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# **ON TIGHT 4-DESIGNS**

To the memory to Otto Grün\*

## NOBORU ITO\*\*

(Received September 12, 1974)

## 1. Introduction

Let v, k and  $\lambda$  be positive integers with v > k. Let X be a v-set and JI a family of k-subsets of X. (X, JI) is called a 4- $(v, k, \lambda)$  design (or simply a 4-design) if for each 4-subset T of X there exist precisely  $\lambda$  elements of JI containing T. By a theorem of Fisher-Petrenjuk [2] the number of elements in  $\mathcal{A}$  is not less than  $\frac{1}{2}v(v-1)$ . If it is equal to  $\frac{1}{2}v(v-1)$ , (X, JI) is called tight.

If  $v \ge 6$  and if Jl is the family of all (v-2)-subsets of X, (X, Jl) is a tight 4-design. Such tight 4-designs are called trivial.

Let (X, Jl) be a 4-design. If  $v-k \ge 4$  and if  $\mathcal{A}c$  is the family of (v-k)-subsets of X each of which is a complement of an element of  $\mathcal{A}$  in X, (X, Jlc) is a 4-design. (X, Jl) and (X, Jlc) are called complementary with each other. Furthermore if (X, Jl) is tight, (X, Jlc) is also tight.

There exist only two known non-trivial tight 4-designs (X, JI) they are a 4-(27, 7, 1) design and a 4-(23, 16, 52) design. They are complementary with each other. We call these designs Witt tight designs, because they are found by Witt [5], [6].

Now the purpose of this paper is to prove the following theorem.

**Theorem.** Let (X, Jl) be a non-trivial tight 4- $(v, s, \lambda)$  design. Then (X, Jl) is a Witt tight design.

Our proof relies on the following theorem of Wilson and Ray-Chaudhuri [4]: Let (X, Jl) be a tight 4- $(v, k, \lambda)$  design. Then a non-negative integer  $\mu$  is called an intersection number of  $(X, \mathcal{A})$ , if there exist two distinct elements A and B of Jl such that  $|A \cap B| = \mu$ . There exist precisely two intersection numbers, say,  $\mu_1$  and  $\mu_2$ , where  $\mu_2 > \mu_1 \quad \mu l$  and  $\mu_2$  are the roots of the polynomial

<sup>\*</sup> After the author submitted this paper, he was informed that Dr. Otto Grün passed away on September 29, 1974 in Berlin.

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$$t^{2} - \left\{ \frac{2(k-1)(k-2)}{v-3} \right\} t \quad \sqrt{t-4} V$$

In particular,

(1) 
$$\frac{2(k-1)(k-2)}{v-3} = x = \mu_2 + \mu_1 - 2$$

and

(2) 
$$\frac{4\lambda}{R-s} = \mu_2 \mu_1 - 2\lambda = y$$

are positive integers.

Furthermore, since  $(X, \mathcal{A})$  is tight, we obtain that

(3) 
$$2\lambda(v-2)(v-3) = k(k-1)(k-2)(k-3)$$

### 2. Eigen-values of adjacency matrices

Let  $X_f(i=1, \dots, v)$  and  $A_j(j=1, \dots, \frac{1}{2}v(v-1))$  be the elements of X and  $\mathcal{A}$  respectively. Let / be the incidence matrix of (X, Jl). So / is the matrix of size  $(v, \frac{1}{2}v(v-1))$  whose (i, j)-component is either 1 or 0 according as  $x_i$  belongs to  $A_j$  or not. Let  $N_k$  be the adjacency matrix of (X, Jl) (k=1, 2). So the (i, j)-component of  $N_k$  is 1 or 0 according as  $|A_i \cap Aj| = \mu_k$  or not (k=1,2). Then we obtain that

(4) 
$$J = E + N_1 + N_2$$
,

(5)  $I^t I = kE + \mu_1 N_2 + \mu_2 N_1$ 

and

$$(6, \qquad II^{t} = \frac{1}{2}k(v-k)E' + \frac{1}{2}k(k-1)J',$$

where E and E' denote the identity matrices of degree  $\frac{1}{2}v(v-1)$  and v respectively, and J and J' are the matrices of degree  $\frac{1}{2}v(v-1)$  and v with every component 1 respectively. Furthermore, by R. Noda [1], we have that

$$(7) N_1 N_2 = N_2 N_1.$$

(7) is equivalent to the fact that the number of elements A. of A such that  $|A_i \cap A|_j = \mu_k$  for a given element  $A_i$  of Jl is independent from the choice of  $A_i$  (k=1, 2). For a proof of this fact see [3].

From (6) we see that the eigen-value distribution of  $II^t$  consists of  $\frac{1}{2}k^2(v-1)$  with multiplicity 1 and  $\frac{1}{2}k(v-k)$  with multiplicity v-1. In particular,  $II^t$  is non-singular. Now there exists an orthogonal matrix U of degree  $\frac{1}{2}v(v-1)$  such that IU has the following shape:

$$(8) IU = (I*0),$$

where 0 denotes the zero matrix of size  $(v, \frac{1}{2}v(v-3))$  and  $I^*$  is a non-singular matrix of degree v. From (8) it follows that

$$IUU^t I^t = II^t = I^* I^{*t}.$$

Hence IP and  $I^*I^{*t}$  have the same eigen-value distribution. On the other hand, from (8) it also follows that

$$(9) \qquad U^t I^t I U = \begin{pmatrix} I^{*t} I^* & 0 \\ 0 & 0 \end{pmatrix},$$

where 0's denote zero matrices. So from (9) we see that the eigen-value distribution of PI consists of  $\frac{1}{2}k^2(v-1)$  with multiplicity 1,  $\frac{1}{2}k(v-k)$  with multiplicity v-1 and 0 with multiplicity  $\frac{1}{2}v(v-3)$ .

Since  $N_1$  and  $N_2$  are commuting real symmetric matrices, there exists an orthogonal matrix V such that  $U'^t N_1 U'$  and  $U'^t N_2 U'$  diagonal matrices:

$$U'^{t}N_{1}U' = [\alpha_{1}, \cdots, \alpha_{\frac{1}{2}v(v-1)}]$$

and

$$U'^t N_2 U' = [\beta_1, \cdots, \beta_{\frac{1}{2}v(v-1)}].$$

Then by (4) and (5) we may assume that

(10)  $1+\alpha_1+\beta_1=\frac{1}{2}v(v-1)$ ,

(11) 
$$1 + \alpha_i + \beta_i = 0$$
 for  $2 \le i \le \frac{1}{2}v(v-1)$ ,

(12) 
$$k + \mu_1 \alpha_1 + \mu_2 \beta_1 = \frac{1}{2} k^2 (v-1)$$
,

(13) 
$$k+\mu_1\alpha_i+\mu_2\beta_i=\frac{1}{2}k(v-k)$$
 for  $2\leq i\leq v$ ,

and

(14) 
$$k + \mu_1 \alpha_j + \mu_2 \beta_j = 0$$
 for  $v + 1 \le j \le \frac{1}{2} v(v-1)$ .

Let A be a fixed element of  $\mathcal{A}$ . Let  $b_k$  be the number of elements of  $\mathcal{A}$  which intersect with A in  $\mu_k$  elements of X(k=1, 2). Then we obtain that

(15) 
$$\alpha_1 = b_1$$
 and  $\beta_1 = b_2$ .

After M. Kano [1] we set

(16) 
$$a = \mu_2 - \mu_1$$
.

Now from (10)-(16) we obtain that

(17) 
$$1+b_1+b_2=\frac{1}{2}v(v-1)$$
,

(18) 
$$k + \mu_1 b_1 + \mu_2 b_2 = \frac{1}{2} k^2 (v-1)$$
,

(19) 
$$\alpha_i = \{-\frac{1}{2}k(v-k) + (k-\mu_2)\}/a, \\ \beta_i = \{\frac{1}{2}k(v-k) - (k-\mu_1)\}/a,$$

where  $2 \leq i \leq v$ , and

(20) 
$$\alpha_j = (k - \mu_2)/a$$
,  
 $\beta_j = -(k - \mu_1)/a$ 

where  $v+1 \leq j \leq \frac{1}{2}v(v-1)$ .

Since the  $\alpha_i$  and the  $\beta_i$  are algebraic integers, the numbers in (19) and (20) are rational integers. Put

(21) 
$$e = (k - \mu_2)/a.$$

Now we show that there exist two non-negative rational integers  $\lambda_{1}$  and  $\lambda_{2}$  such that

(22) 
$$N_1^2 = b_1 E + \lambda_1 N_1 + \lambda_2 N_2$$
.

In fact, it suffices to show the existence of two rational numbers  $\lambda_1$  and  $\lambda_2$  which satisfy (22). Then it suffices to solve

(23) 
$$\begin{cases} \alpha_2^2 = b_1 + \lambda_1 \alpha_2 + \lambda_2 \beta_2 \\ \{ \alpha_{\nu+1}^2 = b_1 + \lambda_1 \alpha_{\nu+1} + \lambda_2 \beta_{\nu+1} \end{cases}$$

in  $\lambda_1$  and  $\lambda_2$ . For, since by (4) the trace of  $N_1^2$  is equal to  $\frac{1}{2}b_1v(v-1)$ , we obtain (22). In particular, we obtain that

(24)  $b_1^2 = b_1 + \lambda_1 b_1 + \lambda_2 b_2$ .

By (11) and (23) we see that  $\alpha_2$  and  $\alpha_{v+1}$  are the roots of the equation

$$X^2-(\lambda_1-\lambda_2)X-(b_1-\lambda_2)=0$$
 .

So we obtain that

(25) 
$$\alpha_2 + \alpha_{v+1} = \lambda_1 - \lambda_2$$
  $\alpha_2 \alpha_{v+1} = -(b_1 - \lambda_2)$   
Since, by (19) and (20),  $-\alpha_2 > \alpha_{v+1} > 0$ , by (25) we obtain that

(26) 
$$\lambda_2 > \lambda_1$$
.

Since  $\alpha_2$  and  $\alpha_{v+1}$  are rational integers,

(27)  $d = (\lambda_1 - \lambda_2)^2 + 4(b_1 - \lambda_2)$ 

is a square. Further we obtain that

(28) 
$$\alpha_2 = \frac{1}{2} (\lambda_1 - \lambda_2 - \sqrt{d}) \quad \alpha_{v+1} = \frac{1}{2} (\lambda_1 - \lambda_2 + \sqrt{d}) .$$

From (20), (21) and (28) we obtain that

(29)  $\sqrt{d} = \lambda_2 - \lambda_1 + 2e$ .

By (1), (16) and (21) we obtain that

(30) 
$$2k = x+1+a(2e+1)$$
.

Put

$$(31) \quad v = \lambda_2 - \lambda_1 \, .$$

Eliminate  $b_2$  from (10), (15) and (24). Then we obtain that

(32) 
$$b_1^2 + (v-1)b_1 = \frac{1}{2}\lambda_2(v+1)(v-2)$$
.

From (27), (29) and (31) we obtain that

$$(33) b_1 = e\nu + e^2 + \lambda_2.$$

Since the trace of  $N_1$  is equal to 0, from (15), (19), (20), (21) and (28) it follows that

(34) 
$$b_1 = (v-1)\nu - \frac{1}{2}(v^2 - 5v + 2)e$$

Eliminate  $b_2$  from (10), (12) and (15). In the resulting equation express  $\mu_1$  and  $\mu_2$  in terms of *a*, *e* and *k* using (16) and (21). Then we obtain that

(35) 
$$ab_1 = \frac{1}{2}(v-1)k(v-k) - \frac{1}{2}ea(v+1)(v-2)$$

Eliminate  $\lambda_2$  and  $b_1$  from (32), (33) and (34). Then after canceling v(v-1) we obtain that

(36) 
$$\nu^2 - \{(v-5)e + \frac{1}{2}v - 1\}\}\nu + ti(\nu^2 - 9\nu + 22)e + \pm (\nu^2 - 5\nu + 2)\}e = 0$$
.

From (36) we obtain that

(37) 
$$\{2\nu - ev - e^2 + 4e\} \{(e+1)v - 2\nu - e^2 - 6e - 1\}$$
$$= (e+2)(e+1)e(e-1) = (e^2 + e - 2)(e^2 + e).$$

# 3. Cases with small *e*

It is convenient to eliminate cases with small e first. For this purpose (37) can play a rather useful role. First we show that both terms of the left-hand side of (37) are positive for e > l. Deny this. Then the sum of the negatives of both terms of the left-hand side of (37) is smaller than that of the right-hand side of (37). Since

$$(e^2+e)-(e^2+e-2)=2$$

we have the following three cases:

(i) 
$$ev + e^2 - 2v - 4e = 2v + e^2 + 6e + 1 - (e+1)v + 1$$
,

(ii) 
$$ev + e^2 - 2v - 4e = 2v + e^2 + 6e + 1 - (e+1)v$$
,

and

(iii) 
$$ev + e^2 - 2v - 4e = 2v + e^2 + 6e + 1 - (e+1)v - 1$$
.

Case (i): We have that  $4\nu = (2e+1)\nu - 10e - 2$ . So from (37) we obtain that

$$(38) v^2 - 2(2e^2 + 2e + 1)v + 12(e^2 + e) = 0$$

Put  $c=e^2+e$ . Then  $c \ge 6$ , because  $e \ge 2$ . From (38) it follows that

$$\{v-(2c+1)\}^2 = 4c^2-8c+1.$$

So put  $4c^2 - 8c + 1 = f^2$ , where f is a positive integer. Then we obtain that

$$(39) \qquad 2c - f = \frac{8c - 1}{2c + f}.$$

Hence the right-hand side of (39) does not exceed 3. If it equals 1, it leads to the absurdity that 6c=1. If it equals 2, it leads to the absurdity that 3=0. If it equals 3, then we get that c=2, which is against the assumption.

Case (ii) and (iii) can be handled with in the similar manner as in case (i).

Case where e=1. We notice that this is the case for all trivial designs. From (37) we obtain that

$$(40) \quad (2\nu - \nu + 3)(\nu - \nu - 4) = 0.$$

If  $\nu = \frac{1}{2}(\nu-3)$ , then from (34) and (33) we obtain that  $b_1 = \frac{1}{2}(\nu+1)$ , and that  $\lambda_2 = 1$ . Then  $\lambda_1 = 0$ . From (31)  $\nu = 1$ . Hence  $\nu = 5$ , which is against the assumption. Thus we obtain that  $\nu = \nu - 4$ . Then from (34) and (33) we obtain that

(41) 
$$b_1 = \frac{1}{2}(v-2)(v-3)$$
,

and that

(42) 
$$\lambda_2 = \frac{1}{2}(v-3)(v-4)$$
.

From (17)  $b_2 = 2(v-2)$ . Thus from (18) we obtain that

(43) 
$$\mu_1(v-2) = k(k-2)$$
.

From (43) and (3) it follows that

(44) 
$$2\lambda(v-3) = \mu_1(k-1)(k-3)$$
.

From (44) and (1) we obtain that

(45)  $4\lambda(k-2) = \mu_1 x(k-3)$ .

Using (21), from (45) and (2) we obtain that

$$\mu_2+1=a(\mu_2+a),$$

which implies that a=1. Now (35) implies that k=v-2. Thus  $(X, \mathcal{A})$  is trivial.

Case where e=2. We notice that this is the case for Witt tight designs. From (37) we obtain that

(46) 
$$(\nu - v + 2)(3v - 2\nu - 17) = 12$$
.

We let the first factor of the left-hand side of (46) run all positive divisors of 12 from 1 to 12. It gives us the values of v and v in each case. Then we can calculate the values of remaining parameters. Anyway we have the following six cases: v=27; v=23, v=23,  $b_1=90$ ,  $b_2=162$ ; v=23, v=24,  $b_1=112$ ,  $b_2=140$ ; v=24; v=27; v=38.

We may assume that  $k \leq \frac{1}{2}v$ . Then using (1) and (3) we can eliminate all but the second and third cases with k=7. Then a=2,  $\mu_2=3$  and  $\mu_1=1$ . A Witt tight design yields the third case and the uniqueness is known. The second case is eliminated by (18).

Cases where  $3 \le e \le 6$ . From now on we are expecting no designs. Furthermore the checking procedure is just the same as in the above case. The following two lemmas are sufficient to take care of all cases where  $3 \le e \le 6$ .

**Lemma 1.** k cannot be a prime for e > 2.

Proof. From (1), (2) and (3) we obtain that

(47) y(v-2) = kx.

(i) The case where k divides v-2. Put v-2=ck, where c is a positive integer. Since  $(X, \mathcal{A})$  is non-trivial,  $c \ge 2$ . From (1) it follows that

(48) 2(k-1)(k-2) = x(ck-1).

Thus x=dk-4, where d is a positive integer. So from (48) we obtain that

$$\left(c-\frac{2}{d}\right)k=\frac{4c}{d}+1-\frac{6}{d}.$$

If  $d \ge 4$ , then  $(c-\frac{1}{2})k < c+1$ . Since  $k \ge 4$ , this is a contradiction. If d=3 or 2, we obtain a similar contradiction. If d=1, then

$$k=4+\frac{3}{c^{\sim}2}.$$

So c=3 or 5. If c=3, then k=7 and v=23. From (17) and (18) it follows that  $\mu_1=2$ . Since  $e \ge 3$ , from (21) it follows that a=1. But the case a=1 can occur only for trivial designs. In fact, from (1), (2), (3) and (16) it follows that

(49) 
$$a^2 = (\mu_2 - \mu_1)^2 = (\mu_2 + \mu_1)^2 - 4\mu_2\mu_1$$
  
-  $\frac{4(k-1)(k-2)(v-k-1)(v-k-2)}{(v-2)(v-3)^2} + 1$ 

(49) implies that

$$(50) (a2-1)(v-2)(v-3)2 = 4(k-1)(k-2)(v-k-1)(v-k-2)$$

If a=1, (50) implies that k=v-2. Then we have the trivial designs. If c=5, then k=5 and v=27. But then (3) shows that  $\lambda$  cannot be an integer.

(ii) The case where k and v-2 are relatively prime. By (47) v-2 divides

x. So put x = z(v-2). From (1) it follows that

$$2(k-l)(k-2) = z(v-2)(v-3).$$

Since v-2 > k, we obtain that z=1. Thus

(51) x - v - 2 and y = k.

Now the equation

$$2(k-1)(k-2) = (v-2)(v-3)$$

can be rewritten as a Pellian equation

$$(2v-5)^2-2(2k-3)^2 = -1$$
.

The solutions of the Pellian equation

$$Y^2 - 2Z^2 = -1$$

are recursively given by

$$(52) \quad (Y_1, Z_1) = (1, 1)$$

and

(53) 
$$\begin{cases} Y_{i+1} = 3Y_i + 4Z_i \\ Z_{i+1} = 2Y_i + 3Z_i \end{cases} \quad (i = 1, 2, \cdots).$$

So there exists an integer m such that

(54) 
$$(Y_m, Z_m) = (2v - 5, 2k - 3)$$
.

By (2), (51) and (54) we obtain that

(55)  $16\lambda = 4k^2 - 12k = Z_m^2 - 9.$ 

(55) implies that

(56)  $Z_m = 3, 5, -5 \text{ or } -3 \pmod{16}$ .

On the other hand, from (52) and (53) we obtain that

$$(57) \quad (Y_{l}, Z_{l}) \pmod{16} = \begin{cases} (1, 1) & \text{for } l \equiv 1 \pmod{8}, \\ (7, 5) & \text{for } l \equiv 2 \pmod{8}, \\ (-7, -3) & \text{for } l \equiv 3 \pmod{8}, \\ (-1, -7) & \text{for } l = 4 \pmod{8}, \\ (1, -7) & \text{for } l = 4 \pmod{8}, \\ (1, -7) & \text{for } l \equiv 5 \pmod{8}, \\ (7, -3) & \text{for } l \equiv 6 \pmod{8}, \\ (-7, 5) & \text{for } l \equiv 7 \pmod{8}, \\ (-1, 1) & \text{for } l \equiv 0 \pmod{8}. \end{cases}$$

From (56) and (57) it follows that  $Z_m = 2k-3=5$  or -3 (mod 16). This implies that k is even.

**Lemma** 2. The greatest common divisor of v - 2 and k - 3 divides 3.

Proof. Let c denote the number of blocks containing three distinct points of X. Then since  $(X, \mathcal{A})$  is tight, we obtain that

2c(v-2) = k(k-1)(k-2).

Thus the greatest common divisor of v-2 and k-3 divides that of 2k and k-3. By (18) if v is even, then k is even.

REMARK. Perhaps the case where e=3, v=47 and k=12 is slightly beyond the powers of Lemmas 1 and 2. In that case, from (1) it follows that  $\mu_2 + \mu_1 = 6$ . Hence  $2\mu_2 - a = 6$ . By (21)  $12 - 3a = \mu_2$ . Hence 7a = 8. This is absurd.

From now on we may assume that  $e \ge 7$ .

# 4. Case where a=2

This case has been already treated by M. Kano in a similar way [1]. We may assume that  $k \leq \frac{1}{2}v$ . Then from (16) and (50) it follows that

(58) 
$$\frac{2(a^2-1)(v-3)}{v-4} > \frac{2(k-1)(k-2)}{v-3} = x = a - 1 + 2\mu_1.$$

Since  $v \ge k+3=ae+a+\mu_1+3\ge 20$ , if a=2, then (58) implies that  $43>16\mu_1$ . Thus  $\mu_1 \le 2$ . Then (2) implies the contradiction that  $k \le 10$ .

# 5. *e*-adic expansions of *v* and related parameters

From (34) it follows that

$$2\nu = e(v-4) + \frac{2b_1-2e}{v-1}$$
.

So further by (33)  $\frac{2b_1}{v-1}$  is a positive rational integer. Put

(59) 
$$C = \frac{2b_1 - 2e}{v - 1}.$$

Then

(60) 
$$2v = e(v-4)+C$$
.

Substituting (60) in (37) we obtain that

(61) 
$$v = C + (e+1)^2 + \frac{(e+2)(e+1)e(e-1)}{C-e^2}$$
.

The numerator of the third term of the right-hand side of (61) equals

 $e^4 + e^3 + (e-2)e^2 + (e-2)e$ .

Now express C in the *e*-adic form

(62) 
$$C = c_4 e^4 + c_3 e^3 + c_2 e^2 + c_1 e + c_0$$
,  
where  $0 \leq c_i < e$  (*i*=0, 1, 2, 3, 4).

Since the first factor of the left-hand side of (37) is positive, by (33) we can easily see that  $C > e^2$ . So the third term of the right-hand side of (61) is a positive integer. We put

(63) 
$$e^4 + e^3 + (e-2)e^2 + (e-2)e = (C-e^2)D$$
,

where

$$D = v - C - (e + \chi) = d_4 e^4 + d_3 e^3 + d_2 e^2 + d_1 e + d_0$$
  
with  $0 \le d_i < e$  (i=0, 1, 2, 3, 4).

First we show that  $c_4=0$  in (62). In fact, clearly we have that  $c_4 \leq 1$ . If  $c_4=1$ , then D=1. So from (61) and (63) we obtain that

(64)  $v = e^4 + 2e^3 + e^2 + 2$ .

Then from (59) it follows that

(65) 
$$2b_1 = e^8 + 4e^7 + 4e^6 + (e-1)e^5 + (e-3)e^4$$

Now from (35) it follows that

(66) 
$$(v-1)k^2 - v(v-1)k + a\{2b_1 + e(v+1)(v-2)\} = 0$$
.

Further from (1) and (30) it follows that

(67) 
$$2k^2 - 2vk + (v+1) + a(2e+1)(v-3) = 0$$
.

So from (66) and (67) we obtain that

$$2a \{2b_1+e(v+1)(v-2)\}=(v+1)(v-1)+a(2e+1)(v-3)(v-1)$$
,

which implies that

(68) 
$$a\{4b_1+2e(3v-5)-(v-3)(v-1)=(v+1)(v-1).$$

From (64), (65) and (68) we obtain that

$$a(e^{4}+2e^{3}-e^{2}+2e+1)=e^{4}+2e^{3}+e^{2}+3.$$

So  $a=3 \pmod{e}$ . This implies that a=3 and that  $2e^4+4e^3-4e^2+6e=0$ . This is a contradiction. Hence we obtain that  $c_4=0$ .

Secondly we show that  $c_3 = 0$  in (62). If  $c_3 \ge 2$ , then we obtain that D < e. So from (61) we obtain that

(69) 
$$v = c_3 e^3 + (c_2 + 1) e^2 + (c_1 + 2) e + c_0 + 1 + D$$

Now from (59) and (68) we obtain that

(70) 
$$a(2C+6e-v+3) = v+1$$
.

Then from (69) and (70) it follows that

$$2C+6e-v+3 = c_3e^3+(c_2-1)e^2+(c_1+4)e+c_0+2-D.$$

Since  $a \ge 3$ , this implies that

$$2c_3e^3 + 2(c_2 - 1)e^2 + 2(c_1 + 4)e + 2c_0 + 4 - 2D - 1$$
  

$$\leq e_3e^3 + (c_2 + 1)e^2 + (c_1 + 2)e + c_0 + 1 + D,$$

which is clearly a contradiction. So we obtain that  $c_3 \leq 1$ . Now assume that  $c_3=1$ . If  $c_2\geq 2$ , then  $D\leq e$ . So from (61) and (70) we obtain that

$$v = e^{3} + (c_{2}+1)e^{2} + (c_{1}+2)e + (c_{0}+1) + D$$

and that

$$2C+6e-v+3 = \frac{v+1}{a}$$
$$= e^{3}+(c_{2}-1)e^{2}+(c_{1}+4)e+c_{0}+2-D.$$

Since  $a \ge 3$ , this implies that

$$2e^{3}+2(c_{2}-1)e^{2}+2(c_{1}+4)e+2(c_{0}+2)-2D-1$$
  

$$\leq e^{3}+(c_{2}+1)e^{2}+(c_{1}+2)e+(c_{0}+1)+D,$$

which is clearly a contradiction. If  $c_2=1$ , then  $D \leq e+1$ . This leads to a contradiction as above. If  $c_2=0$ , then  $D \leq e+3$ . Since  $a \geq 3$ , we get a contradiction as above. Hence we obtain that  $c_3=0$ .

Thirdly we show that  $c_2 = 2$  in (62). If  $c_2 \ge 4$ , then  $D < e^2$ . From (61) and (70) we obtain that

$$v = (c_2+1)e^2+(c_1+2)e+(c_0+1)+D$$

and that

$$2C+6e-v+3 = \frac{v+1}{a}$$
$$= (c_2-1)e^2+(c_1+1)e+(c_0+2)-D .$$

Since  $a \ge 3$ , this implies that

$$3(c_2-1)e^2+3(c_1+4)e+3(c_0+2)-3D-1 \\ \leq (c_2+1)e^2+(c_1+2)e+c_0+1+D,$$

which is clearly a contradiction. Hence we obtain that  $c_2 \leq 3$ . If  $c_2 = 1$ , then  $C - e^2 = c_1 e + c_0$ . From (61) we obtain that

$$v = 2e^{2} + (c_{1}+2)e + c_{0} + 1 + D$$

and that

(71) 
$$2C+6e+3-v = (c_1+4)e+c_0+2-D.$$

Since  $c_1 e + c_0 \leq e^2 - 1$ , from (63) we obtain that

$$D \ge e^2 + 2e$$
.

Since (71) is positive, we obtain that  $c_1 \ge e - 3$ . If  $c_1 = e - 1$ , then

(72) 
$$v = 3e^2 + e + c_0 + 1 + D$$

and

(73) 
$$2C+6e+3 = 2e^2+4e+2c_0+3$$
.

Since (72) is smaller than (73), this is a contradiction. If  $c_1 = e - 2$ , then

 $v = 3e^2 + c_0 + 1 + D$ 

and

$$2C+6e+3 = 2e^2+2e+2c_0+3$$
.

As above this is a contradiction. If  $c_1 = e - 3$ , then

$$v = 2e^2 + (e-1)e + c_0 + 1 + D$$

and

$$2C+6e+3=2e^2+2c_0+3$$
.

As above this is a contradiction. Hence we obtain that  $c_2=3$  or  $c_2=2$ . If  $c_2=3$ , then from (63) and (61) we obtain the following equations:

(74)  

$$(2e^{2}+c_{1}e+c_{0})(d_{2}e^{2}+d_{1}e+d_{0})$$

$$= e^{4}+e^{3}+(e-2)e^{2}+(e-2)e;$$

$$v = (4+d_{2})e^{2}+(c_{1}+2+d_{1})e+c_{0}+1+d_{0};$$

$$2C+6e+3-v$$

$$= (2-d_{2})e^{2}+(c_{1}+4-d_{1})e+c_{0}+2-d_{0}.$$

Further from (70) we obtain that

(75) 
$$a\{(2-d_2)e^2+(_1+4-d_1)e+c_0+2-d_0\} = (4+d_2)e^2+(c_1+2+d_1)e+c_0+2+d_0.$$

Now from (74) we obtain that  $d_2=0$ . If  $a \ge 5$ , from (75) we obtain that

$$10e^{2} + 5(c_{1} + 4 - d_{1})e + 5(c_{0} + 2 - d_{0})$$

$$\leq 4e^{2} + (c_{1} + 2 + d_{1})e + c_{0} + 2 + d_{0},$$

which implies that

$$6e^2 + (4c_1 + 18)e + (4c_0 + 8) \leq 6d_1e + 6d_0 \leq 6e^2 - 6$$

This is a contradiction. If a=4, as above we obtain that

(76)  $4e^2 + (3c_1 + 14)e + (3c_0 + 6) = 5d_1e + 5d_0.$ 

On the other hand, from (74) we obtain that

$$2d_1e^3 + (2d_0 + c_1d_1)e^2 + (c_1d_0 + c_0d_1)e + c_0d_0$$
  
=  $e^4 + e^3 + (e-2)e^2 + (e-2)e$ ,

which implies that  $2d_1 \leq e+1$ . But then we obtain that

$$5d_1e + 5d_0$$

$$\leq 5d_1e + 5(e-1)$$

$$= 5(d_1+1)e - 5$$

$$\leq 5\{\frac{1}{2}(e+1) + 1\}e - 5$$

$$= 3e^2 + \frac{1}{2}(15-e)e - 5$$

Since  $e \ge 7$ , this contradicts (76). If a=3, as above we obtain the following equation:

$$2e^{2}+(2c_{1}+10)e+2c_{0}+5=4d_{1}e+4d_{0}$$
,

which is clearly a contradiction. Hence we obtain that  $c_2 = 2$ .

Now we have the following equations:

(77)

(77) 
$$(e^{2}+c_{1}e+c_{0})(d_{2}e^{2}+d_{1}e+d_{0})$$
  
=  $e^{4}+e^{3}+(e-2)e^{2}+(e-2)e;$   
(78)  $v = (3+d_{2})e^{2}+(c_{1}+2+d_{1})e+c_{0}+1+d_{0};$ 

and

(79) 
$$2C+6e+3-v = (1-d_2)e^2+(c_1+4-d_1)e+c_0+2-d_0.$$

Hence we obtain that either  $d_2 = 1$  or 0.

There seems to be a rather big difference between the cases  $d_2=1$  and  $d_2=0$ . First we assume that  $d_2=1$ . Then from (77) it follows that

(80) 
$$(c_1+d_1)e^3 + (c_0+c_1d_1+d_0)e^2 + (c_0d_1+c_1d_0)e + c_0d_0 = e^3 + (e-2)e^2 + (e-2)e.$$

Hence we obtain that  $c_1 + d_1 \leq 1$ .

Case (i) where  $c_1 = d_1 = 0$ . If  $c_0 = d_0 = e - 1$ , then e = 1. Hence either  $c_0 \leq e - 2$  or  $d_0 \leq e - 2$ . So we obtain that

$$(2e-3)e^2+(e-1)(e-2)\geq e^3+(e-2)e^2+(e-2)e$$
,

which is a contradiction.

Case (ii) where  $c_1 = 0$  and  $d_1 = 1$ . From (80) it follows that

(81) 
$$(c_0+d_0)e^2+c_0e+c_0d_0=(e-2)e^2+(e-2)e$$
.

Put  $c_0 + d_0 = e - l$  and  $c_0 d_0 = em$ . Then  $l \ge 2$  and  $e - 3 \ge m \ge 0$ . Then from (81) we obtain that

$$(e-l)e+c_0+m=(e-2)e+e-2.$$

Now we have that  $2e > c_0 + m + 2 = (l-1)e \ge e$ . Hence  $l=2, c_0 + m + 2 = e$  and

 $c_0+d_0+2=e$ . So  $m=d_0$ . Thus  $d_0=0$  and  $c_0=e-2$ . Then from (78) and (70) we obtain that

 $(82) v = 4e^2 + 4e - 1,$ 

and that

(83) a = e + l.

Now from (67), (82) and (83) we obtain that

$$k^{2} - (4e^{2} + 4e - 1)k + 2(e + 1)^{3}(2e - 1) = 0$$
.

Since

$$(4e^2+4e-1)^2-8(e+1)^3(2e-1)= -8e^3-16e^2+9$$

is negative, this is a contradiction.

Case (iii) where  $c_1=1$  and  $d_1=0$ . From (80) it follows that

(84) 
$$(c_0+d_0)e^2+d_0e+c_0d_0=(e-2)e^2+(e-2)e$$

Put  $c_0 + d_0 = e - l$  and  $c_0 d_0 = me$ . Then  $l \ge 2$  and  $e - 3 \ge m \ge 0$ . Then from (84) it follows that

$$(e-l)e^{2}+d_{0}e+me = (e-2)e^{2}+(e-2)e.$$

Thus we obtain that

$$(e-l)e+d_0+m=(e-2)e+e-2.$$

Now we have that  $2e > d_0 + m + 2 = (l-1)e \ge e$ . Hence l=2,  $c_0 + d_0 = e - 2$  and  $d_0 + m = e - 2$ . So  $m = c_0$ . Thus  $c_0 = 0$  and  $d_0 = e - 2$ . Then from (78) and (70) we obtain that

$$(85) v = 4e^2 + 4e - 1,$$

and that

(86) 
$$a = e$$

Now from (67), (85) and (86) we obtain that

(87) 
$$k^2 - (4e^2 + 4e - 1)k + 2e^3(2e + 3) = 0$$
.

Hence  $8e^3 + 8e^2 - 8e + 1$  is a square. Thus we are confronted with a Diophantine equation

(88) 
$$Y^2 = 8X^3 + 8X^2 - 8X + 1$$
.

It is easy to solve (88) with the restriction  $X \equiv 2 \pmod{3}$ .

**Lemma 3.** The integral solutions of (88) with  $X \equiv 2 \pmod{3}$  are the following: (i) X=0,  $Y=\pm l$  and (ii) X=1,  $Y=\pm 3$ .

Proof. We start with

(89) 
$$Y^2 = (2X-1)(4X^2+6X-1)$$
.

If X is negative, from (89) we obtain that  $-6X \ge 4X^2 - 1$ , which implies that  $X \ge -1$ . So we may assume that X is non-negative. For X=0 and 1 we obtain the solutions (i) and (ii) respectively. Hence we may assume that  $X \ge 3$ . Since

$$4X^{2}+6X-1=(2X-1)(2X+4)+3,$$

2X-1 and  $4X^2+6X-1$  are relatively prime by assumption. Hence we may put

(90) 
$$2X-1=R^2$$
 and  $4X^2+6X-=S^2$ ,

where R and S are positive integers. Then S>2X and S<2X+2. Hence S=2X+1, which implies that X=1 against the assumption.

For  $X \leq 14$  we obtain the further solutions (iii) X = -1,  $Y = \pm 3$ ;(iv) X = 2,  $Y = \pm 9$  and (v) X = 14,  $Y = \pm 153$ .

REMARK 1. We owe to Dr. Jeffrey Leon the following fact that the solution (v) is the only solution of (88) in the interval  $7 \le X \le 25000$ .

If e=14, then v=839. Further we may assume that k=343. But these parameters do not satisfy (3). Hence we may assume that  $e \ge 15$ .

We may put

(91)  $2e-1 = 3R^2$ ,

where R is an integer bigger than 3. Now we show the non-existence of tight 4-designs with parameters (85), (86), (87) and (91).

Assume that 2e-1 has a prime divisor p bigger than 3. Then from (87) we obtain that

(92)  $k \equiv 1 \pmod{p}$ .

Further from (16), (21), (86) and (92) we obtain that

$$\mu_2 = 1 - e^2 \pmod{p} ,$$

and that

$$\mu_1 = 1 - e - e^2 \pmod{p}$$
.

If  $e^2 \equiv 1 \pmod{p}$ , then  $4e^2 \equiv 4=1 \pmod{p}$ , which implies that p=3. If  $e^2+e \equiv 1 \pmod{p}$ , then  $4e^2+4e \equiv 4 \equiv 3 \pmod{p}$ , which is absurd. Hence we obtain that

(93)  $\mu_2 \mu_1 \equiv 0 \pmod{p}$ .

On the other hand, from (2) we obtain that

$$4\lambda = (k-3)(\mu_2\mu_1-2\lambda) = -2(\mu_2\mu_1-2\lambda) \pmod{p},$$

which implies that

(94)  $\mu_2 \mu_1 \equiv 0 \pmod{p}$ .

Obviously (93) and (94) are in contradiction.

Hence we obtain that

 $2e - 1 = 3^{E}$ ,

where, by (91), E is an odd integer bigger than 3. Now from (89) and (91) we obtain that

$$(95) F^2 = 3^{2E-1} + 3^E + 2 \cdot 3^{E-1} + 1$$

is a square, where F is a positive integer. From (95) we obtain that either

(96) 
$$F-l = G3^{E-1}$$
,

or

$$(97) \qquad F+1 = G3^{E-1},$$

where G is a positive integer. If (96) occurs, then

 $F^2 - 1 = G^2 3^{2E-2} + 2G 3^{E-1}$ .

Since G is clearly bigger than 1, we obtain that

$$F^2 - 1 \ge 4.3^{2E-2} + 4.3^{E-1} > 3^{2E-1} + 3^{2E-2}$$
.

Since  $E \ge 5$ , this is a contradiction. If (97) occurs, then

$$(98) F^2 - 1 = G^2 3^{2E-2} - 2G 3^{E-1}.$$

If  $G \geq 3$ , then

$$G^{2}3^{2E-2} - 2G3^{E-1} > 2.3^{2E-1}$$

which contradicts (95). Hence G=2. Then (95) and (98) imply that E=1. This is against assumption.

REMARK 2. We are informed that Dr. Koichi Yamamoto, meantime, has

solved (88) confirming that (i)-(v) are the only solutions. His proof seems to be not entirely elementary.

#### 6. Relation between v and k

From now on we assume that  $d_2=0$ . By (77), (78), (79) and (70) we can start with the following equations:

(99) 
$$(e^2+c_1e+c_0)(d_1e+d_0) = e^4+e^3+(e-2)e^2+(e-2)e;$$

(100) 
$$v = 3e^2 + (c_1 + d_1 + 2)e + c_0 + d_0 + 1;$$

(101) 
$$2C+6e+3-v = e^2+(c_1-d_1+4)e+(c_0-d_0+2),$$

where  $C = 2e^2 + c_1e + c_0$ ;

(102) 
$$a \{e^{2} + (c_{1} - d_{1} + 4)e + c_{0} - d_{0} + 2\} = 3e^{2} + (c_{1} + d_{1} + 2)e + c_{0} + d_{0} + 2 = v + 1.$$

From (99) it follows that

(103) 
$$\begin{aligned} d_1 e^3 + (c_1 d_1 + d_0) e^2 + (c_0 d_1 + c_1 d_0) e + c_0 d_0 \\ = e^4 + e^3 + (e-2) e^2 + (e-2) e \,. \end{aligned}$$

First we show that

(104)  $c_1 + d_1 - e + j$  for some *j* with  $e - 2 \ge j > 0$ .

To show this put  $d_1 = e - A$  and  $d_0 = e - B$ . Then  $A \ge 1$  and  $B \ge 1$ . From (103) it follows that

$$(c_1 - A + 1)e^3 + (c_0 + c_1 - B - c_1 A)e^2 + (c_0 - c_1 B - c_0 A)e - c_0 B$$
  
=  $e^3 + (e - 2)e^2 + (e - 2)e$ .

This implies that  $c_1 - A + 1 > 1$ . Hence  $c_1 + d_1 > e$ . Secondly we show that

(105)  $d_1 < e - 1$ .

Assume that  $d_1 = e - 1$ . Then from (103) it follows that

(106) 
$$c_1 e^3 + (c_0 + d_0 + 1) e^2 + (c_1 d_0 + 2) e + c_0 d_0$$
  
=  $3 e^3 + c_1 e^2 + c_0 e$ .

Hence we obtain that  $c_1 \leq 3$ . By (104)  $c_1 \geq 2$ . If  $c_1 = 3$ , then from (106) it follows that

$$(c_0+d_0)e^2+(3d_0+2)e+c_0d_0=2e^2+c_0e$$
.

If  $c_0 \le 1$ , then we obtain a contradiction that e=5. If  $c_0=2$ , then  $d_0=0$ . Then from (102) it follows that

$$a(2e+1) = e^2 + e + 1$$
.

This implies that  $a \equiv 1 \pmod{e}$ . Then a=1. This is against assumption. If  $c_1=2$ , then from (106) it follows that

(107) 
$$(c_0+d_0)e^2+(2d_0+2)e+c_0d_0=e^3+e^2+c_0e$$
.

If  $c_0=0$ , then  $d_0=e-1$ . Then from (102) and (100) it follows that

$$3a = 2e + 1$$

and that

$$v = 4e(e+1).$$

Further from (67) it follows that

$$3k^2 - 8e(e+1)k + 2e(e+1)(2e+1)^2 = 0$$
.

Since

$$64e^{2}(e+1)^{2}-24e(e+1)(2e+1)^{2} = 8e(e+1)(-4e^{2}-4e-3)$$

is negative, this is a contradiction. Hence  $c_0 > 0$ . If  $d_0 = 0$ , then  $c_0 > e$ . This contradiction shows that  $c_0 d_0 > 0$ . Put  $c_0 d_0 = Ae$  and  $2d_0 + 2 + A - c_0 = Be$ . Then 0 < A < e - 3 and  $0 \le B \le 2$ . Further we have that

(108) 
$$c_0 + d_0 + B = e + 1$$
.

If B=0, then by (107) and (108) we obtain that

(109) 
$$e^{2} - (4d_{0} + 1)e + d_{0}(d_{0} - 1) = 0$$
.

Hence we consider the function  $f(X)=X^2-(4d_0+1)X+d_0(d_0-1)$ . Since  $d_0>1$ ,  $f(0)=d_0(d_0-1)>0$  and  $f(d_0)=-2d_0^2-2d_0<0$ . Further we obtain that  $f(4d_0)=d_0^2-5d_0$ . If  $d_0=5$ , then e=20 and  $c_0=16$ . Then from (102) we obtain that a=11. From (100) we obtain that v=1682. Further from (101) we obtain that C=856. Now from (59) and (66) we obtain that

(110) 
$$k(v-k) = a(C+ev)$$
.

With data above (110) implies that

$$k^2 - 1682k + 379456 = 0$$
.

But then k cannot be integral. If  $d_0 \leq 4$ , (109) has no integral solution. Hence we obtain that  $d_0 > 5$  and that  $e > 4d_0$ . But  $f(4d_0+1) = d_0(d_0-1) > 0$ . This is a contradiction. If B=1, we can follow the above argument to get a contradiction. If B=2, we obtain that  $d_0=e-1$ . This is a contradiction.

Thirdly we observe that

(111)  $c_1 \ge +2 \ge 3$ .

In fact,  $c_1+d_1=e+j$  with i>0. Hence by (105)  $c_1+e-2\ge e+j$ . Now we show that

 $(112) \quad e > 2a$ .

Assume not. Then from (102) it follows that

$$e \{e^{2} + (c_{1} + 4 - d_{1})e + c_{0} + 2 - d_{0}\}$$

$$\leq 6e^{2} + (2c_{1} + 4 + 2d_{1})e + 2c_{0} + 4 + d_{0},$$

which implies that

(113) 
$$e^{3}+c_{1}e^{2}+c_{0}e$$
  
 $\leq (d_{1}+2)e^{2}+(2c_{1}+2+2d_{1}+d_{0})e+2c_{0}+4+d_{0}$   
 $\leq (d_{1}+7)e^{2}+1.$ 

This implies that

(114) 
$$d_1 + 7 \ge e + c_1 + 1$$
,

provided  $c_0 > 0$ . But if  $c_0 = 0$ , (113) becomes

$$e^{3}+c_{1}e^{2}$$

$$\leq (d_{1}+2)e^{2}+(2c_{1}+2+2d_{1}+d_{0})e+4+d_{0}$$

$$\leq (d_{1}+7)e^{2}-2e+3.$$

Hence we recover (114). (114) implies that

(115)  $6 \ge A + c_1$ ,

where  $d_1 = e - A$ . Since  $c_1 \ge 3$  by (111), this is a contradiction for  $A \ge 4$ . If A=3, then  $c_1=3+j\ge 4$ . Hence (115) shows a contradiction. Then by (105) we obtain that A=2. Hence  $c_1=2+j$  with  $j\le 2$ . If j=2, then (113) becomes

$$e^{3}+4e+c_{0}e$$
  
 $\leq e^{3}+2e^{2}+(d_{0}+6)e+2c_{0}+4+d_{0}$   
 $\leq e^{3}+3e^{2}+6e+2c_{0}+3.$ 

Thus  $e \leq 6$ . This is against assumption. Hence we obtain that j=1. Thus  $c_1=3$  and  $d_1=e-2$ . Now (113) becomes

$$e^{3}+3e^{2}+c_{0}e$$
  
 $\leq e^{3}+2e^{2}+(4+d_{0})e+2c_{0}+4+d_{0}$   
 $\leq e^{3}+2e^{2}+(7+d_{0})e+1$ .

This implies that

(116) 
$$d_0 + 7 \ge e + c_0$$
.

If  $c_0=6$ , then  $d_0=e-1$  by (116). Then by (103) e divides 6. This is against assumption. If  $c_0=5$ , then  $d_0 \ge e-2$ . If  $d_0=e-1$ , then by (103) e divides 5. This is against assumption. If  $d_0=e-2$ , then by (103) e=10. Hence  $d_1=8$ ,  $c_1=3$ ,  $d_0=8$  and  $c_0=5$ . Then from (102) it follows that a=5. From (100) and (101) we obtain that z = 444 and C = 235. Then from (110) we obtain that

 $k^2 - 444k + 23375 = 0$ .

But then k cannot be integral. If  $c_0=4$ , then  $d_0 \ge e-3$ . If  $d_0=e-1$ , then by (103) e divides 4. This is against assumption. If  $d_0=e-2$ , then by (103) e=8. Then  $c_1=3$ ,  $d_1=6$ ,  $c_0=4$  and  $d_0=6$ . Then from (102) it follows that a=4. From (100) and (101) we obtain that v=291 and C=156. Then from (110) we obtain that

 $k^2 - 291k + 9936 = 0$ .

But then k cannot be integral. If  $d_0=e-3$ , then by (103) e=12. Thus  $c_1=3$ ,  $d_1=10$ ,  $c_0=4$  and  $d_0=9$ . Then from (102) it follows that 105a=627. This is a contradiction. If  $c_0=3$ , then  $d_0\geq e-4$ . If  $d_0\geq e-2$ , then by (103)  $e\leq 6$ . This is against assumption. If  $d_0=e-3$ , then by (103) e=9. Thus  $c_1=3$ , rfi=7,  $c_0=3$  and rfo-6. Then from (102) it follows that 40a=181. This is a contradiction. If  $d_0=e-4$ , then by (103) e=12. Thus  $c_1=3$ ,  $d_1=10$ ,  $c_0=3$  and rfo=8. Then from (102) it follows that 21a=125. This is a contradiction. If  $c_0=e-4$ , then by (103) e=8. Thus  $c_1=3$ ,  $d_1=6$ ,  $c_0=2$  and  $d_0=4$ . Then from (102) it follows that a=4. From (100) and (101) we obtain that v=287 and C=154. Then from (110) we obtain that

 $k^2 - 287k + 9800 = 0$ .

But then k cannot be integral. If  $d_0 = e - 5$ , then by (103) e = 10. Thus  $c_1 = 3$ ,  $d_1 = 8$ ,  $c_0 = 2$  and  $d_0 = 5$ . Then from (102) it follows that 89a = 439. This is a contradiction. If  $c_0 = 1$ , then  $d_0 \ge e - 6$ . Then by (103)  $e \le 6$ . This is against assumption. If  $c_0 = 0$ , then from (103) we obtain that

$$(d_0-6)e+3d_0=(e-2)e+e-2.$$

This implies that

$$d_0(e+3) = e(e+3)+2(e-1)$$
.

Hence e+3 divides 2(e-1). Thus e+3=2e-2, and e=5. This is against assumption. This completes the proof of (112).

By (100) we may put

$$k = k_2 e^2 + k_1 e + k_0$$
,

where  $e-1 \ge k_i \ge 0$  for i=2, 1, 0. Then from (110) it follows that

(117) 
$$a\{3e^{3}+(c_{1}+d_{1}+4)e^{2}+(c_{1}+c_{0}+d_{0}+1)e+c_{0}\}\$$
$$=(k_{2}e^{2}+k_{1}e+k_{0})\{(3-k_{2})e^{2}+(c_{1}+d_{1}+2-k_{1})e+c_{0}+d_{0}+1-k_{0}\}.$$

Now under the assumption that  $k \leq \frac{1}{2}v$  we show that

(118) 
$$k_2 = 0$$
.

First of all, from (117) and (112) it follows that

(119) 
$$\begin{array}{l} 6k_{2}e^{4}+2\left\{k_{2}(c_{1}+d_{1}+2)+3k_{1}\right\}e^{3} \\ +2\left\{k_{2}(c_{0}+d_{0}+1)+k_{1}(c_{1}+d_{1}+2)+3k_{0}\right\}e^{2} \\ +2\left\{k_{1}(c_{0}+d_{0}+1)+k_{0}(c_{1}+d_{1}+2)\right\}e+k_{0}(c_{0}+d_{0}+1) \\ <\left(3+2k_{2}^{2}\right)e^{4}+\left\{\left(c_{1}+d_{1}+4\right)+4k_{1}k_{2}\right\}e^{3} \\ +\left(c_{0}+d_{0}+1+4k_{0}k_{2}+2k_{1}^{2}\right)e^{2}+\left(c_{0}+4k_{0}k_{1}\right)e+k_{0}^{2}. \end{array}$$

Now from (100) it follows that  $k_2 \leq 2$ . If  $k_2 = 2$ , then from (119) we obtain that

$$\begin{split} & e^{4} + \{3(c_{1}+d_{1})+4\} e^{3} + \{3(c_{0}+d_{0}+1)+2k_{1}(c_{1}+d_{1}+2)\} e^{2} \\ & +2\{k_{1}(c_{0}+d_{0}+1)+k_{0}(c_{1}+d_{1}+2)\} e+k_{0}(c_{0}+d_{0}+1) \\ & <2k_{1}e^{3} + (2k_{0}+2k_{1}^{2})e^{2} + (c_{0}+4k_{1}k_{0})e+k_{0}^{2} \\ & \leq 4e^{4} - 7e^{2} + (c_{0}+2)e+1 . \end{split}$$

By (104) this is a contradiction. If  $k_2=1$ , then from (119) we obtain that

$$\begin{split} & e^{4} + (e_{1} + d_{1} + 2k_{1})e^{3} + \{c_{0} + d_{0} + 1 + k_{1}(c_{1} + d_{1} + 2) + 2k_{0}\}e^{2} \\ & + 2\{k_{1}(c_{0} + d_{0} + 1) + k_{0}(c_{1} + d_{1} + 2)\}e + k_{0}(c_{0} + d_{0} + 1) \\ & < 2k_{1}^{2}e^{2} + (c_{0} + 4k_{1}k_{0})e + k_{0}^{2} \\ & \leq 2e^{4} - 5e^{2} + (c_{0} + 2)e + 1. \end{split}$$

By (104) this is a contradiction. Thus  $k_2 = 0$ .

From (118) it follows that

(120) 4k < v.

In fact, otherwise, from (100) we obtain that

$$3e^2 + (c_1 + d_1 + 2)e + c_0 + d_0 + 1$$
  
 $\leq 4k_1e + 4k_0 \leq 4e^2 - 4$ .

By (104) this is a contradiction.

# 7. Cases with small *a*

Now it is convenient to eliminate cases with small a. From (50) and (120) we obtain that

(121)  $8a^2-9a+1>18\mu_1$ .

On the other hand, let us consider the complementary design  $(X, \mathcal{A}c)$ . Let  $\nu_2$  and  $\nu_1$  be the intersection numbers of  $(X, \mathcal{A}c)$ , where  $\nu_2 > \nu_1$ . Then easily we obtain that

(122) 
$$v-2k = \nu_2 - \mu_2 = \nu_1 - \mu_1$$
.

(122) implies that

(123) 
$$\nu_2 - \nu_1 = \mu_2 - \mu_1 = a$$
,

and that

(124) 
$$v-k-\nu_2 = k-\mu_2 = ea$$
.

Put  $x_c = \nu_2 + \nu_1 - 1$ . Then from (123) and (124) it follows that

(125) 
$$x_c = 2v - 2k - 1 - a(2e+1) = 2v - 4k + x.$$

Furthermore corresponding to (1) we obtain that

(126) 
$$x_c = \frac{2(v-k-1)(v-k-2)}{v-3}$$
.

From (1), (126) and (50) we obtain that

$$(127) \qquad (a^2 - 1)(v - 2) = xx_c \, .$$

Then from (30), (125) and (127) we obtain that

(128) 
$$(a^2-1)(v-2) = (2\mu_1-1+a)(2v-4ae-2\mu_1-3a-1).$$

(128) implies that

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(129)  $4\mu_1 > a^2 - 2a + 1$ .

Now we have to eliminate the case where  $10 \ge a \ge 3$ . For each case we can argue exactly in the same way. So we demonstrate only for the case a=3. If a=3, then from (121) and (129) we obtain that  $\mu_1=2$ . Then  $\mu_2=5$  and x=6. Then from (1) and (125) we obtain that

$$v=6k-3,$$

and that

$$k^2 - 21k + 50 = 0$$
.

But then k cannot be integral. It is easy to proceed by hand up to a=10. Hence from now on we may assume that  $a \ge 11$ .

# 8. Completion of the proof

First we scrutinize (103) a little more. By (103) we may put

$$(130) \quad c_0 d_0 = Ee$$

(131) 
$$c_0d_1+c_1d_0+E=Fe-2;$$

and

$$(132) \quad c_1 d_1 + d_0 + F = Ge - 1$$

where E is a non-negative integer and F and G are positive integers. Then from (130)-(132) and (103) it follows that

(133) 
$$d_1 + G = e + 2$$
.

Further from (133), (105) and (106) it follows that

(134)  $c_1+2=j+G\geq j+4$ .

Secondly we show that

(135)  $k_1 > a$ .

In fact, by (21)  $k_1 \ge a$ . Hence assume that  $k_1 = a$ . Then  $k_0 = \mu_2 < e$ . From (47), (100) and (104) we obtain that

$$4ye^{2} < y(v-2) = kx = (ae+\mu_{2})(2\mu_{2}-a-1) < 2(a+1)e^{2}$$

which implies that

(136)  $y < \frac{1}{2}(a+1)$ .

On the other hand, let d be the greatest common divisor of k and v-2. Then from (1) we obtain that

$$x+4 \equiv 0 \pmod{d}.$$

Now by (47) y is divisible by k/d. Hence in order to get a contradiction to (136), it suffices to show that

(137) 
$$\frac{k}{x+4} \ge \frac{1}{2}(a+1)$$
.

Since

$$2k - (a+1)(x+4) = 2a(e-\mu_2) + a^2 - 2a - 3,$$

(137) holds.

As a consequence of (135) we obtain that

(138) 
$$e < \frac{1}{2}a^2$$
.

In fact, (135) implies that  $\mu_2 > e$ . Hence from (121) it follows that

$$2e < 2\mu_2 < \frac{8a^2 + 9a + 1}{9} < a^2$$
.

Thirdly we show that

$$(139) \quad c_1 \leq \frac{4e}{a}.$$

In fact, first assume that  $ac_1 > 5e$ , which implies that

$$a(d_1-c_1-5) < (a-5)e$$
.

Hence by (102) we obtain that

$$(140) \quad v+i \geq 5e^2.$$

If  $c_1 \ge d_1$ , from (102) it follows that  $ae^2 < 6e^2$ . This contradiction shows that  $c_1 < d_1$ . Now from (102) and (105) it follows that

$$v+1 \leq 5e^2 - e$$
,

which contradicts (140). Thus  $ac_1 \leq 5e$ . Next assume that  $ac_1 > 4e$ . Then by (100) and (102) we obtain that

$$ae^{2}+4ae+a(c_{0}+2)<(j+2)e+c_{0}+d_{0}+2+ad_{1}e+ad_{0}$$

Hence by (105) we obtain that

$$6ae + a(c_0+2) < (j+2)e + c_0+2 + a(d_0+1).$$

Since, by (105), (138), and (139)

$$j = c_1 - (e - d_1) \leq (5e/a) - 2 \leq 3a - 2$$
,

we obtain that

$$3ae + a(c_0+2) < c_0+2 + a(d_0+1)$$
 ,

which is a contradiction.

Now from (30), (125) and (127) we obtain that

$$(141) \qquad (a^2-1)(v-2) = (2k-1-2ae-a)(2v-2k-1-2ae-a)$$

From (141) we obtain that

(142) 
$$4k^2 = 4a^2e^2 + 4a^2e + 4ae + 3a^2 + 2a - 1 \pmod{v}$$
.

On the other hand, from (67) we obtain that

(143) 
$$2k^2 = 6ae + 3a - 1 \pmod{v}$$
.

From (142) and (143) it follows that

(144)  $4a^2e^2+(4a^2-8a)e+3a^2-4a+1=0 \pmod{v}$ .

Further from (100) and (104) we obtain that

$$a^2v = 4a^2e^2 + (j+2)a^2e + (c_0 + d_0 + 1)a^2$$
.

Hence together with (144) we may put

(145) 
$$Yv = \{(j-2)a^2 + 8ae + (c_0 + d_0 - 2)a^2 + 4a - 1, \}$$

where Y is a rational integer. Since  $v \equiv -l \pmod{a}$  by (102), from (145) we obtain that  $Y \equiv 1 \pmod{a}$ , and hence that

(146) 
$$(aZ+1)v = \{(j-2)a^2+8ae+(c_0+d_0-2)a^2+4a-1,$$

where Z is a rational integer.

We show that Z=0 in (146). First we show that Z is non-negative. In fact, assume that Z is negative. Then  $j \leq 2$ . If j=2, then  $c_0+d_0 \leq 1$ . But since e > 2a by (112),  $8ae-2a^2 > 0$ . This is absurd. Hence j=1. Then from (146) we obtain that

$$(a+1)(4e^2+3e+c_0+d_0+1)+8ae+(c_0+d_0)a^2+4a \le a^2e+2a^2+1$$
.

Since e > 2a by (112), this is absurd. Next we show that  $Z \leq 1$ . In fact, otherwise, from (146) we obtain that

$$\{ (j-2)a^2 + 8a \} e + (c_0 + d_0 - 2)a^2 + 4a - 1 \\ \geq (2a+1) \{ 4e^2 + (j+2)e + c_0 + d_0 + 1 \}.$$

Since e > 2a by (112), from the above inequality we obtain that ja > 8e. Hence by (134) we obtain that  $ac_1 > 8e$ . This contradicts (139). Now assume that Z=1. Then as above we obtain that ja > 4e. Hence as above we obtain that  $ac_1 > 4e$ . This contradicts (139). Thus Z=0. Hence we obtain that

(147) 
$$v = \{(j-2)a^2+8a\}e+(c_0+d_0-2)a^2+4a-1.$$

From (100), (104) and (147) we obtain that

(148) 
$$j(a^2-1)e+(c_0+d_0+1)(a^2-1)$$
  
= 4(e-1)e+2(a-1)(a-3)e+(3a-1)(a-1)

From (138) and (148) we obtain that

(149) 
$$j + \frac{c_0 + d_0 + 1}{e} = \frac{4(e-1)}{a^2 - 1} + \frac{2(a-3)}{a+1} + \frac{3a-1}{(a+1)e} + 4$$
.

By (104), (149) implies that

(150)  $1 \le j \le 3$ .

By (148) we may put

(151) 
$$4(e-1)e = (a-1)(He+I)$$
,

where H and / are rational integers such that H>0 and that  $e>I\geq 0$ . Then from (148) we obtain that

$$(152) \quad (c_0+d_0+1)(a+1) = (H-8)e + I - 4 + (2-j)e(a+1) + 3(a+1).$$

By (152) we may put

(153) 
$$(H-8)e+I-4 = J(a+1)$$
,

where J is a rational integer.

From (152) and (153) we obtain that

(154)  $c_0 + d_0 = J + (2-j)e + 2$ .

From (100), (104) and (154) we obtain that

(155)  $v = 4e^2 + 4e + J + 3$ .

Now from (102) it follows that

$$a(c_0-d_0+2) = c_0+d_0+2 \pmod{e}$$
.

So we may put

(156)  $(a-1)(a+2)-(a+1)d_0 = Ke$ ,

where K is a rational integer.

From (102), (133), (134) and (156) we obtain that

(157) 
$$a(2c_1+4-j)+K=4e+j+2.$$

From (102) and (104) it follows that

(158) 
$$v = (2c_1 + 4 - j)ae + ac_0 - ad_0 + 2a - 1$$

Hence from (147) and (158) we obtain that

(159) 
$$(2c_1-ja+2a-j-4) \in (a-1)(c_0-2)+(a+1)d_0$$

From (156) and (159) we obtain that

(160) 
$$(2c_1-ja+2a-j-4+K) = 2(a-1)c_0.$$

From (157) and (160) we obtain that

(161) 
$$(2e+c_1-c_1a-a-1)e=(a-1)c_0.$$

Similarly we obtain that

(162) 
$$(c_1a+c_1+3a-2e-ja-j-3)e=(a+1)d_0-2(a-1).$$

From (161) it follows that

(163) 
$$a = \frac{C-e}{C-2e^2+e}$$

Further from (154) and (162) it follows that

(164) 
$$J = C - 2e^2 + \frac{ae - 4e^2 - 5e - 4}{a + 1}$$
.

Hence from (155), (163) and (164) we obtain that

(165) 
$$v = \frac{C^2 + 2Ce + C - 2e^2 - 2e}{C - e^2}.$$

Now solve (70) in C and use (147), (149) and (154) to obtain

(166) 
$$(a-1)C = \{2ae - (a+1)\} e$$
.

Going back to (1) and (2) once again and using the definition of a and e, we obtain that

(167) 
$$k(k-1)(v-k)(v-k-1) = a^2 e(e+1)(v-2)(v-3)$$
.

From (50) and (167) we obtain that

(168) 
$$\frac{4(k-2)(v-k-2)}{k(v-k)} = \frac{(a^2-1)(v-3)}{a^2e(e+1)}$$

Subtracting 4 from both sides of (168) and using (147) and (154), we obtain that

$$\binom{(169)}{k(v-k)} = \frac{2ae+a-1}{a^2e(e+1)},$$

From (110) and (169) we obtain that

(170) 
$$(2ae+a-1)C+4ae(e+1)=a(e+1)v.$$

From (165) and (170) we obtain that

(171) 
$$(ae-1)C^2 - (2ae^3 + a - ae^2 - ae - e^2)C - 2ae(e+1)(2e^2 - e - \frac{1}{2} 0)$$

Substitute C in (171) by (166). Then we obtain that

(172) 
$$4a^{2}e^{4} - (4a^{3} + 8a)e^{3} + (7a^{2} + 4a - 1)e^{2} + (a^{3} - 7a^{2} + 5a - 1)e + (3a^{3} - 4a^{2} + a) = 0.$$

(172) allows the following factorization:

$$(e-a){4a^2e^3-8ae^2-(a^2-4a+1)e-(3a^2-4a+1)}=0$$
.

Since e > a, we obtain that

$$4a^2e^3 - 8ae^2 - (a^2 - 4a + 1)e - (3a^2 - 4a + 1) = 0$$
.

Since a and e are not so small and hence the first term of the left hand side is too large comparing with other terms, this is a contradiction.

This completes the proof.

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