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Osaka University

ON TIGHT 4-DESIGNS

To the memory to Otto Grün*

NOBORU ITO**

(Received September 12, 1974)

1. Introduction

Let v , k and λ be positive integers with $v > k$. Let X be a v -set and Jl a family of k -subsets of X . (X, Jl) is called a 4 -(v, k, λ) design (or simply a 4-design) if for each 4-subset T of X there exist precisely λ elements of Jl containing T . By a theorem of Fisher-Petrenjuk [2] the number of elements in \mathcal{A} is not less than $\frac{1}{2}v(v-1)$. If it is equal to $\frac{1}{2}v(v-1)$, (X, Jl) is called tight.

If $v \geq 6$ and if Jl is the family of all $(v-2)$ -subsets of X , (X, Jl) is a tight 4-design. Such tight 4-designs are called trivial.

Let (X, Jl) be a 4-design. If $v-k \geq 4$ and if $\mathcal{A}c$ is the family of $(v-k)$ -subsets of X each of which is a complement of an element of \mathcal{A} in X , (X, Jlc) is a 4-design. (X, Jl) and (X, Jlc) are called complementary with each other. Furthermore if (X, Jl) is tight, (X, Jlc) is also tight.

There exist only two known non-trivial tight 4-designs (X, Jl) they are a 4-(27, 7, 1) design and a 4-(23, 16, 52) design. They are complementary with each other. We call these designs Witt tight designs, because they are found by Witt [5], [6].

Now the purpose of this paper is to prove the following theorem.

Theorem. *Let (X, Jl) be a non-trivial tight 4-(v, s, λ) design. Then (X, Jl) is a Witt tight design.*

Our proof relies on the following theorem of Wilson and Ray-Chaudhuri [4]: Let (X, Jl) be a tight 4-(v, k, λ) design. Then a non-negative integer μ is called an intersection number of (X, \mathcal{A}) , if there exist two distinct elements A and B of Jl such that $|A \cap B| = \mu$. There exist precisely two intersection numbers, say, μ_1 and μ_2 , where $\mu_2 > \mu_1$. μ_1 and μ_2 are the roots of the polynomial

* After the author submitted this paper, he was informed that Dr. Otto Grün passed away on September 29, 1974 in Berlin.

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$$t^2 - \left\{ \frac{2(k-1)(k-2)}{v-3} \right\} t \quad / \quad \begin{matrix} 4 & \vee \\ v & \text{ft} - 3/ \end{matrix}$$

In particular,

$$(1) \quad \frac{2(k-1)(k-2)}{v-3} = x = \mu_2 + \mu_1 - 1$$

and

$$(2) \quad \frac{4\lambda}{k-3} = \mu_2\mu_1 - 2\lambda = y$$

are positive integers.

Furthermore, since (X, \mathcal{A}) is tight, we obtain that

$$(3) \quad 2\lambda(v-2)(v-3) = k(k-1)(k-2)(k-3)$$

2. Eigen-values of adjacency matrices

Let $X_j (j=1, \dots, v)$ and $A_j (j=1, \dots, \frac{1}{2}v(v-1))$ be the elements of X and \mathcal{A} respectively. Let I be the incidence matrix of (X, J) . So I is the matrix of size $(v, \frac{1}{2}v(v-1))$ whose (i, j) -component is either 1 or 0 according as x_i belongs to A_j or not. Let N_k be the adjacency matrix of (X, J) ($k=1, 2$). So the (i, j) -component of N_k is 1 or 0 according as $|A_i \cap A_j| = \mu_k$ or not ($k=1, 2$). Then we obtain that

$$(4) \quad J = E + N_1 + N_2,$$

$$(5) \quad I^t I = kE + \mu_1 N_2 + \mu_2 N_1$$

and

$$(6) \quad II^t = \frac{1}{2}k(v-k)E' + \frac{1}{2}k(k-1)J',$$

where E and E' denote the identity matrices of degree $\frac{1}{2}v(v-1)$ and v respectively, and J and J' are the matrices of degree $\frac{1}{2}v(v-1)$ and v with every component 1 respectively. Furthermore, by R. Noda [1], we have that

$$(7) \quad N_1 N_2 = N_2 N_1.$$

(7) is equivalent to the fact that the number of elements A_j of \mathcal{A} such that $|A_i \cap A_j| = \mu_k$ for a given element A_i of J is independent from the choice of A_i ($k=1, 2$). For a proof of this fact see [3].

From (6) we see that the eigen-value distribution of II^t consists of $\frac{1}{2}k^2(v-1)$ with multiplicity 1 and $\frac{1}{2}k(v-k)$ with multiplicity $v-1$. In particular, II^t is non-singular. Now there exists an orthogonal matrix U of degree $\frac{1}{2}v(v-1)$ such that IU has the following shape:

$$(8) \quad IU = (I^* 0),$$

where 0 denotes the zero matrix of size $(v, \frac{1}{2}v(v-3))$ and I^* is a non-singular matrix of degree v . From (8) it follows that

$$IUU^t I^t = II^t = I^* I^{*t}.$$

Hence IP and $I^* I^{*t}$ have the same eigen-value distribution. On the other hand, from (8) it also follows that

$$(9) \quad U^t I^t IU = \begin{pmatrix} I^{*t} I^* & 0 \\ 0 & 0 \end{pmatrix},$$

where 0's denote zero matrices. So from (9) we see that the eigen-value distribution of PI consists of $\frac{1}{2}k^2(v-1)$ with multiplicity 1, $\frac{1}{2}k(v-k)$ with multiplicity $v-1$ and 0 with multiplicity $\frac{1}{2}v(v-3)$.

Since N_1 and N_2 are commuting real symmetric matrices, there exists an orthogonal matrix V such that $U'^t N_1 U'$ and $U'^t N_2 U'$ are diagonal matrices:

$$U'^t N_1 U' = [\alpha_1, \dots, \alpha_{\frac{1}{2}v(v-1)}],$$

and

$$U'^t N_2 U' = [\beta_1, \dots, \beta_{\frac{1}{2}v(v-1)}].$$

Then by (4) and (5) we may assume that

$$(10) \quad 1 + \alpha_1 + \beta_1 = \frac{1}{2}v(v-1),$$

$$(11) \quad 1 + \alpha_i + \beta_i = 0 \quad \text{for } 2 \leq i \leq \frac{1}{2}v(v-1),$$

$$(12) \quad k + \mu_1 \alpha_1 + \mu_2 \beta_1 = \frac{1}{2}k^2(v-1),$$

$$(13) \quad k + \mu_1 \alpha_i + \mu_2 \beta_i = \frac{1}{2}k(v-k) \quad \text{for } 2 \leq i \leq v,$$

and

$$(14) \quad k + \mu_1 \alpha_j + \mu_2 \beta_j = 0 \quad \text{for } v+1 \leq j \leq \frac{1}{2}v(v-1).$$

Let A be a fixed element of \mathcal{A} . Let b_k be the number of elements of \mathcal{A} which intersect with A in μ_k elements of X ($k=1, 2$). Then we obtain that

$$(15) \quad \alpha_1 = b_1 \quad \text{and} \quad \beta_1 = b_2.$$

After M. Kano [1] we set

$$(16) \quad a = \mu_2 - \mu_1.$$

Now from (10)–(16) we obtain that

$$(17) \quad 1 + b_1 + b_2 = \frac{1}{2}v(v-1),$$

$$(18) \quad k + \mu_1 b_1 + \mu_2 b_2 = \frac{1}{2}k^2(v-1),$$

$$(19) \quad \alpha_i = \{-\frac{1}{2}k(v-k) + (k - \mu_2)\} / a, \\ \beta_i = \{\frac{1}{2}k(v-k) - (k - \mu_1)\} / a,$$

where $2 \leq i \leq v$, and

$$(20) \quad \begin{aligned} \alpha_j &= (k - \mu_2)/a, \\ \beta_j &= -(k - \mu_1)/a, \end{aligned}$$

where $v+1 \leq j \leq \frac{1}{2}v(v-1)$.

Since the α_i and the β_i are algebraic integers, the numbers in (19) and (20) are rational integers. Put

$$(21) \quad e = (k - \mu_2)/a.$$

Now we show that there exist two non-negative rational integers λ_1 and λ_2 such that

$$(22) \quad N_1^2 = b_1 E + \lambda_1 N_1 + \lambda_2 N_2.$$

In fact, it suffices to show the existence of two rational numbers λ_1 and λ_2 which satisfy (22). Then it suffices to solve

$$(23) \quad \begin{cases} \alpha_2^2 = b_1 + \lambda_1 \alpha_2 + \lambda_2 \beta_2 \\ \alpha_{v+1}^2 = b_1 + \lambda_1 \alpha_{v+1} + \lambda_2 \beta_{v+1} \end{cases}$$

in λ_1 and λ_2 . For, since by (4) the trace of N_1^2 is equal to $\frac{1}{2}b_1v(v-1)$, we obtain (22). In particular, we obtain that

$$(24) \quad b_1^2 = b_1 + \lambda_1 b_1 + \lambda_2 b_2.$$

By (11) and (23) we see that α_2 and α_{v+1} are the roots of the equation

$$X^2 - (\lambda_1 - \lambda_2)X - (b_1 - \lambda_2) = 0.$$

So we obtain that

$$(25) \quad \alpha_2 + \alpha_{v+1} = \lambda_1 - \lambda_2 \quad \alpha_2 \alpha_{v+1} = -(b_1 - \lambda_2)$$

Since, by (19) and (20), $-\alpha_2 > \alpha_{v+1} > 0$, by (25) we obtain that

$$(26) \quad \lambda_2 > \lambda_1.$$

Since α_2 and α_{v+1} are rational integers,

$$(27) \quad d = (\lambda_1 - \lambda_2)^2 + 4(b_1 - \lambda_2)$$

is a square. Further we obtain that

$$(28) \quad \alpha_2 = \frac{1}{2}(\lambda_1 - \lambda_2 - \sqrt{d}) \quad \alpha_{v+1} = \frac{1}{2}(\lambda_1 - \lambda_2 + \sqrt{d}).$$

From (20), (21) and (28) we obtain that

$$(29) \quad \sqrt{d} = \lambda_2 - \lambda_1 + 2e.$$

By (1), (16) and (21) we obtain that

$$(30) \quad 2k = x + 1 + a(2e + 1).$$

Put

$$(31) \quad v = \lambda_2 - \lambda_1.$$

Eliminate b_2 from (10), (15) and (24). Then we obtain that

$$(32) \quad b_1^2 + (v-1)b_1 = \frac{1}{2}\lambda_2(v+1)(v-2).$$

From (27), (29) and (31) we obtain that

$$(33) \quad b_1 = ev + e^2 + \lambda_2.$$

Since the trace of N_1 is equal to 0, from (15), (19), (20), (21) and (28) it follows that

$$(34) \quad b_1 = (v-1)v - \frac{1}{2}(v^2 - 5v + 2)e$$

Eliminate b_2 from (10), (12) and (15). In the resulting equation express μ_1 and μ_2 in terms of a , e and k using (16) and (21). Then we obtain that

$$(35) \quad ab_1 = \frac{1}{2}(v-1)k(v-k) - \frac{1}{2}ea(v+1)(v-2)$$

Eliminate λ_2 and b_1 from (32), (33) and (34). Then after canceling $v(v-1)$ we obtain that

$$(36) \quad v^2 - \{(v-5)e + \frac{1}{2}v-1\}v + ti(v^2 - 9v + 22)e + \pm (v^2 - 5v + 2)e = 0.$$

From (36) we obtain that

$$(37) \quad \{2v - ev - e^2 + 4e\}\{(e+1)v - 2v - e^2 - 6e - 1\} \\ = (e+2)(e+1)e(e-1) = (e^2 + e - 2)(e^2 + e).$$

3. Cases with small e

It is convenient to eliminate cases with small e first. For this purpose (37) can play a rather useful role. First we show that both terms of the left-hand side of (37) are positive for $e > l$. Deny this. Then the sum of the negatives of both terms of the left-hand side of (37) is smaller than that of the right-hand side of (37). Since

$$(e^2 + e) - (e^2 + e - 2) = 2,$$

we have the following three cases:

$$(i) \quad ev + e^2 - 2v - 4e = 2v + e^2 + 6e + 1 - (e+1)v + 1,$$

$$(ii) \quad ev + e^2 - 2v - 4e = 2v + e^2 + 6e + 1 - (e+1)v,$$

and

$$(iii) \quad ev + e^2 - 2v - 4e = 2v + e^2 + 6e + 1 - (e+1)v - 1.$$

Case (i): We have that $4v = (2e+1)v - 10e - 2$. So from (37) we obtain that

$$(38) \quad v^2 - 2(2e^2 + 2e + 1)v + 12(e^2 + e) = 0.$$

Put $c = e^2 + e$. Then $c \geq 6$, because $e \geq 2$. From (38) it follows that

$$\{v - (2c+1)\}^2 = 4c^2 - 8c + 1.$$

So put $4c^2 - 8c + 1 = f^2$, where f is a positive integer. Then we obtain that

$$(39) \quad 2c - f = \frac{8c-1}{2c+f}.$$

Hence the right-hand side of (39) does not exceed 3. If it equals 1, it leads to the absurdity that $6c=1$. If it equals 2, it leads to the absurdity that $3=0$. If it equals 3, then we get that $c=2$, which is against the assumption.

Case (ii) and (iii) can be handled with in the similar manner as in case (i).

Case where $e=1$. We notice that this is the case for all trivial designs. From (37) we obtain that

$$(40) \quad (2v - v + 3)(v - v - 4) = 0.$$

If $v = \frac{1}{2}(v-3)$, then from (34) and (33) we obtain that $b_1 = \frac{1}{2}(v+1)$, and that $\lambda_2 = 1$. Then $\lambda_1 = 0$. From (31) $v=1$. Hence $v=5$, which is against the assumption. Thus we obtain that $v=v-4$. Then from (34) and (33) we obtain that

$$(41) \quad b_1 = \frac{1}{2}(v-2)(v-3),$$

and that

$$(42) \quad \lambda_2 = \frac{1}{2}(v-3)(v-4).$$

From (17) $b_2 = 2(v-2)$. Thus from (18) we obtain that

$$(43) \quad \mu_1(v-2) = k(k-2).$$

From (43) and (3) it follows that

$$(44) \quad 2\lambda(v-3) = \mu_1(k-1)(k-3).$$

From (44) and (1) we obtain that

$$(45) \quad 4\lambda(k-2) = \mu_1 x(k-3).$$

Using (21), from (45) and (2) we obtain that

$$\mu_2 + 1 = a(\mu_2 + a),$$

which implies that $a=1$. Now (35) implies that $k=v-2$. Thus (X, \mathcal{A}) is trivial.

Case where $e=2$. We notice that this is the case for Witt tight designs. From (37) we obtain that

$$(46) \quad (v-v+2)(3v-2v-17) = 12.$$

We let the first factor of the left-hand side of (46) run all positive divisors of 12 from 1 to 12. It gives us the values of v and v in each case. Then we can calculate the values of remaining parameters. Anyway we have the following six cases: $v=27; v=23, v=23, b_1=90, b_2=162; v=23, v=24, b_1=112, b_2=140; v=24; v=27; v=38$.

We may assume that $k \leq \frac{1}{2}v$. Then using (1) and (3) we can eliminate all but the second and third cases with $k=7$. Then $a=2, \mu_2=3$ and $\mu_1=1$. A Witt tight design yields the third case and the uniqueness is known. The second case is eliminated by (18).

Cases where $3 \leq e \leq 6$. From now on we are expecting no designs. Furthermore the checking procedure is just the same as in the above case. The following two lemmas are sufficient to take care of all cases where $3 \leq e \leq 6$.

Lemma 1. *k cannot be a prime for $e > 2$.*

Proof. From (1), (2) and (3) we obtain that

$$(47) \quad y(v-2) = kx.$$

(i) The case where k divides $v-2$. Put $v-2=ck$, where c is a positive integer. Since (X, \mathcal{A}) is non-trivial, $c \geq 2$. From (1) it follows that

$$(48) \quad 2(k-1)(k-2) = x(ck-1).$$

Thus $x=dk-4$, where d is a positive integer. So from (48) we obtain that

$$\left(c - \frac{2}{d}\right)k = \frac{4c}{d} + 1 - \frac{6}{d}.$$

If $d \geq 4$, then $(c - \frac{1}{2})k < c + 1$. Since $k \geq 4$, this is a contradiction. If $d=3$ or 2, we obtain a similar contradiction. If $d=1$, then

$$k = 4 + \frac{3}{c-2}.$$

So $c=3$ or 5 . If $c=3$, then $k=7$ and $v=23$. From (17) and (18) it follows that $\mu_1=2$. Since $e \geq 3$, from (21) it follows that $a=1$. But the case $a=1$ can occur only for trivial designs. In fact, from (1), (2), (3) and (16) it follows that

$$(49) \quad a^2 = (\mu_2 - \mu_1)^2 = (\mu_2 + \mu_1)^2 - 4\mu_2\mu_1 - \frac{4(k-1)(k-2)(v-k-1)(v-k-2)}{(v-2)(v-3)^2} + 1$$

(49) implies that

$$(50) \quad (a^2 - 1)(v-2)(v-3)^2 = 4(k-1)(k-2)(v-k-1)(v-k-2)$$

If $a=1$, (50) implies that $k=v-2$. Then we have the trivial designs. If $c=5$, then $k=5$ and $v=27$. But then (3) shows that λ cannot be an integer.

(ii) The case where k and $v-2$ are relatively prime. By (47) $v-2$ divides x . So put $x=zs(v-2)$. From (1) it follows that

$$2(k-1)(k-2) = z(v-2)(v-3).$$

Since $v-2 > k$, we obtain that $z=1$. Thus

$$(51) \quad x = v-2 \quad \text{and} \quad y = k.$$

Now the equation

$$2(k-1)(k-2) = (v-2)(v-3)$$

can be rewritten as a Pellian equation

$$(2v-5)^2 - 2(2k-3)^2 = -1.$$

The solutions of the Pellian equation

$$Y^2 - 2Z^2 = -1$$

are recursively given by

$$(52) \quad (Y_1, Z_1) = (1, 1)$$

and

$$(53) \quad \begin{cases} Y_{i+1} = 3Y_i + 4Z_i \\ Z_{i+1} = 2Y_i + 3Z_i \end{cases} \quad (i = 1, 2, \dots).$$

So there exists an integer m such that

$$(54) \quad (Y_m, Z_m) = (2v-5, 2k-3).$$

By (2), (51) and (54) we obtain that

$$(55) \quad 16\lambda = 4k^2 - 12k = Z_m^2 - 9.$$

(55) implies that

$$(56) \quad Z_m = 3, 5, -5 \text{ or } -3 \pmod{16}.$$

On the other hand, from (52) and (53) we obtain that

$$(57) \quad (Y_l, Z_l) \pmod{16} = \begin{cases} (1, 1) & \text{for } l \equiv 1 \pmod{8}, \\ (7, 5) & \text{for } l \equiv 2 \pmod{8}, \\ (-7, -3) & \text{for } l \equiv 3 \pmod{8}, \\ (-1, -7) & \text{for } l \equiv 4 \pmod{8}, \\ (1, -7) & \text{for } l \equiv 5 \pmod{8}, \\ (7, -3) & \text{for } l \equiv 6 \pmod{8}, \\ (-7, 5) & \text{for } l \equiv 7 \pmod{8}, \\ (-1, 1) & \text{for } l \equiv 0 \pmod{8}. \end{cases}$$

From (56) and (57) it follows that $Z_m = 2k - 3 = 5 \text{ or } -3 \pmod{16}$. This implies that k is even.

Lemma 2. *The greatest common divisor of $v - 2$ and $k - 3$ divides 3.*

Proof. Let c denote the number of blocks containing three distinct points of X . Then since (X, \mathcal{A}) is tight, we obtain that

$$2c(v - 2) = k(k - 1)(k - 2).$$

Thus the greatest common divisor of $v - 2$ and $k - 3$ divides that of $2k$ and $k - 3$. By (18) if v is even, then k is even.

REMARK. Perhaps the case where $e = 3$, $v = 47$ and $k = 12$ is slightly beyond the powers of Lemmas 1 and 2. In that case, from (1) it follows that $\mu_2 + \mu_1 = 6$. Hence $2\mu_2 - a = 6$. By (21) $12 - 3a = \mu_2$. Hence $7a = 8$. This is absurd.

From now on we may assume that $e \geq 7$.

4. Case where $a = 2$

This case has been already treated by M. Kano in a similar way [1]. We may assume that $k \leq \frac{1}{2}v$. Then from (16) and (50) it follows that

$$(58) \quad \frac{2(a^2 - 1)(v - 3)}{v - 4} > \frac{2(k - 1)(k - 2)}{v - 3} = x = a - 1 + 2\mu_1.$$

Since $v \geq k+3 = ae+a+\mu_1+3 \geq 20$, if $a=2$, then (58) implies that $43 > 16\mu_1$. Thus $\mu_1 \leq 2$. Then (2) implies the contradiction that $k \leq 10$.

5. e -adic expansions of v and related parameters

From (34) it follows that

$$2v = e(v-4) + \frac{2b_1 - 2e}{v-1}.$$

So further by (33) $\frac{2b_1 - 2e}{v-1}$ is a positive rational integer. Put

$$(59) \quad C = \frac{2b_1 - 2e}{v-1}.$$

Then

$$(60) \quad 2v = e(v-4) + C.$$

Substituting (60) in (37) we obtain that

$$(61) \quad v = C + (e+1)^2 + \frac{(e+2)(e+1)e(e-1)}{C-e^2}.$$

The numerator of the third term of the right-hand side of (61) equals

$$e^4 + e^3 + (e-2)e^2 + (e-2)e.$$

Now express C in the e -adic form

$$(62) \quad C = c_4e^4 + c_3e^3 + c_2e^2 + c_1e + c_0,$$

where $0 \leq c_i < e$ ($i=0, 1, 2, 3, 4$).

Since the first factor of the left-hand side of (37) is positive, by (33) we can easily see that $C > e^2$. So the third term of the right-hand side of (61) is a positive integer. We put

$$(63) \quad e^4 + e^3 + (e-2)e^2 + (e-2)e = (C-e^2)D,$$

where

$$D = v - C - (e+1)^2 = d_4e^4 + d_3e^3 + d_2e^2 + d_1e + d_0$$

with $0 \leq d_i < e$ ($i=0, 1, 2, 3, 4$).

First we show that $c_4=0$ in (62). In fact, clearly we have that $c_4 \leq 1$. If $c_4=1$, then $D=1$. So from (61) and (63) we obtain that

$$(64) \quad v = e^4 + 2e^3 + e^2 + 2.$$

Then from (59) it follows that

$$(65) \quad 2b_1 = e^8 + 4e^7 + 4e^6 + (e-1)e^5 + (e-3)e^4$$

Now from (35) it follows that

$$(66) \quad (v-1)k^2 - v(v-1)k + a\{2b_1 + e(v+1)(v-2)\} = 0.$$

Further from (1) and (30) it follows that

$$(67) \quad 2k^2 - 2vk + (v+1) + a(2e+1)(v-3) = 0.$$

So from (66) and (67) we obtain that

$$2a\{2b_1 + e(v+1)(v-2)\} = (v+1)(v-1) + a(2e+1)(v-3)(v-1),$$

which implies that

$$(68) \quad a\{4b_1 + 2e(3v-5) - (v-3)(v-1)\} = (v+1)(v-1).$$

From (64), (65) and (68) we obtain that

$$a(e^4 + 2e^3 - e^2 + 2e + 1) = e^4 + 2e^3 + e^2 + 3.$$

So $a \equiv 3 \pmod{e}$. This implies that $a=3$ and that $2e^4 + 4e^3 - 4e^2 + 6e = 0$. This is a contradiction. Hence we obtain that $c_4 = 0$.

Secondly we show that $c_3 = 0$ in (62). If $c_3 \geq 2$, then we obtain that $D < e$. So from (61) we obtain that

$$(69) \quad v = c_3e^3 + (c_2+1)e^2 + (c_1+2)e + c_0 + 1 + D$$

Now from (59) and (68) we obtain that

$$(70) \quad a(2C + 6e - v + 3) = v + 1.$$

Then from (69) and (70) it follows that

$$2C + 6e - v + 3 = c_3e^3 + (c_2-1)e^2 + (c_1+4)e + c_0 + 2 - D.$$

Since $a \geq 3$, this implies that

$$\begin{aligned} & 2c_3e^3 + 2(c_2-1)e^2 + 2(c_1+4)e + 2c_0 + 4 - 2D - 1 \\ & \leq e_3e^3 + (c_2+1)e^2 + (c_1+2)e + c_0 + 1 + D, \end{aligned}$$

which is clearly a contradiction. So we obtain that $c_3 \leq 1$. Now assume that $c_3 = 1$. If $c_2 \geq 2$, then $D \leq e$. So from (61) and (70) we obtain that

$$v = e^3 + (c_2+1)e^2 + (c_1+2)e + (c_0+1) + D$$

and that

$$\begin{aligned} 2C+6e-v+3 &= \frac{v+1}{a} \\ &= e^3+(c_2-1)e^2+(c_1+4)e+c_0+2-D. \end{aligned}$$

Since $a \geq 3$, this implies that

$$\begin{aligned} &2e^3+2(c_2-1)e^2+2(c_1+4)e+2(c_0+2)-2D-1 \\ &\leq e^3+(c_2+1)e^2+(c_1+2)e+(c_0+1)+D, \end{aligned}$$

which is clearly a contradiction. If $c_2=1$, then $D \leq e+1$. This leads to a contradiction as above. If $c_2=0$, then $D \leq e+3$. Since $a \geq 3$, we get a contradiction as above. Hence we obtain that $c_3=0$.

Thirdly we show that $c_2=2$ in (62). If $c_2 \geq 4$, then $D < e^2$. From (61) and (70) we obtain that

$$v = (c_2+1)e^2+(c_1+2)e+(c_0+1)+D$$

and that

$$\begin{aligned} 2C+6e-v+3 &= \frac{v+1}{a} \\ &= (c_2-1)e^2+(c_1+2)e+(c_0+2)-D. \end{aligned}$$

Since $a \geq 3$, this implies that

$$\begin{aligned} &3(c_2-1)e^2+3(c_1+4)e+3(c_0+2)-3D-1 \\ &\leq (c_2+1)e^2+(c_1+2)e+c_0+1+D, \end{aligned}$$

which is clearly a contradiction. Hence we obtain that $c_2 \leq 3$. If $c_2=1$, then $C-e^2=c_1e+c_0$. From (61) we obtain that

$$v = 2e^2+(c_1+2)e+c_0+1+D$$

and that

$$(71) \quad 2C+6e+3-v = (c_1+4)e+c_0+2-D.$$

Since $c_1e+c_0 \leq e^2-1$, from (63) we obtain that

$$D \geq e^2+2e.$$

Since (71) is positive, we obtain that $c_1 \geq e-3$. If $c_1=e-1$, then

$$(72) \quad v = 3e^2+e+c_0+1+D$$

and

$$(73) \quad 2C+6e+3 = 2e^2+4e+2c_0+3.$$

Since (72) is smaller than (73), this is a contradiction. If $c_1=e-2$, then

$$v = 3e^2 + c_0 + 1 + D$$

and

$$2C + 6e + 3 = 2e^2 + 2e + 2c_0 + 3.$$

As above this is a contradiction. If $c_1=e-3$, then

$$v = 2e^2 + (e-1)e + c_0 + 1 + D$$

and

$$2C + 6e + 3 = 2e^2 + 2c_0 + 3.$$

As above this is a contradiction. Hence we obtain that $c_2=3$ or $c_2=2$. If $c_2=3$, then from (63) and (61) we obtain the following equations:

$$\begin{aligned} (74) \quad & (2e^2 + c_1e + c_0)(d_2e^2 + d_1e + d_0) \\ & = e^4 + e^3 + (e-2)e^2 + (e-2)e; \\ & v = (4 + d_2)e^2 + (c_1 + 2 + d_1)e + c_0 + 1 + d_0; \\ & 2C + 6e + 3 - v \\ & = (2 - d_2)e^2 + (c_1 + 4 - d_1)e + c_0 + 2 - d_0. \end{aligned}$$

Further from (70) we obtain that

$$\begin{aligned} (75) \quad & a\{(2 - d_2)e^2 + (c_1 + 4 - d_1)e + c_0 + 2 - d_0\} \\ & = (4 + d_2)e^2 + (c_1 + 2 + d_1)e + c_0 + 2 + d_0. \end{aligned}$$

Now from (74) we obtain that $d_2=0$. If $a \geq 5$, from (75) we obtain that

$$\begin{aligned} & 10e^2 + 5(c_1 + 4 - d_1)e + 5(c_0 + 2 - d_0) \\ & \leq 4e^2 + (c_1 + 2 + d_1)e + c_0 + 2 + d_0, \end{aligned}$$

which implies that

$$6e^2 + (4c_1 + 18)e + (4c_0 + 8) \leq 6d_1e + 6d_0 \leq 6e^2 - 6.$$

This is a contradiction. If $a=4$, as above we obtain that

$$(76) \quad 4e^2 + (3c_1 + 14)e + (3c_0 + 6) = 5d_1e + 5d_0.$$

On the other hand, from (74) we obtain that

$$\begin{aligned} & 2d_1e^3 + (2d_0 + c_1d_1)e^2 + (c_1d_0 + c_0d_1)e + c_0d_0 \\ & = e^4 + e^3 + (e-2)e^2 + (e-2)e, \end{aligned}$$

which implies that $2d_1 \leq e + 1$. But then we obtain that

$$\begin{aligned}
& 5d_1e + 5d_0 \\
& \leq 5d_1e + 5(e-1) \\
& = 5(d_1+1)e - 5 \\
& \leq 5\left\{\frac{1}{2}(e+1)+1\right\}e - 5 \\
& = 3e^2 + \frac{1}{2}(15-e)e - 5.
\end{aligned}$$

Since $e \geq 7$, this contradicts (76). If $a=3$, as above we obtain the following equation:

$$2e^2 + (2c_1 + 10)e + 2c_0 + 5 = 4d_1e + 4d_0,$$

which is clearly a contradiction. Hence we obtain that $c_2 = 2$.

Now we have the following equations:

$$(77) \quad (e^2 + c_1e + c_0)(d_2e^2 + d_1e + d_0) \\ = e^4 + e^3 + (e-2)e^2 + (e-2)e;$$

$$(78) \quad v = (3+d_2)e^2 + (c_1+2+d_1)e + c_0 + 1 + d_0;$$

and

$$(79) \quad 2C + 6e + 3 - v = (1-d_2)e^2 + (c_1+4-d_1)e + c_0 + 2 - d_0.$$

Hence we obtain that either $d_2=1$ or 0.

There seems to be a rather big difference between the cases $d_2=1$ and $d_2=0$.

First we assume that $d_2=1$. Then from (77) it follows that

$$(80) \quad (c_1+d_1)e^3 + (c_0+c_1d_1+d_0)e^2 + (c_0d_1+c_1d_0)e + c_0d_0 \\ = e^3 + (e-2)e^2 + (e-2)e.$$

Hence we obtain that $c_1+d_1 \leq 1$.

Case (i) where $c_1=d_1=0$. If $c_0=d_0=e-1$, then $e=1$. Hence either $c_0 \leq e-2$ or $d_0 \leq e-2$. So we obtain that

$$(2e-3)e^2 + (e-1)(e-2) \geq e^3 + (e-2)e^2 + (e-2)e,$$

which is a contradiction.

Case (ii) where $c_1=0$ and $d_1=1$. From (80) it follows that

$$(81) \quad (c_0+d_0)e^2 + c_0e + c_0d_0 = (e-2)e^2 + (e-2)e.$$

Put $c_0+d_0=e-l$ and $c_0d_0=em$. Then $l \geq 2$ and $e-3 \geq m \geq 0$. Then from (81) we obtain that

$$(e-l)e + c_0 + m = (e-2)e + e - 2.$$

Now we have that $2e > c_0 + m + 2 = (l-1)e \geq e$. Hence $l=2$, $c_0+m+2=e$ and

$c_0 + d_0 + 2 = e$. So $m = d_0$. Thus $d_0 = 0$ and $c_0 = e - 2$. Then from (78) and (70) we obtain that

$$(82) \quad v = 4e^2 + 4e - 1,$$

and that

$$(83) \quad a = e + l.$$

Now from (67), (82) and (83) we obtain that

$$k^2 - (4e^2 + 4e - 1)k + 2(e + 1)^3(2e - 1) = 0.$$

Since

$$(4e^2 + 4e - 1)^2 - 8(e + 1)^3(2e - 1) = -8e^3 - 16e^2 + 9$$

is negative, this is a contradiction.

Case (iii) where $c_1 = 1$ and $d_1 = 0$. From (80) it follows that

$$(84) \quad (c_0 + d_0)e^2 + d_0e + c_0d_0 = (e - 2)e^2 + (e - 2)e.$$

Put $c_0 + d_0 = e - l$ and $c_0d_0 = me$. Then $l \geq 2$ and $e - 3 \geq m \geq 0$. Then from (84) it follows that

$$(e - l)e^2 + d_0e + me = (e - 2)e^2 + (e - 2)e.$$

Thus we obtain that

$$(e - l)e + d_0 + m = (e - 2)e + e - 2.$$

Now we have that $2e > d_0 + m + 2 = (l - 1)e \geq e$. Hence $l = 2$, $c_0 + d_0 = e - 2$ and $d_0 + m = e - 2$. So $m = c_0$. Thus $c_0 = 0$ and $d_0 = e - 2$. Then from (78) and (70) we obtain that

$$(85) \quad v = 4e^2 + 4e - 1,$$

and that

$$(86) \quad a = e.$$

Now from (67), (85) and (86) we obtain that

$$(87) \quad k^2 - (4e^2 + 4e - 1)k + 2e^3(2e + 3) = 0.$$

Hence $8e^3 + 8e^2 - 8e + 1$ is a square. Thus we are confronted with a Diophantine equation

$$(88) \quad Y^2 = 8X^3 + 8X^2 - 8X + 1.$$

It is easy to solve (88) with the restriction $X \not\equiv 2 \pmod{3}$.

Lemma 3. *The integral solutions of (88) with $X \not\equiv 2 \pmod{3}$ are the following: (i) $X=0$, $Y=\pm 1$ and (ii) $X=1$, $Y=\pm 3$.*

Proof. We start with

$$(89) \quad Y^2 = (2X-1)(4X^2+6X-1).$$

If X is negative, from (89) we obtain that $-6X \geq 4X^2-1$, which implies that $X \geq -1$. So we may assume that X is non-negative. For $X=0$ and 1 we obtain the solutions (i) and (ii) respectively. Hence we may assume that $X \geq 3$. Since

$$4X^2+6X-1 = (2X-1)(2X+4)+3,$$

$2X-1$ and $4X^2+6X-1$ are relatively prime by assumption. Hence we may put

$$(90) \quad 2X-1 = R^2 \quad \text{and} \quad 4X^2+6X-1 = S^2,$$

where R and S are positive integers. Then $S > 2X$ and $S < 2X+2$. Hence $S=2X+1$, which implies that $X=1$ against the assumption.

For $X \leq 14$ we obtain the further solutions (iii) $X=-1$, $Y=\pm 3$; (iv) $X=2$, $Y=\pm 9$ and (v) $X=14$, $Y=\pm 153$.

REMARK 1. We owe to Dr. Jeffrey Leon the following fact that the solution (v) is the only solution of (88) in the interval $7 \leq X \leq 25000$.

If $e=14$, then $v=839$. Further we may assume that $k=343$. But these parameters do not satisfy (3). Hence we may assume that $e \geq 15$.

We may put

$$(91) \quad 2e-1 = 3R^2,$$

where R is an integer bigger than 3. Now we show the non-existence of tight 4-designs with parameters (85), (86), (87) and (91).

Assume that $2e-1$ has a prime divisor p bigger than 3. Then from (87) we obtain that

$$(92) \quad k \equiv 1 \pmod{p}.$$

Further from (16), (21), (86) and (92) we obtain that

$$\mu_2 \equiv 1-e^2 \pmod{p},$$

and that

$$\mu_1 \equiv 1-e-e^2 \pmod{p}.$$

If $e^2 \equiv 1 \pmod{p}$, then $4e^2 \equiv 4 \equiv 1 \pmod{p}$, which implies that $p = 3$. If $e^2 + e \equiv 1 \pmod{p}$, then $4e^2 + 4e \equiv 4 \equiv 3 \pmod{p}$, which is absurd. Hence we obtain that

$$(93) \quad \mu_2 \mu_1 \not\equiv 0 \pmod{p}.$$

On the other hand, from (2) we obtain that

$$4\lambda = (k-3)(\mu_2 \mu_1 - 2\lambda) = -2(\mu_2 \mu_1 - 2\lambda) \pmod{p},$$

which implies that

$$(94) \quad \mu_2 \mu_1 = 0 \pmod{p}.$$

Obviously (93) and (94) are in contradiction.

Hence we obtain that

$$2e - 1 = 3^E,$$

where, by (91), E is an odd integer bigger than 3. Now from (89) and (91) we obtain that

$$(95) \quad F^2 = 3^{2E-1} + 3^E + 2 \cdot 3^{E-1} + 1$$

is a square, where F is a positive integer. From (95) we obtain that either

$$(96) \quad F - 1 = G3^{E-1},$$

or

$$(97) \quad F + 1 = G3^{E-1},$$

where G is a positive integer. If (96) occurs, then

$$F^2 - 1 = G^2 3^{2E-2} + 2G3^{E-1}.$$

Since G is clearly bigger than 1, we obtain that

$$F^2 - 1 \geq 4 \cdot 3^{2E-2} + 4 \cdot 3^{E-1} > 3^{2E-1} + 3^{2E-2}.$$

Since $E \geq 5$, this is a contradiction. If (97) occurs, then

$$(98) \quad F^2 - 1 = G^2 3^{2E-2} - 2G3^{E-1}.$$

If $G \geq 3$, then

$$G^2 3^{2E-2} - 2G3^{E-1} > 2 \cdot 3^{2E-1},$$

which contradicts (95). Hence $G = 2$. Then (95) and (98) imply that $E = 1$. This is against assumption.

REMARK 2. We are informed that Dr. Koichi Yamamoto, meantime, has

solved (88) confirming that (i)–(v) are the only solutions. His proof seems to be not entirely elementary.

6. Relation between ν and k

From now on we assume that $d_2=0$. By (77), (78), (79) and (70) we can start with the following equations:

$$(99) \quad (e^2+c_1e+c_0)(d_1e+d_0) = e^4+e^3+(e-2)e^2+(e-2)e;$$

$$(100) \quad \nu = 3e^2+(c_1+d_1+2)e+c_0+d_0+1;$$

$$(101) \quad 2C+6e+3-\nu = e^2+(c_1-d_1+4)e+(c_0-d_0+2),$$

where $C=2e^2+c_1e+c_0$;

$$(102) \quad \begin{aligned} & a\{e^2+(c_1-d_1+4)e+c_0-d_0+2\} \\ &= 3e^2+(c_1+d_1+2)e+c_0+d_0+2 \\ &= \nu+1. \end{aligned}$$

From (99) it follows that

$$(103) \quad \begin{aligned} & d_1e^3+(c_1d_1+d_0)e^2+(c_0d_1+c_1d_0)e+c_0d_0 \\ &= e^4+e^3+(e-2)e^2+(e-2)e. \end{aligned}$$

First we show that

$$(104) \quad c_1+d_1 - e+j \quad \text{for some } j \text{ with } e-2 \geq j > 0.$$

To show this put $d_1=e-A$ and $d_0=e-B$. Then $A \geq 1$ and $B \geq 1$. From (103) it follows that

$$\begin{aligned} & (c_1-A+1)e^3+(c_0+c_1-B-c_1A)e^2+(c_0-c_1B-c_0A)e-c_0B \\ &= e^3+(e-2)e^2+(e-2)e. \end{aligned}$$

This implies that $c_1-A+1 > 1$. Hence $c_1+d_1 > e$.

Secondly we show that

$$(105) \quad d_1 < e-1.$$

Assume that $d_1=e-1$. Then from (103) it follows that

$$(106) \quad \begin{aligned} & c_1e^3+(c_0+d_0+1)e^2+(c_1d_0+2)e+c_0d_0 \\ &= 3e^3+c_1e^2+c_0e. \end{aligned}$$

Hence we obtain that $c_1 \leq 3$. By (104) $c_1 \geq 2$. If $c_1=3$, then from (106) it follows that

$$(c_0 + d_0)e^2 + (3d_0 + 2)e + c_0d_0 = 2e^2 + c_0e.$$

If $c_0 \leq 1$, then we obtain a contradiction that $e=5$. If $c_0=2$, then $d_0=0$. Then from (102) it follows that

$$a(2e+1) = e^2 + e + 1.$$

This implies that $a \equiv 1 \pmod{e}$. Then $a=1$. This is against assumption. If $c_1=2$, then from (106) it follows that

$$(107) \quad (c_0 + d_0)e^2 + (2d_0 + 2)e + c_0d_0 = e^3 + e^2 + c_0e.$$

If $c_0=0$, then $d_0=e-1$. Then from (102) and (100) it follows that

$$3a = 2e + 1$$

and that

$$v = 4e(e+1).$$

Further from (67) it follows that

$$3k^2 - 8e(e+1)k + 2e(e+1)(2e+1)^2 = 0.$$

Since

$$64e^2(e+1)^2 - 24e(e+1)(2e+1)^2 = 8e(e+1)(-4e^2 - 4e - 3)$$

is negative, this is a contradiction. Hence $c_0 > 0$. If $d_0=0$, then $c_0 > e$. This contradiction shows that $c_0d_0 > 0$. Put $c_0d_0 = Ae$ and $2d_0 + 2 + A - c_0 = Be$. Then $0 < A < e - 3$ and $0 \leq B \leq 2$. Further we have that

$$(108) \quad c_0 + d_0 + B = e + 1.$$

If $B=0$, then by (107) and (108) we obtain that

$$(109) \quad e^2 - (4d_0 + 1)e + d_0(d_0 - 1) = 0.$$

Hence we consider the function $f(X) = X^2 - (4d_0 + 1)X + d_0(d_0 - 1)$. Since $d_0 > 1$, $f(0) = d_0(d_0 - 1) > 0$ and $f(d_0) = -2d_0^2 - 2d_0 < 0$. Further we obtain that $f(4d_0) = d_0^2 - 5d_0$. If $d_0=5$, then $e=20$ and $c_0=16$. Then from (102) we obtain that $a=11$. From (100) we obtain that $v=1682$. Further from (101) we obtain that $C=856$. Now from (59) and (66) we obtain that

$$(110) \quad k(v-k) = a(C+ev).$$

With data above (110) implies that

$$k^2 - 1682k + 379456 = 0.$$

But then k cannot be integral. If $d_0 \leq 4$, (109) has no integral solution. Hence we obtain that $d_0 > 5$ and that $e > 4d_0$. But $f(4d_0+1) = d_0(d_0-1) > 0$. This is a contradiction. If $B=1$, we can follow the above argument to get a contradiction. If $B=2$, we obtain that $d_0 = e-1$. This is a contradiction.

Thirdly we observe that

$$(111) \quad c_1 \geq +2 \geq 3.$$

In fact, $c_1 + d_1 = e + j$ with $j > 0$. Hence by (105) $c_1 + e - 2 \geq e + j$.

Now we show that

$$(112) \quad e > 2a.$$

Assume not. Then from (102) it follows that

$$\begin{aligned} & e \{e^2 + (c_1 + 4 - d_1)e + c_0 + 2 - d_0\} \\ & \leq 6e^2 + (2c_1 + 4 + 2d_1)e + 2c_0 + 4 + d_0, \end{aligned}$$

which implies that

$$\begin{aligned} (113) \quad & e^3 + c_1 e^2 + c_0 e \\ & \leq (d_1 + 2)e^2 + (2c_1 + 2 + 2d_1 + d_0)e + 2c_0 + 4 + d_0 \\ & \leq (d_1 + 7)e^2 + 1. \end{aligned}$$

This implies that

$$(114) \quad d_1 + 7 \geq e + c_1 + 1,$$

provided $c_0 > 0$. But if $c_0 = 0$, (113) becomes

$$\begin{aligned} & e^3 + c_1 e^2 \\ & \leq (d_1 + 2)e^2 + (2c_1 + 2 + 2d_1 + d_0)e + 4 + d_0 \\ & \leq (d_1 + 7)e^2 - 2e + 3. \end{aligned}$$

Hence we recover (114). (114) implies that

$$(115) \quad 6 \geq A + c_1,$$

where $d_1 = e - A$. Since $c_1 \geq 3$ by (111), this is a contradiction for $A \geq 4$. If $A = 3$, then $c_1 = 3 + j \geq 4$. Hence (115) shows a contradiction. Then by (105) we obtain that $A = 2$. Hence $c_1 = 2 + j$ with $j \leq 2$. If $j = 2$, then (113) becomes

$$\begin{aligned} & e^3 + 4e + c_0 e \\ & \leq e^3 + 2e^2 + (d_0 + 6)e + 2c_0 + 4 + d_0 \\ & \leq e^3 + 3e^2 + 6e + 2c_0 + 3. \end{aligned}$$

Thus $e \leq 6$. This is against assumption. Hence we obtain that $j=1$. Thus $c_1=3$ and $d_1=e-2$. Now (113) becomes

$$\begin{aligned} & e^3 + 3e^2 + c_0e \\ & \leq e^3 + 2e^2 + (4+d_0)e + 2c_0 + 4 + d_0 \\ & \leq e^3 + 2e^2 + (7+d_0)e + 1. \end{aligned}$$

This implies that

$$(116) \quad d_0 + 7 \geq e + c_0.$$

If $c_0=6$, then $d_0=e-1$ by (116). Then by (103) e divides 6. This is against assumption. If $c_0=5$, then $d_0 \geq e-2$. If $d_0=e-1$, then by (103) e divides 5. This is against assumption. If $d_0=e-2$, then by (103) $e=10$. Hence $d_1=8$, $c_1=3$, $d_0=8$ and $c_0=5$. Then from (102) it follows that $a=5$. From (100) and (101) we obtain that $\tau=444$ and $C=235$. Then from (110) we obtain that

$$k^2 - 444k + 23375 = 0.$$

But then k cannot be integral. If $c_0=4$, then $d_0 \geq e-3$. If $d_0=e-1$, then by (103) e divides 4. This is against assumption. If $d_0=e-2$, then by (103) $e=8$. Then $c_1=3$, $d_1=6$, $c_0=4$ and $d_0=6$. Then from (102) it follows that $a=4$. From (100) and (101) we obtain that $v=291$ and $C=156$. Then from (110) we obtain that

$$k^2 - 291k + 9936 = 0.$$

But then k cannot be integral. If $d_0=e-3$, then by (103) $e=12$. Thus $c_1=3$, $d_1=10$, $c_0=4$ and $d_0=9$. Then from (102) it follows that $105a=627$. This is a contradiction. If $c_0=3$, then $d_0 \geq e-4$. If $d_0 \geq e-2$, then by (103) $e \leq 6$. This is against assumption. If $d_0=e-3$, then by (103) $e=9$. Thus $c_1=3$, $rft=7$, $c_0=3$ and $rfo=6$. Then from (102) it follows that $40a=181$. This is a contradiction. If $d_0=e-4$, then by (103) $e=12$. Thus $c_1=3$, $d_1=10$, $c_0=3$ and $rfo=8$. Then from (102) it follows that $21a=125$. This is a contradiction. If $c_0=2$, then $d_0 \geq e-5$. If $d_0 \geq e-3$, then by (103) $e \leq 6$. This is against assumption. If $d_0=e-4$, then by (103) $e=8$. Thus $c_1=3$, $d_1=6$, $c_0=2$ and $d_0=4$. Then from (102) it follows that $a=4$. From (100) and (101) we obtain that $v=287$ and $C=154$. Then from (110) we obtain that

$$k^2 - 287k + 9800 = 0.$$

But then k cannot be integral. If $d_0=e-5$, then by (103) $e=10$. Thus $c_1=3$, $d_1=8$, $c_0=2$ and $d_0=5$. Then from (102) it follows that $89a=439$. This is a contradiction. If $c_0=1$, then $d_0 \geq e-6$. Then by (103) $e \leq 6$. This is against assumption. If $c_0=0$, then from (103) we obtain that

$$(d_0 - 6)e + 3d_0 = (e - 2)e + e - 2.$$

This implies that

$$d_0(e + 3) = e(e + 3) + 2(e - 1).$$

Hence $e + 3$ divides $2(e - 1)$. Thus $e + 3 = 2e - 2$, and $e = 5$. This is against assumption. This completes the proof of (112).

By (100) we may put

$$k = k_2 e^2 + k_1 e + k_0,$$

where $e - 1 \geq k_i \geq 0$ for $i = 2, 1, 0$. Then from (110) it follows that

$$(117) \quad \begin{aligned} & a\{3e^3 + (c_1 + d_1 + 4)e^2 + (c_1 + c_0 + d_0 + 1)e + c_0\} \\ & = (k_2 e^2 + k_1 e + k_0)\{(3 - k_2)e^2 + (c_1 + d_1 + 2 - k_1)e + c_0 + d_0 + 1 - k_0\}. \end{aligned}$$

Now under the assumption that $k \leq \frac{1}{2}v$ we show that

$$(118) \quad k_2 = 0.$$

First of all, from (117) and (112) it follows that

$$(119) \quad \begin{aligned} & 6k_2 e^4 + 2\{k_2(c_1 + d_1 + 2) + 3k_1\} e^3 \\ & + 2\{k_2(c_0 + d_0 + 1) + k_1(c_1 + d_1 + 2) + 3k_0\} e^2 \\ & + 2\{k_1(c_0 + d_0 + 1) + k_0(c_1 + d_1 + 2)\} e + k_0(c_0 + d_0 + 1) \\ & < (3 + 2k_2^2)e^4 + \{(c_1 + d_1 + 4) + 4k_1 k_2\} e^3 \\ & + (c_0 + d_0 + 1 + 4k_0 k_2 + 2k_1^2) e^2 + (c_0 + 4k_0 k_1) e + k_0^2. \end{aligned}$$

Now from (100) it follows that $k_2 \leq 2$. If $k_2 = 2$, then from (119) we obtain that

$$\begin{aligned} & e^4 + \{3(c_1 + d_1) + 4\} e^3 + \{3(c_0 + d_0 + 1) + 2k_1(c_1 + d_1 + 2)\} e^2 \\ & + 2\{k_1(c_0 + d_0 + 1) + k_0(c_1 + d_1 + 2)\} e + k_0(c_0 + d_0 + 1) \\ & < 2k_1 e^3 + (2k_0 + 2k_1^2) e^2 + (c_0 + 4k_1 k_0) e + k_0^2 \\ & \leq 4e^4 - 7e^2 + (c_0 + 2)e + 1. \end{aligned}$$

By (104) this is a contradiction. If $k_2 = 1$, then from (119) we obtain that

$$\begin{aligned} & e^4 + (e_1 + d_1 + 2k_1) e^3 + \{c_0 + d_0 + 1 + k_1(c_1 + d_1 + 2) + 2k_0\} e^2 \\ & + 2\{k_1(c_0 + d_0 + 1) + k_0(c_1 + d_1 + 2)\} e + k_0(c_0 + d_0 + 1) \\ & < 2k_1^2 e^2 + (c_0 + 4k_1 k_0) e + k_0^2 \\ & \leq 2e^4 - 5e^2 + (c_0 + 2)e + 1. \end{aligned}$$

By (104) this is a contradiction. Thus $k_2 = 0$.

From (118) it follows that

$$(120) \quad 4k < v.$$

In fact, otherwise, from (100) we obtain that

$$\begin{aligned} & 3e^2 + (c_1 + d_1 + 2)e + c_0 + d_0 + 1 \\ & \leq 4k_1e + 4k_0 \leq 4e^2 - 4. \end{aligned}$$

By (104) this is a contradiction.

7. Cases with small a

Now it is convenient to eliminate cases with small a . From (50) and (120) we obtain that

$$(121) \quad 8a^2 - 9a + 1 > 18\mu_1.$$

On the other hand, let us consider the complementary design $(X, \mathcal{A}c)$. Let ν_2 and ν_1 be the intersection numbers of $(X, \mathcal{A}c)$, where $\nu_2 > \nu_1$. Then easily we obtain that

$$(122) \quad v - 2k = \nu_2 - \mu_2 = \nu_1 - \mu_1.$$

(122) implies that

$$(123) \quad \nu_2 - \nu_1 = \mu_2 - \mu_1 = a,$$

and that

$$(124) \quad v - k - \nu_2 = k - \mu_2 = ea.$$

Put $x_c = \nu_2 + \nu_1 - 1$. Then from (123) and (124) it follows that

$$(125) \quad x_c = 2v - 2k - 1 - a(2e + 1) = 2v - 4k + x.$$

Furthermore corresponding to (1) we obtain that

$$(126) \quad x_c = \frac{2(v-k-1)(v-k-2)}{v-3}.$$

From (1), (126) and (50) we obtain that

$$(127) \quad (a^2 - 1)(v - 2) = xx_c.$$

Then from (30), (125) and (127) we obtain that

$$(128) \quad (a^2 - 1)(v - 2) = (2\mu_1 - 1 + a)(2v - 4ae - 2\mu_1 - 3a - 1).$$

(128) implies that

$$(129) \quad 4\mu_1 > a^2 - 2a + 1.$$

Now we have to eliminate the case where $10 \geq a \geq 3$. For each case we can argue exactly in the same way. So we demonstrate only for the case $a=3$. If $a=3$, then from (121) and (129) we obtain that $\mu_1=2$. Then $\mu_2=5$ and $x=6$. Then from (1) and (125) we obtain that

$$v = 6k - 3,$$

and that

$$k^2 - 21k + 50 = 0.$$

But then k cannot be integral. It is easy to proceed by hand up to $a=10$. Hence from now on we may assume that $a \geq 11$.

8. Completion of the proof

First we scrutinize (103) a little more. By (103) we may put

$$(130) \quad c_0 d_0 = Ee$$

$$(131) \quad c_0 d_1 + c_1 d_0 + E = Fe - 2;$$

and

$$(132) \quad c_1 d_1 + d_0 + F = Ge - 1$$

where E is a non-negative integer and F and G are positive integers. Then from (130)-(132) and (103) it follows that

$$(133) \quad d_1 + G = e + 2.$$

Further from (133), (105) and (106) it follows that

$$(134) \quad c_1 + 2 = j + G \geq j + 4.$$

Secondly we show that

$$(135) \quad k_1 > a.$$

In fact, by (21) $k_1 \geq a$. Hence assume that $k_1 = a$. Then $k_0 = \mu_2 < e$. From (47), (100) and (104) we obtain that

$$4ye^2 < y(v-2) = kx = (ae + \mu_2)(2\mu_2 - a - 1) < 2(a+1)e^2$$

which implies that

$$(136) \quad y < \frac{1}{2}(a+1).$$

On the other hand, let d be the greatest common divisor of k and $v-2$. Then from (1) we obtain that

$$x+4 \equiv 0 \pmod{d}.$$

Now by (47) y is divisible by k/d . Hence in order to get a contradiction to (136), it suffices to show that

$$(137) \quad \frac{k}{x+4} \geq \frac{1}{2}(a+1).$$

Since

$$2k - (a+1)(x+4) = 2a(e - \mu_2) + a^2 - 2a - 3,$$

(137) holds.

As a consequence of (135) we obtain that

$$(138) \quad e < \frac{1}{3}a^2.$$

In fact, (135) implies that $\mu_2 > e$. Hence from (121) it follows that

$$2e < 2\mu_2 < \frac{8a^2 + 9a + 1}{9} < a^2.$$

Thirdly we show that

$$(139) \quad c_1 \leq \frac{4e}{a}.$$

In fact, first assume that $ac_1 > 5e$, which implies that

$$a(d_1 - c_1 - 5) < (a-5)e.$$

Hence by (102) we obtain that

$$(140) \quad v + 1 \geq 5e^2.$$

If $c_1 \geq d_1$, from (102) it follows that $ae^2 < 6e^2$. This contradiction shows that $c_1 < d_1$. Now from (102) and (105) it follows that

$$v+1 \leq 5e^2 - e,$$

which contradicts (140). Thus $ac_1 \leq 5e$. Next assume that $ac_1 > 4e$. Then by (100) and (102) we obtain that

$$ae^2 + 4ae + a(c_0 + 2) < (j+2)e + c_0 + d_0 + 2 + ad_1e + ad_0.$$

Hence by (105) we obtain that

$$6ae + a(c_0 + 2) < (j+2)e + c_0 + 2 + a(d_0 + 1).$$

Since, by (105), (138), and (139)

$$j = c_1 - (e - d_1) \leq (5e/a) - 2 \leq 3a - 2,$$

we obtain that

$$3ae + a(c_0 + 2) < c_0 + 2 + a(d_0 + 1),$$

which is a contradiction.

Now from (30), (125) and (127) we obtain that

$$(141) \quad (a^2 - 1)(v - 2) = (2k - 1 - 2ae - a)(2v - 2k - 1 - 2ae - a)$$

From (141) we obtain that

$$(142) \quad 4k^2 = 4a^2e^2 + 4a^2e + 4ae + 3a^2 + 2a - 1 \pmod{v}.$$

On the other hand, from (67) we obtain that

$$(143) \quad 2k^2 = 6ae + 3a - 1 \pmod{v}.$$

From (142) and (143) it follows that

$$(144) \quad 4a^2e^2 + (4a^2 - 8a)e + 3a^2 - 4a + 1 = 0 \pmod{v}.$$

Further from (100) and (104) we obtain that

$$a^2v = 4a^2e^2 + (j+2)a^2e + (c_0 + d_0 + 1)a^2.$$

Hence together with (144) we may put

$$(145) \quad Yv = \{(j-2)a^2 + 8ae + (c_0 + d_0 - 2)a^2 + 4a - 1,$$

where Y is a rational integer. Since $v \equiv -1 \pmod{a}$ by (102), from (145) we obtain that $Y \equiv 1 \pmod{a}$, and hence that

$$(146) \quad (aZ + 1)v = \{(j-2)a^2 + 8ae + (c_0 + d_0 - 2)a^2 + 4a - 1,$$

where Z is a rational integer.

We show that $Z=0$ in (146). First we show that Z is non-negative. In fact, assume that Z is negative. Then $j \leq 2$. If $j=2$, then $c_0 + d_0 \leq 1$. But since $e > 2a$ by (112), $8ae - 2a^2 > 0$. This is absurd. Hence $j=1$. Then from (146) we obtain that

$$\begin{aligned} & (a+1)(4e^2 + 3e + c_0 + d_0 + 1) + 8ae + (c_0 + d_0)a^2 + 4a \\ & \leq a^2e + 2a^2 + 1. \end{aligned}$$

Since $e > 2a$ by (112), this is absurd. Next we show that $Z \leq 1$. In fact, otherwise, from (146) we obtain that

$$\begin{aligned} & \{(j-2)a^2 + 8a\}e + (c_0 + d_0 - 2)a^2 + 4a - 1 \\ & \geq (2a+1)\{4e^2 + (j+2)e + c_0 + d_0 + 1\}. \end{aligned}$$

Since $e > 2a$ by (112), from the above inequality we obtain that $ja > 8e$. Hence by (134) we obtain that $ac_1 > 8e$. This contradicts (139). Now assume that $Z=1$. Then as above we obtain that $ja > 4e$. Hence as above we obtain that $ac_1 > 4e$. This contradicts (139). Thus $Z=0$. Hence we obtain that

$$(147) \quad v = \{(j-2)a^2 + 8a\}e + (c_0 + d_0 - 2)a^2 + 4a - 1.$$

From (100), (104) and (147) we obtain that

$$\begin{aligned} (148) \quad & j(a^2 - 1)e + (c_0 + d_0 + 1)(a^2 - 1) \\ & = 4(e-1)e + 2(a-1)(a-3)e + (3a-1)(a-1) \end{aligned}$$

From (138) and (148) we obtain that

$$(149) \quad j + \frac{c_0 + d_0 + 1}{e} = \frac{4(e-1)}{a^2 - 1} + \frac{2(a-3)}{a+1} + \frac{3a-1}{(a+1)e} + 4.$$

By (104), (149) implies that

$$(150) \quad 1 \leq j \leq 3.$$

By (148) we may put

$$(151) \quad 4(e-1)e = (a-1)(He + I),$$

where H and I are rational integers such that $H > 0$ and that $e > I \geq 0$. Then from (148) we obtain that

$$(152) \quad (c_0 + d_0 + 1)(a+1) = (H-8)e + I - 4 + (2-j)e(a+1) + 3(a+1).$$

By (152) we may put

$$(153) \quad (H-8)e + I - 4 = J(a+1),$$

where J is a rational integer.

From (152) and (153) we obtain that

$$(154) \quad c_0 + d_0 = J + (2-j)e + 2.$$

From (100), (104) and (154) we obtain that

$$(155) \quad v = 4e^2 + 4e + J + 3.$$

Now from (102) it follows that

$$a(c_0 - d_0 + 2) = c_0 + d_0 + 2 \pmod{e}.$$

So we may put

$$(156) \quad (a-1)(c_0+2) - (a+1)d_0 = Ke,$$

where K is a rational integer.

From (102), (133), (134) and (156) we obtain that

$$(157) \quad a(2c_1 + 4 - j) + K = 4e + j + 2.$$

From (102) and (104) it follows that

$$(158) \quad v = (2c_1 + 4 - j)ae + ac_0 - ad_0 + 2a - 1$$

Hence from (147) and (158) we obtain that

$$(159) \quad (2c_1 - ja + 2a - j - 4)e = (a-1)(c_0 - 2) + (a+1)d_0$$

From (156) and (159) we obtain that

$$(160) \quad (2c_1 - ja + 2a - j - 4 + K)e = 2(a-1)c_0.$$

From (157) and (160) we obtain that

$$(161) \quad (2e + c_1 - c_1a - a - 1)e = (a-1)c_0.$$

Similarly we obtain that

$$(162) \quad (c_1a + c_1 + 3a - 2e - ja - j - 3)e = (a+1)d_0 - 2(a-1).$$

From (161) it follows that

$$(163) \quad a = \frac{C - e}{C - 2e^2 + e}.$$

Further from (154) and (162) it follows that

$$(164) \quad J = C - 2e^2 + \frac{ae - 4e^2 - 5e - 4}{a+1}.$$

Hence from (155), (163) and (164) we obtain that

$$(165) \quad v = \frac{C^2 + 2Ce + C - 2e^2 - 2e}{C - e^2}.$$

Now solve (70) in C and use (147), (149) and (154) to obtain

$$(166) \quad (a-1)C = \{2ae - (a+1)\}e.$$

Going back to (1) and (2) once again and using the definition of a and e , we obtain that

$$(167) \quad k(k-1)(v-k)(v-k-1) = a^2e(e+1)(v-2)(v-3).$$

From (50) and (167) we obtain that

$$(168) \quad \frac{4(k-2)(v-k-2)}{k(v-k)} = \frac{(a^2-1)(v-3)}{a^2e(e+1)}.$$

Subtracting 4 from both sides of (168) and using (147) and (154), we obtain that

$$(169) \quad \frac{2v-4}{k(v-k)} = \frac{2ae+a-1}{a^2e(e+1)}.$$

From (110) and (169) we obtain that

$$(170) \quad (2ae+a-1)C + 4ae(e+1) = a(e+1)v.$$

From (165) and (170) we obtain that

$$(171) \quad (ae-1)C^2 - (2ae^3 + a - ae^2 - ae - e^2)C - 2ae(e+1)(2e^2 - e - 1) = 0.$$

Substitute C in (171) by (166). Then we obtain that

$$(172) \quad 4a^2e^4 - (4a^3 + 8a)e^3 + (7a^2 + 4a - 1)e^2 + (a^3 - 7a^2 + 5a - 1)e + (3a^3 - 4a^2 + a) = 0.$$

(172) allows the following factorization:

$$(e-a)\{4a^2e^3 - 8ae^2 - (a^2 - 4a + 1)e - (3a^2 - 4a + 1)\} = 0.$$

Since $e > a$, we obtain that

$$4a^2e^3 - 8ae^2 - (a^2 - 4a + 1)e - (3a^2 - 4a + 1) = 0.$$

Since a and e are not so small and hence the first term of the left hand side is too large comparing with other terms, this is a contradiction.

This completes the proof.

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