

Title	Generalizations of Nakayama ring. II
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Citation	Osaka Journal of Mathematics. 1986, 23(3), p. 509-521
Version Type	VoR
URL	<a href="https://doi.org/10.18910/4306">https://doi.org/10.18910/4306</a>
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## GENERALIZATIONS OF NAKAYAMA RING II

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(Received December 18, 1984)

We have defined (right US-3) rings satisfying (\*\*, 3) in [5], which are rings generalized from Nakayama ring (right generalized uni-serial rings). As stated in [5], we shall give, in this note, another generalization of Nakayama rings, which is related to the condition (\*, 3), and give a characterization of those rings.

**1. Preliminary results.** Let  $R$  be a ring with identity. We assume always throughout this note that  $R$  is a right artinian ring and every module is a right unitary  $R$ -module  $M$  with finite length, which we denote by  $|M|$ . We have studied the following conditions in [3] and [5]:

- (\*\*,  $n$ ) Every (non-zero) maximal submodule of a direct sum  $D(n)$  of  $n$  non-zero hollow modules contains a non-trivial direct summand of  $D(n)$ .
- (\* ,  $n$ ) Every (non-zero) maximal submodule of the  $D(n)$  is also a direct sum of hollow modules.

We shall study mainly, in this note, rings satisfying (\*, 3) for any direct sum of three hollow modules. We shall use the same notations as given in [3] and [5].

Let  $e$  be a primitive idempotent in  $R$ .

CONDITION II [3].  $|eJ|/|eJ^2| \leq 2$  for each  $e$ , where  $J$  is the Jacobson radical of  $R$ .

In [3] we have given the structure of rings which satisfy Condition II and

CONDITION I. Every submodule in any direct sum of (three) hollow modules is also a direct of hollow modules.

However, checking carefully each step, we know that we utilize only (\*, 3) for any direct sum of three hollow modules. Thus we have the following theorem.

**Theorem 1.** Let  $R$  be a right artinian ring. Assume that (\*, 3) for any direct sum of three hollow modules and Condition II hold. Then for each primitive idempotent  $e$  in  $R$ , we have the following properties:

- 1)  $eJ = A_1 \oplus B_1$ , where  $A_1$  and  $B_1$  are uniserial modules. Further, if  $A_1/J(A_1) \cong B_1/J(B_1)$ ,  $\alpha A_1 = B_1$  for some unit  $\alpha$  in  $eRe$ .

2) For every submodule  $N$  in  $eJ$ , there exists a trivial submodule  $A_i \oplus B_j$  of  $eJ$  and a unit  $\gamma$  in  $eRe$  such that  $N = \gamma(A_i \oplus B_j)$ , where  $A_i = A_1 J^{i-1} \subset A_1$  and  $B_j = B_1 J^{j-1} \subset B_1$ .

3) If  $A_1 \approx B_1$ , then  $\Delta(A_i \oplus B_i) = \Delta$  and  $[\Delta : \Delta(A_i \oplus B_j)] = 2$  provided  $i \neq j$ ; further  $\Delta(A_1) = \Delta(A_i) = \Delta(A_i \oplus B_j)$  ( $i < j$ ) and  $\Delta(B_1) = \Delta(B_j) = \Delta(A_i \oplus B_j)$  ( $i > j$ ). If  $A_1 \not\approx B_1$ , then  $\Delta(N) = \Delta$  for any submodule  $N$  in  $eJ$ .

Here we shall recall the notations above. Put  $\Delta = \overline{eRe} = eRe/eJe$ . For any right ideal  $A$  in  $eR$ ,  $\Delta(A) = \{\bar{x} \mid \bar{x} \in \Delta, (x+j)A \subset A \text{ for some element } j \text{ in } eJe\}$ . Then  $\Delta(A)$  is a subdivision ring of  $\Delta$  and  $[\Delta : \Delta(A)]$  means the dimension of  $\Delta$  over  $\Delta(A)$  as a right  $\Delta(A)$ -vector space.

**2. Rings with (\*, 3).** We shall study, in this section, the converse of Theorem 1. We assume that  $R$  has the structure given in Theorem 1, unless otherwise stated.

We have given the following lemma in [3], provided Condition II'' in [3] is satisfied. We shall show in the same manner that the lemma is valid under a weaker condition.

**Lemma 1** ([3]). *Let  $R$  be a ring whose structure is given as in Theorem 1, and  $e$  a primitive idempotent. Let  $\{E_i\}_{i=1}^n$  be a family of right ideals in  $eR$  and  $D = \sum_{i=1}^n \oplus eR/E_i$ . Then, if  $\Delta(A_1) = \Delta(B_1) = \Delta$ ,  $D$  satisfies (\*,  $n$ )*

*Proof.* We shall quote the same argument as given in the last part of § 3 in [3], and hence use the induction on the nilpotency of  $J$ . If  $E_i \subset E_j$  for some  $i, j$ , every maximal submodule of  $D$  contains a direct summand of  $D$  by assumption and [3], Lemma 27 (cf. the proof of Lemma 3 below). By induction we may consider the following case:

$$E_0 = A_i, E_k = A_{i_k} \oplus B_{j_k};$$

$$i < i_1 < i_2 < \dots < i_p, j_1 > j_2 > \dots > j_p \text{ and } D = \sum_{i=0}^p \oplus eR/E_i.$$

Assume  $i_i \leq j_i, i_{i+1} > j_{i+1}$ . Let  $M$  be a maximal submodule of  $D$ . We may assume that  $\bar{M} = M/J(D)$  ( $\subset \bar{D} = D/J(D)$ ) has a basis  $\{(0, \dots, 0, \bar{e}, \bar{k}_i, \dots, 0)\}_i^p$ . Since  $\Delta(A_1) = \Delta(B_1) = \Delta$ , we can take  $k_s$  with  $k_s A_1 = A_1$  for  $s \leq t$  and  $k_r B_1 = B_1$  for  $r > t$ . Set  $M^* = A_1/A_i \oplus \sum_{s=1}^p \oplus eR/(A_{i_s} \oplus B_{j_{s-1}}) \oplus B_1/B_{j_p}, (B_{j_0} = 0)$ , then  $|M^*| = |D| - 1$ . Define a homomorphism  $f$  of  $M^*$  to  $D$  by setting

$$f((x+A_i) + \sum_{s=1}^p (y_s + (A_{i_s} \oplus B_{j_{s-1}})) + (z + B_{j_p}))$$

$$= (x + y_1 + A_i) + (ek_1 y_1 + y_2 + (A_{i_1} \oplus B_{j_1})$$

$$+ (ek_2 y_2 + y_3 + (A_{i_2} \oplus B_{j_2})) + \dots + (ek_p y_p + z + (A_{i_p} \oplus B_{j_p})),$$

where  $x \in A_1, y_s \in eR$  and  $z \in B_1. A_{i_a} \oplus B_{j_{a-1}} \subset A_{i_{a-1}} \oplus B_{j_{a-1}}, k_a(A_{i_a} \oplus B_{j_{a-1}})$

$=A_{i_a} \oplus B_{j_{a-1}} \subset A_{i_a} \oplus B_{j_a}$  for  $a \neq t+1$ , and  $k_{t+1}(A_{i_{t+1}} \oplus B_{j_t}) \subset k_{t+1}(A_{i_{t+1}} \oplus B_{j_{t+1}}) = A_{i_{t+1}} \oplus B_{j_{t+1}}$ . Hence  $f$  is well defined. Assume that the right hand side of the above is zero. Since  $x \in A_1$  and  $z \in B_1$ ,  $y_s \in eJ$  for all  $s$ . Put  $y_r = y_{r1} + y_{r2}$ , where  $y_{r1}$  is in  $A_1$  and  $y_{r2}$  in  $B_1$ . Now  $x + y_1 = x + y_{11} + y_{12} \in A_i$ ,  $x \in A_1$  and so  $y_{12} = 0$ .  $ek_1(y_{11} + y_{12}) + (y_{21} + y_{22}) \in A_{i_1} \oplus B_{j_1}$ . Since  $k_1 A_1 \subset A_1$ ,  $y_{22} \in B_{j_1}$ . Repeating those arguments, we assume by induction that  $y_{l+12} \in B_{j_l}$  for  $l < t'$   $\leq t$ . Put  $w = ek_{t'}(y_{t'1} + y_{t'2}) + (y_{t'+11} + y_{t'+12}) \in A_{i_{t'}} \oplus B_{j_{t'}}$ . Since  $ek_{t'}$  is an isomorphism of  $eJ$ ,  $ek_{t'} B_{j_{t'-1}} \subset A_{i_{t'-1}} \oplus B_{j_{t'-1}}$ . Now  $B_{j_{t'-1}} \subset B_{j_{t'}}$  and  $k_{t'} A_1 \subset A_1$ . Let  $\pi_2: eJ \rightarrow B_1$  be the projection. Then  $B_{j_{t'}} \in \pi_2(w) = \pi_2(ek_{t'} y_{t'2} + y_{t'+12}) = \pi_2(ek_{t'} y_{t'2}) + y_{t'+12}$ . Since  $\pi_2(ek_{t'} y_{t'2}) \in B_{j_{t'}}$ ,  $y_{t'+12} \in B_{j_{t'}}$ . Consider next from the bottom side.  $ek_p(y_{p1} + y_{p2}) + z \in A_{i_p} \oplus B_{j_p}$ . Since  $k_p B_1 \subset B_1$  and  $x \in B_1$ ,  $y_{p1} \in A_{i_p}$  from the same argument above (take  $\pi_1: eJ \rightarrow A_1$ ). Repeating those arguments inductively, we obtain  $y_{s1} \in A_{i_s}$  for  $s \geq t+1$ . Consider  $ek_{t+1}(y_{t+11} + y_{t+12}) + (y_{t+21} + y_{t+22}) \in A_{i_{t+1}} \oplus B_{j_{t+1}}$ . Since  $y_{t+12} \in B_{j_t}$ ,  $y_{t+11} \in A_{i_{t+1}}$  and  $i_{t+1} > j_{t+1}$ ,  $j_t > j_{t+1}$ ,  $y_{t+22} \in B_{j_{t+1}}$ . Similarly, from  $ek_i(y_{i1} + y_{i2}) + (y_{t+11} + y_{t+12}) \in A_{i_i} \oplus B_{j_i}$ ,  $y_{i1} \in A_{i_i}$ . Combining the above two steps, we know that  $f$  is a monomorphism. Hence  $M \approx M^*$ .

It is remained for us, from Lemma 1, to study a case of  $\Delta(A_1) \neq \Delta$ , i.e.,  $\bar{A}_1 \approx \bar{B}_1$ . We have shown in [3] that if a right artinian ring  $R$  has the structure in Theorem 1, then  $(*, n)$  is satisfied for any  $D(n)$ , provided  $J^3 = 0$ . We shall show that  $(*, 3)$  is satisfied without the assumption  $J^3 = 0$ .

**Lemma 2** ([3], Lemma 24). *We assume the above situation. Let  $\delta$  be an element in  $eRe$  such that  $\delta \in \Delta(A_1)$ . Then  $\pi_2 \delta A_i = B_i$ , where  $\pi_2: eJ \rightarrow B_1$  is the projection.*

Proof. Since  $[\Delta: \Delta(A_1)] = 2$ ,  $\delta = \alpha_1 + \alpha_2 \alpha_2$ ; the  $\alpha_i \in \Delta(A_1)$  and  $\alpha A_1 = B_1$ . Set  $\delta = \alpha_1 + \alpha_2 \alpha_2 + j$ ;  $\alpha_i A_1 \subset A_1$ ,  $j \in eJe$ . Since  $j A_i \subset A_{i+1} \oplus B_{i+1}$ ,  $\pi_2 \delta A_i = B_i$ .

**Lemma 3.** *Assume that  $R$  has the structure 1), 2) and 3) given in Theorem 1. Then  $(*, 2)$  is fulfilled for any  $D(2)$ .*

Proof. The assumption 2) in Theorem 1 gives us a guarantee of  $(*, 1)$  for any hollow module. Let  $eJ = A_1 \oplus B_1$ . If  $A_1 \approx B_1$ ,  $\Delta(C) = \Delta$  for any submodule  $C$  of  $eR$  by assumption. Then we have shown by Lemma 1 that  $(*, 2)$  is fulfilled for any  $D(2)$ . Assume that  $A_1 \not\approx B_1$ . Then  $\Delta = \Delta(A_1) \oplus \alpha \Delta(A_1)$ , where  $\alpha$  is the element given in 1). Set  $D = eR/N_1 \oplus eR/N_2$ , where the  $N_i$  are submodules of  $eR$ . We shall show the lemma by induction on the nilpotency of  $J$ . If  $J^3 = 0$ , we are done in [3], § 4. Assume  $eJ^n \neq 0$  and  $eJ^{n+1} = 0$ . If  $N_i \supset eJ^n$  for  $i = 1, 2$ ,  $eR/N_i$  is a hollow  $R/J^n$ -module. Hence we may assume that  $N_1 = A_i = A_1 J^{i-1}$  by induction. Let  $\bar{M}$  be a maximal submodule of  $D$ , and put  $\bar{D} = D/J(D) \supset \bar{M} = M/J(D)$ . We may assume that  $\bar{M}$  has a basis  $\{(e + J(D), \delta + J(D))\}$ , where  $\delta$  is a unit element in  $eRe$  (it is sufficient to show

the lemma in case  $R$  is basic; see [2] and [3]).

i)  $N_2 = A_k \oplus B_j$ , (we may assume  $k \leq i$  [3]) a)  $i \leq k \leq i$ .  $F = A_i \cap \delta^{-1}(A_k \oplus B_j) = \delta^{-1}(\delta A_i \cap (A_k \oplus B_j))$ . a)-i) If  $\bar{\delta} \in \Delta_1 = \Delta(A_1)$ ,  $F = A_k$  (we may assume  $\delta A_1 \subset A_i$ ). a)-ii). If  $\bar{\delta} \notin \Delta_1$ ,  $F = A_j$  by Lemma 2. Put  $M^* = eR/A_k \oplus A_1/A_i \oplus B_1/B_j$  for the case a-i). Define a homomorphism  $f$  of  $M^*$  to  $D$  by setting

$$f((x+A_k)+(y+A_i)+(z+B_j)) = (x+y+A_i) + (\delta x + z + (A_k \oplus B_j)),$$

where  $x$  is in  $eR$ ,  $y$  in  $A_1$  and  $z$  in  $B_1$ . Then  $f$  is well defined. It is easy to check that  $f$  is a monomorphism, since  $\delta A_1 = A_1$ . Put  $M^* = eR/A_j \oplus A_1/A_i \oplus A_1/A_k$  for the case a)-ii). Define a homomorphism  $f$  of  $M^*$  into  $D$  by setting

$$f((x+A_j)+(y+A_i)+(z+A_k)) = (x+y+A_i) + (\delta x + z + A_k \oplus B_j),$$

where  $x$  is in  $eR$  and  $y, z$  are in  $A_1$ . We can show from the fact:  $\bar{\delta} \notin \Delta_1$ , that  $f$  is a monomorphism (cf. the proof of Lemma 4 below). Hence  $M \approx M^*$ , since  $|M| = |M^*|$  and  $f(M^*) \subset M$ .

b)  $k \leq i \leq j$ . If  $\bar{\delta} \in \Delta_1$  (resp.  $\notin \Delta_1$ ),  $F = A_i$  (resp.  $F = A_j$ ). We obtain the same result as in a-ii) for  $F = A_j$ . If  $F = A_i$ , put  $M^* = eR/A_i \oplus A_1/A_k \oplus B_1/B_j$ . Then  $M \approx M^*$  as above.

c)  $k \leq j \leq i$ . Since  $eReA_i \subset A_k \oplus B_j$ ,  $M$  contains a direct summand of  $D$ .

ii)  $N_2 = A_j$ ,  $i \geq j$ . If  $\bar{\delta} \notin \Delta_1$ ,  $\delta A_i \cap A_j = 0$  and  $M$  is isomorphic to  $eR \oplus A_1/A_i \oplus A_1/A_j$ . If  $\bar{\delta} \in \Delta_1$ ,  $\delta A_i \subset A_j$ . Hence we obtain the same situation as in i)-c).

**Lemma 4.**  $(*, 3)$  is satisfied for any three hollow modules.

Proof. We may assume  $\Delta(A_1) = \Delta_1 \neq \Delta$  by Lemma 1. From induction on the nilpotency of  $J(R)$ , it is sufficient to study the case:

$$E_0 = A_i, E_1 = A_{i_1} \oplus B_{j_1} \quad \text{and} \quad E_2 = A_{i_2} \oplus B_{j_2}$$

with  $i_k \leq j_k$  for  $k=1, 2$ , and  $D = \sum \oplus eR/E_i$ . Here  $B_{j_k}$  may be equal to zero (cf. [3], § 3).

If  $B_{j_1} = B_{j_2} = 0$ ,  $D$  satisfies  $(*, 3)$  by [4], Corollary 3. Let  $M$  be a maximal submodule of  $D$ . If  $M$  contains a non-zero direct summand  $D_1$  of  $D$ ,  $M = D_1 \oplus M_1$  where  $M_1$  is a maximal submodule of  $N_1 \oplus N_2$ ; the  $N_i$  are isomorphic to some of  $\{eR/E_i\}_{i=1}^3$ . Then  $M_1$  is a direct sum of hollow modules by Lemma 3, and hence so is  $M$ . Therefore we consider  $M$  not containing a direct summand of  $D$ . Put  $\bar{D} = D/J(D) \supset \bar{M} = M/J(D)$ , and  $D = (\bar{e}\Delta, \bar{e}\Delta, \bar{e}\Delta)$ . Then the above  $\bar{M}$  has a basis  $\{(\bar{e}, \bar{\delta}_1, 0), (0, \bar{e}, \bar{\delta}_2)\}$ , where  $\bar{\delta}_i$  are in  $\Delta$  and  $\bar{\delta}_1 \bar{\delta}_2 \neq 0$  (cf. [3]). We consider the following situation:

1)  $i \leq i_1 \leq i_2 \leq j_1 \leq j_2$ .

a)  $\bar{\delta}_2 \in \Delta_1$ . Then  $\delta_2 E_2 \subset E_1$ . Hence  $M$  contains a direct summand of  $D$  by [1], Theorem 2. (1)

- b)  $\bar{\delta}_1 \in \Delta_1$  and  $\bar{\delta}_2 \notin \Delta_1$ .  $M \approx A_1/A_i \oplus eR/A_{i_1} \oplus eR/(A_{j_1} \oplus B_{j_2}) \oplus A_1/A_{i_2}$ . (2)
- c)  $\bar{\delta}_1$  and  $\bar{\delta}_2 \notin \Delta_1$ .  $M \approx A_1/A_i \oplus eR/A_{i_2} \oplus eR/(A_{j_1} \oplus B_{j_2}) \oplus A_1/A_{i_1}$ . (3)
- 2)  $i_1 \leq i \leq i_2 \leq j_1 \leq j_2$ .
  - a)  $\bar{\delta}_1$  or  $\bar{\delta}_2 \in \Delta_1$ . We obtain (1).
  - b)  $\bar{\delta}_1$  and  $\bar{\delta}_2 \notin \Delta_1$ . We obtain (3).
  - 3)  $i_1 \leq i_2 \leq i \leq j_1 \leq j_2$ . Since  $E_1 \supset E_2 \supset E_3$ . We obtain (1) by [4], Corollary 3.
  - 4)  $i_1 \leq j_1 \leq i_2 \leq j_2$ . Since  $eReE_2 \subset E_1$ , we obtain (1).
  - 5)  $i \leq i_1 \leq i_2 \leq j_2 \leq j_1$ .
    - a)  $\bar{\delta}_1$  and  $\bar{\delta}_2 \in \Delta_1$ . We obtain the same situation as in the proof of Lemma 1. i.e.,  $M \approx A_1/A_i \oplus eR/A_{j_1} \oplus eR/(A_{i_2} \oplus B_{j_1}) \oplus B_1/B_{j_2}$ . (4)
    - b)  $\bar{\delta}_1 \in \Delta_1$  and  $\bar{\delta}_2 \notin \Delta_1$ .  $M \approx A_1/A_i \oplus eR/A_{i_1} \oplus eR/(A_{j_2} \oplus B_{j_1}) \oplus A_1/A_{i_2}$ . (5)
    - c)  $\bar{\delta}_1 \notin \Delta_1$  and  $\bar{\delta}_2 \in \Delta_1$ .  $M \approx A_1/A_i \oplus eR/A_{j_2} \oplus eR/(A_{i_2} \oplus B_{j_1}) \oplus A_1/A_{i_1}$ . (6)
    - d)  $\bar{\delta}_1$  and  $\bar{\delta}_2 \notin \Delta_1$ .  $M \approx A_1/A_i \oplus eR/A_{i_2} \oplus eR/(A_{j_2} \oplus B_{j_1}) \oplus A_1/A_{i_1}$ . (7)
    - 6)  $i_1 \leq i \leq i_2 \leq j_2 \leq j_1$ .
      - a)  $\bar{\delta}_1 \in \Delta_1$ . We obtain (1).
      - b)  $\bar{\delta}_1 \notin \Delta_1$  and  $\bar{\delta}_2 \in \Delta_1$ . We obtain (6).
      - c)  $\bar{\delta}_1$  and  $\bar{\delta}_2 \notin \Delta_1$ . We obtain (7).
      - 7)  $i_1 \leq i_2 \leq i \leq j_2 \leq j_1$ .
        - a)  $\bar{\delta}_1 \in \Delta_1$  or  $\bar{\delta}_2 \notin \Delta_1$  and  $\bar{\delta}_2 = \bar{\delta}_1 \bar{x}_2$ ;  $\bar{x}_2 \in \Delta_1$ . We obtain (1).
        - b)  $\bar{\delta}_1 \notin \Delta_1$  and  $\bar{\delta}_2 \in \Delta_1$ . We obtain (6).
        - c)  $\bar{\delta}_1$  and  $\bar{\delta}_2 \notin \Delta_1$  and  $\{\bar{\delta}_1, \bar{\delta}_2\}$  is linearly independent over  $\Delta_1$ .  
 $M \approx A_1/A_{i_1} \oplus eR/A_i \oplus eR/(A_{j_2} \oplus B_{j_1}) \oplus A_1/A_{i_2}$ . (8)
      - 8)  $i_1 \leq i_2 \leq j_2 \leq i \leq j_1$  or  $i_1 \leq i_2 \leq j_2 \leq j_1 \leq i$ . Since  $eReE_0 \subset E_2$ , we obtain (1).

We shall give a sample of proofs.

1)-c). Put  $\xi' = (\bar{e}, \bar{\delta}_1, 0)$  and  $\eta' = (0, \bar{e}, \bar{\delta}_2)$ . Consider  $\{\xi', \eta' = (0, \bar{\delta}'_2, \bar{e})\}$ , where  $\bar{\delta}'_2 = \bar{\delta}_2^{-1} \in \Delta_1$ . If  $\{\bar{\delta}_1, \bar{\delta}_2\}$  is linearly independent, there exist  $\bar{a}'_1$  and  $\bar{a}'_2$  in  $\Delta_1$  such that  $\bar{e} = \bar{\delta}_1 \bar{a}'_1 + \bar{\delta}_2 \bar{a}'_2$  and  $\bar{a}'_2 \neq 0$ , since  $[\Delta: \Delta_1] = 2$ . Then  $\bar{M}$  has a basis  $\{\xi = \xi' + \eta' \bar{a}'_2 \bar{a}'_1^{-1} = (\bar{e}, \bar{a}_1, \bar{a}_2)$  and  $\eta = \eta' = (0, \bar{\delta}'_2, \bar{e})\}$ , where  $\bar{a}_1 = \bar{a}'_1^{-1}$  and  $\bar{a}_2 = \bar{a}'_2 \bar{a}'_1^{-1}$ . On the other hand, if  $\bar{\delta}_1 = \bar{\delta}'_2 \bar{a}'_2'$ ,  $\bar{M}$  has a basis  $\{\xi = \xi - \eta' \bar{a}'_2' = (\bar{e}, 0, \bar{a}_2)$  ( $\bar{a}'_2' = \bar{a}_2$ ) and  $\eta = \eta' = (0, \bar{\delta}'_2, \bar{e})\}$ . In either case,  $\bar{a}_2 \neq 0$  and define a homomorphism  $f$  of  $M^* = A_1/A_i \oplus eR/A_{i_2} \oplus eR/(A_{j_1} \oplus B_{j_2}) \oplus A_1/A_{i_1}$  to  $D$  by setting

$$f((x+A_i)+(y+A_{i_2})+(z+(A_{j_1} \oplus B_{j_2}))+(w+A_{i_1})) = (x+y+A_i)+(a_1y+\delta'_2z+w+(A_{i_1} \oplus B_{j_1}))+(a_2y+z+(A_{i_2} \oplus B_{j_2})),$$

where  $x$  is in  $A_1$ ,  $y$  and  $z$  in  $eR$  and  $w$  in  $A_1$ .

Since  $A_i \cap a_1^{-1}(A_{i_1} \oplus B_{j_1}) \cap a_2^{-1}(A_{i_2} \oplus B_{j_2}) = A_{i_2} (0^{-1}(A_{i_1} \oplus B_{j_1}) = eR)$  and  $\delta_2^{-1}(A_{i_1} \oplus B_{j_1}) \cap (A_{i_2} \oplus B_{j_2}) = A_{j_1} \oplus B_{j_2}$  by Lemma 2,  $f$  is well defined. Assume that the latter term of the above equation is zero, i.e.,

- 0)  $x, w \in A_1$ .
- 1)  $x+y \in A_i$ .
- 2)  $a_1y + \delta'_2z + w \in A_{i_1} \oplus B_{j_1}$ .
- 3)  $a_2y + z \in A_{i_2} \oplus B_{j_2}$ .

Since  $x$  is in  $A_1 \subset eJ$ ,  $y$  and  $z$  are in  $eJ$  by 1) and 3). Put  $z = a + b$ ;  $a \in A_1$ ,  $b \in B_1$ . Since we may assume  $a_1 A_1 = A_1$ ,  $b$  is in  $B_{j_2}$  by 3).  $\delta'_2 z = \delta'_2 a + \delta'_2 b$  and  $\delta'_2 b \in A_{j_2} \oplus B_{j_2} \subset A_{i_1} \oplus B_{j_1}$ . Hence  $a$  is in  $A_{j_1}$  by 2) and Lemma 2, and so  $z$  is in  $A_{j_1} \oplus B_{j_2} \subset A_{i_2} \oplus B_{j_2}$ . Therefore  $y$  is in  $A_{i_2}$  by 3), since  $a_2 \neq 0$ , and so  $x$  is in  $A_i$ ,  $w$  in  $A_{i_1}$ . We have shown that  $f$  is a monomorphism. On the other hand,  $|D| = n + i + i_1 + i_2 + j_1 + j_2 - 2$  and  $|M^*| = n + i + i_1 + i_2 + j_1 + j_2 - 3 = |D| - 1$ , where  $eJ^n \neq 0$ ,  $eJ^{n+1} = 0$ . Hence  $f(M^*) = M$ , for  $M \supset J(D)$  and  $f(M^*) = M$ .

Now let  $eJ = A_1 \oplus B_1$  be as before and  $eJ^n \neq 0$  and  $eJ^{n+1} = 0$ . We consider here together all cases: a)  $B_1 = 0$ , b)  $\bar{A} \approx \bar{B}_1$  and c)  $\bar{A}_1 \approx \bar{B}_1$ . We obtain the following three hollow modules;

1)  $S_i(e) = eR/(A_1 \oplus B_i)$ , 2)  $T_i(e) = eR/A_i$  (or  $eR/B_j$ ) and 3)  $U_{ij}(e) = eR/(A_i \oplus B_j)$  (we denote those modules by  $H(e)$ ).

Now  $S_i$  and  $U_{ij}$  are  $R/J^t$ -modules, where  $t = i$  and  $\max\{i, j\}$ , respectively. We shall give a weight for each hollow module  $H$  as follows;  $w(H) = |J(H)/J^2(H)|$ , i.e.,  $w(S_i) = 1$ ,  $w(T_i) = 2$  ( $i \neq 1$ ),  $w(T_1) = 1$  and  $w(U_{ij}) = 2$  ( $i \neq 1$  and  $j \neq 1$ ).

**Lemma 5.** *Let  $S(e)$ ,  $T(e)$  and  $U(e)$  be as above. Then for a maximal submodule  $M$  of  $D$  below, we obtain the following:*

- 1)  $D = S(e) \oplus S'(e)$ .  $M \approx S(f_1) \oplus S(e)$ ,  $S(e) \oplus S(f_2)$  or  $U(e)$ .
  - 2)  $D = T(e) \oplus S(e)$ .  $M \approx S(f_1) \oplus S(f_2) \oplus S(e)$  or  $T'(e) \oplus S(f)$ .
  - 3)  $D = U(e) \oplus S(e)$ .  $M \approx S(f_1) \oplus S(f_2) \oplus S(e)$ ,  $U(e) \oplus S(f)$  or  $U'(e) \oplus S(f)$ ,
- where  $e$  and  $f$  are primitive idempotents.

Proof. We can show the lemma from Lemmas 1 and 3 (consider  $D$  as  $R/J^t$ -modules for 3);  $t \leq n$ ).

Assume that

$$C = \sum_{i=1}^b \sum_{j=1}^{i_i} \oplus H_j(e_i),$$

where  $1 = \sum e_i$ ,  $\{e_i\}$  is a set of mutually orthogonal primitive idempotents (and  $R$  is basic). Let  $M$  be a maximal submodule of  $C$ . Since  $H_j(e_i)/J(H_j(e_i)) \approx H_{j'}(e_{i'})/J(H_{j'}(e_{i'}))$  for  $i \neq i'$ ,  $M = \sum_i \oplus M_i$ , where  $M_k = \sum_j \oplus H_j(e_k)$  for all  $k$  except some  $q$  and  $M_q$  is a maximal one in  $\sum_j \oplus H_j(e_q)$ . Put  $w(C) = \sum_i \sum_j w(H_j(e_i))$ .

**Lemma 6.** *Every submodule  $F$  of  $D(q)$  is a direct sum of hollow modules  $H_i$  and  $w(F) \leq 2q$  ( $q \leq 3$ ).*

Proof. We shall show the lemma for a case  $q = 3$ . The remaining parts are same. In order to prove the lemma, we may show that any maximal submodule  $M$  of  $C$  above with  $t = w(C) \leq 6$  has a similar direct decomposition and  $w(M) \leq t$ . Further, from the argument before Lemma 6, we may assume  $e_i = e$ , and show that

$$M = \sum_{s=1}^m \oplus H_s \quad \text{and} \quad w(M) \leq t \tag{\#}$$

We note that if  $w(H_i(e))=2$ ,  $J(H_i(e))$  is a direct sum of two uniserial modules. If  $H_i(e)=eR$  for some  $i$ ,  $M$  contains a direct summand of  $C$  by [1], Theorem 2. Hence  $M$  satisfies  $(\#)$  by induction on  $m$  and the above remark. We shall show  $(\#)$  by induction on  $n$  ( $J^{n+1}=0$ ). If  $n=0$ , then  $(\#)$  is trivial. We assume that every maximal submodule  $M$  satisfies  $(\#)$  for  $k \leq n-1$ . Start from

$$D = H(e_1) \oplus H(e_2) \oplus H(e_3).$$

$w(D)=6$  provided no-one of  $\{H(e_i)\}$  is uniserial, and  $w(D) \leq 5$  for other cases. Further, if no-one of  $\{H(e_i)\}$  is isomorphic to  $T_i(e)$ , the  $H(e_i)$  are  $R/J^t$ -modules for some  $t \leq n$ . Then we can show  $(\#)$  by the induction hypothesis. Hence assume  $H(e_1)=T_1(e_1)$ . We may further assume  $e=e_i$  for all  $i$  from the remark before Lemma 6. Let  $M$  be a maximal submodule of  $D$ . Then from Lemma 4  $M = \sum_{i=1}^4 \oplus H(f_i)$ ;  $f_i=e$  if  $H(f_i) \approx T$  or  $U$ , and  $w(D) \geq w(M)$ . Put  $M_0 = \sum_{f_i \neq e} \oplus H(f_i)$ . First we remark that the  $M_0$  is an  $R/J^t$ -module, and hence  $(\#)$  is satisfied for  $M_0$ . Further, if no-one of  $\{H(e_i)\}$  is isomorphic to  $T_1(e) = eR/A_1$ , the same for  $\{H(f_i)\}$ . Now let  $M$  be the maximal submodule in  $C(\subset D)$  given in the beginning. Remarking the above fact (the case  $H(e) = T_1(e)$ ), we have the following cases:

I)  $C = T_{i_1} \oplus T_{i_2} \oplus T_{i_3}, T_{i_1} \oplus T_{i_2} \oplus U_{k_1j_1},$  or  $T_{i_1} \oplus U_{k_1j_1} \oplus U_{k_2j_2}.$

In the first case  $M$  contains a direct summand of  $C$ , and hence we have  $(\#)$  by Lemmas 1 and 3. For the remaining cases we can use Lemmas 1 and 4.

II)  $C = T_{i_1} \oplus T_{i_2} \oplus S_{k_1} \oplus S_{k_2}.$

$M$  contains a direct summand of  $C$  by [1], Theorem 2. Repeating this argument, we can reduce  $M$  to a case  $M = M_1 \oplus S_{k_1} \oplus S_{k_2}$  ( $M_1$  is a maximal in  $T_{i_1} \oplus T_{i_2}$ ),  $M = M_2 \oplus T_{i_2} \oplus S_{k_2}$  ( $M_2$  is maximal in  $T_{i_1} \oplus S_{k_1}$ ) or  $M = M_3 \oplus T_{i_1} \oplus T_{i_2}$  ( $M_3$  is maximal in  $S_{k_1} \oplus S_{k_2}$ ). Therefore  $M$  satisfies  $(\#)$  by Lemma 5.

III)  $C = T_{i_1} \oplus U_{k_1j_1} \oplus S_{h_1} \oplus S_{h_2},$  or  $T_1 \oplus T_1 \oplus U_{i_1j_1} \oplus S_{k_1}.$

We can make use of the same argument as in I).

IV)  $T_i$  does not appear in a direct summand of  $C$ , for instance  $C = U_{i_1j_1} \oplus U_{i_2j_2} \oplus U_{i_3j_3}.$

We can use the induction hypothesis.

V) Some of  $T, U$  and  $S$  are equal to zero.

We have the same result as above.

Thus we have

**Theorem 2.** *Let  $R$  be a right artinian ring satisfying Condition II. Then the following conditions are equivalent:*

- 1) Every submodule of any  $D(3)$  is a direct sum of hollow modules.
- 2)  $(*, 3)$  holds for any  $D(3)$ .

3)  $eR$  has the structure given in Theorem 1 for each primitive idempotent  $e$ .  
 In this case every submodule of  $D(i)$  is a direct sum of at most  $2i$  hollow modules for  $i \leq 3$ .

REMARK. If  $R$  is an algebra of finite dimension over a field  $K$ , then H. Asashiba has shown that  $(*, 3)$  implies Condition II. Further, if  $K$  is algebraically closed,  $\Delta(N) = \Delta = K$  for any submodule  $N$  of  $eR$ . If  $\Delta(N) = \Delta$  for  $N$ ,  $(*, 3)$  implies Condition II by [2], Proposition 10.

**Theorem 3.** *Let  $R$  be as above. Assume that  $\Delta(N) = \Delta$  for any submodule  $N$  of  $eR$ . Then the following statements are equivalent:*

- 1) *Every submodule of a finite direct sum of any hollow modules is also a direct sum of hollow modules.*
- 2) *Every submodule of a direct sum of any three hollow modules is also a direct sum of hollow modules.*
- 3)  *$(*, 3)$  holds for any  $D(3)$ .*

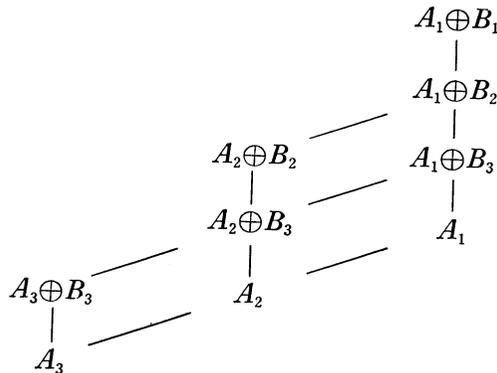
*In this case every submodule  $M$  of  $D(i)$  is a direct sum of at most  $2i$  hollow modules.*

The author believes that Theorem 3 will be true without assumption  $\Delta(N) = \Delta$ . However, he can not find a systematic proof. We have studied this problem in [3], § 4, provided  $J^3 = 0$ . We shall extend this manner to the case  $J^4 = 0$ .

**Proposition 4.** *Let  $R$  be a right artinian ring with  $J^4 = 0$  and assume that Condition II. Then the following conditions are equivalent:*

- 1) *Condition I for any direct sum of hollow modules holds.*
- 2) *Condition I for any direct sum of three hollow modules holds.*
- 3)  *$eR$  has the structure given in Theorem 1.*

Proof. We may consider the proposition in case of  $\Delta(A_1) \neq \Delta$ . Under the assumption above, we obtain the diagram of submodules in  $eJ$  up to isomorphism:



Let  $\{E_i\}_{i=1}^4$  be a family of the modules above. Put  $D = \sum_{i=1}^4 eR/E_i$ . Then, since  $\Delta(A_2 \oplus B_2) = \Delta$ , every maximal submodule  $M$  of  $D$  contains a non-trivial direct summand of  $D$  by [1], Theorem 2 and [4], Corollary 3, except  $D_1 = eR/A_1 \oplus eR/A_1 \oplus eR/(A_2 \oplus B_3) \oplus eR/(A_2 \oplus B_3)$ . Let  $M$  be the maximal submodule such that  $\bar{M} = M/J(D) = \xi\Delta \oplus \eta\Delta \oplus \zeta\Delta$ , where  $\xi = (\bar{e}, \bar{\delta}_1, 0, 0)$ ,  $\eta = (0, \bar{e}, \bar{\delta}_2, 0)$  and  $\zeta = (0, 0, \bar{e}, \bar{\delta}_3)$ . If  $\bar{\delta}_1$  or  $\bar{\delta}_3$  is in  $\Delta_1$ ,  $M$  contains a direct summand of  $D$ . Assume  $\bar{\delta}_1$  and  $\bar{\delta}_3 \notin \Delta_1$ . If  $\bar{\delta}_2 \in \Delta_1$ , there exist  $a_2, a_3 (\neq 0)$  in  $\Delta_1$  such that  $\bar{\delta}_2 a_2^{-1} - \bar{\delta}_2^{-1} \bar{\delta}_3 a_3 = -\bar{\delta}_1$ , for  $[\Delta: \Delta_1] = 2$ . Then  $\bar{M}$  has a basis  $\{\xi(-\bar{\delta}_1^{-1} \bar{\delta}_2^{-1}(a_2 - \bar{\delta}_3^{-1}) + \eta \bar{\delta}_2^{-1}(a_2 - \bar{\delta}_3^{-1} a_3) + \zeta \bar{\delta}_3^{-1} a_3 = (\bar{e}, 0, a_2, a_3), \eta, \zeta\}$ . Then  $M \approx eR/A_2 \oplus eR/A_2 \oplus eR/(A_3 \oplus B_3)$  as in the proof of Lemma 4. Next assume  $\bar{\delta}_2 \notin \Delta_1$ . If  $\bar{\delta}_1 = \bar{\delta}_2^{-1} a_2$ ,  $\bar{M}$  has a basis  $\{(\bar{e}, 0, a_2, 0), \eta, \zeta\}$ . If  $\{\bar{\delta}_1, \bar{\delta}_2^{-1}\}$  is linearly independent, there exist  $a_1, a_2$  in  $\Delta_1$  with  $a_1 a_2 \neq 0$  such that  $\bar{e} = \bar{\delta}_1 a_1 + \bar{\delta}_2^{-1} a_2$ . Then  $\bar{M}$  contains a basis  $\{(\bar{e}, a_1, a_2, 0), \eta, \zeta\}$ . Repeating this argument for  $\eta$  and  $\zeta$ , we obtain a basis  $\{(\bar{e}, a_1, a_2, 0), (0, \bar{e}, \bar{b}_1, \bar{b}_2), \zeta\}$ , where  $a_2 \bar{b}_2 \neq 0$ . In this case we obtain also the same result. Therefore every maximal submodule of  $D$  is a direct sum of hollow modules. Finally, if  $D$  is a direct sum of  $m$  hollow modules ( $m \geq 5$ ),  $M$  contains a non-trivial direct summand of  $D$  by [1], Theorem 2 and [4], Corollary 3. Hence we can prove the proposition by induction on  $m$ .

**3. Right US-3 rings with  $(*, n)$ .** We have defined right US-3 rings in [5], i.e., rings satisfying  $(**, 3)$ . In this section we shall study the structure of right US-3 rings with  $(*, 1)$  or  $(*, 2)$ .

**Lemma 7.** *If a right US-3 ring satisfies  $(*, 2)$  for any  $D(2)$ , then Condition I is satisfied for any  $D(n)$ .*

Proof. Let  $\{N_i\}_{i=1}^n$  be a set of hollow modules, and put  $D = \sum_{i=1}^n N_i$ . If  $n \geq 3$ , every maximal submodule  $M$  of  $D$  is of a form  $M_1 \oplus \sum_{i=3}^n N_i$ , where  $M_1$  is a maximal submodule of  $N'_1 \oplus N'_2$  and the  $N'_i$  are isomorphic to some in  $\{N_i\}$ . Hence  $M_1$  is a direct sum of hollow modules by  $(*, 2)$ .

**Theorem 5.** *Let  $R$  be a right artinian ring. Then  $R$  is a right US-3 ring and  $(*, 2)$  holds for any  $D(2)$  if and only if, for each primitive idempotent  $e$ ,  $eJ$  has the following structure:*

I)  $eJ^2 = 0$ . 1)  $eJ = A_1 \oplus B_1$  with  $A_1, B_1$  simple or zero. 2) If  $A_1 \approx B_1$ ,  $[\Delta: \Delta(A_1)] = 2$  and, for any simple submodule  $C$  in  $eJ$ ,  $A_1 \sim C$ , i.e., there exists a unit  $x$  in  $eRe$  such that  $x C \subset A_1$ .

II)  $eJ^2 \neq 0$ . 1)  $eJ = A_1 \oplus B_1$  with  $A_1$  uniserial and  $B_1$  simple or zero. 2)  $\Delta = \Delta(E)$  and 3)  $x E = A_i$  or  $x E = A_i \oplus B_1$ , where  $E$  is a submodule of  $eJ$ ,  $A_i$  is a submodule of  $A_1$  and  $x$  is a unit in  $eRe$ .

Proof. If  $(*, 2)$  and  $(**, 3)$  hold,  $|eJ/eJ^2| \leq 2$  by [5], Proposition 1, and

Condition I holds for any  $D(n)$  by Lemma 7. Hence  $eR$  has the structure in Theorem 1. If  $eJ^2=0$ , we are done. Assume that  $eJ^2\neq 0$ , and  $eJ=A_1\oplus B_1$  with  $A_1, B_1$  uniserial. Put  $A_i=A_1J^{i-1}$  and  $B_j=B_1J^{j-1}$ . If  $A_1\approx B_1$ ,  $[\Delta: \Delta(A_1)]=2$  by [3], Theorem 2. Since  $A_2\neq 0$  and hence  $B_2\neq 0$ ,  $eR/A_1\oplus eR/A_1\oplus eR/(A_2\oplus B_2)$  does not satisfy (\*, 3) from [4], Corollary 2. Hence  $A_1\not\approx B_1$ . If  $A_1\neq 0$  and  $B_2\neq 0$ , any two modules of  $\{A_1, A_2\oplus B_2, B_1\}$  are not related by  $\sim$ , which contradicts [5], Lemma 1 (note that  $\Delta=\Delta(E)$  and  $eJ=A_1\oplus B_1$ ). Hence  $B_1$  (or  $A_1$ ) is simple or zero. The remaining parts are clear from [3], Theorem 1. Conversely, if the case I) occurs, Condition I and (\*\*, 3) hold by [2], Theorem 12 and [3], Theorem 2 (note that  $\Delta=\Delta(A)$  provided  $A_1\approx B_1$ ). Assume the case II). Then (\*, 2) holds for any  $D(2)$  by Lemma 3. Further, since  $\Delta=\Delta(E)$ ,  $N_1\oplus N_2$  satisfies (\*\*, 2) provided  $N_i=eR/C_i$  and  $C_1\sim C_2$  by [4], Corollary 1. If  $\{E_i\}_{i=1}^3$  is a family of submodules in  $eJ$ , then  $E_{i_1}\sim E_{i_2}$  by the assumption 3). Hence (\*\*, 3) holds for any  $D(3)$ .

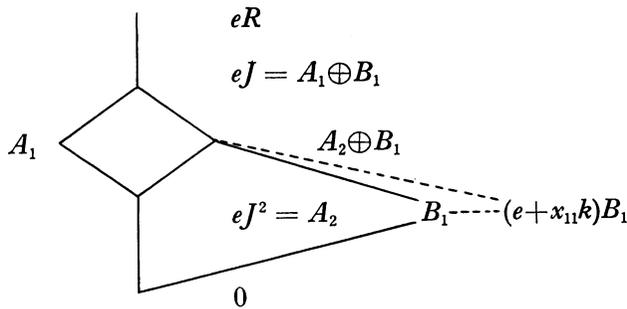
**Theorem 6.** *Let  $R$  be a right US-3 ring. Then (\*, 1) holds for any hollow module if and only if  $eR$  has one of the following structure for each primitive idempotent  $e$ :*

- 1)  $|eJ/eJ^2|\leq 1$ .
- 2)  $|eJ/eJ^2|=2$ 
  - i)  $eJ^2=0$
  - ii)  $eJ^2\neq 0$ ,  $eJ=A_1\oplus B_1$  has the structure as in Theorem 1, where  $A_1$  is uniserial and  $B_1$  is simple ( $A_1\approx B_2$ ).

Proof. Since  $R$  is a right US-3 ring,  $|eJ/eJ^2|\leq 2$  by [5], Theorem 2. Assume that (\*, 1) holds and  $|eJ/eJ^2|=2$ . Then  $eJ=A_1\oplus B_1$  by assumption, where  $A_1$  and  $B_1$  are hollow. If  $\bar{A}_1=A_1/A_1J\approx\bar{B}_1$ ,  $eJ^2$  is a waist and  $A_1\approx B_1$  by [5], Theorem 2. Hence, if  $eJ^2\neq 0$ ,  $A_1J\subseteq eJ^2$ . Then  $eR/A_1J$  contains a non-trivial waist  $eJ^2/A_1J$  and  $eJ/A_1J$  is not hollow. Accordingly,  $eJ/A_1J$  is not a direct sum of hollow modules. Therefore  $eJ^2=0$ . Next assume  $eJ^2\neq 0$ , and hence  $A_1\approx B_1$ . Then  $\Delta(A_1)=\Delta(B_1)=\Delta$  and  $A_1\sim B_1$ . From the proof of Theorem 5, we can show that either  $A_1$  or  $B_1$  is simple (note  $|eJ/eJ^2|=2$ ), say  $B_1$ . We shall show that  $A_1$  is uniserial. We know from the proof of [5], Theorem 2 that if  $\Delta(C)\neq\Delta$  for some submodule  $C$  of  $eJ$ , then  $eJ$  contains a non-trivial waist module  $eJ^i$  with  $|eJ^i/eJ^{i+1}|=2$ . Then (\*, 1) does not hold from the observation of the case  $eJ^2=0$ . Hence  $\Delta(C)=\Delta$  for all  $C$  in  $eR$ . Now  $J(A_1)=A_2\oplus A'_2\oplus A''_2\oplus\cdots$  from (\*, 1), where  $A_2, A'_2, \cdots$  are hollow (actually  $A'_2=\cdots=0$  from [5], Theorem 2). Being  $A_2\sim B_1$  and  $A'_2\sim B_1$ , we know that  $A_2\sim A'_2$ . Let  $a_2$  be in  $A_2-A_2J$ . Since  $\Delta(A_2)=\Delta$  and  $A_2\sim A'_2$ , there exist a unit  $x$  in  $eRe$  and  $j$  in  $eJe$  such that  $xA_2=A_2$  and  $(x+j)A_2=A'_2$ . Put  $a'_2=(x+j)a_2\in A'_2$ . Since  $x$  is an isomorphism of  $A_2$ ,  $xa_2\in A_2J$ ,  $ja_2\in eJeeJ\subset eJ^2=A_2J\oplus A'_2\oplus\cdots$ , which is a contradiction. Hence  $A_2=A'_2$ . Repeating this

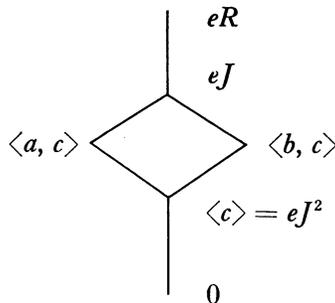
procedure, we know that  $A_1$  is uniserial. Therefore every submodule of  $eJ$  is one of the following: 1)  $A_i$ , 2)  $A_i \oplus B_1$ , and 3)  $A_i(f)$ , where  $A_i = A_1 J^{i-1}$  and  $A_i(f) = \{a_i + f(a_i) \mid a_i \in A_i, f \in \text{Hom}_R(A_i, B_1)\}$ . Assume  $A_n \neq 0$  and  $A_{n+1} = 0$ . Then considering  $\{A_i, A_i(f), B_1\}$  ( $i < n$ ),  $A_i \sim A_i(f)$ . It is clear from [5], Lemma 1 that  $A_n \sim A_n(f)$  or  $A_n(f) \sim B_1$  (if  $A_n \sim A_n(f)$ ,  $A_n = A_n(f)$  for  $eJ e A_n = 0$ ). Therefore  $eJ$  has the structure in Theorem 5. Conversely, assume that  $eR$  has the structure of the theorem. If  $|eJ/eJ^2| \leq 1$ ,  $eJ^2$  is a waist, and hence, for any submodule  $C \subseteq eJ^2$ ,  $J(eR/C) = eJ/C$  contains a unique maximal submodule  $eJ^2/C$ . If  $eJ^2 = 0$ , (\*, 2) holds for any two hollow modules by [3], Proposition 3. It is clear for the last case to show that (\*, 1) holds.

**4. Examples.** 1. Let  $R$  be the algebra over a field  $K$  given in [3], Example 2. Then the lattice of submodules of  $eR$  is the following:



where  $k$  are in  $K$ . Hence (\*\*, 3) and (\*, 2) are satisfied by Theorem 5.

2. Let  $R$  be a vector space over  $K$  with basis  $\{e, f, a, b, c, d\}$ . Define the multiplication among these elements as follows:  $e^2=e, f^2=f, ef=fe=0, ea=ae=a, eb=bf=b, ec=cf=c, fd=df=d, ab=bd=c$  and other products are equal to zero. Then the lattice of submodules of  $eR$  is the following:



Then  $R$  is a right US-3 ring with Condition II'. However,  $eJ$  is indecomposable, but not hollow. Hence (\*, 1) is not satisfied.

3. Let  $L, K$  be fields with  $[L: K]=2$ . Put

$$R = \begin{pmatrix} L & L & L \\ 0 & L & L \\ 0 & 0 & K \end{pmatrix}.$$

Then  $R$  is a right US-3 ring with  $(*, 1)$ , but without  $(*, 2)$  (note that  $\Delta(e_{13}K) = K \neq L = \Delta$ ).

4. Assume that a right artinian ring  $R$  has a decomposition  $R = eR \oplus fR$  and  $J^2 = 0$ , where  $\{e, f\}$  is a set of mutually orthogonal primitive idempotents. Then  $(*, 2)$  holds for any  $D(2)$  by [3], Proposition 3. We shall give the complete list of such rings with  $(**, 3)$  and Condition II. If  $R$  is the ring mentioned above,  $eJ = A_1 \oplus A_2$  and  $fJ = B_1 \oplus B_2$ , where the  $A_i$  and the  $B_i$  are simple or zero. We always assume, in the following observation, that

$$\alpha) \quad \begin{pmatrix} T_1 & A_1 \oplus A_2 \\ B_1 \oplus B_2 & T_2 \end{pmatrix}$$

means that  $T_1$  and  $T_2$  are local right artinian rings, the  $A_i$  (resp. the  $B_i$ ) are right  $T_2$  and left  $T_1$  (resp. right  $T_1$  and left  $T_2$ ) simple module,  $(A_1 \oplus A_2)J(T_2) = J(T_1)(A_1 \oplus A_2) = 0$  (the same for  $B_1 \oplus B_2$ ), and  $(A_1 \oplus A_2)(B_1 \oplus B_2) = (B_1 \oplus B_2)(A_1 \oplus A_2) = 0$ .

$\beta)$   $\Delta$  means a division ring.

$\gamma)$   $S$  means a local serial ring.

$\delta)$   $L$  means the following local ring:

$$J(L) = A_1 \oplus A_2, A_1 \approx A_2 \text{ as right } L\text{-modules,}$$

$\xi)$   $[L/J(L): L/J(L)(A_1)] = 2$ , and for any simple  $L$ -module  $A'_1$  in  $J(L)$ , there exists a unit  $\alpha$  in  $L$  such that  $A'_1 = \alpha A_1$  (see [1] for such a ring).

i)  $A_1 \approx A_2 \approx e\bar{R}$  and  $B_1 \approx B_2 \approx f\bar{R}$ . Then  $eJf = fJe = 0$ . Hence

$$R = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$

ii)  $A_1 \approx A_2 \approx f\bar{R}$ ,  $B_1 \approx B_2 \approx e\bar{R}$ . Then

$$R = \begin{pmatrix} \Delta_1 & A_1 \oplus A_2 \\ B_1 \oplus B_2 & \Delta_2 \end{pmatrix},$$

where the  $A_i$  (resp.  $B_i$ ) satisfy  $\xi$ ) as  $\Delta_1 - \Delta_2$  (resp.  $\Delta_2 - \Delta_1$ ) bimodules.

iii)  $A_1 \approx A_2 \approx B_1 \approx B_2 \approx e\bar{R}$  (resp.  $\approx f\bar{R}$ ). Then

$$R = \begin{pmatrix} L_1 & 0 \\ B_1 \oplus B_2 & \Delta_2 \end{pmatrix} \quad (\text{resp. } R = \begin{pmatrix} \Delta_1 & A_1 \oplus A_2 \\ 0 & L_2 \end{pmatrix}),$$

where the  $B_i$  (resp.  $A_i$ ) satisfy  $\xi$ ) as  $\Delta_2 - L_1/J(L_1)$  (resp.  $\Delta_1 - L_2/J(L_2)$ ) bimodules.

iv)  $A_1 \approx A_2 \approx B_1 \approx e\bar{R}$  and  $B_2 \approx f\bar{R}$ .

Then

$$R = \begin{pmatrix} \Delta_1 & A_1 \oplus A_2 \\ B_2 & S_2 \end{pmatrix},$$

where the  $A_i$  are similar to iii).

v)  $A_1 \approx A_2$  and  $B_1 \approx B_2$ . Then

$$R = \begin{pmatrix} S_1 & A_2 \\ B_2 & S_2 \end{pmatrix}.$$

vi) Other cases. We may put  $A_i=0$  or  $B_i=0$  in the above. The right serial rings appear in v) by setting  $S_2=\Delta_2$  or  $S_i=\Delta_i$  and  $B_2=0$  (or  $A_2=0$ ).

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