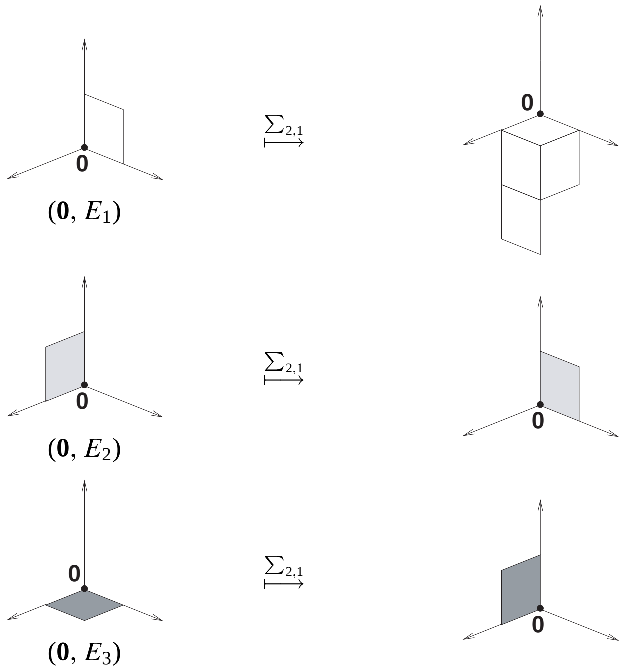

Fig. 3. Figure of $\Sigma_{2,1}(0, E_i)$, $i = 1, 2, 3$.

on the family of patches of the stepped surface of $\mathbf{P}$ by

$$\Sigma_{k_1,k_2} : \begin{array}{l} (\mathbf{0}, E_1) \longrightarrow (\mathbf{0}, E_3) + \displaystyle\sum_{k=1}^{k_1} (\mathbf{e}_1 - k\mathbf{e}_3, E_1) + \sum_{k=1}^{k_2} (\mathbf{e}_2 - k\mathbf{e}_3, E_2) \\ (\mathbf{0}, E_2) \longrightarrow (\mathbf{0}, E_1) \\ (\mathbf{0}, E_3) \longrightarrow (\mathbf{0}, E_2) \end{array},$$

$$\Sigma_{k_1,k_2}(\mathbf{x}, E_i) := M^{-1}\mathbf{x} + \Sigma_{k_1,k_2}(\mathbf{0}, E_i)$$

(see Fig. 3). Then, the domains X and $X_i, i = 1, 2, 3$ are given by

$$X = \lim_{n\to\infty} M^n \pi \left(\Sigma_{k_1,k_2}\right)^n \left(\bigcup_{i=1,2,3} (\mathbf{e}_i, E_i) \right),$$

and

$$X_i = \lim_{n\to\infty} M^n \pi \left(\Sigma_{k_1,k_2}\right)^n (\mathbf{e}_i, E_i)$$

(see Fig. 4).
The boundaries of the domains X, $X_i, i = 1, 2, 3$ are given by the following manner: Let $\theta_{k_1,k_2} : G\langle 1, 2, 3\rangle \to G\langle 1, 2, 3\rangle$ be the endomorphism of the free group of rank 3

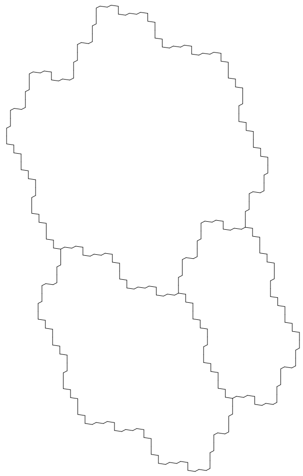

Fig. 4. Figure of $X = \bigcup_{i=1,2,3} X_i$ on $(k_1, k_2 = (1, 1))$.

given by

$$\theta_{k_1,k_2} : \begin{array}{l} 1 \longrightarrow 3 \\ 2 \longrightarrow 1\,\overbrace{3^{-1}3^{-1}\ldots 3^{-1}}^{k_1}, \\ 3 \longrightarrow 2\,\overbrace{3^{-1}3^{-1}\ldots 3^{-1}}^{k_2} \end{array}$$

then the boundaries are given by

$$\partial X = \lim_{n\to\infty} M^n \pi \left(\mathbf{f}_1^{(n)} + \mathcal{K}\left(\theta^n\left(21^{-1}32^{-1}13^{-1}\right)\right)\right),$$
$$\partial X_i = \lim_{n\to\infty} M^n \pi \left(\mathbf{f}_i^{(n)} + \mathcal{K}\left(\theta^n\left(jkj^{-1}k^{-1}\right)\right)\right),$$

where $\left(\mathbf{f}_1^{(n)}, \mathbf{f}_2^{(n)}, \mathbf{f}_3^{(n)}\right) = M^{-n}$, $\mathcal{K}$ is the polygonal realization map on $G\langle 1, 2, 3\rangle$, and $\{i, j, k\} = \{1, 2, 3\}$ (see [6], [3] in detail).

From the fact in Theorem (Arnoux-Ito) and the property $M\pi\mathbf{e}_3 = \pi M\mathbf{e}_3 = \pi\mathbf{e}_1$, we know

$$X = \bigcup_{j=0}^{k_1}(MX_1 - j\pi\mathbf{e}_1) \cup \bigcup_{j=0}^{k_2}(MX_2 - j\pi\mathbf{e}_1) \cup MX_3 \quad \textit{(disjoint)}.$$

On the notation

$$\begin{aligned} P_j^{(1)} &= MX_1 - j\pi\mathbf{e}_1, \quad j = 0, 1, \ldots, k_1, \\ P_j^{(2)} &= MX_2 - j\pi\mathbf{e}_1, \quad j = 0, 1, \ldots, k_2, \\ P_0^{(3)} &= MX_3, \end{aligned}$$

the set $\xi = \{P_0^{(1)}, \ldots, P_{k_1}^{(1)}, P_0^{(2)}, \ldots, P_{k_2}^{(2)}, P_0^{(3)}\}$ is a partition of X. Using the partition ξ, let us define the transformation T_λ^* on X by

$$T_\lambda^* \mathbf{x} = M^{-1}\mathbf{x} + b^*\pi\mathbf{e}_3 \quad \text{if} \quad \mathbf{x} \in P_{b^*}^{(i)} \text{ for some } i \text{ and } b^*.$$

Then, for each $\mathbf{x} \in X$ we have the following sequence $\left(b_1^*, b_2^*, \ldots\right)$ by

$$T_\lambda^{*\,k-1}\mathbf{x} \in P_{b_k^*}^{(i)} \quad \text{for some } i,$$

and we have the expansion: for each $\mathbf{x} \in X$,

$$\mathbf{x} = -\sum_{k=1}^{\infty} b_k^* M^{k-1}\pi\mathbf{e}_1.$$

We see that the sequence $\left(b_1^*, b_2^*, \ldots\right)$ is obtained from the labeled graph G^*, which is dual of the graph G (see Fig. 5). Therefore, we say the transformation T_λ^* is a *dual* transformaiton of T_λ.

Let us define the three dimensional domains $\widehat{X} = \bigcup_{i=1,2,3} \widehat{X}_i$ as follows: for $i = 1, 2, 3$

$$\widehat{X}_i := \left\{ t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + \mathbf{x} \;\middle|\; 0 \le t < t_i^0, \mathbf{x} \in X_i \right\},$$

where $\left(t_1^0, t_2^0, t_3^0\right) = (1, \gamma, \delta)$ and $\omega = 1/(1 + \alpha\gamma + \beta\delta)$ (see Fig. 6). Let us define the transformation $\widehat{T}_\lambda : \widehat{X} \to \mathbf{R}^3$ by

$$\widehat{T}_\lambda \left(t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + \mathbf{x} \right) := \left(\lambda t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} - [\lambda t]\,\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + M\mathbf{x} - [\lambda t]\pi\mathbf{e}_1 \right).$$

Then we have the proposition.

Proposition 1.1. *The transformation $\widehat{T}_\lambda$ is surjective and a.e. injective transformation on $\widehat{X}$.*

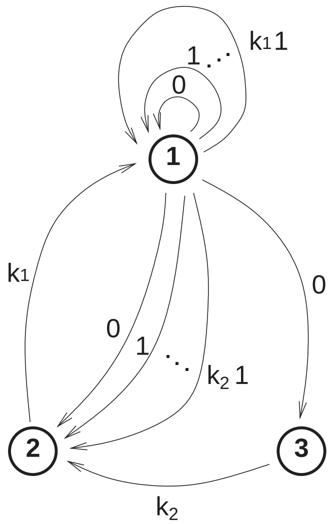

Fig. 5. Dual graph G^*.

Proof. By the Theorem (Arnoux-Ito), the domains $X_i, i = 1, 2, 3$ are decomposed in the following way:

$$\begin{aligned} X_1 &= \bigcup_{j=0}^{k_1-1} (MX_1 - j\pi\mathbf{e}_1) \cup \bigcup_{j=0}^{k_2-1} (MX_2 - j\pi\mathbf{e}_1) \cup MX_3, \\ X_2 &= MX_1 - k_1\pi\mathbf{e}_1, \\ X_3 &= MX_2 - k_2\pi\mathbf{e}_1. \end{aligned}$$

On the other hand, the sets $\widehat{X}_i, i = 1, 2, 3$ are transformed by M

$$M\widehat{X}_i = \left\{ \lambda t\omega \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + M\mathbf{x} \;\middle|\; 0 \le t < t_i^0, \mathbf{x} \in X_i \right\}.$$

By using the fact that $\lambda \cdot 1 = k_1 + \gamma$, $\lambda \cdot \gamma = k_2 + \delta$, and $\lambda \cdot \delta = 1$, we cut the cylinder $M\widehat{X}_1$ to k_1 pieces of the length ω and one piece of length $\gamma\omega$. Analogously, we cut the cylinder $M\widehat{X}_2$ to k_2 pieces of the length ω and one piece of length $\delta\omega$. Then, applying $\widehat{T}_\lambda$ shows that $\widehat{T}_\lambda$ is surjective and injective except the boundary on $\widehat{X}$ (see Fig. 6). $\square$

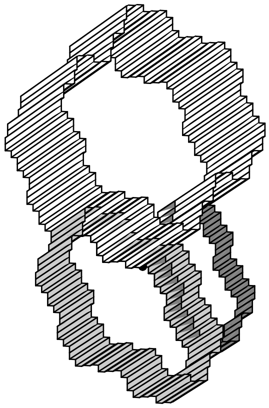

Fig. 6. Figure of $\widehat{X}$.

2. Preliminaries from algebra

We know the vector $\langle 1, \alpha, \beta \rangle$ is the basis of $\mathbf{Q}(\lambda)$, that is, for any $x \in \mathbf{Q}(\lambda)$ there exist rational numbers c_0, c_1 and c_2 such that

$$x = c_0 + c_1\alpha + c_2\beta,$$

and we denote x' and x'' which are algebraic conjugates of x, that is,

$$\begin{aligned} x' &= c_0 + c_1\alpha' + c_2\beta' \in \mathbf{Q}\left(\lambda'\right), \\ x'' &= c_0 + c_1\alpha'' + c_2\beta'' \in \mathbf{Q}\left(\lambda''\right). \end{aligned}$$

First, let us assume that the cubic field is not totally real. We will begin with introducing two maps $\eta : \mathbf{Q}(\lambda) \to \mathbf{R} \times \mathbf{C}^2$ and $\rho : \mathbf{Q}(\lambda) \to \mathbf{R}^3$ by

$$\eta(x) := \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix} \quad \text{and} \quad \rho(x) := \begin{pmatrix} x \\ x' + x'' \\ \left(x' - x''\right) i \end{pmatrix}.$$

We get a few primitive lemmas and corollaries.

Lemma 2.1. *Let*

$$P := \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{bmatrix} \quad \textit{and} \quad Q := [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2],$$

where

$$\mathbf{u}_0 := \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \mathbf{u}_1 := \frac{1}{2}\left(\begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix} + \begin{pmatrix} 1 \\ \alpha'' \\ \beta'' \end{pmatrix}\right), \mathbf{u}_2 := \frac{1}{2i}\left(\begin{pmatrix} 1 \\ \alpha' \\ \beta' \end{pmatrix} - \begin{pmatrix} 1 \\ \alpha'' \\ \beta'' \end{pmatrix}\right).$$

Then, we have
(1) *for any* $x \in \mathbf{Q}(\lambda)$

$$P(\eta(x)) = Q(\rho(x)),$$

(2) *the inverse matrix of* P *is given by*

$$P^{-1} = \begin{bmatrix} \omega & \mu & \nu \\ \omega' & \mu' & \nu' \\ \omega'' & \mu'' & \nu'' \end{bmatrix},$$

where

$$\omega = \frac{1}{D}\begin{vmatrix} \alpha' & \alpha'' \\ \beta' & \beta'' \end{vmatrix}, \quad \mu = \frac{1}{D}\left(\beta' - \beta''\right),$$
$$\nu = \frac{1}{D}\left(\alpha'' - \alpha'\right), \quad D = \det P$$

(*In* Corollary 2.3*, we see* $\omega = 1/(1 + \alpha\gamma + \beta\delta)$).

Proof. (1) is easily obtained.
(2) By Cramer's rule, we have

$$\left(P^{-1}\right)_{11} = \omega, \quad \left(P^{-1}\right)_{12} = \mu, \quad \text{and} \quad \left(P^{-1}\right)_{13} = \nu.$$

In Corollary 2.3, we can see that ω, μ, and ν are elements of $\mathbf{Q}(\lambda)$. Consider the matrix $P \cdot {}^tP$:

$$P \cdot {}^tP = \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{bmatrix} \begin{bmatrix} 1 & \alpha & \beta \\ 1 & \alpha' & \beta' \\ 1 & \alpha'' & \beta'' \end{bmatrix} = \begin{bmatrix} 3 & Tr(\alpha) & Tr(\beta) \\ Tr(\alpha) & Tr(\alpha^2) & Tr(\alpha\beta) \\ Tr(\beta) & Tr(\alpha\beta) & Tr(\beta^2) \end{bmatrix},$$

where

$$Tr(\theta) = \theta + \theta' + \theta'',$$

for any algebraic number θ. Since $Tr(\theta)$ is rational, each element of $P \cdot {}^tP$ is a rational number. Then there exists a right elementary transformation U whose elements are rational numbers such that

$$\left(P \cdot {}^tP\right) \cdot U = I,$$

where I indicates the identity matrix. So that,

$$P \cdot \left({}^tP \cdot U\right) = I.$$

Therefore we know

$$P^{-1} = {}^tP \cdot U = \begin{bmatrix} \omega & \mu & \nu \\ \omega' & \mu' & \nu' \\ \omega'' & \mu'' & \nu'' \end{bmatrix}. \qquad \square$$

Lemma 2.2. *The inverse matrix of Q is given by*

$$Q^{-1} = \begin{bmatrix} \omega & \mu & \nu \\ \omega' + \omega'' & \mu' + \mu'' & \nu' + \nu'' \\ i\left(\omega' - \omega''\right) & i\left(\mu' - \mu''\right) & i\left(\nu' - \nu''\right) \end{bmatrix}.$$

Therefore, the canonical basis $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle$ of $\mathbf{R}^3$ is given by

$$\begin{aligned} \mathbf{e}_1 &= \omega \mathbf{u}_0 + \left(\omega' + \omega''\right)\mathbf{u}_1 + i\left(\omega' - \omega''\right)\mathbf{u}_2, \\ \mathbf{e}_2 &= \mu \mathbf{u}_0 + \left(\mu' + \mu''\right)\mathbf{u}_1 + i\left(\mu' - \mu''\right)\mathbf{u}_2, \\ \mathbf{e}_3 &= \nu \mathbf{u}_0 + \left(\nu' + \nu''\right)\mathbf{u}_1 + i\left(\nu' - \nu''\right)\mathbf{u}_2. \end{aligned}$$

Corollary 2.3. *The projections of $\mathbf{e}_i, i = 1, 2, 3$ by π are given by*

$$\begin{aligned} \pi\mathbf{e}_1 &= \left(\omega' + \omega''\right)\mathbf{u}_1 + i\left(\omega' - \omega''\right)\mathbf{u}_2, \\ \pi\mathbf{e}_2 &= \left(\mu' + \mu''\right)\mathbf{u}_1 + i\left(\mu' - \mu''\right)\mathbf{u}_2, \\ \pi\mathbf{e}_3 &= \left(\nu' + \nu''\right)\mathbf{u}_1 + i\left(\nu' - \nu''\right)\mathbf{u}_2. \end{aligned}$$

Moreover, we have

$$Q^{-1}\pi\mathbf{e}_1 = \begin{pmatrix} 0 \\ \omega' + \omega'' \\ i\left(\omega' - \omega''\right) \end{pmatrix}, \; Q^{-1}\pi\mathbf{e}_2 = \begin{pmatrix} 0 \\ \mu' + \mu'' \\ i\left(\mu' - \mu''\right) \end{pmatrix},$$

$$Q^{-1}\pi\mathbf{e}_3 = \begin{pmatrix} 0 \\ \nu' + \nu'' \\ i\left(\nu' - \nu''\right) \end{pmatrix}, \; \textit{and} \; (\omega, \mu, \nu) = \frac{1}{1 + \alpha\gamma + \beta\delta}(1, \gamma, \delta).$$

Proof. Recall that π is the projection on the plane $\mathbf{P}$ along $\mathbf{u}_0$. The plane $\mathbf{P}$ which is orthogonal ${}^t(1, \gamma, \delta)$ is spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence, we can get the above from Lemma 2.2. The second assertion is obtained by $Q^{-1}\mathbf{u}_1 = \mathbf{e}_2$, $Q^{-1}\mathbf{u}_2 = \mathbf{e}_3$. Put

$$\mathbf{e}_i = \pi \mathbf{e}_i + c_i \mathbf{u}_0, \qquad i = 1, 2, 3.$$

Then from the relation

$$\left\langle \mathbf{e}_i, \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} \right\rangle = \left\langle \pi \mathbf{e}_i, \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} \right\rangle + c_i \left\langle \mathbf{u}_0, \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} \right\rangle,$$

we have

$$(c_1, c_2, c_3) = \frac{1}{1 + \alpha\gamma + \beta\delta} (1, \gamma, \delta).$$

On the other hand, we know from Lemma 2.2 that

$$(c_1, c_2, c_3) = (\omega, \mu, \nu).$$

Therefore, we arrive at the conclusion. □

Lemma 2.4. *The following relation holds*:

$$MQ = Q \begin{bmatrix} \lambda & 0 & 0 \\ 0 & (\lambda' + \lambda'')/2 & -i\left(\lambda' - \lambda''\right)/2 \\ 0 & i\left(\lambda' - \lambda''\right)/2 & (\lambda' + \lambda'')/2 \end{bmatrix}.$$

Proof. The proof is easily obtained from the equations

$$M\mathbf{u}_0 = \lambda \mathbf{u}_0, \quad M\mathbf{u}_1 = \frac{\lambda'}{2}(\mathbf{u}_1 + i\mathbf{u}_2) + \frac{\lambda''}{2}(\mathbf{u}_1 - i\mathbf{u}_2),$$
$$M\mathbf{u}_2 = \frac{\lambda'}{2i}(\mathbf{u}_1 + i\mathbf{u}_2) - \frac{\lambda''}{2i}(\mathbf{u}_1 - i\mathbf{u}_2),$$

and the definitions of $\mathbf{u}_1$ and $\mathbf{u}_2$. □

Secondly, let us assume that the cubic field is totally real. We use the same notation ρ as the map from $\mathbf{Q}(\lambda)$ to $\mathbf{R}^3$ by

$$\rho(x) := \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}.$$

Lemma 2.5. *Let*

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{bmatrix} = [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2].$$

Then the inverse matrix of Q is given by

$$Q^{-1} = \begin{bmatrix} \omega & \mu & \nu \\ \omega' & \mu' & \nu' \\ \omega'' & \mu'' & \nu'' \end{bmatrix},$$

where ω, μ, ν is given as (2) *in* Lemma 2.1.

In stead of Lemma 2.2, Corollary 2.3, and Lemma 2.4, we have

Lemma 2.6. *The following formula hold*:

$$\begin{aligned} \pi\mathbf{e}_1 &= \omega'\mathbf{u}_1 + \omega''\mathbf{u}_2, \\ \pi\mathbf{e}_2 &= \mu'\mathbf{u}_1 + \mu''\mathbf{u}_2, \\ \pi\mathbf{e}_3 &= \nu'\mathbf{u}_1 + \nu''\mathbf{u}_2. \end{aligned}$$

Therefore, we have

$$Q^{-1}\pi\mathbf{e}_1 = \begin{pmatrix} 0 \\ \omega' \\ \omega'' \end{pmatrix}, Q^{-1}\pi\mathbf{e}_2 = \begin{pmatrix} 0 \\ \mu' \\ \mu'' \end{pmatrix}, Q^{-1}\pi\mathbf{e}_3 = \begin{pmatrix} 0 \\ \nu' \\ \nu'' \end{pmatrix},$$

and

$$(\omega, \mu, \nu) = \frac{1}{1 + \alpha\gamma + \beta\delta}(1, \gamma, \delta).$$

Moreover, we know trivially

$$MQ = Q \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda'' \end{bmatrix}.$$

We have the following corollary in the both cases:

Corollary 2.7. *Let us define R_λ by*

$$R_\lambda := \begin{cases} \begin{bmatrix} (\lambda' + \lambda'')/2 & -i\left(\lambda' - \lambda''\right)/2 \\ i\left(\lambda' - \lambda''\right)/2 & (\lambda' + \lambda'')/2 \end{bmatrix} & \textit{if} \quad \mathbf{Q}(\lambda) \textit{ is not totally real,} \\ \begin{bmatrix} \lambda' & 0 \\ 0 & \lambda'' \end{bmatrix} & \textit{if} \quad \mathbf{Q}(\lambda) \textit{ is totally real,} \end{cases}$$

then we have

$$MQ = Q \left[\begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{R_λ}} \\ 0 & & \end{array}\right].$$

Let us define the domains $\widehat{Y}$ and $\widehat{Y}_i, i = 1, 2, 3$ as follows:

$$\widehat{Y} := Q^{-1}\left(\widehat{X}\right) \quad \text{and} \quad \widehat{Y}_i := Q^{-1}\left(\widehat{X}_i\right).$$

Then the domains $\widehat{Y}$ and $\widehat{Y}_i$ have explicit forms (see Fig. 1). From now on, remark that $\widehat{Y}$ and $\widehat{Y}_i$ are written as domains in $\mathbf{R} \times \mathbf{R}^2$.

Lemma 2.8. *The domains $\widehat{Y}$ and $\widehat{Y}_i, i = 1, 2, 3$ are given by*

$$\widehat{Y}_i = \left\{ \left(t\omega, -\sum_{k=1}^{\infty} b_k^* R_\lambda^{k-1} \mathbf{v} \right) \;\middle|\; 0 \le t < t_i^0, \right.$$
$$\left. (b_1^*, b_2^*, \ldots) \textit{ is an admissible sequence starting at } i \textit{ in } G^* \right\},$$

where

$$\mathbf{v} = \begin{cases} \begin{pmatrix} \omega' + \omega'' \\ i\left(\omega' - \omega''\right) \end{pmatrix} & \textit{if} \quad \mathbf{Q}(\lambda) \textit{ is not totally real,} \\ \begin{pmatrix} \omega' \\ \omega'' \end{pmatrix} & \textit{if} \quad \mathbf{Q}(\lambda) \textit{ is totally real.} \end{cases}$$

Proof. From the definitions of $\widehat{X}_i$ and $\widehat{Y}_i$, $\widehat{Y}_i$ is given by

$$\widehat{Y}_i = \left\{ t\omega Q^{-1}\mathbf{u}_0 + Q^{-1}\mathbf{x} \;\middle|\; 0 \le t < t_i^0, \mathbf{x} \in X_i \right\}.$$

Using the formula of X_i and $Q^{-1}\mathbf{u}_0 = \mathbf{e}_1$, we have

$$\widehat{Y}_i = \left\{ t\omega\mathbf{e}_1 - \sum_{k=1}^{\infty} b_k^* Q^{-1} M^{k-1} \pi \mathbf{e}_1 \;\middle|\; 0 \le t < t_i^0, \right.$$
$$\left. (b_1^*, b_2^*, \ldots) \text{ is an admissible sequence starting at } i \text{ in } G^* \right\}.$$

From the fact that

$$Q^{-1} M^k = \left[\begin{array}{c|c} \lambda^k & 0 \; 0 \\ \hline 0 & \\ 0 & R_\lambda^k \end{array} \right] Q^{-1} \quad \text{and} \quad Q^{-1} \pi \mathbf{e}_1 = \mathbf{v},$$

we have the conclusion. □

Now, let us define the transformation $\widehat{S}_\lambda$ on $\widehat{Y}$ by

$$\widehat{S}_\lambda := Q^{-1} \circ \widehat{T}_\lambda \circ Q.$$

Proposition 2.9. *The transformation $\widehat{S}_\lambda$ on $\widehat{Y}$ is given explicitly by*

$$\widehat{S}_\lambda \left(t\omega, -\sum_{k=1}^{\infty} b_k^* R_\lambda^{k-1} \mathbf{v} \right) = \left((\lambda t \omega - [\lambda t]\, \omega), -[\lambda t]\, \mathbf{v} - \sum_{k=1}^{\infty} b_k^* R_\lambda^k \mathbf{v} \right)$$

and $\widehat{S}_\lambda$ on $\widehat{Y}$ is surjective.

Proof. The proof is obtained from Proposition 1.1. □

3. Reduction theorem

Let $\widehat{\mathbf{Z}} := [0, \omega) \times \mathbf{R}^2$ and let us define the transformation $\widetilde{S}_\lambda$ on $\widehat{\mathbf{Z}}$ by

$$\widetilde{S}_\lambda \left(x, \begin{pmatrix} y \\ z \end{pmatrix} \right) = \left(\left(\lambda x - \omega \left[\frac{\lambda x}{\omega} \right] \right), -\left[\frac{\lambda x}{\omega} \right] \mathbf{v} + R_\lambda \begin{pmatrix} y \\ z \end{pmatrix} \right).$$

Then, the restriction of the map $\widetilde{S}_\lambda$ on the set $\widehat{Y}$ coincides with $\widehat{S}_\lambda$.
In stead of the transformation $T_\lambda : [0, 1) \to [0, 1)$

$$T_\lambda x = \lambda x - [\lambda x],$$

let us introduce the following transformation $T'_\lambda : [0, \omega) \to [0, \omega)$ by

$$T'_\lambda x = \lambda x - \omega \left[\frac{\lambda x}{\omega} \right].$$

Then dynamical systems $([0,1), T_\lambda)$ and $\left([0,\omega), T'_\lambda\right)$ are isomorphic by the map : $x \to \omega x$ and for any $x \in [0,\omega)$ can be expressed by

$$x = \omega \sum_{k=1}^{\infty} \frac{b_k}{\lambda^k},$$

where $(b_1, b_2, \ldots)$ is the admissible sequence of $x/\omega \in [0,1)$ by the transformation T_λ. And we can say that the transformation $\widehat{S}_\lambda$ is the natural extension of T'_λ and T_λ.

Hereafter, we denote $\rho(x)$ by the map from $\mathbf{Q}(\lambda)$ to $\mathbf{R} \times \mathbf{R}^2$, that is,

$$\rho(x) = \begin{cases} \left(x, \begin{pmatrix} x' + x'' \\ (x' - x'')\, i \end{pmatrix}\right) & \text{if } \mathbf{Q}(\lambda) \text{ is not a totally real cubic field,} \\ \left(x, \begin{pmatrix} x' \\ x'' \end{pmatrix}\right) & \text{if } \mathbf{Q}(\lambda) \text{ is a totally real cubic field.} \end{cases}$$

Lemma 3.1. *For any real number $x \in \mathbf{Q}(\lambda) \cap [0,\omega)$, we have*

$$\widetilde{S}_\lambda \rho(x) = \rho(y),$$

where $y = T'_\lambda x$.

Proof. From the definitions of $T'_\lambda x$, $\widetilde{S}_\lambda$ and ρ, we know

$$\widetilde{S}_\lambda \rho(x) = \begin{cases} \left(\lambda x - \omega \left[\dfrac{\lambda x}{\omega}\right], -\left[\dfrac{\lambda x}{\omega}\right] \begin{pmatrix} \omega' + \omega'' \\ (\omega' - \omega'')\, i \end{pmatrix} + R_\lambda \begin{pmatrix} x' + x'' \\ (x' - x'')\, i \end{pmatrix}\right) \\ \qquad \text{if } \mathbf{Q}(\lambda) \text{ is not totally real,} \\ \left(\lambda x - \omega \left[\dfrac{\lambda x}{\omega}\right], -\left[\dfrac{\lambda x}{\omega}\right] \begin{pmatrix} \omega' \\ \omega'' \end{pmatrix} + R_\lambda \begin{pmatrix} x' \\ x'' \end{pmatrix}\right) \\ \qquad \text{if } \mathbf{Q}(\lambda) \text{ is totally real.} \end{cases}$$

On the other hand, by $y = \lambda x - \omega\left[\lambda x/\omega\right]$, we see $\rho(y) = \widetilde{S}_\lambda \rho(x)$. □

Let us introduce the concept of *reduced*.

DEFINITION 3.2. A real number $x \in \mathbf{Q}(\lambda) \cap [0,\omega)$ is said to be reduced if $\rho(x) \in \widehat{Y}$.

Lemma 3.3. *Let $x \in \mathbf{Q}(\lambda) \cap [0,\omega)$ be reduced. Then*

(1) *$T'_\lambda x$ is reduced,*

(2) *there exists x^* such that x^* is reduced and $T'_\lambda x^* = x$.*

Proof. (1) is easily obtained from Lemma 3.1.
(2) From Proposition 2.9, the transformation $\widetilde{S}_\lambda|\widehat{Y} = \widehat{S}_\lambda$ is surjective. Hence, there exists $\mathbf{x}^* \in \widehat{Y}$ such that

$$\widehat{S}_\lambda(\mathbf{x}^*) = \rho(x).$$

We put

$$\mathbf{x}^* = \left(x^*, \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}\right).$$

Then

$$T'_\lambda x^* = x.$$

Thus it suffice to show that

$$\mathbf{x}^* = \rho(x^*).$$

Here we only show this in the case $\mathbf{Q}(\lambda)$ is not totally real field. In the case of totally real, it is easy to show this relation. From $\widehat{S}_\lambda(\mathbf{x}^*) = \rho(x)$, we have

$$\lambda x^* - \omega \left[\frac{\lambda x^*}{\omega}\right] = x,$$

and

$$-\left[\frac{\lambda x^*}{\omega}\right] \mathbf{v} + R_\lambda \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x' + x'' \\ (x' - x'')i \end{pmatrix}.$$

In the two equations above, we take algebraic conjugates of the former one and substitute it to $-\left[(\lambda x^*/\omega)(\mathbf{v})\right]$ of the latter one. From the fact that $\lambda' \neq \lambda''$, we have

$$x_2 = x^{*\prime} + x^{*\prime\prime} \quad \text{and} \quad x_3 = (x^{*\prime} - x^{*\prime\prime})i.$$

We can get the result. □

Lemma 3.4. *For $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$ we put*

$$x = \frac{1}{q}\left(u + v\frac{1}{\lambda} + w\frac{1}{\lambda^2}\right), \quad q, u, v, w \in \mathbf{Z},$$

and

$$\omega = \frac{1}{q_0}\left(u_0 + v_0\frac{1}{\lambda} + w_0\frac{1}{\lambda^2}\right), \quad q_0, u_0, v_0, w_0 \in \mathbf{Z}.$$

Let $T_\lambda'^k y = x$, then there exist integers u_k, v_k and w_k such that

$$y = \frac{1}{qq_0}\left(u_k + v_k\frac{1}{\lambda} + w_k\frac{1}{\lambda^2}\right).$$

Proof. From $T_\lambda'^k y = x$, y is represented by

$$y = \omega\left(\sum_{i=1}^{k} b_i\lambda^{-i}\right) + x\lambda^{-k}.$$

Therefore, using the equation $1/\lambda^3 = 1 - k_1/\lambda - k_2/\lambda^2$, we can get the above. □

We call qq_0 the *quotient* of $T_\lambda'^k(x)$. We claim that the quotient is independent of k.

Proposition 3.5. *Let $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$ be reduced. Then λ-expansion of x/ω is purely periodic, that is, there exists an integer k such that $T_\lambda'^k x = x$.*

Proof. We put

$$x = \frac{1}{q}\left(u + v\frac{1}{\lambda} + w\frac{1}{\lambda^2}\right), \quad q, u, v, w \in \mathbf{Z}.$$

Lemma 3.3 shows that there exists a sequence $(x_0^*, x_1^*, \ldots)$ such that x_i^* is reduced and $T_\lambda' x_i^* = x_{i-1}^*$ for $i \in \mathbf{N}$ where $x_0^* := x$. We know the finiteness of the cardinarity of the set $\left\{x_i^* \mid x_i^* \text{ is reduced and } T_\lambda' x_i^* = x_{i-1}^* \text{ for } i \in \mathbf{N}\right\}$ since $\widehat{Y}$ is a bounded set and the quotient of $T_\lambda' x$ is invariant. Hence, there exist integers j and k $(j - k > 0)$ such that

$$x_j^* = x_{j-k}^*.$$

Then we have

$$x_k^* = x_0^*.$$

Consequently, we get

$$T_\lambda'^k x = x.$$

□

Proposition 3.6. *Let $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$. Then there exists $N_1 > 0$ such that $T_\lambda'^N x$ is reduced for any $N > N_1$.*

Proof. For any $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$, the point $\left(x, {}^t(0,0)\right)$ is in $\widehat{Y}$. We consider the Euclidean distance d between $\widetilde{S}_\lambda^k \rho(x)$ and $\widetilde{S}_\lambda^k\left(x, {}^t(0,0)\right)$ for all $k \in \mathbf{N}$. The first coordinates are equal to each other for all $k \in \mathbf{N}$. Hence, we have

$$d\left(\widetilde{S}_\lambda^k(\rho(x)), \widetilde{S}_\lambda^k\left(x, {}^t(0,0)\right)\right) \leq u^k d\left(\rho(x), \quad \left(x, {}^t(0,0)\right)\right)$$

where

$$u = \max\left(|\lambda'|, |\lambda''|\right).$$

On the other hand, from the fact $\left(x, {}^t(0,0)\right) \in \widehat{Y}$ and $\widetilde{S}_\lambda|\widehat{Y} = \widehat{S}_\lambda$ we know

$$\widetilde{S}_\lambda^k\left(x, {}^t(0,0)\right) \in \widehat{Y}$$

for all k. Therefore $\widehat{S}_\lambda^k \rho(x)$ must exponentially comes near the set $\widehat{Y}$. Since the quotient of $T_\lambda'^k x$ is also invariant, using Lemma 3.1, we have

$$\widetilde{S}_\lambda^N \rho(x) = \rho\left(T_\lambda'^N x\right) \in \widehat{Y}$$

for sufficiently large N. Then $T_\lambda'^N x$ is reduced. And, from Lemma 3.3 (1) we can get the above. □

We can get the following result:

Theorem 3.7. *Let $x \in [0, \omega)$, then*
(1) *$x \in \mathbf{Q}(\lambda)$ if and only if λ-expansion of x/ω is eventually periodic,*
(2) *$x \in \mathbf{Q}(\lambda)$ is reduced if and only if λ-expansion of x/ω is purely periodic.*

Proof. (1) Assume that $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$. By Proposition 3.6, there exists $N > 0$ such that $T_\lambda'^N x$ is reduced. Proposition 3.5 says that $T_\lambda'^N x/\omega = T_\lambda^N(x/\omega)$ has a purely periodic λ-expansion. Hence, λ-expansion of x/ω is eventually periodic. The opposite direction is trivial.
(2) Necessity is obtained by Proposition 3.5. Conversely, assume that λ-expansion of x/ω is purely periodic. From (1), we see $x \in \mathbf{Q}(\lambda) \cap [0, \omega)$. According to Proposition 3.6, there exists $N > 0$ such that $T_\lambda'^N x$ is reduced. Therefore, we know that x is reduced by Lemma 3.3 (1) because of purely periodicity. □

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