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ON MAXIMAL SUBMODULES OF A FINITE DIRECT SUM OF HOLLOW MODULES I

Dedicated to Professor Hirosi Nagao on his 60th birthday

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Introduction

We have given, in [5], a characterization of a right (upper) serial ring in terms of submodules of finite direct sums of serial modules. In this paper we shall replace the serial module by hollow modules in the above. Then it is clear that we shall be able to obtain a new class of rings R generalized from the right serial rings.

However, it is difficult for the author to give a complete characterization of those rings. We shall restrict ourselves to a particular case where the Jacobson radical J of R is square zero. It is not still easy to find the characterization of such rings. If R is either a commutative artinian ring or an algebra of finite dimension over an algebraically closed field, then we can show the structure of R as follows: $|eR|$, the composition length of eR, is equal to or less than three and if two simple right ideals A_1 and A_2 in eJ are isomorphic to each other, then there exists a unit element x in eRe such that $A_2=xA_1$ and $[eRe/eJe:\Delta(A_1)]_r=2$, where e is a primitive idempotent of R and $\Delta(A_1)=\{\bar{x}\in eRe/eJe \mid xA_1\subseteq A_1\}$. We shall give the similar structure for any right artinian ring R under an assumption that $|eR|\leqslant 5$. We do not know any examples of rings which have the property mentioned above (see Condition I in § 3) and $|eR|\geqslant 4$ for some primitive idempotent e. We shall study the similar problem without the assumption $eJ^2=0$ in the forthcomming paper.

1 Right serial rings

Let R be a ring with identity. Every module in this paper is a unitary right R-module. For an R-module M, $|M|$ means the length of the composition series of M. We shall denote the *Jacobson radical* and the *socle* of M by $J(M)$ and $S(M)$, respectively. Put $J^n(M)=J(J^{n-1}(M))$ and $S_n(M)/S_{n-1}(M)=S(M/S_{n-1}(M))$ inductively. Then $M\supseteq J(M)\supseteq J^2(M)\supseteq\cdots$ and $0\subseteq S_1(M)\subseteq S_2(M)\subseteq\cdots$ are called the *upper Loewy series* and the *lower Loewy series* of M, respectively. If each factor module $J^n(M)/J^{n+1}(M)$ $(S_{n+1}(M)/S_n(M))$ is simple or zero, the upper (lower) Loewy series is a unique composition series such that $|M/J^n(M)|=n$ ($|S_n(M)|=n$) and if $|M/N|=m<\infty$ ($|N|=m<\infty$) for some submodule N, $N=J^m(M)$ ($N=S_m(M)$) provided that $J^k(M)\neq J^{k+1}(M)$ ($S_k(M)\neq S_{k+1}(M)$) for all $k\leqslant m-1$. If M has the unique chain as above, we call M an *upper* (*lower*) *serial module*. An upper (lower) serial module M with $J^t(M)=0$ ($S_t(M)=M$) for some t is called a *serial module* and in this case $S_r(M)=J^{t-r}(M)$.

Let R be a semi-perfect ring. If, for each primitive idempotent e, eR is an upper serial module, then R is called a *right upper serial ring* (cf. [5]). Next we assume that R is a right semiartinian ring. If, for each indecomposable injective module E, E is a lower serial module, then R is called a *right lower coserial ring*.

We have shown in [5], Theorem 2 that if R is an artinian right (upper) serial ring, R satisfies the following condition: every submodule of a direct sum of hollow modules is also a direct sum of hollow modules. We shall study, in this section, a similar property for a quasi-projective module. The following result is well known provided R/J is a simple ring (cf. [1], p. 75). We shall give a proof for the sake of completeness.

Proposition 1. *Let R be a semi-perfect ring. If R/J^2 is a right serial ring, then R is a right upper serial ring.*

Proof. We may assume that R is basic. We shall show by induction on t that $eR\supset eJ\supseteq\cdots\supseteq eJ^t$ is serial for each primitive idempotent e. Assume that the above fact is true for $i\leqslant t$. Then $t\geqslant 2$ by assumption. If $eJ^2=eJ$, the proposition is trivial. We assume that eJ/eJ^2 is a non-zero simple module. Then $eJ/eJ^2\approx fR/fJ$ for some primitive idempotent f. Hence there exists an element x in eJf such that $eJ=xR+eJ^2$. Then $eJ^t=xJ^{t-1}+eJ^{t+1}$, and so eJ^t/eJ^{t+1} is a homomorphic image of fJ^{t-1}/fJ^t, since $x=exf\in eJ$. Hence eJ^t/eJ^{t+1} is either simple or zero. Therefore R is right upper serial by induction.

We obtain the following proposition as dual to the above.

Proposition 1.′ *Let R be a right semi-artinian ring. If $S_2(E)$ is serial for every indecomposable and injective module E, then R is a right lower coserial ring.*

Proof. Let $S_i(E)\supseteq S_{i-1}(E)\supseteq\cdots\supseteq S_1(E)\supset 0$ be a serial chain of E. We may assume $i\geqslant 2$. Then $E/S_{i-1}(E)$ is a uniform module. Hence $\tilde{E}=E(E/S_{i-1}(E))$, injective hull of $E/S_{i-1}(E)$, is indecomposable. Therefore $S_{i+1}(E)/S_i(E)\subseteq S_2(\tilde{E})/S_1(\tilde{E})$, and so $S_{i+1}(E)/S_i(E)$is either simple or zero. Hence R is right lower coserial·

It is clear that eR is not serial even if eR/eJ^2 is serial for a primitive idempotent e (cf. Example 1 below). Concerning this fact, we have the following proposition.

Proposition 2. *Let R be a semi-perfect ring and J the Jacobson radical of R. Let P be a hollow module eR/B with B a character right ideal of eR; i.e. P is a cyclic quasi-projective module. Assume P/PJ^2 is a serial R-module. Then P is upper serial if and only if every maximal submodule of a finite (two) direct sum of homomorphic images P_i of P with $|P_i|<\infty$ is also a direct sum of hollow modules.*

Proof. "Only if" part is clear from [3], Theorem 2 and [4], Theorem 1.

"If" part. Assume that the last condition of the proposition is satisfied and that, for a primitive idempotent e, $eR\supset eJ\supset(eJ^2+B)\supset\cdots\supset(eJ^t+B)$ is the chain with $(eJ^i+B)/(eJ^{i+1}+B)$ simple for all $i\leqslant t-1$. Then we may assume $t\geqslant 2$ by assumption. Let N_1 and N_2 be maximal submodules, containing $(eJ^{t+1}+B)$, of (eJ^t+B). Put $D=eR/N_1\oplus eR/N_2$ and $\bar{D}=D/J(D)=eR/eJ\oplus eR/eJ$. Let $M'=\{\bar{x}+\bar{x}\mid \bar{x}\in\bar{e}\bar{R}\}$ be a submodule of $\bar{D}$. Then there exists the maximal submodule M of D such that $M\supset J(D)$ and $M/J(D)=M'$. Since $|S(D)|=|(eJ^t+B)/N_1\oplus(eJ^t+B)/N_2|=2$, $M=M_1\oplus M_2$ by assumption, where the M_i are either hollow or zero. Assume $M_i\neq 0$ for $i=1, 2$. Then we know that each M_i is uniform, for $|S(D)|=2$. Let π_i: $D\to eR/N_i$ be the projection for $i=1, 2$. Since $\bar{M}=M'$, $\pi_i(M_{j(i)})=eR/N_i$ for some $j(i)$ of $\{1, 2\}$, and so $|M_{j(i)}|\geqslant t+1$. On the other hand, $|M|=2t+1$. Hence $j(1)=j(2)$ $(=1)$. M_1 containing the simple socle, either $\pi_1|M_1$ or $\pi_2|M_1$ is an isomorphism. Assume $\pi|M_1$ is an isomorphism. Then $D=M_1\oplus eR/N_2$. Now take the composition mapping f: $eR/N_1\xrightarrow{i}D\xrightarrow{p}eR/N_2$, where i is the injection and p is the projection of the above decomposition. Let m be an element in M_1. Then $m=\pi_1(m)+\pi_2(m)$, and so $\pi_1(m)=m-\pi_2(m)$. Hence $f(\pi_1(m))=-\pi_2(m)$. On the other hand, $\overline{\pi_1(m)}=\overline{m-\pi_2(m)}=(\bar{x}+\bar{x})-\overline{\pi_2(m)}=\bar{x}+(\bar{x}-\overline{\pi_2(m)})$, where $\bar{m}=\bar{x}+\bar{x}$; $\bar{x}\in\bar{e}\bar{R}$. Hence $\overline{\pi_1(m)}=\bar{x}=\overline{\pi_2(m)}=-\overline{f(\pi_1(m))}$. Accordingly, since $\pi_1|M_1$ is an isomorphism, the identity mapping of eR/eJ is liftable to $-f\in\mathrm{Hom}_R(eR/N_1, eR/N_2)$. Then there exists an element x in eRe such that $-f(\bar{r})=x\bar{r}$ for $\bar{r}\in eR/N_1$. Hence $xN_1\subseteq N_2$ and $e-x=j$ is an element in eJe, for f is the identity of eR/eJ. On the other hand, $jN_1\subseteq j(eJ^t+B)\subseteq(eJ^{t+1}+B)\subseteq N_2$. Hence $N_1=eN_1=(x+j)N_2\subseteq N_2$ and so $N_1=N_2$ (cf. the proof of [3], Theorem 4). Therefore $(eJ^n+B)/(eJ^{n+1}+B)$ is either simple or zero. Next assume that $M_2=0$ and so M is hollow. Then $J(M)=J(D)$ and $M/J(M)\approx eR/eJ$. Hence we may assume $M=eR/eA$ for some right ideal A. $|D|=2t+2$ implies $|eR/eA|=2t+1$. Since $M/J(D)=M'$, $\pi_i|eR/eA$ is an epimorphism for $i=1, 2$. Put $B_i=\ker(\pi_i|eR/eA)$. Then $|eR/B_i|=t+1$. On the other hand, since $N_i\subseteq(eJ^t+B)$, we have the natural epimorphism ν_i of eR/N_i onto $eR/(eJ^t+B)$. Hence $\nu_i\pi_i$ is an epimorphism of eR/N_i onto $eR/(eJ^t+B)$.

Therefore there exists a unit element y in eRe such that $y\bar{r}=\nu_i\pi_i\bar{r}$ for $\bar{r}\in eR/N_i$. Since $B_i\subseteq\ker\nu_i\pi_i$, $yB_i\subseteq(eJ^t+B)$, and so $B_i\subseteq y^{-1}(eJ^t+B)\subseteq(eJ^t+B)$. Now $|eR/(eJ^t+B)|=t$ and $|eR/B_i|=t+1$. Hence $|(eJ^t+B)/B_i|=1$. Furthermore, $B_1\cap B_2=eA$. Therefore $|eR/eA|=|eR/(eJ^t+B)|+|(eJ^t+B)/B_1|+|B_1/eA|=t+1+|(B_1+B_2)/B_2|\leqslant t+2$, for $B_1+B_2\subseteq eJ^t+B$. On the other hand, $|eR/eA|=2t+1$, which contradicts the assumption $t\geqslant 2$.

From the first half of the above proof we obtain the following:

Proposition 2′. *Let R and P be as above (not necessarily P/PJ^2 is serial). Then P is an upper serial module if and only if every finite (two) direct sum of homomorphic images P_i of P with $|P_i|<\infty$ has the lifting property of simple modules modulo the radical.*

Proof. If $D=eR/N_1\oplus eR/N_2$ has the lifting property of simple modules modulo the radical, then every maximal submodule of D contains a non-zero direct summand of D by definition. Hence we have the first case of the above proof.

We note that the assumption on P/PJ^2 is inevitable in Proposition 2 (cf. § 3).

2. Maximal submodules

i) General case

From now on we always assume that R is a right artinian ring. We shall study the similar situation to Proposition 2. Hence we may assume that R is basic. Let e be a primitive idempotent, then eRe/eJe is a division ring. We consider hollow modules N_i of the form eR/B_i, where B_i is a submodule of eJ. Put $D=\sum_{j=1}^{k}\oplus N_j$ and $\bar{D}=D/J(D)=\sum\oplus eR/eJ$. Now R is basic. Then $(eR/eJ)R=(eR/eJ)(eRe)=(eR/eJ)(eRe/eJe)$ and $R(eJ/eJ^2)=(eRe/eJe)(eJ/eJ^2)$. Put $eRe/eJe=\overline{eRe}=\Delta$. Then $\bar{D}$ is a right Δ-vector space of dimension k. Let $\bar{x}=\sum\bar{x}_j$ be an element in $\bar{D}$, where the $\bar{x}_j$ are in $N_j/J(N_j)=eR/eJ$ and we denote $\bar{x}_j^{-1}$ by $\overline{x_j^{-1}}$, where x_j^{-1} is an element in eRe. Then $\bar{x}_j\bar{x}_j^{-1}=\bar{e}$. Let M be a maximal submodule of D. Then $M\supseteq J(D)$, and put $\bar{M}=M/J(D)$. It is clear that either $\bar{M}=\sum_{j\neq i}\oplus N_j$ for some i or $\bar{M}$ has the following basis: $\{\bar{\alpha}_1=(\bar{\delta}, \bar{e}, o, \cdots, o, o), \bar{\alpha}_2=(\bar{\delta}_2, o, \bar{e}, o, \cdots, o), \cdots, \bar{\alpha}_{k-1}=(\bar{\delta}_{k-1}, o, \cdots, \bar{e})\}$ and that M is generated by $\{\alpha_j=(\tilde{\delta}_j, o, \cdots, \tilde{e}, \cdots, o)\}$ and $J(D)$ for the latter case, where the δ_j are in eRe and $\tilde{\delta}_j$ is an element in N_j. Conversely, if we take the set $\{\alpha_i\}_{i=1}^{k-1}$, the module generated by the α_i and $J(D)$ is a maximal submodule of D. We consider the condition in Proposition 2.

(∗) *Any maximal submodule of D is a direct sum of hollow modules.*

Lemma 3. *Let $\{N_1, N_2, \cdots, N_{t+1}\}$ be a set of hollow modules with $|N_i|=t$.*

Put $D=\sum_{i=1}^{t+1}\oplus N_i$. *If* D *satisfies* $(*)$, *then every maximal submodule* M *of* D *contains a direct summand of* D, *which is isomorphic to some* N_i.

Proof. Let $\pi_i\colon D\to N_i$ be the projection of D onto N_i. Assume that M is a direct sum of hollow modules M_i: $M=\sum\oplus M_i$. If $|M_i|<t$, $M_i\subseteq J(D)=\sum\oplus J(N_i)$. Furthermore, since $|\bar{M}|=t$ and $\bar{M}_i=(M_i+J(D))/J(D)\approx M_i/(M_i\cap J(D)$ is either simple or zero, there exist at least t M_{i_j} among $\{M_i\}$ such that $|\bar{M}_{i_j}|=1$, and hence $|M_{i_j}|\geqslant t$, since $\pi_k|M_{i_j}$ is an epimorphism for some k. If $|M_{i_j}|\geqslant$t+1 for all j, $|M|\geqslant t(t+1)=|D|$. Hence there exists some M_{i_0} with $|M_{i_0}|=t$. $\pi_k|M_{i_0}$ is an epimorphism for some k as above. Hence M_{i_0} is a direct summand of D for $|M_{i_0}|=|N_k|$.

Assume that M in Lemma 3 is generated by $\{\alpha_1, \alpha_2, \cdots, \alpha_t\}$ as above and $J(D)$. Then M_1 is generated by $\beta=\sum \alpha_i y_i+j$, where j is an elements in $J(D)$. Since M_1 is a direct summand of D, $M_1\not\subseteq J(D)$. Hence some y_i of $\{y_i\}$ is not contained in eJe, Therefore we may assume $\beta=\alpha_1+\sum_{i\geqslant 2}\alpha_i y_i+j$. Put

$$(1)\qquad \gamma=\alpha_1+\sum_{i\geqslant 2}\alpha_i y_i\,,\quad \delta=\delta_1+\sum_{i\geqslant 2}\delta_i y_i \quad\text{and}\quad \beta=\gamma+j\,,$$

where $\alpha_i=(\tilde{\delta}_i, o, \cdots, o, \tilde{e}, o, \cdots, o)$ is in D.

We frequentely use the method of the proof of [3], Theorem 2, and so we summarize here its content. Let N_1 and N_2 be hollow modules and f' an element in $\mathrm{Hom}_R(N_1/J(N_1), N_2/J(N_2))$. If there exists an element f, which induces f', in $\mathrm{Hom}_R(N_1, N_2)$, we say that *f' is lifted to f.*

Lemma 4. *Let* $D=\sum_{i=1}^{n}\oplus N_i$ *be a direct sum of hollow modules. Let* $\bar{M}=\{\bar{x}+\bar{f}_2(\bar{x})+\cdots+\bar{f}_n(\bar{x})\,|\,\bar{x}\in\bar{N}_1$ *and* $\bar{f}_i\in\mathrm{Hom}_R(\bar{N}_1, \bar{N}_i)\}$ *be a submodule of* $\bar{D}=D/J(D)$. *If each* $\bar{f}_i$ *is liftable,* D *contains a direct summand* D_1 *such that* $\bar{D}_1=\bar{M}$.

From Lemma 3, we are interested in the condition:

$(**)$ *Every maximal submodule of* $D(k)=N_1\oplus N_2\oplus\cdots\oplus N_k$ *contains a direct summand of* D, *which is isomorphic to some* N_i.

Let B be a submodule in eJ contained in $r_R(eJe)=\{x\in R\,|\,eJex=0\}$ and put $\Delta(B)=\{x\in\Delta\,|\,xB\subseteq B\}$.

Lemma 5. *Let* B *be a submodule in* eJ *contained in* $r_R(eJe)$[1] *and* $N_i=eR/B$ *for* $i=1, 2, \cdots, k+1$. *Then* $[\Delta\colon \Delta(B)]=k$ *as a right* $\Delta(B)$-*module if and only if* $D(k+1)=\sum_{i=1}^{k+1}\oplus N_i$ *satisfies* $(**)$, *but* $D(k)$ *does not.*

Proof. Assume $[\Delta\colon \Delta(B)]=$k. Let M be a maximal submodule of $D=D(k+1)$. Then we may assume that $\bar{M}$ has the basis $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k\}$ as before. Since $[\Delta\colon \Delta(B)]=k$, there exists a set of elements $\{\bar{y}_i\}\neq o$ of $\Delta(B)$ such

1) We shall remove this assumption in the forth comming paper.

that $\sum \delta_i \bar{y}_i \in \Delta(B)$. Then $\theta = \sum \alpha_i y_i = (\sum \tilde{\delta}_i y_i, \tilde{y}_1, \cdots, \tilde{y}_k)$ is an element in M. Now all components of θ modulo eJ are elements in $\Delta(B)$. Therefore there exists a direct summand M_1 of D such that $\bar{M}_1 = (\theta R + J(D))/J(D)$ by Lemma 4. Hence M_1 is a submodule of M, for $M \supseteq J(D)$. It is clear that M_1 is isomorphic to some N_i, for $\bar{M}_1 = (\theta R + J(D))/J(D)$. Assume that $D(k+1)$ satisfies (**). Let $\{\delta_0 = \bar{e}, \delta_1, \delta_2, \cdots, \delta_k\}$ be any set of elements in Δ and M a maximal submodule generated by $\{\alpha_i = (\tilde{\delta}_i, o, \cdots, \overset{i+1}{\tilde{e}}, o, \cdots, o)\}_{i=1}^{k}$ and $J(D)$. Then M contains a direct summand M_1 of D with $|M_1| = |eR/B|$ by assumption. We may assume that M_1 is generated by $\beta = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_k y_k + j$, where the $\bar{y}_i$ are in Δ and j in $J(D)$. We may assume that there exists an integer i such that $\tilde{y}_j \neq o$ for all $j \leqslant i$ and $\tilde{y}_{j'} = o$ for all $j' > i$. Then $\beta = (\tilde{\delta}_1 y_1 + \tilde{\delta}_2 y_2 + \cdots + \tilde{\delta}_i y_i + \tilde{j}_1, \tilde{e} y_1 + \tilde{j}_2, \tilde{e} y_2 + \tilde{j}_3, \cdots, \tilde{e} y_i + \tilde{j}_{i+1}, \tilde{j}_{i+2}, \cdots, \tilde{j}_{k+1})$, where the j_p are in eJ/B. Consider the natural epimorphism φ of eR onto $\beta eR \subseteq M_1$ by setting $\varphi(r) = \beta r$ for $r \in eR$. Let x be in $\ker \varphi$. Then $(e + j_2)x \in B$. We may assume that $j_2 \in eJe$ and $(e + j_2)^{-1} = e + j_2'$, where j_2' is in eJe. Hence $x \in (e + j_2')B = B$, and so $\ker \varphi \subseteq B$, which implies $|eR/B| = |M_1| \geqslant |\beta eR| = |eR/\ker \varphi| \geqslant |eR/B|$, and so $\ker \varphi = B$. Hence $\delta B \subseteq B$, $(ey_2)B = B$, $\cdots$ and $(ey_i)B = B$ provided $\delta = (\delta_1 + \delta_2 \bar{y}_2 + \cdots \delta_i \bar{y}_i) \neq o$ (note that $j_p B = 0$). Therefore $[\Delta : \Delta(B)] \leqslant k$. Thus we obtain the lemma from the above.

We note that if $D(i)$ satisfies (**), then $D(i)$ does for all $i \geqslant j$ (cf. § 3). Hence we have the following corollary.

Corollary. *$[\Delta : \Delta(B)] = k$ implies that k is the minimal integer among k' such that $D(k'+1)$ satisfies (**), where $D(k'+1) = \sum_{i=1}^{k+1} \oplus N_i$ and $N_i = eR/B$ for all i.*

Proposition 6. *Let A and B be submodules of eJ contained in $r_R(eJe)$ and with $|A| = |B|$ and $[\Delta : \Delta(B)] = k$.*

i) *There exists a unit element x in eRe such that $xB = A$ if and only if $D(k+1) = eR/A \oplus \overbrace{eR/B \oplus \cdots \oplus eR/B}^{k}$ satisfies (**).*

ii) *If eJ is an irredundant sum of $\{B = B_1, B_2, \cdots, B_{t+1} \mid |B_i| = |B|$ for all $i\}$ and* i) *is satisfied for any pair (B_i, B_j), then $k \geqslant t+1$.*

Proof. i). If there exists a unit x such that $xB = A$, $eR/A \approx eR/B$. Hence $D(k+1)$ satisfies (**) by Lemma 5. Conversely, we assume that $D(k+1) = eR/A \oplus \overbrace{eR/B \oplus \cdots \oplus eR/B}^{k}$ satisfies (**). Let M be a maximal submodule of D such that $\bar{M} = \langle \bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_k \rangle$, where $\alpha_i = (\delta_i, o, \cdots, o, e_i, o, \cdots, o)$ and $\{\bar{\delta}_1, \bar{\delta}_2, \cdots, \bar{\delta}_k\}$ is linearly independent over $\Delta(B)$. Then M contains a direct summand M_1 generated by $\beta + j$, where $\beta = (\sum \tilde{\delta} y_i, \tilde{e}, \tilde{y}_2, \cdots, \tilde{y}_k)$. From the similar argument to the proof of Lemma 5, we have $(\sum \delta_i y_i)B \subseteq A$ and $\sum \bar{\delta}_i y_i = \bar{\delta}_i + \sum_{i \geqslant 2} \bar{\delta}_i y_i \neq o$. ii). Let $eJ = B_1 + B_2 + \cdots + B_{t+1}$ be an irredundant sum.

Then there exists a unit element x_i of eRe such that $x_iB_1=B_i$ and $eJ=eB_1+\sum_{i=2}^{t+1}x_iB_1$. Therefore $\{\bar{x}_1=\bar{e}, \bar{x}_2, \cdots, \bar{x}_{s+1}\}$ is linearly independent over $\Delta(B)$, and so $[\Delta: \Delta(B)]\geqslant t+1$.

ii) Case $eJ^2=0$.

From now on we assume $eJ^2=0$. Then $eJ=\sum_{i=1}^{t}\oplus A_i$, where the A_i are simple. We shall study the case $t=2$ in the above. In this case (**) is equivalent to

$(**)_2$ *Every maximal submodule of a direct sum* $D(3)$ *of three serial modules* (*of length two*) *contains a direct summand of* $D(3)$.

(2 in $(**)_2$ means the length of the serial modules in $D(3)$.)

Case I. $A_1\approx A_2$.

Then there exists a simple right ideal A_3 such that $A_1\oplus A_2=A_1\oplus A_3=A_2\oplus A_3$. Put $eJ=A_1\oplus A_2\oplus B$, $N_2=eR/(A_1\oplus B)$, $N_2=eR/(A_3\oplus B)$ and $N_3=eR/(A_1\oplus B)$.

Case II. $|eJ|\geqslant 3$.

Put $eJ=A_1\oplus A_2\oplus A_3\oplus B$, $N_1=eR/(A_2\oplus A_3\oplus B)$, $N_2=eR/(A_1\oplus A_3\oplus B)$ and $N_3=eR/(A_1\oplus A_2\oplus B)$.

In either case we put $D=N_1\oplus N_2\oplus N_3$. We take a maximal submodule M' of D generated by $(\tilde{e}, \tilde{k}_1, \tilde{o})$ and $(\tilde{o}, \tilde{e}, \tilde{k}_2)$, where $\bar{k}_1\neq o$ and $\bar{k}_2\neq o$. M' being maximal in D, there exists a unique maximal submodule M of D such that $M\subseteq J(D)$ and $\bar{M}=M'$. From now on, we assume $(**)_2$. Then M contains a direct summand M_1 with $|M_1|=2$. M is generated by $(\tilde{e}, \tilde{k}_1, \tilde{o})$, $(\tilde{o}, \tilde{e}, \tilde{k}_2)$ and $J(D)$. Hence M_1 is generated by an element

$$\alpha=(\tilde{e}, \tilde{k}_1, \tilde{o})x+(\tilde{o}, \tilde{e}, \tilde{k}_2)y+j\,, \tag{2}$$

where x or y is not in J and j is in $J(D)$. Since R is basic, we may assume that x and y are in eRe as above. Here we shall observe the element α of the form $k_1=k_2=e$ in (2), dividing into three cases:

i) y is in eJe, ii) x is in eJe and iii) x and y are units in eRe.

Case i). $M_1\supseteq\alpha x^{-1}A_1=\{\tilde{a}, \tilde{k}_1\tilde{a}, o)\,|\,a\in A_1\}\neq 0$ and $M_1\supseteq\alpha x^{-1}A_2\supseteq(0\ \tilde{A}_2, -)\neq 0$. Hence $|M_1|\geqslant 3$.

Case ii). We have similarly $|M_1|\geqslant 3$.

Thus we have the following lemma.

Lemma 7. *Assume* $(**)_2$. *Let* $D=N_1\oplus N_2\oplus N_3$ *be as in* Cases I *and* II *and* M *the maximal submodule of* D *given as above. Then there exists a hollow direct summand* M_1 *of* M *with* $|M_1|=2$, *whose generator* α *is of the form in* Case iii).

Lemma 8. *Assume* $(**)_2$. *In Case* I, *there exists a unit* x *in eRe satisfying* $xB=B$ *and* $xA_1\equiv A_2$ *(mod* B*), where* $eJ=A_1\oplus A_2\oplus B$.

Proof. Let A_i, N_i and B be as in Case I. Then there exists α of Case iii) by Lemma 7, which generates a hollow direct summand M_1 of M with $|M_1|=2$. We may assume that M_1 contains $\alpha=(\tilde{e}, \tilde{e}+\tilde{y}, \tilde{y})+j$ and y is a unit in eRe. Then $\alpha A_1=\{(\tilde{a}_1, (\tilde{e}+\tilde{y})a_1, \tilde{y}a_1)\,|\,a_1\in A_1\}\neq 0$. Assume $yB\neq B$. Then there exists $b\neq o$ in B such that $yb=a_1+a_2+b'$, where a_i is in A_i, b' in B and $a_1+a_2\neq o$. Then $\alpha b=(\tilde{o}, (\tilde{e}+\tilde{y})b, \tilde{y}b)=(\tilde{o}, \tilde{a}_1+\tilde{a}_2, \tilde{a}_2)\neq o$, and so $|M_1|\geqslant 3$, a contradiction. Therefore $yB=B$. Similarly, $\alpha A_2=(0, -, \tilde{y}A_2)=0$, and so $yA_2\subseteq A_1\oplus B$. Since $yB=B$, $yA_2\equiv A_1$ (mod B).

Lemma 9. *If* $|eJ|\geqslant 3$, *all the simple right ideals in eR are isomorphic to one another provided that* $(**)_2$ *is satisfied.*

Proof. Assume $|eJ|\geqslant 3$ and $A_1\not\approx A_3$ for simple right ideals A_1 and A_3 in eJ. Since $|eJ|\geqslant 3$, we have $eJ=A_1\oplus A_2\oplus A_3\oplus B$ as in Case II. Let M_1 and α be as above. Then $\alpha A_1\neq 0$. Let $ya_3=\beta_1+\beta_2+\beta_3+b'$, where b' is in B and β_i, a_i are in A_i. Since $A_1\not\approx A_3$, $\beta_1=o$. Hence, since $|M_1|=2$, $\alpha a_3=(o, \tilde{\beta}_2+\tilde{a}_3, \tilde{\beta}_3)=o$ implies

$$\beta_1=\beta_2=\beta_3=o\,. \tag{3}$$

Therefore $yA_3\subseteq B$. Let $yb=a_1+a_2+a_3+b'$, where b, b' are in B and a_i in A_i. Then $\alpha b=(\tilde{o}, \tilde{a}_2, \tilde{a}_3)=o$ by the assumption $|M_1|=2$. Hence $a_2=a_3=o$, and so $yB\subseteq B\oplus A_1$. Let π be the projection of $B\oplus A_1$ onto A_1. Since $\pi yB=A_1$ (note that $yA_3\subseteq B$ implies $yB\neq B$), $B=A_1'\oplus B_1$, where $A_1'\approx A_1$ and $B_1=\ker\pi y$. Hence $yB_1\subseteq B$. Since $yA_3\subseteq B$ as above, and $A_3\approx yA_3\not\approx A_1'$, $yA_3\subseteq B_1$. Put $B_2=yA_3+y^2A_3+\cdots+y^nA_3=yA_3+\cdots+y^nA_3+y^{n+1}A_3$ for some n. Since $yB_1\subseteq B$ and $y^kA_3\not\approx A_1'$, $y^kA_3\subseteq B_2=yB_2\subseteq B_1$ by the above fact and induction on k of y^kA_3, which is a contradiction, for $A_3\not\subseteq B_2$, $yA_3\subset B_2$ and $yB_2=B_2$.

Proposition 10. *Assume* $(**)_2$ *and* $eJ^2=0$. *If* $|eJ|\geqslant 3$, $eRA_1=eJ$, *where* A_1 *is a simple right ideal in* eJ. *Hence* eJ *is a simple two-sided ideal of* R.

Proof. Put $eJ=eRA_1\oplus B$. Assume $B\neq 0$. Put $B=A_2\oplus B_0$ and $eRA_1=A_1\oplus C_0$. Then $eJ=A_1\oplus A_2\oplus(B_0\oplus C_0)$. By Lemma 8, there exists an x in eRe such that $xA_1\subseteq A_2\oplus B_0\oplus C_0$; $xa_1=a_2+b_0+c_0$ $(a_2\neq o)$ for $a_1\neq o$ in A_1. Hence $xa_1-c_0=a_2+b_0\in eRA_1\cap B=0$, which is a contradiction.

Proposition 11. *Assume* $(**)_2$ *and* $eJ^2=0$. *There exist two simple right ideals not isomorphic to each other in* eJ *or* $|eJ|=1$ *if and only if* $\Delta=\Delta(A_1)$ *for a simple right ideal* A_1. *In this case,* $eJ=A_1\oplus A_2$ *or* $A_2=0$.

Proof. This is clear from Lemmas 8 and 9.

3. Main theorems

Let R be a right artinian ring with identity. We have shown in [5], Corollary 3 that R is a right serial ring if and only if

I *Every submodule of a finite direct sum of hollow* (*serial*) *modules is also a direct sum of hollow modules, and*

II *R is a right QF-2 ring.*

We shall study, in this section, a ring R satisfying Condition I. It is clear that Condition I is preserved by Morita equivalence, and hence we may assume that R is a basic ring. Then if $R=\sum\oplus e_iR$ for primitive idempotents e_i, $e_jRe_i=e_iJe_j$ for $i\neq j$ and e_iRe_i/e_iJe_i is a division ring.

If every finitely generated indecomposable R-module is hollow, R is called a ring of *right local type* following Tachikawa [8] (see [7]). Now we assume that every indecomposable injective module is finitely generated (e.g. R is an algebra of finite dimension over a field, cf. [6]). It is clear that if R is of right local type, then R satisfies Condition I and every indecomposable injective module is hollow. Conversely, we assume the above two conditions. Let M be a finitely generated indecomposable module. Then the injective envelope of M is a finite direct sum of indecomposable injectives, which are hollow. Hence M is hollow by Condition I, and so R is a ring of right local type.

It is not easy for the author to give a characterization of R with Condition I. Hence we shall restrict ourselves to a case $J^2=0$. From now on, we always assume $J^2=0$. In this section we shall add one more assumption: $|eJ|\leqslant 4$ for every primitive idempotent e.

Theorem 12. *Let R be a right artinian ring with $J^2=0$. Assume $|eJ|\leqslant 4$ for every primitive idempotent e. Then R satisfies Condition* I *if and only if eJ has one of the following forms*:

i) $eJ=A_1$.

ii) $eJ=A_1\oplus A_2$; $A_1\not\approx A_2$.

iii) $eJ=A_1\oplus A_2$; $A_1\approx A_2$ *and* a), *for any right ideals A and A' with $|eR/A|=|eR/A'|=2$, $eR/A\approx eR/A'$; i.e. $A=xA'$ for a unit element x in eRe and* b) $[\Delta\colon \Delta(A)]=2$.

iv) $eJ=A_1\oplus A_2\oplus A_3$; $A_1\approx A_2\approx A_3$ *and* iii–a) *and* iii)–b) *are satisfied for right ideals A' in eJ with $|eR/A'|=2$ and* a), *for any right ideals B, B' in eJ with $|eR/B|=|eR/B'|=3$, $eR/B\approx eR/B'$ and* b) $[\Delta\colon \Delta(B_1)]=3$, *where $B_1=A_1\oplus A_2$.*

v) $eJ=A_1\oplus A_2\oplus A_3\oplus A_4$; $A_1\approx_2 A\approx A_3\approx A_4$ *and* iii)–a), iii)–b) *and* iv)–a), iv)–b) *are satisfied for right ideals A and B with $|eR/A|=2$ and $|eR/B|=3$, respectively and* a), *for any right ideals C, C' of eJ with $|eR/C|=|eR/C'|=4$, $eR/C\approx eR/C'$,* b) $[\Delta\colon \Delta(C)]=4$ *and* c) $\mathrm{End}_R(A_1)=\Delta(A_1)$, *where the A_i are simple right ideals in eR, $\Delta=eRe/eJe$ and $\Delta(A)=\{x\in\Delta\,|\,xA\subseteq A\}$.*

Proof of "Only if" part.

i), ii) and iii).

We assume Condition I, and hence $(*)$ and $(**)_2$. We may assume $eJ=A_1\oplus A_2$ and $A_1\approx A_2$. Let A_3 be a simple right ideal in eJ. Then $A_1=A_3$ or $eJ=A_1\oplus A_3$. It is clear that $A_1\approx A_3$. Hence there exists a unit element x in eRe such that $xA_1=A_3$ by Lemma 8. Therefore $[\Delta\colon \Delta(A_1)]=2$ by Lemma 3 and Corollary to Lemma 5.

iv) From now on, in this paragraph, we shall assume that R satisfies Condition I and that $eJ=A_1\oplus A_2\oplus\cdots\oplus A_n$; $A_1\approx A_i$ for all i. Hence $D=\sum_{i=1}^{t+1}\oplus N_i$ satisfies $(**)$ by Lemma 3, where $N_i\approx eR/C_i$ and $|N_i|=t$.

Lemma 13. *Let A_1 and B_1 be right ideals in eJ such that $|A_1|=1$ and $|B_1|=n-1$. Then $[\Delta\colon \Delta(A_1)]=n$ and $[\Delta\colon \Delta(B_1)]=2$.*

Proof. Let $[\Delta\colon \Delta(A_1)]=m$ and $\Delta=\Delta(A_1)\oplus x_2\Delta(A_1)\oplus\cdots\oplus x_m\Delta(A_1)$. Then $eJ=\Delta A_1=A_1+\sum_{i\leqslant 2}^{m}x_iA_1$ by Proposition 10, provided $n\geqslant 3$ (note if $n=2$, $A_1\approx B_1$). Hence $m\geqslant n$. On the otner hand, $m\leqslant n$ by Lemma 3 and Corollary to Lemma 5, and hence $m=n$. It is clear from Lemma 5 that $[\Delta\colon \Delta(B_1)]\leqslant 2$. If $n=2$ and $\Delta(B_1)=\Delta$, B_1 is a character submodule of eJ, which contradictions the assumption: $A_1\approx A_i$ by Lemma 8. Hence if $n=2$, $[\Delta\colon \Delta(B_1)]=2$. If $n\geqslant 3$, $eJ=\Delta A_1=\Delta B_1$ by Proposition 10. Hence $[\Delta\colon \Delta(B_1)]=2$.

Lemma 14. *Let A_1, A_2 and B_1, B_2 be as in Lemma 13, respectively. Then there exists a unit x in eRe such that $xA_1=A_2$ $(xB_1=B_2)$.*

Proof. This is clear from Lemma 13 and Proposition 6.

Thus we have shown the "Only if" part for the case $|eJ|=3$.

v). We shall show the "Only if" part for $|eJ|=4$. It is remained to show $[\Delta\colon \Delta(B_1)]=3$ and $\mathrm{End}_R(A_1)=\Delta(A_1)$, where $B_1=A_1\oplus A_2$. $[\Delta\colon \Delta(B_1)]\leqslant 3$ by Lemma 3 and Corollary to Lemma 5. Hence we may show by Proposition 6 that, for B in eJ with $|B|=2$, there exists a unit element y in eRe such that $yB_1=B$, for eJ is an irredundant sum of $B_1+(A_1\oplus A_3)+(A_1\oplus A_4)$. Since $\Delta B_1=eJ$ by Proposition 10, $\Delta\neq\Delta(B_1)=\Delta_2$. Then there exist two elements $\bar{b}$, $\bar{c}$ in Δ which are independent over Δ_2 and $\bar{a}$ in $\Delta-\Delta_2'$, where $\Delta_2'=\Delta(B)$. Put $D=eR/B\oplus eR/B_1\oplus eR/B_1$ (not necessarily $n=4$, but $|B_1|=|B|=n-2$). We shall consider three elements in D as before: $\alpha_1=(\tilde{a}, \tilde{e}, \tilde{o}, \tilde{o})$, $\alpha_2=(\tilde{b}, \tilde{o}, \tilde{e}, \tilde{o})$ and $\alpha_3=(\tilde{c}, \tilde{o}, \tilde{o}, \tilde{e})$. By Lemma 3 we can find a cyclic submodule M_1 with $|M_1|=3$ containing $\beta'=\beta+j$, where $\beta=\sum_{i=1}^{3}\alpha_iy_i$ and $j\in J(D)$. We shall show from the choice of $\{\bar{a}, \bar{b}, \bar{c}\}$ that we may assume that two elements of $\{y_i\}$ are not in eJe. If $\bar{y}_2=\bar{y}_3=o$, assuming $\bar{y}_1=\bar{e}$, we have $|\beta C|=|\{(\tilde{a}c, \tilde{o}, \tilde{o}, \tilde{o})\,|\,c\in C\}|\geqslant 2$, where $eJ=B\oplus C$. On the other hand, $\beta B=(\tilde{a}B, 0, 0, 0)\neq 0$, and hence

$|J(M_1)| \geqslant |\beta C| + |\beta B| \geqslant 3$, which is a contradiction. If $\bar{y}_1 = \bar{y}_3 = o$, $|\beta C'| \geqslant 2$, where $eJ = B_1 \oplus C'$, and hence $\beta B_1 = (\tilde{b}B_1, 0, 0, 0)$ must be zero by the similar argument as above. Hence $bB_1 = B$. Similarly, if $\bar{y}_1 = \bar{y}_2 = o$, $cB_1 = B$. Hence we may assume that some two elements of $\{y_i\}$ are not in eJe. Assume $\bar{y}_1 \neq o$ and $\bar{y}_2 \neq o$. Then we may assume $\beta = (\tilde{a} + \tilde{b}y_2 + \tilde{c}y_3, \tilde{e}, \tilde{y}_2, \tilde{y}_3)$. Since $\beta C = (-, \tilde{C}, -, -)$, $|\beta C| \geqslant 2$. Hence $\beta B = (-, 0, \tilde{y}_2 B, -) = 0$. Therefore $y_2 B = B_1$. We have the same situation for $\bar{y}_1 \neq o$ and $\bar{y}_3 \neq o$. Finally, assume $\bar{y}_1 = o$, $\bar{y}_2 \neq o$ and $\bar{y}_3 \neq o$. Then $\beta = (\tilde{b} + \tilde{c}y_3, \tilde{o}, \tilde{e}, \tilde{y}_3)$. As above, $\beta B_1 = 0 = \{(\tilde{b} + \tilde{c}y_3)b_1, \tilde{o}, \tilde{o}, \tilde{y}_3 b_1) \mid b_1 \in B_1\}$. Hence $(\tilde{b} + \tilde{c}y_3)B_1 \subseteq B$ and $y_3 B_1 \subseteq B_1$, which implies that $\bar{y}_3$ is in Δ_2. Since $\bar{b}$ and $\bar{c}$ are independent over Δ_2, $\bar{b} + \bar{c}\bar{y}_3 \neq o$ and so $b + cy_3$ is a unit element. Therefore $[\Delta \colon \Delta_2] = 3$ by Proposition 6.

From the above argument we have the following lemma.

Lemma 15. $[\Delta \colon \Delta(B)] = 3$, *where* $B \subseteq eJ$ *and* $|B| = n-2$.

Lemma 16. $\mathrm{End}_R(A_1) = \Delta(A_1)$ *if* $|eJ| = 4$, *where* A_1 *is a simple right ideal in* eJ.

Proof. Let f be a non-zero element in $\mathrm{Hom}_R(A_1, A_3)$. Put $A_3' = \{f^{-1}(a_3) + a_3 \mid a_3 \in A_3\} \subseteq A_1 \oplus A_3$. Then $B \oplus A_3' = B \oplus A_3$, where $B = A_1 \oplus A_2$. There exist $\bar{a}$ and $\bar{b}$ in Δ such that $a(B) = A_4 \oplus A_3'$ and $b(A_1) = A_3$ by Proposition 6 and Lemmas 14 and 15. Let M be a maximal submodule of $D = eR/B \oplus eR/B \oplus eR/A_1$ whose basis modulo $J(D)$ is $\{\bar{\alpha}_1 = (\bar{a}, \bar{e}, o), \bar{\alpha}_2 = (\bar{b}, o, \bar{e})\}$. We define an epimorphism $\varphi \colon eR \oplus eR \to \alpha_1 R + \alpha_2 R \subset M$ by setting $\varphi(r' + s') = \alpha_1 r' + \alpha_2 s' = (\tilde{a}r' + \tilde{b}s', \tilde{r}', \tilde{s}')$ for $r', s' \in eR$. Let $r+s$ be in $\ker \varphi$. Then r is in B and s in A_1. Since $a(B) = A_4 \oplus A_3'$, $B = B_1 \oplus B_2$, where $B_1 = a^{-1}(A_4)$ and $B_2 = a^{-1}(A_3')$. Now $ar + bs \in B = B_1 \oplus B_2$. Put $r = b_1 + b_2$; $b_i \in B_i$. Then $ar + bs = ab_1 + ab_2 + bs \in B$, where $ab_2 \in A_3'$ and $bs \in A_3$. Since $ab_2 \in A_3'$, $ab_2 = f^{-1}(x) + x$ for some $x \in A_3$. Hence $o \equiv ab_1 + ab_2 + bs = f^{-1}(x) + (x + bs) + ab_1 \pmod B$ implies $x = -bs$ and $b_1 = o$, and so $-ab_2 = f^{-1}(bs) + bs$. Thus we obtain an isomorphism $g \colon A_1 \to B_2$ by setting $g(s) = b_2$. Hence $\ker \varphi = \{g(s) + s \mid s \in A_1\} \subseteq eR \oplus eR$ and $(\alpha_1 R + \alpha_2 R) \approx (eR \oplus eR)/\ker \varphi$. On the other hand, $|M| = 9 = |(eR \oplus eR)/\ker \varphi|$. Hence $M \approx (eR \oplus eR)/\ker \varphi$. Now M is a direct sum of hollow modules by assumption and M is decomposable for $\bar{M} = \bar{\alpha}_1 R \oplus \bar{\alpha}_2 R$. Therefore g is extendible to an element in $\mathrm{Hom}_R(eR, eR) = eRe$, say $g(s) = cs$; $c \in \Delta$ by [2], Theorem 2.5 and [7], Lemma 1.2. $g(s) = b_2 = cs$ and $(-ac - b)s = f^{-1}(bs)$. Accordingly, f is given by the left-sided multiplication of $(-acb^{-1} - e)^{-1}$. Let h be in $\mathrm{Hom}_R(A_1, A_1)$. Then $bh \in \mathrm{Hom}_R(A_1, A_3)$. Hence bh is given by an element c' in Δ from the above.

Proof of "If" part.

We assume the conditions in Theorem 12 and we shall show that R satisfies Condition I. In order to see this we need several lemmas below.

Let N_1 and N_2 be hollow modules. Assume that $N_1=eR$ and $N_2=eR/A$ for a right ideal A in eJ. Let f' be an element in $\mathrm{Hom}_R(eR/eJ, eR/eJ)$. Then f' is given by the left-sided multiplication of an element $\bar{k}$ in eRe/eJe, where k is in eRe. Let ν be the natural epimorphism of eR to eR/eA. Then $\nu k\in \mathrm{Hom}_R(eR, eR/eA)$ induces f'. Hence we have the following lemma.

Lemma 17. *Every element in* $\mathrm{Hom}_R(eR/eJ, eR/eJ)$ *is lifted to an element in* $\mathrm{Hom}_R(eR, eR/eA)$.

By N_i we denote the hollow module of the form e/e_iB, where B is a right ideal. Let $T=\sum_{I_1}\oplus N_{1i}\oplus\sum_{I_2}\oplus N_{2i}\oplus\cdots\oplus\sum_{I_n}\oplus N_{ni}$, and let M be a maximal submodule of T. Then $M\supseteq J(T)$ and $\bar{T}=T/J(T)\supseteq\bar{M}=M/J(T)$. We shall show by induction on $\sum_i|I_i|$ that M is a direct sum of hollow modules. Since $\sum_{I_i}\oplus\bar{N}_{ij}$ is the homogeneous component of $\bar{T}$ and $\bar{M}$ is maximal, $\bar{M}=\sum\oplus(\bar{M}\cap\sum_{I_i'}\oplus\bar{N}_{ij})$ and $\bar{M}\supseteq\sum_{I_i'}\oplus\bar{N}_{i'j}$ except some i, say $i=1$. Therefore $M=M_1\oplus\sum_{i\geqslant 2}\sum_{I_i}\oplus N_{ij}$ and M_1 is a maximal submodule of $\sum_{I_1}\oplus N_{1i}$. Hence we may assume $n=1$; i.e. $T=\sum_I\oplus N_i$ and $N_i\approx e_1R/\epsilon_1B_i$. Let π_i be the projection of T onto N_i. If $\bar{\pi}_i(\bar{M})=0$ for some i, $M=J(N_i)\oplus\sum_{j\neq i}\oplus N_j$ and $J(N_j)$ is a direct sum of simple modules by the form of N_i. Hence M is a direct sum of hollow modules. Therefore we assume $\bar{\pi}_i(\bar{M})\neq 0$ for all i. Then we have the following lemma.

Lemma 18. *Let T, M and N_i be as above. If N_1 is either isomorphic to eR or eR/eJ, then M contains a non-zero direct summand of T.*

Proof. Assume that N_1 is simple. Since M is maximal, $M\supseteq N_1$ or $M\oplus N_1=T$. Next assume $N_1=eR$. Since $\bar{\pi}_1(\bar{M})\neq 0$, $\bar{M}$ contains a simple submodule $\bar{C}$ such that $\bar{\pi}_i(\bar{C})\neq 0$. Furthermore every element in $\mathrm{Hom}_R(\bar{N}_1, \bar{N}_i)$ is liftable by Lemma 17 for all i. Therefore T contains a direct summand T_1 isomorphic to N_1 such that $\bar{T}_1=\bar{C}$ by Lemma 4. Since $M\supseteq J(T)$, $M\supseteq T_1$.

Let T_1 be a direct summand of T as in Lemma 18. Then $T=T_1\oplus T_2\supseteq M=T_1\oplus(M\cap T_2)$ and $M\cap T_2$ is a maximal submodule of T_2.

i), ii) and iii).

Now $\{eR, eR/eJ$ and $eR/A_1\}_e$ or $\{eR, eR/eJ, eR/A_1$ and $eR/A_2\}_e$ is the representative set of hollow modules, accordingly as $\{A_2=0$ or $A_1\approx A_2\}$ or $\{A_1\not\approx A_2\}$. Hence it is sufficient to consider the case $N_i=eR/A_1$ or eR/A_2 for all i by Lemma 18. Under this assumption we shall show by induction on $|I|$ that M is a direct sum of hollow modules.

1) $A_2=0$.

Then A_1 is a character submodule of eR. Hence T has the lifting property of direct decompositions modulo the radical by [3], Theorem 3 (cf. Lemma 4), and so $M \approx J(N_1)\oplus\sum_{i\neq 1}\oplus N_i$.

2) $|I|=1$.

This is trivial.

3) $|I|=2$.

Let M be a maximal submodule of $D=eR/N_1\oplus eR/N_2$. Then $\bar{M}=\bar{\alpha}R$ for some $\alpha\in M$. Since $\alpha R\not\subseteq J(D)$ and $J(D)$ is semi-simple, $\alpha R+J(D)=\alpha R\oplus C_1\oplus\cdots\oplus C_i$, where the C_i are simple. Hence $M=\alpha R\oplus C_1\oplus\cdots\oplus C_i$ for $\bar{M}=\bar{\alpha}R$. We shall show the explicit form of αR in the following.

α) $A_1\not\approx A_2$.

Then A_1 and A_2 are character submodules.

a) $T=eR/A_1\oplus eR/A_1$ or $eR/A_2\oplus eR/A_2$.

We have the same situation as in 1).

b) $T=eR/A_1\oplus eR/A_2$.

We may assume $\bar{M}=\{\bar{x}+f(\bar{x})\,|\,x\in eR/A_1,\ f\in \mathrm{Hom}_R(\overline{eR}/A_1,\ \overline{eR}/A_2)=\mathrm{Hom}_R(eR/eJ,\ eR/eJ)\}$. f is given by the left-sided multiplication of an element $\bar{z}$, where z is in eRe. Take a mapping θ: $eR\to T$ given by setting $\theta(a)=\nu_1(a)+\nu_2(za)$, where ν_i: $eR\to eR/A_i$ is the natural epimorphism for $i=1$, 2. Then $\ker\theta=A_1\cap A_2=0$ provided $z\notin eJe$, and $\overline{\mathrm{im}\,\theta}=\bar{M}$. Hence $M=\mathrm{im}\,\theta\approx eR$ for $|M|=|eR|=3$. (If $z\in eJe$, it is clear that $M=eR/A_1\oplus eJ/A_2$.)

β) $A_1\approx A_2$ and hence $eR/A_1\approx eR/A_2$.

Let M and z be as above. If z is in $\Delta(A_1)$, f is liftable. Hence $M\approx eR/A_1\oplus eJ/A_1$. If $z\notin\Delta(A_1)$, $\bar{z}^{-1}(A_1)\neq A_1$, and so $A_1\cap\bar{z}^{-1}(A_1)=0$. We can define the θ: $eR\to T$ as in i). Then $\ker\theta=A_1\cap\bar{z}^{-1}(A_1)=0$. Therefore $M\approx eR$.

4) $|I|=3$. α) $A_1\not\approx A_2$.

a) $T=eR/A_1\oplus eR/A_1\oplus eR/A_1$ or $T=eR/A_2\oplus eR/A_2\oplus eR/A_2$.

b) $T=eR/A_1\oplus eR/A_1\oplus eR/A_2$.

Since $\bar{\pi}_3(\bar{M})\neq 0$, $\bar{M}$ contains a simple submodule $\bar{C}$ contained in $\overline{eR}/A_1\oplus\overline{eR}/A_1$. Then $eR/A_1\oplus eR/A_1$ contains a direct summand T_1 of T such that $\bar{T}_1=\bar{C}$ by 3)-a). Then $M\supseteq T_1$. Let $T=T_1\oplus T_2\oplus eR/A_2\supseteq M=T_1\oplus(M\cap(T_2\oplus eR/A_2))$. Then M is a direct sum of hollow modules by 3).

β) $A_1\approx A_2$ and hence $[\Delta\colon\Delta(A_1)]=2$.

Let $T=eR/A_1\oplus eR/A_1\oplus eR/A_1\supseteq M$ be as above. Then M contains a direct summand of T by Proposition 6.

From the above argument we obtain the following lemma.

Lemma 19. *Let* $D=\sum_{i=1}^{3}\oplus N_i$ *and* $N_i=eR/A_i$ *as above such that* $|N_i|=2$ *for* $i=1$, 2, 3, *and* e *a primitive idempotent. Then under the conditions of Theorem* 12, *every maximal submodule of* D *contains a direct summand* D_1 *of* D *with* $|D_1|=2$.

We note that if $\bar{\pi}_i(\bar{M})=0$, $M=J(N_1)\oplus N_2\oplus N_3$, and so M contains the direct summand N_2 of D. We shall show by induction on t that the content of Lemma 19 is true for t direct summands ($t\geqslant 3$).

Let N_i be as in Lemma 19 and $D=\sum_{i=1}^{t}\oplus N_i$. Assume $t\geqslant 4$. Let M be a maximal submodule of D. As is well known, $\bar{M}$ contains a maximal submodule $\bar{M}_1$ in $\bar{N}_1\oplus\bar{N}_1\oplus\bar{N}_2\oplus\overset{i}{\vee}\bar{N}_t=\bar{D}_0$. Hence M contains a maximal submodule M_1 in $N_1\oplus N_2\oplus\overset{i}{\vee}\oplus N_t=D_0$. Since M_1 contains a direct summand D_0 of D_1 by the induction hypothesis, M contains a direct summand D_1 of D with $|D_1|=2$. Thus we can show, by 1)$\sim$4), Lemma 19 and induction on the number of direct summands of D, that every maximal submodule is a direct sum of hollow modules.

Now let P be a submodule of D and let M be a maximal submodule of D containing P. Then M is also a direct sum of hollow modules. Repeating this manner, we can show that P is a direct sum of hollow modules.

iv) We shall show the "If" part for $|eJ|=3$.

Put $\Delta(i, k)=\{x\in\Delta\,|\,x(A_1\oplus A_2\oplus\cdots\oplus A_i)\subseteq(A_1\oplus A_2\oplus\cdots\oplus A_k)\}$ for $i<k$.

Lemma 20. *We assume that i) $eR/(A_1\oplus A_2\oplus\cdots\oplus A_i)\approx eR/B$ for any right ideal B in eJ with $|eR/B|=n-i+1$, where $n=|eJ|$ and ii) $[\Delta\colon \Delta(A_1\oplus A_2\oplus\cdots\oplus A_i)]=n-i+1$. Then $[\Delta(i, k)\colon \Delta(A_1\oplus A_2\oplus\cdots\oplus A_i)]=k-i+1$ as a right $\Delta(A_1\oplus A_2\oplus\cdots\oplus A_i)$-module for $k>i$* (cf. Proposition 6).

Proof. Put $B_0=A_1\oplus\cdots\oplus A_{i-1}$, $B=B_0\oplus A_i$. Then there exists a unit element x_{i+j} in eRe such that $x_{i+j}B=B_0\oplus A_{i+j}$ by i). Since $B+\sum_{t=i+1}^{n}x_tB$ is an irredundant sum, $(\Delta(B)+\sum x_t\Delta(B))$ is a direct sum. Hence $\Delta=\Delta(B)\oplus\sum\oplus x_t\Delta(B)$ by ii). Let $x=\delta_0+\sum x_t\delta_t$ be an element in $\Delta(i, k)$, where the δ_i are in $\Delta(B)$. Note that $x_{i+j}\delta_{i+j}(B)=B_0\oplus A_{i+j}$. Then $\delta_l=o$ for $l<k$, and so $x\in\Delta(B)\oplus x_{i+1}\Delta(B)\oplus\cdots\oplus x_k\Delta(B)\subseteq\Delta(i, k)$.

For the latter use, we assume that $|eJ|=n$ and $B_1=A_1\oplus\cdots\oplus A_{n-2}$. Then the following cases 1) and 2) are trivial from the remark of case $|eJ|=2$.

1) $D=eR/B_1\oplus eR/B_1$.

Let M be a maximal submodule of D and $\bar{\alpha}_1=(\bar{e}, \bar{a})$, a basis of $\bar{M}$. If $\bar{a}$ is in $\Delta_{n-2}=\Delta(B_1)$ ($[\Delta\colon \Delta_{n-2}]=3$), M contains a direct summand of D by Lemma 4. If $\bar{a}$ is not in Δ_{n-2}, $\bar{a}^{-1}B_1\cap B_1=C$, where $|eR/C|=4$ or 5. Then M contains an isomorphic image M_1 of eR/C with $\bar{M}=\bar{M}_1$. Hence $M=M_1\oplus M_2$, where M_2 is simple.

2) $D=eR/B\oplus eR/(B_1\oplus A_{n-1})$, where $B=B_1$ or $B=B_1\oplus A_{n-1}$.

Let M and α be as above. If $aB\not\subseteq B_1\oplus A_{n-1}$, M contains a direct summand of D. Assume $aB\subseteq B_1\oplus A_{n-1}$. If $B=B_1\oplus A_{n-1}$, $\bar{a}^{-1}(B_1\oplus A_{n-1})\cap(B_1\oplus A_{n-1})=C$, where C is a submodule of $B_1 \oplus A_{n-1}$ with $|C|=d-2$. Then M contains an isomorphic image of eR/C, and hence $M \approx eR/C$. If $B=B_1$, $B_1 \subseteq \bar{a}^{-1}(B_1 \oplus A_{n-1})$ and so $|B_1 \cap \bar{a}^{-1}(B_1 \oplus A_{n-1})|=n-3$. Then M contains an isomorphic image of eR/C', where $|C'|=n-1$, and hence $M \approx eR/C'$.

3) $D=eR/B_1 \oplus eR/B_1 \oplus eR/B_1$.

Let M be a maximal submodule of D. We shall show that M contains a non-zero direct summand of D. We may assume that $\bar{M}$ has a basis $\{\bar{\alpha}_1=(\bar{a}', o, \bar{e}), \bar{\alpha}_2=(\bar{b}', \bar{e}, o)\}$ with $\bar{a}'\bar{b}' \neq o$. Assume $\bar{a}'$, $\bar{b}'$ and $\bar{e}$ are dependent over $\Delta_{n-2}=\Delta(B_1)$. Then there exist $\bar{x}$ and $\bar{y}$ in Δ_{n-2} such that $\bar{a}'x+\bar{b}'y \in \Delta_{n-2}$ and $\bar{x} \neq o$ or $\bar{y} \neq o$. $\theta=\alpha_1 x+\alpha_2 y=(\tilde{w}, \tilde{x}, \tilde{y})$ is in M, where $w=a'x+b'y$. Since all components of θ modulo eJ belong to Δ_{n-2}, M contains a direct summand M_1 with $\bar{M}=\bar{\theta}R$ by Lemma 4. Next assume $\bar{a}'$, $\bar{b}'$ and $\bar{e}$ are independent over Δ_{n-2}. We consider special elements β_1, β_2 in D. Let $\beta_1=(\tilde{a}, \tilde{x}, \tilde{y})$ and $\beta_2=(\tilde{b}, \tilde{x}', \tilde{y}')$ be two elements in D, where neither $\bar{a}$ nor $\bar{b}$ belongs to Δ_{n-2}. Assume $\begin{pmatrix} \bar{x} & \bar{x} \\ \bar{y} & \bar{y} \end{pmatrix}$ is a unit matrix in $(\Delta_{n-2})_2$. Then $\bar{\beta}_1$ and $\bar{\beta}_2$ are independent over Δ. First we consider the following case: $a(B_1) \subseteq B_1 \oplus A_{n-1}$ and $b(B_1) \subseteq B_1 \oplus A_n$. Since $B_1 \oplus A_{n-1}=B_1+a(B_1)$ and $B_1 \oplus A_n=B_1+b(B_1)$, $a(B_1) \cap B_1=C_1$ and $b(B_1) \cap B_1=C_2$ are of length $n-3$. We shall define a homomorphism φ_1 of eR to D by setting $\varphi_1(r)=(\tilde{a}r, \tilde{x}r, \tilde{y}r)$. Let r be in $\ker \varphi_1$. Then $ar \in B_1$, $xr \in B_1$ and $yr \in B_1$. Since $\bar{x}$ and $\bar{y}$ are in Δ_{n-2} and $\bar{x} \neq o$ or $\bar{y} \neq o$, $r \in B_1 \cap \bar{a}^{-1}(B_1)$ $=\bar{a}^{-1}(C_1)$.Hence φ_1 induces a monomorphism of $eR/a^{-1}(C_1)$ to D. Similarly, we obtain a monomorphism φ_2 of $eR/b^{-1}(C_1)$ to D. Next we shall show that $\beta_1 R+\beta_2 R=M'$ is a maximal submodule of D. $\beta B=(\tilde{a}B_1, 0, 0)=(\tilde{E}_1, 0, 0)$, where $a(B_1)=C_1 \oplus E_1$ and E_1 is simple and $\beta_2 B_1=(\tilde{b}B_1, 0, 0)=(\tilde{E}_2, 0, 0)$, where $b(B_1)=C_2 \oplus E_2$. Since $B_1+a(B_1)=B_1 \oplus A_{n-1} \neq B_1 \oplus A_n=B_1+b(B_1)$, $\tilde{E}_1 \neq \tilde{E}_2$. Let u and v be any elements in $A_{n-1} \oplus A_n$. Then $\beta_1 u+\beta_2 v=(\tilde{a}u+\tilde{b}v, \tilde{x}u+\tilde{x}'v,$ $\tilde{y}u+\tilde{y}'v)$. Since $\begin{pmatrix} \bar{x} & \bar{x}' \\ \bar{y} & \bar{y}' \end{pmatrix}$ is a unit, $\beta_1 u+\beta_2 v=o$ if and only if $u=v=o$. Hence $M' \supseteq J(M') \supseteq (\beta_1 B \oplus \beta_1 B \oplus \beta_1(A_{n-1} \oplus A_n) \oplus \beta_2(A_{n-1} \oplus A_n))$, and so $M' \supseteq J(D)$ for $|J(D)|=6$. Since $\bar{M}'=\bar{\beta}_1 R+\bar{\beta}_2 R$, M' is a maximal submodule of D. Hence $|M'|=8$. On the other hand, we have an epimorphism $\tilde{\varphi}_1 \oplus \tilde{\varphi}_2$: $eR/a^{-1}(C_1) \oplus$ $eR/b^{-1}(C_2) \to M'$. $|eR/\bar{a}^{-1}(C_1) \oplus eR/\bar{b}^{-1}(C_2)|=8$ for $|\bar{a}^{-1}(C_1)|=|\bar{b}^{-1}(C_2)|=n-3$. Hence $\tilde{\varphi}_1 \oplus \tilde{\varphi}_2$ is an isomorphism and M' is a direct sum of hollow modules. Now we shall come back to the beginning. Assume $\bar{a}'$, $\bar{b}'$ and $\bar{e}$ are independent over Δ_{n-2}. There exist $\bar{a}''$, $\bar{b}''$ in Δ such that $a''(B_1)=A_1 \oplus \cdots \oplus A_{n-3} \oplus A_{n-1}$ and $b''(B_1)=A_1 \oplus \cdots \oplus A_{n-3} \oplus A_n$ by assumption. Then $\bar{e}$, $\bar{a}''$ and $\bar{b}''$ are independent over Δ_{n-2}, for $e(B_1)+a''(B_1)+b''(B_1)$ is an irredundant sum. There exist x, y, z and x', y', z' in Δ_{n-2} such that $\bar{a}''=\bar{e}z+\bar{a}'x+\bar{b}'y$ and $\bar{b}''=\bar{e}z'+$ $\bar{a}'x'+\bar{b}'y'$ by the assumption $[\Delta: \Delta_{n-2}]=3$. It is clear that $\begin{pmatrix} \bar{x} & \bar{x}' \\ \bar{y} & \bar{y}' \end{pmatrix}$ is a unit matrix in $(\Delta_{n-2})_2$. Then M contains $\beta_1=\alpha_1 x+\alpha_2 y=(\tilde{a}''-\tilde{e}z, \tilde{x}, \tilde{y})$ and $\beta_2=$

$\alpha_1 x' + \alpha_2 y' = (\tilde{b}''-\tilde{e}z', \tilde{x}', \tilde{y}')$. It is clear that $(a''-ez)(B_1) \subseteq B_1 \oplus A_{n-1}$, $(b''-ez')(B_1) \subseteq B_1 \oplus A_n$ and neither $(\bar{a}''-\bar{e}z)$ nor $(\bar{b}''-\bar{e}z')$ belongs to Δ_{n-2}. Hence, as was shown in the initial part, $M=\beta_1 R+\beta_2 R$ is a direct sum of hollow modules M_i with $|M_i|=4$.

4) $D=eR/B_1 \oplus eR/B_1 \oplus eR/(B_1 \oplus A_{n-1})$.

We may assume that the maximal submodule $\bar{M}$ has the basis $\{\bar{\alpha}_1=(\bar{e}, o, \bar{a}), \bar{\alpha}_2=(o, \bar{e}, \bar{b})\}$. Now $[\Delta/\Delta(n-2, n-1): \Delta_{n-2}]=1$ by Lemma 20 and the assumptions in Theorem 12. Hence there exists an element $\bar{z}$ in Δ_{n-2} such that $\bar{a}+\bar{b}z \in \Delta(n-2, n-1)$. Then $\theta=\alpha_1+\alpha_2 z=(\tilde{e}, \tilde{z}, \tilde{a}+\tilde{b}z)$ is an element in M. Therefore M contains a direct summand of D by Lemma 4.

5) $D=eR/B_1 \oplus eR/(B_1 \oplus A_{n-1}) \oplus eR/(B_1 \oplus A_{n-1})$.

Let $\{\bar{\alpha}_1=(\bar{a}, o, \bar{e}), \bar{\alpha}_2=(\bar{b}, \bar{e}, o)\}$ be the basis of $\bar{M}$. Then there exist $\bar{x}$, $\bar{y}$ in $\Delta_{n-1}=\Delta(B_1 \oplus A_{n-1})$ such that $\bar{x} \neq o$ or $\bar{y} \neq o$ and $\bar{a}x+\bar{b}y \in \Delta_{n-1}$ by the assumptions in Theorem 12. Since each component of $\theta=\alpha_1 x+\alpha_2 y=(\tilde{a}x+\tilde{b}y, \tilde{y}, \tilde{x})$ modulo eJ belongs to Δ_{n-1}, M contains a direct summand of D (consider two cases $\bar{a}x+\bar{b}y=o$ and $\bar{a}x+\bar{b}y \neq o$).

6) $D=eR/(B_1 \oplus A_{n-1}) \oplus eR/(B_1 \oplus A_{n-1}) \oplus eR/(B_1 \oplus A_{n-1})$.

Every maximal submodule M of D contains a direct summand M_1 of D by Lemma 5.

7) $D=eR/B_1 \oplus eR/B_1 \oplus eR/B_1 \oplus eR/B_1$.

This is similar to 6).

8) $D=eR/B_1 \oplus eR/B_1 \oplus eR/B_1 \oplus eR/(B_1 \oplus A_{n-1})$ and
$D=eR/B_1 \oplus eR/B_1 \oplus eR/(B_1 \oplus A_{n-1}) \oplus eR/(B_1 \oplus A_{n-1})$.

They are reduced to 4) or 5) (cf. the proof of Case $|eJ|=2$).

9) $D=eR/B_1 \oplus eR/(B_1 \oplus A_{n-1}) \oplus eR/(B_1 \oplus A_{n-1}) \oplus eR/(B_1 \oplus A_{n-1})$.

This is reduced to 6).

10) Let $D=\sum_{i=1}^{t} \oplus eR/C_i$, where $C_i=B_1$ or $B_1 \oplus A_{n-1}$. If $t \geqslant 6$, every maximal submodule M of D has a direct summand of D by the assumptions of Theorem 12 and Proposition 6. Hence M is a direct sum of hollow modules from 1)~9) and by induction. Therefore R satisfies Condition I for $|eJ|=3$ from the similar argument to Case $|eJ|=2$.

v). We assume $|eJ|=4$ and that the conditions of Theorem 12 are satisfied. We have many situations similar to those in Case iv), and so we shall give only some remarks in those cases. Let $\{N_i\}$ be a set of hollow modules such that $N_i \approx eR/C_i$ for some right ideal C_i in eJ. Put $D=\sum_I \oplus N_i$. We shall show that every maximal submodule M of D is a direct sum of hollow modules. We shall do this by induction on $|I|$.

1) $|I| \leqslant 2$.

This is clear from the remark given in Case $|eJ|=2$

Put $B_1=A_1 \oplus A_2$, $C_1=A_1 \oplus A_2 \oplus A_3$, $\Delta_1=\Delta(A_1)$, $\Delta_2=\Delta(B_1)$ and $\Delta_3=\Delta(C_1)$.

2) $|I|=3$.

a) $D=eR/A_1\oplus eR/A_1\oplus eR/A_1$.

Assume that $\bar{M}$ has a basis $\{\bar{\alpha}_1=(\bar{a}, \bar{e}, o), \bar{\alpha}_2=(\bar{b}, o, \bar{e})$ with $\bar{a}\bar{b}\neq o\}$. If $\bar{a}$, $\bar{b}$ and $\bar{e}$ are dependent over Δ_1, M contains a non-zero direct summand of D as in Case iv). Assume $\bar{a}$, $\bar{b}$ and $\bar{e}$ are independent over Δ_1. Then $A_1+a(A_1)+b(A_1)=A_1\oplus a(A_1)\oplus b(A_1)$, since the sum is irredundant and A_1 is simple. We obtain a homomorphism φ: $eR\oplus eR\to M$ given by setting $\varphi(r_1+r_2)=\alpha_1r_1+\alpha_2r_2=(\tilde{a}r_1+\tilde{b}r_2, \tilde{r}_1, \tilde{r}_2)$. It is clear from the above direct sum that φ is an isomorphism. Since $\overline{\mathrm{im}\,\varphi}=\bar{M}$, $M=\mathrm{im}\,\varphi\oplus M_2$, where M_2 is simple.

b) $D=eR/B_1\oplus eR/B_1\oplus eR/B_1$.

This is Case 3) of iv).

c) $D=eR/C_1\oplus eR/C_1\oplus eR/C_1$.

Since $[\Delta: \Delta_3]=2$, M contains a direct summand of D by Lemma 5.

d) $D=N_1\oplus eR/A_1\oplus eR/A_1$, where $N_1=eR/B_1$ or eR/C_1.

Since $[\Delta: \Delta_1]=4$ and $[\Delta(1, 2): \Delta_1]=2$, $[\Delta/\Delta(1, 2): \Delta_1]=2$. Let $D=eR/B_1\oplus eR/A_1\oplus eR/A_1$. We shall use the same notations as in 3) of iv). Let $\beta_1=(\tilde{a}, \tilde{x}, \tilde{y})$ and $\beta_2=(\tilde{b}, \tilde{x}', \tilde{y}')$, where $\begin{pmatrix}\bar{x} & \bar{x}'\\ \bar{y} & \bar{y}'\end{pmatrix}$ is a unit matrix in $(\Delta_1)_2$, $a(A_1)\subseteq B_1\oplus A_3$ and $b(A_1)\subseteq B_1\oplus A_4$. We define a homomorphism φ: $eR\oplus eR\to D$ by setting $\varphi(r+s)=\beta_1r+\beta_2s=(\tilde{a}r+\tilde{b}s, \tilde{x}r+\tilde{x}'s, \tilde{y}r+\tilde{y}'s)$. Assume $\tilde{x}r+\tilde{x}'s\in A_1$ and $\tilde{y}r+\tilde{y}'s\in A_1$. Since $\begin{pmatrix}\bar{x} & \bar{x}'\\ \bar{y} & \bar{y}'\end{pmatrix}$ is a unit matrix in $(\Delta_1)_2$, r and s belong to A_1. Further $\tilde{a}r+\tilde{b}s=\pi_3(\tilde{a}r)+\pi_4(\tilde{b}s)\in A_3\oplus A_4$, where π_i is the projection of eJ onto A_i. Hence $\ker\varphi$ is equal to one of the following: (0), $(A_1+(0))$ and $(A_1\oplus A_1)$. Therefore, for a maximal submodule M^* containing β_1 and β_2, $M^*=((eR+eR)/\ker\varphi)\oplus M_1\oplus M_2\oplus\cdots$, where the M_i are simple or zero. Now let M be any maximal submodule of D and $\{\bar{\alpha}_1=(\bar{a}', \bar{e}, o), \bar{\alpha}_2=(\bar{b}', o, \bar{e})\}$ a basis of $\bar{M}$. First we consider the elements $\bar{\bar{a}}'$ and $\bar{\bar{b}}'$ in $\Delta/\Delta(1, 2)$. If $\bar{\bar{a}}'$ and $\bar{\bar{b}}'$ are dependent over Δ_1, there exist $\bar{x}$ and $\bar{y}$ in Δ_1 such that $\bar{a}'x+\bar{b}'y\in\Delta_2$ and $\bar{x}\neq o$ or $\bar{y}\neq o$. $\theta=\alpha_1x+\alpha_2y$ is in M and θR is a direct summand of D by Lemma 4. Next we assume that $\bar{\bar{a}}'$, $\bar{\bar{b}}'$ are independent over Δ_1. There exist a'', $\bar{b}''$ in Δ such that $a''(B_1)=A_1\oplus A_3$ and $b''(B_1)=A_1\oplus A_4$ by assumption. Then $\bar{\bar{a}}''$ and $\bar{\bar{b}}''$ are independent over Δ_1. Since $[\Delta/\Delta(1, 2)]=2$, there exists a unit matrix $\begin{pmatrix}\bar{x} & \bar{y}\\ \bar{x}' & \bar{y}'\end{pmatrix}$ in $(\Delta_1)_2$ such that $\bar{\bar{a}}''=\bar{\bar{a}}'x+\bar{\bar{b}}'y$ and $\bar{\bar{b}}''=\bar{\bar{a}}''x'+\bar{\bar{b}}'y'$. Then M contains $\beta_1=\alpha_1x+\alpha_2y=(\tilde{a}''+\tilde{w}_1, \tilde{x}, \tilde{y})$ and $\beta_2=\alpha_1x'+\alpha_2y'=(\tilde{b}''+\tilde{w}_2, \tilde{x}', \tilde{y}')$, where $\bar{w}_1$ and $\bar{w}_2$ are in $\Delta(1, 2)$. Hence M is a direct sum of hollow modules by the beginning of d). If $N_3=eR/C_1$, the above argument is valid, for $C_1\supseteq B_1$.

e) $D=eR/B_1\oplus eR/B_1\oplus eR/A_1$.

We need the following lemma.

Lemma 21. *Assume* $\mathrm{End}_R(A_1)=\Delta(A_1)$ *and that there exists a unit element* *x in Δ such that $xA_1=A$ for any simple right ideal A in eJ. Let S_1 and S_2 be simple submodules of eR and eR/A_1, respectively. Then every f in* $\mathrm{Hom}_R(S_1, S_2)$ *is extendible to an element in* $\mathrm{Hom}_R(eR, eR/A_1)$.

Proof. Let ν be the natural epimorphism of eR onto eR/A_1. Then we may assume from the assumption that f is given by the left-sided multiplication of an element $\bar{x}$ in Δ. Then νx induces f.

Let M be a maximal submodule and $\{\bar{\alpha}_1=(\bar{a}, \bar{e}, \mathrm{o}), \bar{\alpha}_2=(\bar{b}, \mathrm{o}, \bar{e})\}$ the basis of $\bar{M}$. If $b(A_1)\subseteq B_1$, M contains a direct summand M_1 of D such that $\bar{M}_1=\bar{\alpha}_2R$ by Lemma 4. Hence we assume $b(A_1)\not\subseteq B_1$.

e–1). $a(B_1)\cap b(A_1)=0$ and $eJ=B_1\oplus a(B_1)$.

Then $b(A_1)=X(f)=\{x+f(x)\,|\,x\in X\}\subseteq B_1\oplus a(B_1)$, where X is a simple submodule of $a(B_1)$ and $f\in\mathrm{Hom}_R(X, B_1)$. Let φ: $eR\oplus eR\to\alpha_1R+\alpha_2R$ be an epimorphism given by setting $\varphi(r+s)=\alpha_1r+\alpha_2s=(\tilde{a}r+\tilde{b}s, \tilde{r}, \tilde{s})$. Ker $\varphi=\{r_1+s_1\,|\,r_1\in a^{-1}(X),\ s_1\in A_1$ and $ar_1+f(ar_1)=bs_1\}$. Hence we obtain an isomorphism $g\colon A_1\to a^{-1}(X)$ such that $g(a_1)=r_1$. Therefore $\alpha_1R+\alpha_2R\approx(eR\oplus eR)/\ker\varphi$ $\approx eR\oplus(eR/A_1)$ by Lemma 21, [7], Lemma 2.1 and [2], Theorem 2.5 (cf. Lemma 4). Accordingly, $M\approx eR\oplus eR/A_1$ for $|M|=9=|eR\oplus eR/A_1|$.

e–2) $a(B_1)\cap b(A_1)=0$ and $B_1\cap a(B_1)=X$ is simple.

e–2.1) $a(B_1)\oplus b(A_1)\supseteq B_1$.

Let $a(B_1)=X\oplus Y$. Since $a(B_1)\oplus b(A_1)=X\oplus Y\oplus b(A_1)$, $B_1=X\oplus Z$, where $Z=B_1\cap(Y\oplus b(A_1))$. For z in Z, $z=y+b(a_1)$; $y\in Y$ and $a_1\in A_1$. $b(A_1)\not\subset B_1$ implies that the mapping: $g(y)=a_1$ is an isomorphism of Y onto A_1. Let φ be as above. Then $\ker\varphi=(a^{-1}(X)\oplus 0)\oplus A_1(k)$, where k: $A_1\to a^{-1}(Y)$ is given by $k(a_1)=a^{-1}g^{-1}(a_1)$. Hence $\alpha_1R+\alpha_2R\approx eR/a^{-1}(X)\oplus eR/A_1$ by Lemma 21. Since $\bar{M}=\bar{\alpha}_1R\oplus\bar{\alpha}_2R$, $M\approx eR/a^{-1}(X)\oplus eR/A_1\oplus M_1$, where M_1 is a simple submodule.

e–2.2) $a(B_1)\oplus b(A_1)\not\supset B_1$.

Then $eJ=a(B_1)\oplus b(A_1)\oplus Z=X\oplus Y\oplus b(A_1)\oplus Z=Y\oplus b(A_1)\oplus B_1$. Hence $M\approx eR/a^{-1}(X)\oplus eR$.

e–3) $a(B_1)\supset b(A_1)$ and $eJ=a(B_1)\oplus B_1$.

Then we have an isomorphism $a^{-1}b$ of A_1 onto a simple submodule X of B_1. Hence $M\approx eR/A_1\oplus eR$ by [7], Lemma 2.1.

e–4). $a(B_1)\supset b(A_1)$ and $a(B_1)\cap B_1=b(A_1)$.

This is contained in Case $B_1\supset b(A_1)$.

e–5). $a(B_1)\supset b(A_1)$ and $a(B_1)\cap B_1=X$ is a simple module not equal to $b(A_1)$.

Let $a(B_1)=X\oplus Y$. Since $B_1\not\supset b(A_1)$, $b(A_1)=Y(f)$ for some f: $Y\to X$. Hence $M\approx eR/a^{-1}(X)\oplus eR/A_1\oplus M_1$ by Lemma 21.

e–6). $a(B_1)\supset b(A_1)$ and $a(B_1)=B_1$.

Then M contains a direct summand M_1 of D such that $\bar{M}_1=\bar{\alpha}_1R$ by Lemma 4.

f) $D=eR/C_1\oplus N_2\oplus eR/A_1$, where $N_2=eR/B_1$ or eR/C_1.

Let $\{\bar{\alpha}_1=(\bar{a}, \bar{e}, o), \bar{\alpha}_2=(\bar{b}, o, \bar{e})\}$ be the basis of $\bar{M}$. Since $[\Delta/\Delta(1, 3)\colon \Delta_1]=1$ by Lemma 20 and assumption, there exist x and y in Δ_1 such that $\bar{a}x+\bar{b}y\in\Delta(1, 3)$. Put $\theta=\alpha_1x+\alpha_2y=(\tilde{a}x+\tilde{b}y, \tilde{x}, \tilde{y})$. Then M contains a direct summand M_1 of D such that $\bar{M}_1=\bar{\theta}R$ by Lemma 4.

3) $|I|=4$.

a). $D=eR/A_1\oplus eR/A_1\oplus eR/A_1\oplus eR/A_1$.

Since $|D|=16$, $|M|=15$. Hence M contains a direct summand of D, which is isomorphic to $eR\oplus eR$ by the similar argument to 3) of Case $|eJ|=3$.

b) Other cases.

Since $[\Delta/\Delta(1, 2)\colon \Delta_1]=3$ and $[\Delta/\Delta(2, 3)\colon \Delta_2]=2$, we can use the same argument as above.

4). The remaining part is similar to Case $|eJ|=3$.

Thus we have completed the proof of Theorem 12.

4. Rings with $|eJ|\geqslant 5$

We shall study the ring R with $|/eJ|\geqslant 5$ under the assumption: $J^2=0$.

Theorem 22. *Let R be a right artinian ring with $J^2=0$. Then*

$(*)_2$ *Every maximal submodule of a finite direct sum of serial modules is a direct sum of hollow modules if and only if, for each primitive idempotent e,*

i) $eJ=A_1\oplus A_2$, $A_1\not\approx A_2$ *or* $A_1=0$, *or*

ii) $eJ=A_2\oplus A_2\oplus\cdots\oplus A_n$; $A_1\approx A_i$ *for all* i $(n\geqslant 2)$,

a) $[\Delta\colon \Delta(A_1\oplus A_2\oplus\cdots\oplus A_{n-1})]=2$ *and*

b) *there exists a unit x in Δ, for any right ideal B in eJ with $|B|=n-1$, such that $B=x(A_1\oplus A_2\oplus\cdots\oplus A_{n-1})$; i.e. $eR/A\approx eR/B$.*

Proof. "Only if" part. Put $B=A_1\oplus A_2\oplus\cdots\oplus A_{n-1}\subseteq eJ$. Then $[\Delta\colon \Delta(B)]\leqslant 2$ by Lemma 3 and Corollary to Lemma 5. Assume $\Delta=\Delta(B)$. Then $n\leqslant 2$ by Lemma 8 and hence $A_2=0$ or $A_1\not\approx A_2$ by Lemma 8. ii)–b) is obtained from Proposition 6.

"If" part. Let $D=\sum\oplus N_i$ be a direct sum of serial modules. Then N_i is isomorphic to either eR/eJ or eR/B (or eR if $eJ=A_1$). Let M be a maximal submodule of D. Then, from Proposition 6 and the proof of Theorem 12, M is isomorphic to either $J(N_1)\oplus\sum_{j\geqslant 2}\oplus N_j$ or $M_1\oplus\sum_{j\geqslant 3}\oplus N_j$, where M_1 is a maximal submodule of $N_1\oplus N_2$. It is clear from the proof of 3) of Case $|eJ|=2$ in Theorem 12 that M_1 is isomorphic to $N_1\oplus N_2/J(N_2)$, $N_1/J(N_1)\oplus N_2$ or eR/C, where $|eR/C|=3$. Thus M is a direct sum of hollow modules.

Theorem 23. *Let R be as above. Then*

$(*)_3$ *Every maximal submodule of a finite direct sum of hollow modules whose length is equal to or less than three is a direct sum of hollow modules if and only if*

R satisfies $(*)_2$ *and*

i) *if* $eJ=A_1\oplus A_2\oplus\cdots\oplus A_n$ *and* $A_1\approx A_i$ *for all* i $(n\geqslant 3)$, *then*

a) $[\Delta\colon \Delta(A_1\oplus A_2\oplus\cdots\oplus A_{n-2})]=3$, *and*

b) *for a right ideal* B *with* $|B|=n-2$, *there exists a unit* x *in* Δ *such that* $B=x(A_1\oplus A_2\oplus\cdots\oplus A_{n-1})$; i.e. $eR/A\approx eR/B$.

Proof. Since the proof of Theorem 12 for Case $|eJ|=3$ is given in a general form, we have the theorem from Theorem 22.

Theorem 24. *Let* R *be a right artinian ring with* $J^2=0$. *Then* R *satisfies Condition* I *and* eJ *is simple or contains a proper character submodule for each* e *if and only if* $eJ=A_1\oplus A_2$ *and* $A_1\not\approx A_2$.

Proof. Assume that R satisfies the first two conditions in the theorem. If $|eJ|\geqslant 3$, by Proposition 11 $eJ=eRA_1$ for any simple submodule A_1. If B is a proper character submodule containing A_1 of eJ, $eJ=eReB=B$, which is a contradiction. Hence $|eJ|\leqslant 2$. Further, if $A_1\approx A_2$, $A_2=xA_1$ by Lemma 8. Therefore $|eJ|=1$ or $A_1\not\approx A_2$.

Similarly, by Lemma 5 and Proposition 6, we have the following:

Proposition 25. *Let* R *be a right artinian ring with* $J^2=0$. *Assume the conditions* i)–a) *and* b) *in Theorem* 23. *Then the following condition is satisfied.*

$(**)_3$ *Every maximal submodule of a direct sum* $D(4)=\sum_{i=1}^{4}\oplus N_i$ *of hollow modules* N_i *with* $|N_i|=3$ *contains a non-zero direct summand of* D. *Conversely, if* $D(4)$ *satisfies* $(**)_3$ *and* $D(3)$ *does not, then* i)–a) *and* b) *in Theorem* 23 *are satisfied.*

Proposition 26. *Let* R *be as above. Assume* R *satisfies* $(*)_3$. *If* $|eJ|\geqslant 3$, eRe/eJe *is not commutative for a primitive idempotent* e.

Proof. Assume $|eJ|\geqslant 3$ and $eJ=A_1\oplus\cdots\oplus A_{n-2}\oplus A_{n-1}\oplus A_n$, where the A_i are simple. Put $B=A_1\oplus\cdots\oplus A_{n-2}$. Then there exist unit elements x, y in Δ such that $xB=A_1\oplus\cdots\oplus A_{n-3}\oplus A_{n-1}$ and $yB=B$, $yA_{n-1}\equiv A_n$ (mod B) by Lemma 8 and Theorem 23. $xyB=A_1\oplus\cdots\oplus A_{n-3}\oplus A_{n-1}\not\subseteq B\oplus A_n\subseteq yxB$. Hence $xy\neq yx$.

Corollary 1. *Let* R *be a commutative artinian ring or an algebra of finite dimension over an algebraically closed field. Then* R *satisfied Condition* I *if and only if* i) *or* ii) *in Theorem* 12 *is satisfied.*

Proof. This is clear from Proposition 26 and Lemma 8.

Corollary 2. *Let* R *be as in Corollary* 1 (*not necessarily* $J^2=0$). *Assume* R/J *is a simple ring. Then* R *is a right serial ring if and only if* R *satisfies Condition* I.

Proof. This is clear from Corollary 1 and Proposition 1.

5. Examples

Proposition 26 suggests us very much the possibility of $|eJ|\leq 2$. We shall study this situation. Let D_1 and D_2 be two division rings and V a left D_1 and right D_2 vector space. For a right D_2-subspace V' of V, we denote the dimension by $|V'|_{D_2}$. Put $n=|V|_{D_2}$ and consider the following conditions.

a) *If* $|V_1|_{D_2}=|V_2|_{D_2}$ *for subspaces* V_1, V_2 *of* V, *then there exists an element* d *in* D_1 *such that* $dV_1=V_2$.

b) $[D_1: D_1(V_1)]_r=n-|V_1|_{D_2}+1$, *where* $D_1(V_1)=\{d\in D_1 \mid dV_1\subseteq V_1\}$.

c) *Let* d *be a fixed non-zero element of* V. *There exists a monomorphism* σ *of* D_2 *into* D_1 *such that* $dx=\sigma(x)d$ *for* $x\in D_2$.

Theorem 27. *There exists an artinian ring* R *with* $J^2=0$ *satisfying the conditions* iv) (*resp.* v)) *in* Theorem 12 *if and only if there exists a vector space* V *as above satisfying* 1) $|V|_{D_2}=3$ (*resp.* $|V|_{D_2}=4$), 2) a) *and* b) *are satisfied for any* V_1 *with* $|V_1|_{D_2}\leqslant 2$ (*resp.* $|V_1|_{D_2}\leq 3$), (*resp.* 3) c) *is satisfied*).

Proof. If there exists a D_1-D_2 vector space V, then $R=\begin{pmatrix} D_1 & V \\ 0 & D_2\end{pmatrix}$ satisfies Condition I by Theorem 12. Conversely, assume that there exists an artinian ring R satisfying Condition I. Let $eJ=A_1\oplus A_2\oplus A_3$ (resp. $A_1\oplus A_2\oplus A_3\oplus A_4$). Since A_1 is simple and R may be assumed basic, $A_1\approx fR/fJ=fRf/fJf$ for a primitive idempotent f. Put $D_1=eRe/eJe$ and $D_2=fRf/fJf$. Then eJ is a left D_1 and right D_2 vector space. Hence eJ satisfies 1), 2) (resp. 1)$\sim$3)) by Theorem 12.

We note that if such a D_1-D_2 vector space exists, D_1 should be non-commutative. Finally let R be an algebra of finite dimension over a field K. Assume R satisfies Condition I and put $\Delta=eRe/eJe$ for a primitive idempotent e.

Proposition 28. *Let* R *and* Δ *be as above. Then* $[\Delta: K]$ *is divisible by* $|eJ|$ *provided* $|eJ|\geqslant 3$. *If* $[\Delta: K]$ *is not divisible by* 2 *or* 3, *then* $|eJ|\leqslant 2$.

Proof. This is clear from Lemmas 13 and 15.

EXAMPLES 1. Let $L\supset K$ be distinct fields and put

$$R=\begin{pmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K\end{pmatrix}.$$

Then $e_{11}R/e_{11}J^2$ is serial but $e_{11}R$ is not serial.

2. Let T be a field and x an indeterminate. Let $L=T(x)$ and $K=T(x^n)$. Then we have an isomorphism σ of L onto K given by setting $\sigma(x)=x^n$. Put $R=R(n)=L\oplus Lu$ a vector space over L. R is a ring by the following product:

$(x_1+x_2u)(y_1+y_2u)=x_1y_1+(x_1y_2+x_2\sigma(y_1))u$. Then $J(R)=Lu$ and $Lu=Ku\oplus Kxu\oplus\cdots\oplus Kx^{n-1}u$. Every simple right ideal is isomorphic to Ku via the left-sided multiplication of an element in L. Hence $\{R, R/J, R/Ku\}$ is the representative set of hollow modules if $n=2$. Therefore $R(1)$ and $R(2)$ satisfy Condition I (note that $J=A_1\oplus A_2$ and $A_1\approx A_2$ for $R(2)$), but R(n) does not for n$\geqslant$3 by Theorem 12. $R(3)$ satisfies $(**)_3$ by Proposition 25.

3. Let $L\supset K$ be as in Example 1. Put

$$R=\begin{pmatrix} K & L & L \\ 0 & L & 0 \\ 0 & 0 & L \end{pmatrix}.$$

Then $e_{11}J=(0, L, 0)\oplus(0, 0, L)$ and R satisfies Condition I (note that $e_{11}J=A_1\oplus A_2$ and $A_1\not\approx A_2$).

4. Let $D_1\subset D$ be division rings with $[D: D_1]_r=2$. Then

$$R=\begin{pmatrix} D & D \\ 0 & D_1 \end{pmatrix}$$

satisfies Condition I. If $[D: D_1]_l\geqslant 3$, R is not of right local type (see [7]).

5. Put

$$R=\begin{pmatrix} K & K\oplus K \\ 0 & K \end{pmatrix}.$$

Then $e_{11}J=K\oplus K$ and $(K, 0)\approx(0, K)\subseteq e_{11}J$. Hence R does not satisfy Condition I.

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