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Osaka University
We have defined (right US-3) rings satisfying (**, 3) in [5], which are rings generalized from Nakayama ring (right generalized uni-serial rings). As stated in [5], we shall give, in this note, another generalization of Nakayama rings, which is related to the condition (*, 3), and give a characterization of those rings.

1. Preliminary results. Let $R$ be a ring with identity. We assume always throughout this note that $R$ is a right artinian ring and every module is a right unitary $R$-module $M$ with finite length, which we denote by $|M|$. We have studied the following conditions in [3] and [5]:

$(**)$ Every (non-zero) maximal submodule of a direct sum $D(n)$ of $n$ non-zero hollow modules contains a non-trivial direct summand of $D(n)$.

$(*)$ Every (non-zero) maximal submodule of the $D(n)$ is also a direct sum of hollow modules.

We shall study mainly, in this note, rings satisfying $(*)$ for any direct sum of three hollow modules. We shall use the same notations as given in [3] and [5].

Let $e$ be a primitive idempotent in $R$.

**Condition II** [3]. $|eJ/eJ^2| \leq 2$ for each $e$, where $J$ is the Jacobson radical of $R$.

In [3] we have given the structure of rings which satisfy Condition II and

**Condition I.** Every submodule in any direct sum of (three) hollow modules is also a direct of hollow modules.

However, checking carefully each step, we know that we utilize only $(*)$ for any direct sum of three hollow modules. Thus we have the following theorem.

**Theorem 1.** Let $R$ be a right artinian ring. Assume that $(*)$ for any direct sum of three hollow modules and Condition II hold. Then for each primitive idempotent $e$ in $R$, we have the following properties:

1) $eJ = A_1 \oplus B_1$, where $A_1$ and $B_1$ are uniserial modules. Further, if $A_1/J(A_1) \cong B_1/J(B_1)$, $\alpha A_1 = B_1$ for some unit $\alpha$ in $eRe$. 

2) For every submodule \( N \) in \( eJ \), there exists a trivial submodule \( A_i \oplus B_j \) of \( eJ \) and a unit \( \gamma \) in \( eRe \) such that \( N = \gamma (A_i \oplus B_j) \), where \( A_i = A_I J^{i-1} \subset A \) and \( B_j = B_I J^{j-1} \subset B \).

3) If \( A_i \sim B_j \), then \( \Delta(A_i \oplus B_j) = \Delta \) and \( [\Delta : \Delta(A_i \oplus B_j)] = 2 \) provided \( i \neq j \); further \( \Delta(A_i) = \Delta(A_i \oplus B_j) \) (\( i < j \)) and \( \Delta(B_j) = \Delta(A_i \oplus B_j) \) (\( i > j \)). If \( A_i \sim B_j \), then \( \Delta(N) = \Delta \) for any submodule \( N \) in \( eJ \).

Here we shall recall the notations above. Put \( \Delta = eRe = eRe/Je \). For any right ideal \( A \) in \( eR \), \( \Delta(A) = \{ x \mid \in \Delta, (x+j)A \subset A \) for some element \( j \) in \( eJe \}. \) Then \( \Delta(A) \) is a subdivision ring of \( \Delta \) and \( [\Delta : \Delta(A)] \) means the dimension of \( \Delta \) over \( \Delta(A) \) as a right \( \Delta(A) \)-vector space.

2. Rings with \((*, 3)\). We shall study, in this section, the converse of Theorem 1. We assume that \( R \) has the structure given in Theorem 1, unless otherwise stated.

We have given the following lemma in [3], provided Condition II’ in [3] is satisfied. We shall show in the same manner that the lemma is valid under a weaker condition.

**Lemma 1 ([3])**. Let \( R \) be a ring whose structure is given as in Theorem 1, and \( e \) a primitive idempotent. Let \( \{ E_i \}_{i=1}^p \) be a family of right ideals in \( eR \) and \( D = \sum_{i=1}^p eR/E_i \). Then, if \( \Delta(A_i) = \Delta(B_i) = \Delta \), \( D \) satisfies \((*, n)\)

Proof. We shall quote the same argument as given in the last part of §3 in [3], and hence use the induction on the nilpotency of \( J \). If \( E_i \subset E_j \) for some \( i, j \), every maximal submodule of \( D \) contains a direct summand of \( D \) by assumption and [3], Lemma 27 (cf. the proof of Lemma 3 below). By induction we may consider the following case:

\[
E_0 = A_i, E_k = A_i \oplus B_j;
\]

\[
i < i_1 < i_2 < \cdots < i_p, j_1 > j_2 > \cdots > j_p \quad \text{and} \quad D = \sum_{i=1}^p eR/E_i.
\]

Assume \( i_t < j_s, i_{t+1} > j_{s+1} \). Let \( M \) be a maximal submodule of \( D \). We may assume that \( \bar{M} = M/J(D) \) (\( \subset D = D/J(D) \)) has a basis \( \{(0, \ldots, 0, e, \bar{e}, \ldots, 0)\} \). Since \( \Delta(A_i) = \Delta(B_j) = \Delta \), we can take \( k_s \) with \( k_s A_i = A_i \) for \( s \leq t \) and \( k_s B_j = B_j \) for \( r > t \). Set \( M^* = A_i \oplus \sum_{i=1}^p eR/(A_i \oplus B_{j_{s-1}}) \oplus B_j / B_{j_s} \) \( (B_{j_0} = 0) \), then \( |M^*| = |D| - 1 \). Define a homomorphism \( f \) of \( M^* \) to \( D \) by setting

\[
f((x + A_i) + \sum_{s=1}^p (y_s + (A_i \oplus B_{j_{s-1}})) + (z + B_{j_s})) = (x + y_1 + A_i) + (ek_1 y_1 + y_2 + (A_i \oplus B_{j_2})) + \cdots + (ek_p y_p + z + (A_i \oplus B_{j_p})),
\]

where \( x \in A_i, y_s \in eR \) and \( z \in B_j \). \( A_i \oplus B_{j_{s-1}} \subset A_{i_{s-1}} \oplus B_{j_{s-1}} \) \( k_s(A_i \oplus B_{j_{s-1}}) \)
\[ A_i \oplus B_{i-1} \subset A_i \oplus B_i \] for \( a = t + 1 \), and \( k_{t+1}(A_{i+1} \oplus B_i) \subset k_{t+1}(A_{i+1} \oplus B_{i+1}) = A_{i+1} \oplus B_{i+1} \). Hence \( f \) is well defined. Assume that the right hand side of the above is zero. Since \( x \in A_i \) and \( z \in B_i \), \( y_t \in eJ \) for all \( s \). Put \( y_t = y_{t_1} + y_{t_2} \), where \( y_{t_1} \) is in \( A_i \) and \( y_{t_2} \) in \( B_i \). Now \( x + y_1 = x + y_{t_1} + y_{t_2} \in A_i \), \( x \in A_i \) and so \( y_{t_2} = 0 \).

\[ ek_i(y_{t_1} + y_{t_2}) + (y_{t_1} + y_{t_2}) \in A_i \oplus B_i. \]

Since \( k_i A_i \subset A_i \), \( y_{t_2} \in B_i \).

Repeating those arguments, we assume by induction that \( y_{t+1} \in B_j \) for \( t < t' \).

Put \( w = ek_{t'}(y_{t'_{1}} + y_{t'_{2}}) + (y_{t'_{1}+1} + y_{t'_{1}+2}) \in A_{i_{t'}} \oplus B_{j_{t'}}. \)

Since \( ek_{t'} \) is an isomorphism of \( eJ_i \), \( ek_{t'}(y_{t'_{1}} + y_{t'_{2}}) + (y_{t'_{1}+1} + y_{t'_{1}+2}) \) is in \( A_{i_{t'}} \oplus B_{j_{t'}}. \)

Consider next from the bottom side. \( ek_{t}(y_{t_1} + y_{t_2}) \in A_{i_{t}} \oplus B_{j_{t}}. \)

Since \( k_i A_i \subset A_i \) and \( x \in B_i \), \( y_{t_1} \in A_{i_{t}} \) from the same argument above (take \( \pi_1 : eJ \rightarrow A_i \)). Repeating those arguments inductively, we obtain \( y_{t_1} \in A_{i_{s}} \) for \( s \geq t + 1 \). Consider \( ek_{t+1}(y_{t+1} + y_{t+2}) + (y_{t+1} + y_{t+2}) \in A_{i_{t+1}} \oplus B_{j_{t+1}}. \)

Since \( y_{t+1} \in B_{j_{t}} \), \( y_{t+1} \in A_{i_{t+1}}, \) and \( i_{t+1} > j_{t} \), \( y_{t+1} \in B_{j_{t}}. \)

Similarly, from \( ek_{t}(y_{t_1} + y_{t_2}) \in A_{i_{t}} \oplus B_{j_{t}} \), \( y_{t_1} \in A_{i_{t}} \).

Combining the above two steps, we know that \( f \) is a monomorphism. Hence \( M \cong M^*. \)

It is remained for us, from Lemma 1, to study a case of \( \Delta(A_i) \neq \Delta, \) i.e., \( A_i \cong B_i. \) We have shown in [3] that if a right artinian ring \( R \) has the structure in Theorem 1, then \( (\ast, n) \) is satisfied for any \( D(n), \) provided \( J^3 = 0. \) We shall show that \( (\ast, 3) \) is satisfied without the assumption \( J^3 = 0. \)

**Lemma 2** ([3], Lemma 24). We assume the above situation. Let \( d \) be an element in \( eRe \) such that \( d \in \Delta(A_i) \). Then \( \pi_2 d A_i = B_i, \) where \( \pi_2 : eJ \rightarrow B_i \) is the projection.

**Proof.** Since \( \{ \Delta : \Delta(A_i) \} = 2, \) \( \delta = a_i + \alpha a_i; \) the \( a_i \in \Delta(A_i) \) and \( \alpha A_i = B_i. \)

Set \( \delta = a_i + \alpha a_{i+j} \); \( a_i A_i \subset A_i, j \in eJ. \) Since \( j A_i \subset A_i \oplus B_{i+1}, \) \( \pi_2 \delta A_i = B_i. \)

**Lemma 3.** Assume that \( R \) has the structure 1), 2) and 3) given in Theorem 1. Then \( (\ast, 2) \) is fulfilled for any \( D(2). \)

**Proof.** The assumption 2) in Theorem 1 gives us a guarantee of \( (\ast, 1) \) for any hollow module. Let \( eJ = A_i \oplus B_i. \) If \( A_i \cong B_i, \) \( \Delta(C) \neq \Delta \) for any submodule \( C \) of \( eR \) by assumption. Then we have shown by Lemma 1 that \( (\ast, 2) \) is fulfilled for any \( D(2). \) Assume that \( A_i \cong B_i. \) Then \( \Delta = \Delta(A_i) \oplus \alpha \Delta(A_i), \) where \( \alpha \) is the element given in 1). Set \( D = eR/N_i \oplus eR/N_i, \) where the \( N_i \) are submodules of \( eR. \) We shall show the lemma by induction on the nilpotency of \( J. \) If \( J^3 = 0, \) we are done in [3], § 4. Assume \( eJ^* \neq 0 \) and \( eJ^{3+1} = 0. \)

If \( N_i \rightarrow eJ^* \) for \( t = 1, 2, \) \( eR/N_i \) is a hollow \( R/J^* \)-module. Hence we may assume that \( N_i = A_i = A_i J^{-1} \) by induction. Let \( M \) be a maximal submodule of \( D, \) and put \( D = D \mid (D) \supset M = M \mid (D) \). We may assume that \( M \) has a basis \( \{ e + [D), \delta + [D] \}, \) where \( \delta \) is a unit element in \( eRe \) (it is sufficient to show
the lemma in case $R$ is basic; see [2] and [3]).

i) $N_i = A_i \oplus B_j$, (we may assume $k \leq i$ [3]) a) $i \leq k \leq i$. $F = A_i \cap B_j = \delta^{-1}(A_i \cap B_j) = \delta^{-1}(\delta A_i \cap (A_k \oplus B_j))$. a)–i) If $\delta \in \Delta_i = \Delta_i(A_i)$, $F = A_k$ (we may assume $A_i \subseteq A_k$). a)–ii) If $\delta \in \Delta_i$, $F = A_j$ by Lemma 2. Put $M^* = eR/A_i \oplus A_i/A_k \oplus B_j/B_j$ for the case a)–i). Define a homomorphism $f$ of $M^*$ to $D$ by setting

$$f((x + A_k) + (y + A_i) + (z + B_j)) = (x + y + A_i) + (\delta x + z + (A_k \oplus B_j)),$$

where $x$ is in $eR$, $y$ in $A_i$ and $z$ in $B_j$. Then $f$ is well defined. It is easy to check that $f$ is a monomorphism, since $A_i = A_k$. Put $M^* = eR/A_i \oplus A_i/A_k \oplus A_i/A_k$ for the case a)–ii). Define a homomorphism $f$ of $M^*$ into $D$ by setting

$$f((x + A_j) + (y + A_i) + (z + B_j)) = (x + y + A_i) + (\delta x + z + A_k \oplus B_j),$$

where $x$ is in $eR$ and $y$, $z$ are in $A_i$. We can show from the fact: $\delta \in \Delta_i$, that $f$ is a monomorphism (cf. the proof of Lemma 4 below). Hence $M \cong M^*$, since $|M| = |M^*|$ and $f(M^*) \subseteq M$.

b) $k \leq i \leq j$. If $\delta \in \Delta_i$ (resp. $\in \Delta_j$), $F = A_i$ (resp. $F = A_j$). We obtain the same result as in a)–ii) for $F = A_i$. If $F = A_i$, put $M^* = eR/A_i \oplus A_i/A_k \oplus B_j/B_j$. Then $M \cong M^*$ as above.

c) $k \leq j \leq i$. Since $eReA_i \subseteq A_k \oplus B_j$, $M$ contains a direct summand of $D$.

ii) $N_i = A_j$, $i \geq j$. If $\delta \in \Delta_i$, $A_i \cap A_j = 0$ and $M$ is isomorphic to $eR \oplus A_i/A_i \oplus A_j/A_j$. If $\delta \in \Delta_i$, $A_i \subseteq A_j$. Hence we obtain the same situation as in i)–c).

Lemma 4. (*, 3) is satisfied for any three hollow modules.

Proof. We may assume $\Delta(A_i) = \Delta_i \neq \Delta$ by Lemma 1. From induction on the nilpotency of $J(R)$, it is sufficient to study the case:

$$E_0 = A_i, \quad E_i = A_i \oplus B_i, \quad \text{and} \quad E_2 = A_i \oplus B_i,$$

with $i_k \leq j_k$ for $k = 1, 2$, and $D = \sum eRe/E_i$. Here $B_i$ may be equal to zero (cf. [3], §3).

If $B_i = B_i = 0$, $D$ satisfies (*, 3) by [4], Corollary 3. Let $M$ be a maximal submodule of $D$. If $M$ contains a non-zero direct summand $D_1$ of $D$, $M = D_1 \oplus M_1$ where $M_1$ is a maximal submodule of $N_1 \oplus N_2$; the $N_i$ are isomorphic to some of $\{eR/E_i\}^*_{i=1}$. Then $M_1$ is a direct sum of hollow modules by Lemma 3, and hence so is $M$. Therefore we consider $M$ not containing a direct summand of $D$. Put $\bar{D} = D/J(D) \supset \bar{M} = M/J(D)$, and $D = (\bar{e \Delta}, \bar{e \Delta}, \bar{e \Delta})$. Then the above $\bar{M}$ has a basis $\{(\bar{e}, \bar{d}_i, 0), (0, \bar{e}, \bar{d}_i), \}$, where $\bar{d}_i$ are in $\Delta$ and $\bar{d}_i \bar{d}_i = 0$ (cf. [3]). We consider the following situation:

1) $i \leq i_1 \leq i_2 \leq j_1 \leq j_2$.

a) $\delta \in \Delta_i$. Then $\delta_2 E_2 \subseteq E_1$. Hence $M$ contains a direct summand of $D$ by [1], Theorem 2.
b) $\delta_1 \in \Delta_1$ and $\delta_2 \in \Delta_1$. $M \approx A_1/A_i \oplus eR/A_i \oplus eR/(A_i \oplus B_i) \oplus A_i/A_i$. (2)

c) $\delta_1$ and $\delta_2 \in \Delta_1$. $M \approx A_1/A_i \oplus eR/A_i \oplus eR/(A_i \oplus B_i) \oplus A_i/A_i$. (3)

2) $i_1 \leq i \leq i_2 \leq j \leq j_1$.

a) $\delta_1$ or $\delta_2 \in \Delta_1$. We obtain (1).

b) $\delta_1$ and $\delta_2 \not\in \Delta_1$. We obtain (3).

3) $i_1 \leq i \leq i_2 \leq j_1 \leq j$. Since $E_1 \supseteq E_2 \supseteq E_3$. We obtain (1) by [4], Corollary 3.

4) $i \leq j \leq i_2 \leq j_2$. Since $eReE_2 \subseteq E_3$, we obtain (1).

5) $i \leq i_1 \leq i_2 \leq j_1$. $\delta_1$ and $\delta_2 \in \Delta_1$. We obtain the same situation as in the proof of Lemma 1. i.e., $M \approx A_1/A_i \oplus eR/A_i \oplus eR/(A_i \oplus B_i) \oplus A_i/A_i$. (4)

b) $\delta_1 \not\in \Delta_1$ and $\delta_2 \not\in \Delta_1$. We obtain (6).

c) $\delta_1 \not\in \Delta_1$ and $\delta_2 \not\in \Delta_1$. $M \approx A_1/A_i \oplus eR/A_i \oplus eR/(A_i \oplus B_i) \oplus A_i/A_i$. (5)

d) $\delta_1$ and $\delta_2 \not\in \Delta_1$. $M \approx A_1/A_i \oplus eR/A_i \oplus eR/(A_i \oplus B_i) \oplus A_i/A_i$. (6)

6) $i_1 \leq i \leq i_2 \leq j_1$. $\delta_1$ or $\delta_2 \in \Delta_1$. We obtain (1).

7) $i_1 \leq i_2 \leq i \leq j_1$. $\delta_1$ or $\delta_2 \in \Delta_1$. We obtain (1).

8) $i_1 \leq i_2 \leq i \leq j_1$ or $i_1 \leq i_2 \leq j_2 \leq j_1$. Since $eReE_0 \subseteq E_2$, we obtain (1).

We shall give a sample of proofs.

1)–c). Put $\xi'=(\varepsilon, \delta_1, 0)$ and $\eta'=(0, \varepsilon, \delta_2)$. Consider $\{\xi', \eta'=(0, \varepsilon, \delta_2), \}$, where $\delta_2'=\delta_2^{-1} \in \Delta_1$. If $\{\delta_1, \delta_2\}$ is linearly independent, there exist $a_1'$ and $a_2'$ in $\Delta_1$ such that $\varepsilon=\delta_1 a_1'+\delta_2 a_2'$ and $a_1 a_2=0$, since $[\Delta: \Delta_1]=2$. Then $\tilde{M}$ has a basis $\{\xi'=\varepsilon'-\eta' a_2 a_2^{-1}=\varepsilon, a_1, a_2\}$ and $\eta=\eta'=0, \delta_2, \delta_2'\}$, where $a_1=-\delta_1^{-1}$ and $a_2=-\delta_2^{-1}$. On the other hand, if $\delta_1=\delta_2^{-1}$, $\tilde{M}$ has a basis $\{\xi'=\varepsilon-\eta' a_2'=\varepsilon, 0, a_2\}$ and $\eta=\eta'=0, \delta_2, \delta_2'\}$. In either case, $a_2 \neq 0$ and define a homomorphism $f$ of $M^*=A_1/A_i \oplus eR/A_i \oplus eR/(A_i \oplus B_i) \oplus A_i/A_i$ to $D$ by setting

$$f((x+A_1)+(y+A_1)+(z+(A_i \oplus B_i))+(w+A_1)),$$

where $x$ is in $A_1$, $y$ and $z$ in $eR$ and $w$ in $A_1$.

Since $A_i \cap a^{-1}_1(A_i \oplus B_i) \cap a^{-1}_2(A_i \oplus B_i)=A_{i_2}$ and $\delta_2^{-1}(A_i \oplus B_i) \cap (A_i \oplus B_i)=A_{i_2}$ and $\delta_2^{-1}(A_i \oplus B_i) \cap (A_i \oplus B_i)=A_{i_2}$ by Lemma 2, $f$ is well defined. Assume that the latter term of the above equation is zero, i.e.,

0) $x, w \in A_1$.
1) $x+y \in A_1$.
2) $a_1 y+\delta_2 z+w \in A_i \oplus B_i$.
3) $a_2 y+z \in A_i \oplus B_i$. 
Since $x$ is in $A_i \subset eJ$, $y$ and $z$ are in $eJ$ by 1) and 3). Put $z=a+b$; $a \in A_i$, $b \in B_i$. Since we may assume $a, b \in eJ$ by 3). $\delta z = \delta a + \delta b$ and $\delta a, \delta b \in A_i \oplus B_i \subseteq A_i \oplus B_i$. Hence $a$ is in $A_i$ by 2) and Lemma 2, and so $z$ is in $A_{i_2} \oplus B_{i_2} \subseteq A_i \oplus B_i$. Therefore $y$ is in $A_{i_2}$ by 3), since $a_2 \neq 0$, and so $x$ in $A_i$, $w$ in $A_i$. We have shown that $f$ is a monomorphism. On the other hand, $|D| = n + i_1 + i_2 + j_1 + j_2 - 2$ and $|M^*| = n + i_1 + i_2 + j_1 + j_2 - 3 = |D| - 1$, where $eJ^n \neq 0$, $eJ^{n+1} = 0$. Hence $f(M^*) = M$, for $M \supset J(D)$ and $f(M^*) = M$.

Now let $e = A_i \oplus B_i$ be as before and $eJ^n \neq 0$ and $eJ^{n+1} = 0$. We consider here together all cases: a) $B_i = 0$, b) $A_i \approx B_i$ and c) $A_i \approx B_i$. We obtain the following three hollow modules;

1) $S_i(e) = eR/(A_i \oplus B_i)$, 2) $T_i(e) = eR/A_i$ (or $eR/B_i$) and 3) $U_{ij}(e) = eR/(A_i \oplus B_j)$ (we denote those modules by $H(e)$).

Now $S_i$ and $U_{ij}$ are $R/J'$-modules, where $t = i$ and max $\{i, j\}$, respectively. We shall give a weight for each hollow module $H$ as follows; $w(H) = |J(H)||J'(H)|$, i.e., $w(S_i) = 1$, $w(T_i) = 2$ ($i \neq 1$), $w(T_i) = 1$ and $w(U_{ij}) = 2$ ($i \neq 1$ and $j \neq 1$).

**Lemma 5.** Let $S(e)$, $T(e)$ and $U(e)$ be as above. Then for a maximal submodule $M$ of $D$ below, we obtain the following:

1) $D = S(e) \oplus S'(e)$. $M \approx S(f_1) \oplus S(e)$, $S(e) \oplus S(f_2) \oplus U(e)$.
2) $D = T(e) \oplus S(e)$. $M \approx S(f_1) \oplus S(f_2) \oplus S(e)$ or $T'(e) \oplus S(f)$.
3) $D = U(e) \oplus S(e)$. $M \approx S(f_1) \oplus S(f_2) \oplus S(e)$, $U(e) \oplus S(f)$ or $U'(e) \oplus S(f)$,

where $e$ and $f$ are primitive idempotents.

**Proof.** We can show the lemma from Lemmas 1 and 3 (consider $D$ as $R/J'$-modules for 3); $t \leq n$.

Assume that

$$C = \sum_{i=1}^{k} \sum_{j=1}^{j_i} \oplus H_j(e_i),$$

where $1 = \sum e_i$, $\{e_i\}$ is a set of mutually orthogonal primitive idempotents (and $R$ is basic). Let $M$ be a maximal submodule of $C$. Since $H_j(e_i) \mid (H_j(e_i)) \approx H_j(e_i) \mid (H_j(e_i))$ for $i \neq i'$, $M = \sum \oplus M_i$, where $M_k = \sum \oplus H_j(e_k)$ for all $k$ except some $q$ and $M_q$ is a maximal one in $\sum \oplus H_j(e_q)$. Put $w(C) = \sum w(H_j(e_q))$.

**Lemma 6.** Every submodule $F$ of $D(q)$ is a direct sum of hollow modules $H_i$ and $w(F) \leq 2q$ ($q \leq 3$).

**Proof.** We shall show the lemma for a case $q = 3$. The remaining parts are same. In order to prove the lemma, we may show that any maximal submodule $M$ of $C$ above with $t = w(C) \leq 6$ has a similar direct decomposition and $w(M) \leq 4$. Further, from the argument before Lemma 6, we may assume $e_i = e$, and show that
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\[ M = \sum_{i=1}^{m} \oplus H_i \quad \text{and} \quad w(M) \leq t \quad (\#). \]

We note that if \( w(H_i(e)) = 2 \), \( J(H_i(e)) \) is a direct sum of two uniserial modules. If \( H_i(e) = eR \) for some \( i \), \( M \) contains a direct summand of \( C \) by [1], Theorem 2. Hence \( M \) satisfies (\#) by induction on \( m \) and the above remark. We shall show (\#) by induction on \( n \) \( (J^{n+1} = 0) \). If \( n = 0 \), then (\#) is trivial. We assume that every maximal submodule \( M \) satisfies (\#) for \( k \leq n - 1 \). Start from

\[ D = H(e_1) \oplus H(e_2) \oplus H(e_3). \]

\( w(D) = 6 \) provided no-one of \( \{H(e_i)\} \) is uniserial, and \( w(D) \leq 5 \) for other cases. Further, if no-one of \( \{H(e_i)\} \) is isomorphic to \( T_i(e) \), the \( H(e_i) \) are \( R/J_i \)-modules for some \( t \leq n \). Then we can show (\#) by the induction hypothesis. Hence assume \( H(e_i) = T_i(e) \). We may further assume \( e = e_i \) for all \( i \) from the remark before Lemma 6. Let \( M \) be a maximal submodule of \( D \). Then from Lemma 4 \( M = \sum_{i=1}^{4} \oplus H(f_i); f_i = e \) if \( H(f_i) \approx T \) or \( U \), and \( w(D) \geq w(M) \). Put \( M_0 = \sum_{i=1}^{4} \oplus H(f_i) \). First we remark that the \( M_0 \) is an \( R/J_i \)-module, and hence (\#) is satisfied for \( M_0 \). Further, if no-one of \( \{H(e_i)\} \) is isomorphic to \( T_i(e) = eR/A_i \), the same for \( \{H(f_i)\} \). Now let \( M \) be the maximal submodule in \( C(D) \) given in the beginning. Remarking the above fact (the case \( H(e) = T_i(e) \)), we have the following cases:

I) \( C = T_{i_1} \oplus T_{i_2} \oplus T_{i_3}, T_{i_1} \oplus T_{i_2} \oplus U_{i_1,i_2}, \) or \( T_{i_1} \oplus U_{i_1,i_2} \oplus U_{i_3,i_4} \).

In the first case \( M \) contains a direct summand of \( C \), and hence we have (\#) by Lemmas 1 and 3. For the remaining cases we can use Lemmas 1 and 4.

II) \( C = T_{i_1} \oplus T_{i_2} \oplus S_{i_1} \oplus S_{i_2} \).

\( M \) contains a direct summand of \( C \) by [1], Theorem 2. Repeating this argument, we can reduce \( M \) to a case \( M = M_1 \oplus S_{i_1} \oplus S_{i_2} \) (\( M_1 \) is a maximal in \( T_{i_1} \oplus T_{i_2} \)), \( M = M_2 \oplus T_{i_1} \oplus S_{i_2} \) (\( M_2 \) is maximal in \( T_{i_1} \oplus S_{i_2} \)) or \( M = M_3 \oplus T_{i_1} \oplus T_{i_2} \) (\( M_3 \) is maximal in \( S_{i_1} \oplus S_{i_2} \)). Therefore \( M \) satisfies (\#) by Lemma 5.

III) \( C = T_{i_1} \oplus U_{i_1,i_2} \oplus S_{i_1} \oplus S_{i_2}, \) or \( T_{i_1} \oplus T_{i_2} \oplus U_{i_1,i_2} \oplus S_{i_1} \).

We can make use of the same argument as in I).

IV) \( T_i \) does not appear in a direct summand of \( C \), for instance \( C = U_{i_1,i_2} \oplus U_{i_2,i_3} \oplus U_{i_3,i_4} \).

We can use the induction hypothesis.

V) Some of \( T, U \) and \( S \) are equal to zero.

We have the same result as above.

Thus we have

**Theorem 2.** Let \( R \) be a right artinian ring satisfying Condition II. Then the following conditions are equivalent:

1) Every submodule of any \( D(3) \) is a direct sum of hollow modules.
2) \((*, 3)\) holds for any \( D(3) \).
3) \( eR \) has the structure given in Theorem 1 for each primitive idempotent \( e \).
In this case every submodule of \( D(i) \) is a direct sum of at most \( 2i \) hollow modules
for \( i \leq 3 \).

**Remark.** If \( R \) is an algebra of finite dimension over a field \( K \), then H. Asashiba has shown that \((*, 3)\) implies Condition II. Further, if \( K \) is algebraically closed, \( \Delta(N) = \Delta = K \) for any submodule \( N \) of \( eR \). If \( \Delta(N) = \Delta \) for \( N \),
\((*, 3)\) implies Condition II by [2], Proposition 10.

**Theorem 3.** Let \( R \) be as above. Assume that \( \Delta(N) = \Delta \) for any submodule \( N \) of \( eR \). Then the following statements are equivalent:
1) Every submodule of a finite direct sum of any hollow modules is also a
direct sum of hollow modules.
2) Every submodule of a direct sum of any three hollow modules is also a
direct sum of hollow modules.
3) \((*, 3)\) holds for any \( D(3) \).
In this case every submodule \( M \) of \( D(i) \) is a direct sum of at most \( 2i \) hollow
modules.

The author believes that Theorem 3 will be true without assumption
\( \Delta(N) = \Delta \). However, he can not find a systematic proof. We have studied
this problem in [3], §4, provided \( J^3 = 0 \). We shall extend this manner to the
case \( J^4 = 0 \).

**Proposition 4.** Let \( R \) be a right artinian ring with \( J^4 = 0 \) and assume that
Condition II. Then the following conditions are equivalent:
1) Condition I for any direct sum of hollow modules holds.
2) Condition I for any direct sum of three hollow modules holds.
3) \( eR \) has the structure given in Theorem 1.

Proof. We may consider the proposition in case of \( \Delta(A_1) \neq \Delta \). Under
the assumption above, we obtain the diagram of submodules in \( eJ \) up to isomorphism:

\[
\begin{array}{ccc}
A_1 \oplus B_1 & \rightarrow & A_1 \oplus B_2 \\
& & \\
& & A_1 \oplus B_3 \\
& & \\
& & A_2 \\
& & \\
& & A_3 \\
\end{array}
\]
Let \( \{E_i\}_{i=1}^n \) be a family of the modules above. Put \( D = \bigoplus_{i=1}^n eR/E_i \). Then, since \( \Delta(A_2 \oplus B_2) = \Delta \), every maximal submodule \( M \) of \( D \) contains a non-trivial direct summand of \( D \) by [1], Theorem 2 and [4], Corollary 3, except \( D_1 = eR/A_1 \oplus eR/(A_2 \oplus B_3) \oplus eR/(A_3 \oplus B_3) \). Let \( M \) be the maximal submodule such that \( \overline{M} - \overline{M}(D) = \xi \Delta \oplus \eta \Delta \oplus \zeta \Delta \), where \( \xi = (\xi_0, 0, 0), \eta = (0, \xi_2, 0, 0), \) and \( \zeta = (0, 0, \xi_3) \). If \( \xi_1 \) or \( \xi_2 \) is in \( \Delta \), \( M \) contains a direct summand of \( D \). Assume \( \overline{D}_1 \) and \( \overline{D}_2 \) are \( \Delta \)-finite. If \( \xi_2 = \Delta \), for \( [\Delta: \Delta_1] = 2 \). Then \( \overline{M} \) has a basis \( \{ (\xi_0, 0, 0, 0), \eta, \xi \} \). Then \( M \cong eR/(A_2 \oplus eR/(A_3 \oplus B_3)) \) as in the proof of Lemma 4. Next assume \( \xi_1 = \Delta \). If \( \xi_1 = \Delta \), \( \overline{M} \) has a basis \( \{ (\xi_0, 0, 0, 0), \eta, \xi \} \). If \( \{ \xi_0, \xi_3 \} \) is linearly independent, there exist \( a_0, a_3 \) in \( \Delta \) such that \( \xi = \xi_0 a_0 + \xi_3 a_3 \). Then \( M \) contains a basis \( \{ (\xi_0, 0, 0, 0), \eta, \xi \} \). Repeating this argument for \( \eta \) and \( \xi \), we obtain a basis \( \{ (\xi_0, 0, 0, 0), \eta, \xi \} \), where \( \xi_2 = \Delta \). In this case we obtain also the same result. Therefore every maximal submodule of \( D \) is a direct sum of hollow modules. Finally, if \( D \) is a direct sum of \( m \) hollow modules \( (m \geq 5) \), \( M \) contains a non-trivial direct summand of \( D \) by [1], Theorem 2 and [4], Corollary 3. Hence we can prove the proposition by induction on \( m \).

3. Right US-3 rings with \((*, n)\). We have defined right US-3 rings in [5], i.e., rings satisfying \((**, 3)\). In this section we shall study the structure of right US-3 rings with \((*, 1)\) or \((*, 2)\).

**Lemma 7.** If a right US-3 ring satisfies \((*, 2)\) for any \( D(2) \), then Condition 1 is satisfied for any \( D(n) \).

**Proof.** Let \( \{N_i\}_{i=1}^n \) be a set of hollow modules, and put \( D = \bigoplus_{i=1}^n N_i \). If \( n \geq 3 \), every maximal submodule \( M \) of \( D \) is of a form \( M = \bigoplus_{i=1}^m N_i \), where \( M_i \) is a maximal submodule of \( N_i \) and the \( N_i \) are isomorphic to some in \( \{N_i\} \). Hence \( M_i \) is a direct sum of hollow modules by \((*, 2)\).

**Theorem 5.** Let \( R \) be a right artinian ring. Then \( R \) is a right US-3 ring and \((*, 2)\) holds for any \( D(2) \) if and only if, for each primitive idempotent \( e \), \( ef \) has the following structure:

I) \( ef = 0 \). 1) \( ef = A_1 \oplus B_1 \) with \( A_1, B_1 \) simple or zero. 2) If \( A_1 \simeq B_1 \), \( [\Delta: \Delta(A_1)] = 2 \) and, for any simple submodule \( C \) in \( ej \), \( A_1 \simeq C \), i.e., there exists a unit \( x \) in \( eRe \) such that \( xC \subset A_1 \).

II) \( ef = 0 \). 1) \( ef = A_1 \oplus B_1 \) with \( A_1 \) uniserial and \( B_1 \) simple or zero. 2) \( \Delta = \Delta(E) \) and \( 3) \ x = A_1 \) or \( x = A_1 \oplus B_1 \), where \( E \) is a submodule of \( ej \), \( A_1 \) is a submodule of \( A_1 \) and \( x \) is a unit in \( eRe \).

**Proof.** If \((*, 2)\) and \((**, 3)\) hold, \( |ef| |ej| = 2 \) by [5], Proposition 1, and
Condition I holds for any $D(n)$ by Lemma 7. Hence $eR$ has the structure in Theorem 1. If $eJ=0$, we are done. Assume that $ef^2\neq0$, and $eJ=A_1\oplus B_1$ with $A_1$, $B_1$ uniserial. Put $A_1=A_1J^{-1}$ and $B_1=B_1J^{-1}$. If $A_1\cong B_1$, $[\Delta: \Delta(A_1)]=2$ by [3], Theorem 2. Since $A_2\neq0$ and hence $B_2\neq0$, $eR|A_1\oplus eR/A_1\oplus eR/(A_2\oplus B_2)$ does not satisfy $(\ast, 3)$ from [4], Corollary 2. Hence $A_1\cong B_1$. If $A_1\neq0$ and $B_2\neq0$, any two modules of $\{A_1$, $A_2\oplus B_2$, $B_2\}$ are not related by $\sim$, which contradicts [5], Lemma 1 (note that $\Delta=\Delta(A)$ and $eJ=A_1\oplus B_1$). Hence $B_1$ (or $A_1$) is simple or zero. The remaining parts are clear from [3], Theorem 1. Conversely, if the case I) occurs, Condition I and $(\ast\ast, 3)$ hold by [2], Theorem 12 and [3], Theorem 2 (note that $\Delta=\Delta(A)$ provided $A=A(E)$ and $eJ=A_1\oplus B_1$).

Theorem 6. Let $R$ be a right US-3 ring. Then $(\ast, 1)$ holds for any hollow module if and only if $eR$ has one of the following structure for each primitive idempotent $e$:

1) $|eJ/eJ^2|\leq1$.
2) $|eJ/eJ^2|=2$
   i) $eJ^2=0$
   ii) $eJ^2\neq0$, $eJ=A_1\oplus B_1$ has the structure as in Theorem 1, where $A_1$ is uniserial and $B_1$ is simple $(A_1\cong B_1)$.

Proof. Since $R$ is a right US-3 ring, $|eJ/eJ^2|\leq2$ by [5], Theorem 2. Assume that $(\ast, 1)$ holds and $|eJ/eJ^2|\neq1$. Then $eJ=A_1\oplus B_1$ by assumption, where $A_1$ and $B_1$ are hollow. If $A_1=A_1J\cong B_1$, $eJ^2$ is a waist and $A_1\cong B_1$ by [5], Theorem 2. Hence, if $eJ^2\neq0$, $A_1J\cong eJ^2$. Then $eR/A_1J$ contains a non-trivial waist $eJ^2/A_1J$ and $eJ/A_1J$ is not hollow. Accordingly, $eJ/A_1J$ is not a direct sum of hollow modules. Therefore $eJ^2=0$. Next assume $eJ^2\neq0$, and hence $A_1\cong B_1$. Then $\Delta(A_1)=\Delta(B_1)=\Delta$ and $A_1\cong B_1$. From the proof of Theorem 5, we can show that either $A_1$ or $B_1$ is simple (note $|eJ/eJ^2|=2$), say $B_1$. We shall show that $A_1$ is uniserial. We know from the proof of [5], Theorem 2 that if $\Delta(C)\neq\Delta$ for some submodule $C$ of $eJ$, then $eJ$ contains a non-trivial waist module $eJ^i$ with $|eJ^i/eJ^{i+1}|=2$. Then $(\ast, 1)$ does not hold from the observation of the case $eJ^2=0$. Hence $\Delta(C)=\Delta$ for all $C$ in $eR$. Now $J(A_1)=A_2\oplus A_3\oplus A_4\oplus \cdots$ from $(\ast, 1)$, where $A_2$, $A_3$, ... are hollow (actually $A_2=\cdots=0$ from [5], Theorem 2). Being $A_2\cong B_1$ and $A_3\cong B_1$, we know that $A_1\sim A_2$. Let $a_2$ be in $A_2-A_1J$. Since $\Delta(A_3)=\Delta$ and $A_1\sim A_2$, there exist a unit $x$ in $eRe$ and $j$ in $eJe$ such that $xA_2=A_2$ and $(x+j)A_2=A_2$. Put $a_2'=(x+j)a_2\in A_2$. Since $x$ is an isomorphism of $A_2$, $xa_2\in A_2J$, $ja_2\in eJ\subset eJ^2= A_2J\oplus A_3\oplus \cdots$, which is a contradiction. Hence $A_2=A_2$. Repeating this
procedure, we know that $A_1$ is uniserial. Therefore every submodule of $eJ$ is one of the following: 1) $A_i$, 2) $A_i \oplus B_i$, and 3) $A_i(f)$, where $A_i = A_iJ^{i-1}$ and $A_i(f) = \{a_i + f(a_i) | a_i \in A_i, f \in \text{Hom}_K(A_i, B_i)\}$. Assume $A_n \neq 0$ and $A_{n+1} = 0$. Then considering $\{A_i, A_i(f), B_i\}$ ($i < n$), $A_i \sim A_i(f)$. It is clear from [5], Lemma 1 that $A_n \sim A_n(f)$ or $A_n(f) \sim B_1$ (if $A_n \sim A_n(f)$, $A_n = A_n(f)$ for $eJeA_n = 0$). Therefore $eJ$ has the structure in Theorem 5. Conversely, assume that $eR$ has the structure of the theorem. If $|eJ/eJ^2| \leq 1$, $eJ^2$ is a waist, and hence, for any submodule $C \subset eJ^2$, $J(eR/C) = eJ/C$ contains a unique maximal submodule $eJ^2/C$. If $eJ^2 = 0$, $(\ast, 2)$ holds for any two hollow modules by [3], Proposition 3. It is clear for the last case to show that $(\ast, 1)$ holds.

4. Examples. 1. Let $R$ be the algebra over a field $K$ given in [3], Example 2. Then the lattice of submodules of $eR$ is the following:

```
   eR
   eJ = A_1 \oplus B_1
   A_1

   A_2 \oplus B_1

   eJ^2 = A_2

   B_1 \rightarrow (e + x_1k)B_1

   0
```

where $k$ are in $K$. Hence $(\ast\ast, 3)$ and $(\ast, 2)$ are satisfied by Theorem 5.

2. Let $R$ be a vector space over $K$ with basis $\{e, f, a, b, c, d\}$. Define the multiplication among these elements as follows: $e^2 = e$, $f^2 = f$, $ef = fe = 0$, $ea = ae = a$, $eb = bf = b$, $ec = cf = c$, $fd = df = d$, $ab = bd = c$ and other products are equal to zero. Then the lattice of submodules of $eR$ is the following:

```
   eR

   eJ

   \langle a, c \rangle

   \langle a, c \rangle

   \langle b, c \rangle

   \langle c \rangle = eJ^2

   0
```

Then $R$ is a right US-3 ring with Condition II'. However, $eJ$ is indecomposable, but not hollow. Hence $(\ast, 1)$ is not satisfied.

Then $R$ is a right US-3 ring with $(\ast, 1)$, but without $(\ast, 2)$ (note that $\Delta(e_3K) = K \neq L = \Delta$).

4. Assume that a right artinian ring $R$ has a decomposition $R = eR \oplus fR$ and $J^2 = 0$, where \{e, f\} is a set of mutually orthogonal primitive idempotents. Then $(\ast, 2)$ holds for any $D(2)$ by [3], Proposition 3. We shall give the complete list of such rings with $(\ast \ast, 3)$ and Condition II. If $R$ is the ring mentioned above, $ef = A_1 \oplus A_2$ and $ff = B_1 \oplus B_2$, where the $A_i$ and the $B_i$ are simple or zero. We always assume, in the following observation, that

\begin{align*}
\alpha) \quad & \begin{pmatrix}
T_1 & A_1 \oplus A_2 \\
B_1 \oplus B_2 & T_2
\end{pmatrix}
\end{align*}

means that $T_1$ and $T_2$ are local right artinian rings, the $A_i$ (resp. the $B_i$) are right $T_2$ and left $T_1$ (resp. right $T_1$ and left $T_2$) simple module, $(A_1 \oplus A_2)(T_2) = J(T_1)(A_1 \oplus A_2) = 0$ (the same for $B_1 \oplus B_2$), and $(A_1 \oplus A_2)(B_1 \oplus B_2) = (B_1 \oplus B_2)
(A_1 \oplus A_2) = 0$.

\begin{align*}
\beta) \quad & \Delta \text{ means a division ring.} \\
\gamma) \quad & S \text{ means a local serial ring.} \\
\delta) \quad & L \text{ means the following local ring:} \\
& J(L) = A_1 \oplus A_2, A_1 \approx A_2 \text{ as right } L\text{-modules,} \\
& \left[ L/J(L) : L/J(L)(A_i) \right] = 2, \text{ and for any simple } L\text{-module } A'_i \text{ in } J(L), \text{ there exists a unit } \alpha \text{ in } L \text{ such that } A'_i = \alpha A_i \text{ (see [1] for such a ring).}
\end{align*}

\begin{itemize}
    \item[i)] \( A_1 \approx A_2 \approx eR \) and \( B_1 \approx B_2 \approx fR. \) Then \( eR = fR = 0. \) Hence
\begin{align*}
R = \begin{pmatrix}
L_1 & 0 \\
0 & L_2
\end{pmatrix}
\end{align*}
\end{itemize}

\begin{itemize}
    \item[ii)] \( A_1 \approx A_2 \approx fR, B_1 \approx B_2 \approx eR. \) Then
\begin{align*}
R = \begin{pmatrix}
\Delta_1 & A_1 \oplus A_2 \\
B_1 \oplus B_2 & \Delta_2
\end{pmatrix}
\end{align*}
\end{itemize}

where the $A_i$ (resp. $B_i$) satisfy $\xi$ as $\Delta_1 - \Delta_2$ (resp. $\Delta_2 - \Delta_1$) bimodules.

\begin{itemize}
    \item[iii)] \( A_1 \approx A_2 \approx B_1 \approx B_2 \approx eR \) (resp. \( \approx fR \)). Then
\begin{align*}
R = \begin{pmatrix}
L_1 & 0 \\
B_1 \oplus B_2 & \Delta_2
\end{pmatrix}
\end{align*}
\end{itemize}

where the $B_i$ (resp. $A_i$) satisfy $\xi$ as $\Delta_2 - L_1/J(L_1)$ (resp. $\Delta_1 - L_2/J(L_2)$) bimodules.
iv) $A_1 \simeq A_2 \approx B_1 \approx eR$ and $B_2 \approx fR$.

Then

$$R = \begin{pmatrix} \Delta_1 & A_1 \oplus A_2 \\ B_2 & S_2 \end{pmatrix},$$

where the $A_i$ are similar to iii).

v) $A_1 \not\approx A_2$ and $B_1 \not\approx B_2$. Then

$$R = \begin{pmatrix} S_1 & A_2 \\ B_2 & S_2 \end{pmatrix}.$$

vi) Other cases. We may put $A_i = 0$ or $B_i = 0$ in the above. The right serial rings appear in v) by setting $S_2 = \Delta_2$ or $S_1 = \Delta_1$ and $B_2 = 0$ (or $A_2 = 0$).

References


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